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Quantum Gravity with Purely Virtual Particles from Asymptotically Local Quantum Field Theory

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Abstract

We investigate the relationship between nonlocal and local quantum field theories, and search for a viable notion of "local limit" to relate the unitary models. In Euclidean space it is relatively easy to have nonlocal theories with well-behaved local limits. In Minkowski spacetime, instead, singular behaviors are generically expected. Relaxing some assumptions on the "form factors" considered in the literature, we identify a class of models that have regular local limits in Minkowski spacetime. We call the models "asymptotically local" quantum field theories (AL-QFTs) and show that their limits are theories with physical and purely virtual particles (PVPs). In the bubble diagram, the nonlocal deformation generates PVPs straightforwardly. In the triangle diagram, it does so possibly up to multi-threshold corrections, which may be adjusted by tuning the deformation itself. We also build an asymptotically local deformation of quantum gravity with purely virtual particles. AL-QFT can serve various purposes, such as suggesting innovative approaches to off-shell physics, providing an alternative formulation for theories with PVPs, or smoothing out nonanalytic behaviors. We discuss its inherent arbitrariness and the implications for renormalizability.

1 Introduction

Locality is a guiding principle of quantum field theory (QFT). At the same time, quantum gravity (QG) challenges us to reconsider the principles that worked successfully so far for the standard model of particle physics. Adjusting the concept of locality is a possibility that must be taken into consideration by physicists aiming to classify the viable theoretical options.

In this paper we consider various possibilities in this regard. The main goal is to work out a practically viable notion of "local limit" to relate local and nonlocal theories with physically desirable properties, such as renormalizability and unitarity. Besides deepening our knowledge on quantum fields, this investigation may have interesting physical applications. Nonlocal deformations of the standard model, for example, may be suitable to describe off-shell physics. To our knowledge, the relationship between nonlocal and local theories has not been investigated, so far, to the extent we aim to explore in this paper.

Various classes of nonlocal theories have been studied in the literature, starting from Efimov [1]. Krasnikov [2] had the idea to remove the ghosts of local theories from the spectrum by means of nonlocal deformations. Specifically, the unphysical poles of the free propagators are replaced by certain form factors (entire functions with no zeros) to ensure unitarity [3, 4].

If the form factors are generic, they originate nonrenormalizable gauge and gravity interactions. Concentrating on gravity, Kuz'min [5] showed that renormalizability can be obtained by choosing entire functions that tend to polynomials at high energies. The parent local and the deformed nonlocal theories are super-renormalizable. Precisely, the propagators and vertices of the latter tend sufficiently fast to those of a higher-derivative (HD) theory in the ultraviolet limit [6].

Later, Tomboulis [7] revived this idea and applied it to gauge theories. More recently, Modesto and others [8] elaborated it to a greater extent and built a variety of finite models. Properties and implications of more general form factors were studied in ref.s [9].

The class of super-renormalizable nonlocal theories just mentioned provides an interesting arena for testing candidate notions of local limit. Naively, we may expect that the limit is just the higher-derivative local theory (HD-QFT) that describes the ultraviolet behavior. It turns out that the relationship between locality and nonlocality is more subtle than that.

Let $\mathcal{M}_{\mathrm{NL}}$ and $\mathcal{M}_{\mathrm{L}}$ denote the manifolds of nonlocal and local unitary theories (NL-QFTs and L-QFTs, from now on), respectively. We look for families of models in $\mathcal{M}_{\mathrm{NL}}$, parametrized by some variable $\lambda > 0$, that become local when $\lambda \to \infty$. This way, the space $\mathcal{M}_{\rm L}$ may be understood as a subspace of the boundary $\partial\mathcal{M}_{\rm NL}$.

It is relatively easy to provide a meaningful notion of local limit in Euclidean space. It is more difficult to extend the concept to Minkowski spacetime. For example, in a class of treatable theories inspired by the ones of the literature mentioned above, we find that the limit generates severe singularities inside the correlation functions, such as integrals $\simeq \int \mathrm{d}^4p/|p^2 - m^2|$. We are unable, at this very moment, to say whether the nonlocal models of refs. [5, 7, 8] admit proper local limits in Minkowski spacetime or not.

This leads us to search for new, *ad hoc* nonlocal models that do have meaningful local limits in Minkowski spacetime when λ tends to infinity. We find that the form factors attached to the "propagators of nonpropagating fields" – those which eliminate the unwanted poles – must have some zeros. Apart from that, they are still entire functions. This generalization does not do too much harm, since the associated fields can be integrated out with no loss. The Lagrangian is singular, strictly speaking, but the action is regular.

Another direction to pursue to shed light on the matter is the relation between the local limit $\lambda \rightarrow \infty$ and the continuation E $\rightarrow$ M from Euclidean space (E) to Minkowski spacetime (M). Do these operations commute? If we take the local limit in E and then continue from E to M, do we find the local limit in M?

The continuation E $\rightarrow$ M is not unique. In $\mathcal{M}_{\rm L}$, for example, the analytic continuation of higher-derivative theories gives ghosts (fields with kinetic terms multiplied by the wrong sign), while the "average continuation" [10] gives theories with purely virtual particles (PVPs), i.e., particles that are always off shell. By unitarity, the ghosts are unreachable from $\mathcal{M}_{\rm NL}$, so the continuation E $\rightarrow$ M, which is analytic for $\lambda < \infty$, cannot stay analytic when $\lambda \rightarrow \infty$. The possibility that the target local limits contain PVPs is compatible with this.

Summarizing, in this paper we build "asymptotically local" quantum field theories (AL-QFTs) – models that have regular local limits in Minkowski spacetime. Such limits contain purely virtual particles in addition to physical particles (PVP-QFTs).

We recall two additional methods of introducing purely virtual particles, besides the one already mentioned: one uses diagrammatic spectral optical identities [11] and the other one is based on non-time ordered correlation functions of a special type [12]. Generically speaking, PVPs are defined by tweaking the diagrammatics so as to eradicate the ghosts, while remaining in the realm of local theories. That is to say, the PVP and Krasnikov approaches share the same goals, in local and nonlocal settings, respectively. We can also view AL-QFT as a fourth formulation of theories with PVPs.

We also mention that, taking advantage of the PVP concept, it is possible to build a local theory of quantum gravity [13] that is unitary and renormalizable at the same time. Its main prediction is a constrained window ($4/10000 \lesssim r \lesssim 3/1000$) for the value of the tensor-to-scalar ratio r [14] of primordial fluctuations, which essentially confirms the prediction of the Starobinsky $R + R^2$ model [15] ($r \simeq 3/1000$) within less than an order of magnitude. Other predictions can be derived as well (the running of scalar and tensor tilts, higher-order corrections, etc. [16]), but require more effort to be tested, in the realm of current or planned observations [17]. Among the other things, in this paper we provide a nonlocal deformation of quantum gravity with PVPs, which returns the theory of [13] in the local limit.

The local limit is straightforward at the tree level, where most of its intriguing aspects are not visible. We push the investigation to the simplest nontrivial diagrams, that it to say, the bubble and the triangle. We show that in the case of the bubble diagram the nonlocal deformation "already knows", so to speak, how to generate PVPs, with no need of *ad hoc* adjustments. In the cases of triangle and more complicated diagrams, the local limit is "PVP ready" in the null and single-threshold sectors. The multi-threshold sectors, instead, are more difficult to handle. Possibly, the arbitrariness of the deformation must be invoked to fine-tune the limit appropriately.

The weakness of nonlocal quantum field theory is that it lacks a fundamental principle to select the nonpolynomial functions it is built upon, remove its inherent arbitrariness and identify a unique candidate theory of nature. However, we do not propose AL-QFT as a framework for fundamental theories. We just claim that AL-QFT is interesting *per se*, broadens our knowledge of QFT, and provides useful tools in support of local QFT. For example, as an alternative formulation of PVPs it may allow us to address peculiar aspects of these "non particles" (such as the violation of microcausality [18] and the so called "peak uncertainty" [19]).

Another instance where a comparable degree of arbitrariness plagues (local) QFT (without jeopardizing it as a framework for fundamental interactions) is off-the-mass shell physics [20]. There, the extra parameters describe the environment where the phenomenon under observation is placed. AL-QFT may provide an alternative way to describe these aspects. In this respect, the lack of uniqueness of AL-QFT is not an issue.

Another way to settle the non uniqueness of AL-QFT is as follows. We have remarked that nonlocal theories lack a selection principle for the form factors. The local limit might serve as one. Then, the candidate theories to describe nature from AL-QFT are nothing but their local limits[1].

The paper is organized as follows.

1. In section 2 we recall the basic features of the nonlocal theories considered in the literature and explain how we are going to generalize them to have well-defined local limits in Minkowski spacetime.

2. In section 3 we study the local limit in Euclidean space, which is relatively straightforward.

3. In section 4 we describe the local limit in Minkowski spacetime, including explicit calculations for the bubble and triangle diagrams. We compare the behaviors of the nonlocal theories of the liturature with the properties of the new models built in section 2.

4. In section 5 we formulate an asymptotically local deformation of quantum gravity with purely virtual particles along the lines of section 2.

Section 6 contains the conclusions. In appendix A we build the key functions that are needed for the form factors used in the paper. In appendix B we recall the calculation of the bubble diagram with PVPs. When necessary, the dimensional regularization [21] is used for explicit calculations, where $D = 4 - \varepsilon$ denotes the continued dimension around dimension 4.

2 The nonlocal theories we consider here

We want to study the local limits of nonlocal theories. There is no guarantee that such a limit exists, in general, so we may have to construct *ad hoc* models that admit one.

In this section we lay out the basic features of the classes of theories under consideration. We include those mostly studied in the current literature, as well as extensions that have well-behaved local limits.

We first work in Euclidean space and later switch to Minkowski spacetime by means of the analytic continuation. The models we focus on have Euclidean Lagrangians of the form

$$\mathcal{L}_{\mathrm{NL}} = \frac{\tau}{2}\phi(-p)Q(P(p))\phi(p) + \mathcal{O}(\phi^3), \tag{2.1}$$

[1]Strictly speaking, they are not contained in the manifold $\mathcal{M}_{\mathrm{AL}}$ of asymptotically local theories, but in its border $\partial\mathcal{M}_{\mathrm{AL}}$.

in momentum space, where ϕ denotes the fields, $Q(P)$ is a certain function of a polynomial P of the Euclidean momentum p, and $\tau = \pm 1$ is there so that we can describe nonlocal deformations of both physical particles and ghosts.

We take

$$P(p) = p^2 + m^2 \tag{2.2}$$

where m is the "mass" in the local limit. Polynomials of higher-derivative theories, such as

$$P_{\mathrm{HD}}(p) = (p^2 + m^2)^2 + M^4, \tag{2.3}$$

are also interesting, but will not be treated in detail here.

For a while, we concentrate on the propagator and study the local limit there. We assume that the vertices are local or their limits do not invalidate the derivations we make. We discuss the vertices in detail in subsection 5.4.

In this context, the key quantity that encodes the nonlocal model is the function $Q(P)$. If we want a degree of freedom from ϕ, we know how to obtain it: $Q = P$. The challenge is to *not* have a degree of freedom from ϕ. That is to say, ϕ should be purely virtual[2]. To achieve this, we require that $\tau/Q(P)$ is entire in the complex P plane, so that $Q(P)$ has no zeros. In addition, we require that $Q(P)$ tends to P in the local limit defined below. This makes ϕ a natural candidate to become an ordinary PVP in that limit.

The main difference with respect to the assumptions commonly adopted in the literature [5, 7, 8] is that the propagator $\tau/Q(P)$ is not required to never vanish in the complex P plane. Specifically, we allow $Q(P(p))$ to have a singularity proportional to $(-p_{\mathrm{M}}^2 + M^2)^{-1}$, where p_{M} is the Minkowskian momentum and M is some mass. The Minkowskian action is then defined by means of the Cauchy principal value.

The Lagrangian (2.1) can describe the ϕ subsector of a more general theory, which may contain ordinary physical particles φ as well, as described by the extension

$$\mathcal{L}'_{\mathrm{NL}} = \frac{\tau}{2}\phi(-p)Q(P(p))\phi(p) + \frac{1}{2}\varphi(-p)(p^2 + m^2)\varphi(p) + \mathcal{L}'_{\mathrm{int}}, \tag{2.4}$$

where $\mathcal{L}'_{\mathrm{int}}$ collects the interactions. We focus our attention on ϕ here, since the propagators of the physical particles φ do not need to be nonlocally deformed.

2.1 Local limit

To begin with, we define the local limit as follows: we rescale P and Q by λ and λ^{-1}, respectively, where λ is a positive factor, and then let λ tend to infinity. We assume that

[2] We could call it "nonlocal purely virtual particle" (NL-PVP).

Q tends to P on the real axis:

$$\lim_{\lambda \to +\infty} \lambda^{-1} Q(\lambda P) = P, \qquad P \in \mathbb{R}. \tag{2.5}$$

The functions Q we consider in this paper are

$$Q(P) = h(P), \qquad Q(P) = \frac{h^2(P)}{P}, \tag{2.6}$$

where $h(P)$ is the entire function defined in formula (A.1), taken from the current literature on nonlocal theories [5, 7, 8].

In appendix A we show that $h(P)$ tends to the absolute value, $P/h(P)$ tends to the sign function and $P/h^2(P)$ tends to the Cauchy principal value in the local limit. The second option of (2.6) is the one that allows us to build the asymptotically local models.

With choices like (2.6), the property (2.5) extends to a double cone $\mathcal{C}$ around the real axis (see appendix A).

2.2 Propagator

The propagator of the Euclidean theories (2.1) is

$$G_{\mathrm{nl}}(p) = \frac{\tau}{Q(P(p))}, \tag{2.7}$$

where the polynomial $P(p)$ is given by (2.2), the function Q is given in (2.6) and h given by (A.1). In the local limit, both choices (2.6) give τ/P in Euclidean space. As we are going to show, the two options lead to crucially different results in Minkowski spacetime.

Associated with $G_{\mathrm{nl}}(p)$, we have a double cone $\mathcal{C}$ where the expansion (A.2) applies.

2.3 Loop integrals

Now we study general properties of the Feynman diagrams of nonlocal theories. Let p, p_i denote the internal momenta and k, k_a the external ones. The loop integral associated with a diagram F is the integral of a product of propagators $G_{\mathrm{nl}}(p_i)$ times a product of functions $V(p_i, k_a)$ originated by the vertices. The latter are also entire functions, as we explain in subsection 5.4, so we concentrate on the propagators.

For example, the bubble diagram with circulating ϕ fields has the form

$$\mathcal{B}(k) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} V_1(p,k) V_2(p,k) G_{\mathrm{nl}}(p) G_{\mathrm{nl}}(p-k). \tag{2.8}$$

The integrals must be defined in Euclidean space. This means that both p_i and k_a are Euclidean momenta. Only the external momenta k_a are later analytically continued to Minkowski spacetime. We denote their Minkowski versions by k_{M}.

It is easy to show that the integrals are convergent for Euclidean k_a, in the sense of the dimensional regularization. Consider, for example, the expression (2.8) for $\mathcal{B}(k)$. For Euclidean p and k, the arguments $P(p)$ and $P(p-k)$ are located inside the cones $\mathcal{C}$ and $\mathcal{C}'$ associated with the propagators $G_{\mathrm{nl}}(p)$ and $G_{\mathrm{nl}}(p-k)$. Formula (A.2) then ensures that (2.8) tends to the same integral with $G_{\mathrm{nl}}(p) \rightarrow P(p)$ and $G_{\mathrm{nl}}(p-k) \rightarrow P(p-k)$, which is indeed convergent in the sense of the dimensional regularization (assuming that the vertices do not invalidate the argument, see 5.4). What this means is that there exists an open set Ω of the complex D plane where the integral is convergent in the standard sense, or that the integral can be split into a finite sum of integrals that admit convergence domains Ω_i in the standard sense [22].

Now we prove that the loop integrals are convergent, again in the sense of the dimensional regularization, for every *complex* external momenta k_a. Actually, they are entire functions of k_a, so it is possible to replace the external momenta k with their Minkowski versions k_{M} directly inside the integrals.

It is important to stress that the integrated momenta p_i remain Euclidean till the very end. The reason is that it is not convenient to make a Wick rotation on them. When we close integration paths by including arcs at infinity, we cross regions where the functions $G_{\mathrm{nl}}(p)$ and $G_{\mathrm{nl}}(p-k)$ behave in ways that are hard to control.

Consider (2.8) again. When k is deformed to complex values, the cone $\mathcal{C}'$, which is equal to $\mathcal{C}$ translated by k, is no longer centered along the Euclidean domain, but somewhere else, depending on the imaginary part of k:

$$P(p-k) = (p-k)^2 + m^2 = \rho \mathrm{e}^{i\theta}, \qquad \theta = \arctan \frac{\mathrm{Im}[k_4(k_4 - 2p_4)]}{\mathrm{Re}[(p-k)^2 + m^2]}.$$

The phase θ of $P(p-k)$ tends to zero when the integrated momentum p tends to infinity, in any direction. This implies that the argument of $G_{\mathrm{nl}}(p-k)$ falls off fast enough inside $\mathcal{C}'$ for sufficiently large p, which makes the integral (2.9) convergent.

Together with the fact that the integrand is regular everywhere (both the vertices and the propagators being entire functions, see section 5.4), this argument proves that the function $\mathcal{B}(k)$ is entire. Then we can perform the analytic continuation E $\rightarrow$ M by replacing $k = (k_4, \mathbf{k})$ with its Minkowskian version $k_{\mathrm{M}} = (-ik^0, \mathbf{k})$ directly inside. The Minkowskian bubble diagram is thus

$$\mathcal{B}_{\mathrm{M}}(k_{\mathrm{M}}) = i \int \frac{\mathrm{d}^D p}{(2\pi)^D} V_1(p, k_{\mathrm{M}}) V_2(p, k_{\mathrm{M}}) G_{\mathrm{nl}}(p) G_{\mathrm{nl}}(p - k_{\mathrm{M}}), \tag{2.9}$$

the factor i being due to the fact that p remains Euclidean.

The property just proved extends to more complex loop integrals, which also define entire functions of k_a. Note that it does not apply, on the contrary, to local quantum field theory. There, the entire functions G_{nl} are replaced by the reciprocals of polynomials. The integrands have poles, which can cross the integration path when k is deformed to complex values. Direct replacements $k \to k_{\text{M}}$ may jump *over* the integration paths, and give incorrect results.

When the theory contains physical particles, besides NL-PVPs, as in the example (2.4), one proceeds as usual in the physical sector, i.e., by Wick rotating the external momenta and keeping the internal ones Euclidean [3]. This procedure is also safe in loops that contain both physical particles and NL-PVPs.

To conclude this section, the analytic continuation E→M gives a well-defined map

$$\mathcal{M}_{\text{NL}}^{\text{E}} \quad \overset{\text{E}\rightarrow\text{M}}{\longrightarrow} \quad \mathcal{M}_{\text{NL}}^{\text{M}} \tag{2.10}$$

between the nonlocal Euclidean theories and their Minkowskian versions. It is sufficient to replace the external momenta k with their Minkowski versions k_{M} inside the loop integrals (or Wick rotate them), and adjust the overall factor.

3 Local limit in Euclidean space

In this section we study the local limit in Euclidean space, which does not pose significant challenges.

At the tree level, the assumption (2.5) ensures that the limit of the Lagrangian (2.1) is the Lagrangian of a local theory:

$$\mathcal{L}_{\text{NL}}(\lambda) \equiv \frac{\tau}{2}\phi(-p)\lambda^{-1}Q(\lambda P(p))\phi(p) + \mathcal{O}(\phi^3) \underset{\lambda\rightarrow+\infty}{\rightarrow} \mathcal{L}_{\text{loc}} = \frac{\tau}{2}\phi(-p)P(p)\phi(p) + \mathcal{O}(\phi^3)\big|_{\text{loc}}. \tag{3.1}$$

We assume that the vertices tend to local limits smoothly enough to not affect our arguments. The theories we have in mind satisfy this assumption, as we demonstrate in subsection 5.4.

The propagator tends to the one of the local theory in both cases (2.6):

$$\lim_{\lambda\rightarrow+\infty} \frac{\tau\lambda}{Q(\lambda P(p))} = \frac{\tau}{P(p)}. \tag{3.2}$$

It is also straightforward to show that the correlation functions of the nonlocal theory tend to those of the local theory (3.1). Indeed, the loop integrals just involve the Euclidean domain. There, the values of the rescaled polynomial P are always real and larger than a positive number, $\lambda P(0) = \lambda m^2$, so the expansion (A.2) can be used to prove the statement.

Take for example (2.8), with the Green functions (2.7). Its local limit is

$$\lim_{\lambda\rightarrow+\infty} \lambda^2 \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V_1(p,k)V_2(p,k)}{Q(\lambda P(p))Q(\lambda P(p-k))}. \tag{3.3}$$

Since k is Euclidean, both $P(p)$ and $P(p-k)$ have Euclidean arguments.

Assume that the integral

$$\mathcal{B}_{\mathrm{L}}(k) \equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V_1(p,k)V_2(p,k)}{P(p)P(p-k)} \tag{3.4}$$

is convergent in the physical dimension $D = 4$. The bounds (A.6) and (A.11) allow us to apply the dominated convergence theorem and conclude that the limit (3.3) gives $\mathcal{B}_{\mathrm{L}}(k)$.

If (3.4) is not convergent in $D = 4$, we use the dimensional regularization technique. This means that we continue the spacetime dimension to complex values D, move to a domain Ω where $\mathcal{B}_{\mathrm{L}}(k)$ is convergent[3], take the limit $\lambda \rightarrow \infty$ there and then analytically continue the result to $D = 4$.

This proves that the local limit is well defined in E. In other words, we have a map

$$\mathcal{M}_{\mathrm{NL}}^{\mathrm{E}} \xrightarrow{\mathrm{loc}} \mathcal{M}_{\mathrm{L}}^{\mathrm{E}} \tag{3.5}$$

between nonlocal theories and local theories (3.1) in Euclidean space.

4 Local limit in Minkowski spacetime

The next task is to investigate the local limit of the Minkowskian correlation functions. In this section we

1. describe the problems we face;

2. show that, on general grounds, if it exists, the limit cannot contain ghosts, while it can contain purely virtual particles;

3. study the limit explicitly in correlation functions, concentrating on the bubble and triangle diagrams.

The outcome is that the local limit is singular with the left option of (2.6), while it gives a local theory with PVPs with the right option (with some caveats).

[3]If $\mathcal{B}_{\mathrm{L}}(k)$ does not admit a convergence domain Ω, it can be split into a finite sum of terms $\mathcal{B}_{\mathrm{L}}^{(i)}(k)$ that separately admit convergence domains Ω_i. See [22]. Then the argument can be applied to each $\mathcal{B}_{\mathrm{L}}^{(i)}(k)$ separately.

4.1 The problem

We first note that it is not sufficient to compose the map (3.5) with the continuation E $\rightarrow$ M, because the second step admits a plurality of outcomes:

$$
\mathcal{M}_{\mathrm{NL}}^{\mathrm{E}} \overset{\mathrm{loc}}{\longrightarrow} \mathcal{M}_{\mathrm{L}}^{\mathrm{E}} \overset{\mathrm{E}\rightarrow\mathrm{M}}{\longrightarrow} \begin{array}{cc} \nearrow & \mathcal{M}_{\mathrm{L}}^{\text{M-1}} \\ \longrightarrow & \mathcal{M}_{\mathrm{L}}^{\text{M-2}} \\ \searrow & \mathcal{M}_{\mathrm{L}}^{\text{M-3}}\cdots \end{array} \tag{4.1}
$$

The analytic continuation E $\rightarrow$ M is excluded by unitarity, as we show below in detail. Among the other options, a special place is reserved to the "average continuation" [10], which defines purely virtual particles. There is a worse possibility, though: that the local limit in Minkowski spacetime does not even exist. In particular, we cannot expect that the limit of (2.9) is just (3.4) with $k \rightarrow k_{\mathrm{M}}$, which is ill defined.

Ultimately, the task of identifying the right limit $\mathcal{M}_{\mathrm{L}}^{\mathrm{M}}$ of $\mathcal{M}_{\mathrm{NL}}^{\mathrm{M}}$ amounts to building the diagram

$$
\begin{array}{ccc} \mathcal{M}_{\mathrm{NL}}^{\mathrm{E}} & \overset{\mathrm{loc_E}}{\longrightarrow} & \mathcal{M}_{\mathrm{L}}^{\mathrm{E}} \\ \mathrm{E}\downarrow\mathrm{M} & & \mathrm{E}\downarrow\mathrm{M} \\ \mathcal{M}_{\mathrm{NL}}^{\mathrm{M}} & \overset{\mathrm{loc_M}}{\longrightarrow} & \mathcal{M}_{\mathrm{L}}^{\mathrm{M}} \end{array} \tag{4.2}
$$

by combining (2.10) and (3.5) with further maps that close the bottom right. The correct $\mathcal{M}_{\mathrm{L}}^{\mathrm{M}}$ must follow from the compatibility between the local limit and the continuation E $\rightarrow$ M.

4.2 No-ghost, pro-PVP theorem

Now we show that $\mathcal{M}_{\mathrm{L}}^{\mathrm{M}}$ cannot contain models with ghosts, while it can contain theories with purely virtual particles.

Assume for the moment that the vertices V_1 and V_2 are identically one. Making a reflection $p_4 \rightarrow -p_4$ in (2.9) we infer that $B_{\mathrm{M}}(k_{\mathrm{M}}) = B_{\mathrm{M}}(k_{\mathrm{M}}^*)$. Because of the property (A.5) and the overall factor i, the conjugate $B_{\mathrm{M}}^*(k_{\mathrm{M}})$ coincides with $-B_{\mathrm{M}}(k_{\mathrm{M}}^*)$. Hence, $B_{\mathrm{M}}(k_{\mathrm{M}}) = -[B_{\mathrm{M}}(k_{\mathrm{M}})]^*$. Given that the real part of $B_{\mathrm{M}}(k_{\mathrm{M}})$ vanishes identically, its local limit must vanish as well, if it exists.

Thus, the local limit of $B_{\mathrm{M}}(k_{\mathrm{M}})$ is purely imaginary. If it contained propagating particles or ghosts, its real part would be nontrivial. On the contrary, it is allowed to contain PVPs, because they give a purely imaginary bubble diagram (check appendix B).

If V_1 and V_2 are nontrivial, by unitarity they must be Hermitian in Minkowski spacetime. The argument just outlined extends straightforwardly when they are polynomial.

More generally, the vertices may involve incremental ratios of the entire functions $1/Q$, as explained in subsection 5.4. Yet, the operations described above apply to those cases as well, and lead to the same conclusion.

A more general version of the argument is based on the optical theorem, which reads $-iT + iT^\dagger = TT^\dagger$, where $S = 1 + iT$ is the S matrix and iT collects the loop diagrams in matrix form. Consider the "empty" theory (2.1). We call it this way, because it does not propagate any degree of freedom, differently from (2.4). This means $TT^\dagger = 0$, hence the matrix T is Hermitian: all the loop diagrams are purely imaginary.

Since Re$[iT]$= 0 for arbitrary finite λ, the local limit $\lambda \to \infty$ can only return a local theory with Re$[iT]$= 0, that is to say, still an empty theory. No physical particles or ghosts can be generated by the limit. Purely virtual particles are allowed, precisely because they satisfy Re$[iT]$= 0.

Coming back to the bubble diagram, we separate the tentative (λ independent) local limit from the rest by writing

$$\rangle\!\bigcirc\!\langle_{\text{nloc}} = \rangle\!\bigcirc\!\langle_{\text{loc}} + \rangle\!\bigcirc\!\langle_{\text{rest}} . \tag{4.3}$$

The rest is supposed to tend to zero when λ tends to infinity.

We know that the real part of the left-hand side vanishes, so

$$0 = \mathrm{Re}\left[\rangle\!\bigcirc\!\langle_{\text{nloc}}\right] = \mathrm{Re}\left[\rangle\!\bigcirc\!\langle_{\text{loc}}\right] + \mathrm{Re}\left[\rangle\!\bigcirc\!\langle_{\text{rest}}\right] . \tag{4.4}$$

Assume, ad absurdum, that the local limit adds degrees of freedom. Then the optical theorem $-iT + iT^\dagger = TT^\dagger = -2\mathrm{Re}[iT]$, applied to the limit itself, tells us that $TT^\dagger$ is nonzero. For example, in the case of the ordinary bubble diagram with circulating physical particles, formula (B.1) gives

$$\mathrm{Re}\left[\rangle\!\bigcirc\!\langle_{\text{loc}}\right] = -\int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\pi}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left[\delta(k^0 + \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{p}}) + \delta(k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}})\right] .$$

Since the first term on the right hand side of (4.4) is nonvanishing and λ independent,

$$\mathrm{Re}\left[\rangle\!\bigcirc\!\langle_{\text{loc}}\right] \neq 0, \qquad \frac{\partial}{\partial\lambda}\mathrm{Re}\left[\rangle\!\bigcirc\!\langle_{\text{loc}}\right] = 0,$$

the rest is not negligible in the limit $\lambda \to +\infty$, which invalidates the assumption.

Assuming, instead, that the local limit gives a theory of PVPs,

$$\rangle\!\bigcirc\!\langle_{\text{nloc}} = \rangle\!\bigcirc\!\langle_{\text{PVP}} + \rangle\!\bigcirc\!\langle_{\text{rest}} , \tag{4.5}$$

and recalling that the bubble diagram (B.2) with circulating PVPs is purely imaginary, the real part of the rest vanishes,

$$0 = \mathrm{Re}\,[\rangle\!\bigcirc\!\langle_{\mathrm{nloc}}] = \mathrm{Re}\,[\rangle\!\bigcirc\!\langle_{\mathrm{rest}}]\,,$$

which causes no problem when λ tends to ∞.

Taking the imaginary part of (4.5), we find

$$\mathrm{Im}\,[\rangle\!\bigcirc\!\langle_{\mathrm{nloc}}] = \mathrm{Im}\,[\rangle\!\bigcirc\!\langle_{\mathrm{PVP}}] + \mathrm{Im}\,[\rangle\!\bigcirc\!\langle_{\mathrm{rest}}]\,.$$

The left-hand side is nonzero and λ dependent. The first term on the right-hand side is nonzero and λ independent. Hence the rest can be nonzero, λ dependent and tend to zero when $\lambda \to \infty$ with no contradiction.

We stress once again that these results are predicated on the assumption that the local limit exists. It turns out that it does exist with the second choice of (2.6), but it does not with the first choice. In the rest of this section we show these facts in detail.

4.3 Local limit: tree diagrams

The Euclidean and Minkowskian actions S_{E} and S_{M} are related by $S_{\mathrm{E}} = -iS_{\mathrm{M}}$ (since $\mathrm{e}^{-S_{\mathrm{E}}} \to \mathrm{e}^{iS_{\mathrm{M}}}$ inside the functional integral). We know that the integrated coordinates and momenta remain Euclidean. Thus, the propagator $G_{\mathrm{nl}}^{\mathrm{M}}(p_{\mathrm{M}})$ of the nonlocal Minkowski theory is (2.7) times $-i$:

$$G_{\mathrm{nl}}^{\mathrm{M}}(p_{\mathrm{M}}) = -\frac{i\tau}{Q(P(p_{\mathrm{M}}))}, \tag{4.6}$$

where $p_{\mathrm{M}} = (-ip^0, \mathbf{p})$.

At the tree level, formulas (A.3) and (A.12) give

$$\lim_{\lambda\to+\infty} -\frac{i\tau\lambda}{Q(\lambda P(p_{\mathrm{M}}))} = \begin{cases} -\frac{i\tau}{|P(p_{\mathrm{M}})|} & \text{for } Q(P) = h(P), \\ -\mathcal{P}\frac{i\tau}{P(p_{\mathrm{M}})} & \text{for } Q(P) = \frac{h^2(P)}{P}, \end{cases} \tag{4.7}$$

where $\mathcal{P}$ denotes the Cauchy principal value. The top result illustrates the problem with the left choice of (2.6): since we are assuming that the polynomial P is the one of (2.2), the $\lambda \to +\infty$ limit does not give an acceptable propagator for a local theory. The second choice for (2.6), on the other hand, gives the answer we expect for a PVP.

A polynomial like (2.3) changes the picture. However, we do not have further results to report on this here.

4.4 Local limit: bubble diagram

Now we study the local limit of (2.9) in detail. We work with the second option of (2.6) for the function Q and later comment on the problems of the first option.

As recalled in appendix B, PVPs are defined by a thoroughlly new diagrammatics – i.e., not just the usual diagrams with propagators replaced by the principal value appearing in (4.7). Because of this, before concluding that the local limit with the second choice of (2.6) gives a theory with PVPs, we must prove that the loop diagrams somehow "know what to do" by themselves. That is to say, for some mysterious reason they are already equipped with the instructions to implement the diagrammatic rules of [10, 11, 12], instead of those that lead to (B.3). Luckily, this turns out to be true[4].

It is convenient to split the calculation of the bubble diagram in two steps:

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = i \lim_{\substack{\lambda\rightarrow+\infty \\ \lambda'\rightarrow+\infty}} \lambda\lambda' \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{Q(\lambda' P(p))Q(\lambda P(p-k_{\mathrm{M}}))},$$

where V denotes the vertices. As before, we focus on the propagators and assume that the vertices behave sufficiently well in the limit, so as not to invalidate the arguments. We recall that the polynomial P is the one of (2.2) and $k_{\mathrm{M}} = (-ik^0, \mathbf{k})$.

Since the loop momentum p is Euclidean, the limit $\lambda' \rightarrow +\infty$ can be evaluated immediately by means of (A.3), (A.12) or (3.2). We obtain

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = i \lim_{\lambda\rightarrow+\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{p^2+m^2} \frac{\lambda}{Q(\lambda P(p-k_{\mathrm{M}}))}. \tag{4.8}$$

If $k^0 = 0$ the Euclidean and Minkowskian loop integrals coincide, so we can take the second limit right away, which gives the expected result.

If $k^0 \neq 0$, we can restrict to the case $k^0 > 0$, by symmetry. Then the second limit cannot be evaluated directly, since we do not have an easy control on the behavior of $Q(\lambda P(p-k_{\mathrm{M}}))$ away from the cone $\mathcal{C}$.

We briefly outline the strategy of the calculation. The function $Q(\lambda P(p-k_{\mathrm{M}}))$ is centered in a region translated by k_{M}. We would like to re-center it on the Euclidean domain by means of an opposite translation. The translation in question is complex, so it cannot be expressed as a mere change of variables in the integral.

We overcome the difficulty by adding and subtracting integration paths. Consider the p_4 integral. Its integration domain is the line C (see fig. 1 – left side). We can rewrite the integral on C as the sum of the integral on the closed curve γ, plus the integral on

[4]Besides, we already know that they cannot give (B.3), because unitarity rules out local limits with ghosts.

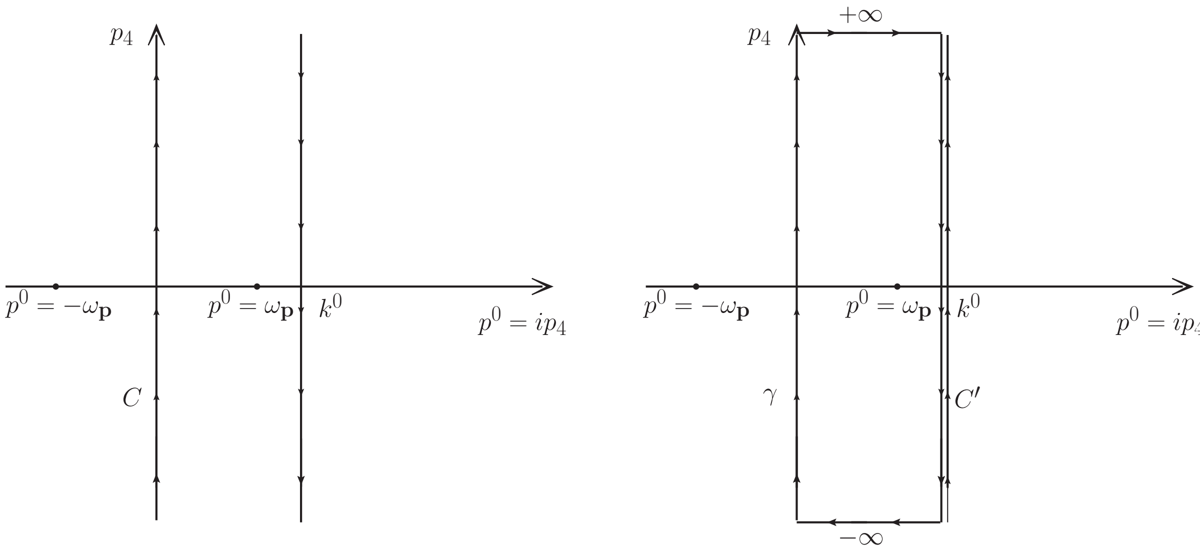

Figure 1: Dealing with the second limit (assuming $k^0 > 0$)

the line C' (fig. 1 – right side). The segments at infinity do not contribute, since they are located inside the cone $\mathcal{C}'$ where $\lambda^{-1}Q(\lambda P(p - k_{\mathrm{M}}))$ converges to $P(p - k_{\mathrm{M}})$, as follows from formula (A.7)[5].

The integrand of (4.8) has poles at $p^2 + m^2 = 0$ in the complex p_4 plane. One of them may fall inside the curve γ. We need to distinguish the case where this happens from the opposite case. The integral on C', instead, is centered on the Euclidean region, so we can take the limit $\lambda \to +\infty$ straightforwardly on it by means of (A.12).

Defining $p^0 = ip_4$, the poles of $p^2 + m^2 = 0$ are located at $p^0 = \pm\omega_{\mathbf{p}}$, where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, so $p^0 = \omega_{\mathbf{p}}$ is inside γ if $k^0 > \omega_{\mathbf{p}}$. No pole is inside γ if $k^0 < \omega_{\mathbf{p}}$. We can write

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = (a) + (b),$$

where

$$(a) = i \lim_{\lambda \to +\infty} \int_{\gamma} \frac{\mathrm{d}p_4}{2\pi} \int_{\omega_{\mathbf{p}} \leqslant k^0} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{p^2 + m^2} \frac{\lambda}{Q(\lambda P(p - k_{\mathrm{M}}))}, \tag{4.9}$$

$$(b) = i \lim_{\lambda \to +\infty} \int_{C'} \frac{\mathrm{d}p_4}{2\pi} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{p^2 + m^2} \frac{\lambda}{Q(\lambda P(p - k_{\mathrm{M}}))}. \tag{4.10}$$

Let us begin by evaluating (a) and (b) in the simplified case $\mathbf{k} = m = 0$. The residue

[5] Recall that we are using the dimensional regularization. This means that when an integral is ultraviolet divergent, we split it into a sum of integrals that admit convergence domains in the complex plane of the dimension D [22]. Then the arguments are applied to each of them separately.

theorem gives[6]

$$(a) = i \lim_{\lambda\to+\infty} \int_{|\mathbf{p}|\leqslant k^0} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{\lambda}{Q(\lambda k^0(2|\mathbf{p}|-k^0))}. \tag{4.11}$$

Since the argument of Q is real, we can take the limit $\lambda \to +\infty$ by means of formula (A.12). We obtain

$$(a) = -i\mathcal{P} \int_{|\mathbf{p}|\leqslant k^0} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0-2|\mathbf{p}|)}. \tag{4.12}$$

The integral (b) is centered on the Euclidean domain. It is convenient to make this fact apparent by relabeling the integration variable p_4 (which is not a change of variables) and writing

$$(b) = i \lim_{\lambda\to+\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p+k_{\mathrm{M}})^2} \frac{\lambda}{Q(\lambda P(p))}. \tag{4.13}$$

The limit $\lambda \to +\infty$ is now straightforward and gives

$$(b) = i \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p+k_{\mathrm{M}})^2} \frac{1}{p^2}. \tag{4.14}$$

Note that two more poles are created by the limit. Moreover,(4.14) does not coincide with the loop integral of the usual bubble diagram, because when we apply the residue theorem to the p_4 integral, we end up including different poles.

Closing the path on the right side, we encircle two poles ($p^0 = |\mathbf{p}| - k^0$ and $p^0 = |\mathbf{p}|$) for $k^0 < |\mathbf{p}|$ and one ($p^0 = |\mathbf{p}|$) for $k^0 > |\mathbf{p}|$. The result is

$$(b) = -i \int_{k^0<|\mathbf{p}|} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0-2|\mathbf{p}|)} - i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0+2|\mathbf{p}|)}. \tag{4.15}$$

This expression does not need particular prescriptions, since the integrand is never singular.

Summing to (a), we find the total

$$\begin{aligned} \mathcal{B}^{\mathrm{loc}}_{\mathrm{M}}(k_{\mathrm{M}}) &= -i\mathcal{P} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0-2|\mathbf{p}|)} - i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0+2|\mathbf{p}|)} \\ &= i\mathcal{P} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{4|\mathbf{p}|^2} \left(\frac{1}{k^0+2|\mathbf{p}|} - \frac{1}{k^0-2|\mathbf{p}|} \right), \end{aligned} \tag{4.16}$$

which is precisely the result obtained in the case of PVPs (apart from the factor V), as given in formula (B.2).

[6] Note that the integral is in $p_4 = -ip^0$, not p^0, so there is an extra factor $-i$.

We have obtained the result (4.16) by adopting the second option of (2.6). Let us see what happens, instead, when we adopt the first option. Everything proceeds as above in (b), because the arguments of Q are positive. The difference is visible in (a), where we obtain

$$(a)=i\int_{|\mathbf{p}|\leqslant k^0}\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{V}{2|\mathbf{p}|}\frac{1}{k^0|k^0-2|\mathbf{p}||}, \tag{4.17}$$

instead of (4.12), which is clearly singular. Thus, the local limit is not well defined with the first option of (2.6) in Minkowski spacetime.

For completeness, let us treat the general case $\mathbf{k}\neq 0$, $m\neq 0$ with the second option (2.6) for Q. The residue theorem gives (4.9) again, but now on the pole $p^0=\omega_{\mathbf{p}}$ we have

$$(p-k_{\mathrm{M}})^2+m^2=-(k^0-\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{k}})(k^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}). \tag{4.18}$$

What is important is that the argument of Q remains real, so we can still use formula (A.12) to evaluate the limit $\lambda\rightarrow+\infty$. The result is

$$(a)=-i\mathcal{P}\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{V\theta(k^0-\omega_{\mathbf{p}})}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left(\frac{1}{k^0-\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{k}}}-\frac{1}{k^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}}\right). \tag{4.19}$$

The singularity due to the first term inside the parentheses is regulated by the principal value inherited from (A.12). Instead, the second term is nonsingular for $\omega_{\mathbf{p}}\leqslant k^0$.

Centering the (b) integral as in (4.13), taking λ to infinity and translating $\mathbf{p}$ back to $\mathbf{p}-\mathbf{k}$, we obtain

$$(b)=i\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{V}{(p^0+k^0-\omega_{\mathbf{p}})(p^0+k^0+\omega_{\mathbf{p}})}\frac{1}{(p^0-\omega_{\mathbf{p}-\mathbf{k}})(p^0+\omega_{\mathbf{p}-\mathbf{k}})}. \tag{4.20}$$

Repeating the argument above, we close the path on the right side, thereby picking two poles ($p^0=\omega_{\mathbf{p}}-k^0$ and $p^0=\omega_{\mathbf{p}-\mathbf{k}}$) for $k^0<\omega_{\mathbf{p}}$ and one ($p^0=\omega_{\mathbf{p}-\mathbf{k}}$) for $k^0>\omega_{\mathbf{p}}$. The result is

$$(b)=-i\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{V}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left(\frac{\theta(\omega_{\mathbf{p}}-k^0)}{k^0-\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{k}}}+\frac{\theta(k^0-\omega_{\mathbf{p}})}{k^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}}-\frac{1}{k^0+\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}}\right).$$

Again, the integrand is never singular.

Summing the outcome to the (a) of (4.19), we finally get

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}})=-i\mathcal{P}\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{V}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left(\frac{1}{k^0-\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{k}}}-\frac{1}{k^0+\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}}\right),$$

which is the same as in the case of PVPs, formula (B.2).

If one particle circulating in the loop is physical, like φ in (2.4), and the other one is ϕ with the second option of (2.6) for Q, the result does not change. The simplest parametrization of the circulating momenta gives (4.8) directly, whence the rest follows as before. If we start from the parametrization

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = i \lim_{\lambda\rightarrow+\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p+k_{\mathrm{M}})^2+m^2} \frac{\lambda}{Q(\lambda P(p))}, \tag{4.21}$$

we have to note that the integration on p_4 must include complex values in order to leave the right pole on one side and the left pole on the other side. Then the argument of Q is not everywhere real, so the limit $\lambda \rightarrow +\infty$ cannot be taken directly. On the other hand, one can easily see that (4.21) is equivalent to (4.8), because the difference is a closed path that encircles no pole.

4.5 Local limit: triangle diagram

Now we study the loop integral

$$\mathcal{T}_{\mathrm{M}}(k_{\mathrm{M}}, q_{\mathrm{M}}) = i \int \frac{\mathrm{d}^D p}{(2\pi)^D} V(p, k_{\mathrm{M}}, q_{\mathrm{M}}) G_{\mathrm{nl}}(p) G_{\mathrm{nl}}(p-k_{\mathrm{M}}) G_{\mathrm{nl}}(p-q_{\mathrm{M}}) \tag{4.22}$$

of the triangle diagram with the second option of (2.6) for Q. After possibly a translation of the internal momentum p, and assuming that the external momenta are generic, we can arrange the expression so that, say, $q^0 > k^0 > 0$.

We introduce the parameter λ and split the local limit $\lambda \rightarrow +\infty$ in three steps. First, we take the limit inside $G_{\mathrm{nl}}(p)$, then in $G_{\mathrm{nl}}(p-k_{\mathrm{M}})$ and finally in $G_{\mathrm{nl}}(p-q_{\mathrm{M}})$. The calculation will tell us to what extent it is legitimate to do so.

As before, the first limit is straightforward, since $G_{\mathrm{nl}}(p)$ does not depend on k_{M} and q_{M}. We remain with

$$\mathcal{T}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}, q_{\mathrm{M}}) = i \lim_{\lambda'\rightarrow+\infty} \lim_{\lambda\rightarrow+\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{p^2+m^2} \frac{\lambda'}{Q(\lambda' P(p-k_{\mathrm{M}}))} \frac{\lambda}{Q(\lambda P(p-q_{\mathrm{M}}))}.$$

Now we use the residue theorem to integrate the loop energy $p^4 = -ip^0$ along the closed curve γ made of the lines $p^0 = 0$ and $p^0 = k^0$, plus segments at infinity. On the pole $p^0 = \omega_{\mathbf{p}}$, which contributes for $\omega_{\mathbf{p}} < k^0$, the arguments of the functions Q are real, so we can use formula (A.12). We then obtain the first contribution, which is

$$(a) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\theta(k^0-\omega_{\mathbf{p}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}} \mathcal{Q}^{12}\mathcal{Q}^{13},$$

where

$$\mathcal{Q}^{ab} = \mathcal{P}^{ab} - \mathcal{P}\frac{1}{e_a - e_b - \omega_a + \omega_b}, \qquad \mathcal{P}^{ab} = \mathcal{P}\frac{1}{e_a - e_b - \omega_a - \omega_b},$$

and the subscripts $a, b, \ldots$ range over the values 1, 2 and 3, while $\{e_a\} = \{0, -k^0, -q^0\}$, $\{\omega_a\} = \{\omega_{\mathbf{p}}, \omega_{\mathbf{p}-\mathbf{k}}, \omega_{\mathbf{p}-\mathbf{q}}\}$.

We are left with the integral on the line $p^0 = k^0$, which we center on $p^0 = 0$ by means of a relabelling of the loop energy. After that, the limit $\lambda' \to +\infty$ acts on $\lambda'/Q(\lambda' P(p))$ and becomes straightforward. We remain with

$$(b) = i \lim_{\lambda \to +\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p^2 + m^2)((p + k_{\mathrm{M}})^2 + m^2)} \frac{\lambda}{Q(\lambda P(p + k_{\mathrm{M}} - q_{\mathrm{M}}))}. \tag{4.23}$$

At this point, we write $(b) = (b_1) + (b_2)$, where (b_1) is the integral on the closed curve γ' made by the lines $p^0 = 0$ and $p^0 = q^0 - k^0$, plus segments at infinity, and (b_2) is the integral on the line $p^0 = q^0 - k^0$.

Using the residue theorem once more on (b_1), we obtain an integral on $\mathbf{p}$ that we do not report here. We just remark that, before taking the limit $\lambda \to \infty$, its integrand is regular. Specifically, the residues at the poles mutually compensate for every $\lambda < \infty$. This means that we can adopt the prescription we want for those poles. We choose the principal value. Then, we take the limit $\lambda \to \infty$ by means of (A.12), noting that the arguments of Q are real on the poles. We find that the contribution of γ' is

$$(b_1) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\theta(q^0 - \omega_{\mathbf{p}})\theta(\omega_{\mathbf{p}} - k^0)\mathcal{Q}^{12}\mathcal{Q}^{13} + \theta(q^0 - k^0 - \omega_{\mathbf{p}-\mathbf{k}})\mathcal{Q}^{23}\mathcal{Q}^{21}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}.$$

Centering the integral (b_2) on $p^0 = 0$ by means of a further relabelling of the loop energy, we obtain

$$(b_2) = i \lim_{\lambda \to +\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{((p + q_{\mathrm{M}})^2 + m^2)((p - k_{\mathrm{M}} + q_{\mathrm{M}})^2 + m^2)} \frac{\lambda}{Q(\lambda P(p))}. \tag{4.24}$$

Now the limit $\lambda \to +\infty$ acts on $\lambda/Q(\lambda P(p))$. Nevertheless, we cannot evaluate it directly by means of (A.12), because if we do so, we find an unprescribed expression:

$$(b_2) = i \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p^2 + m^2)((p - k_{\mathrm{M}} + q_{\mathrm{M}})^2 + m^2)((p + q_{\mathrm{M}})^2 + m^2)}. \tag{4.25}$$

Recall that p is Euclidean, so this formula does not correspond (for, e.g., $V = 1$) to the triangle diagram of local theories, due to the different sets of contributing poles. Using the residue theorem, the result of (4.25) is

$$(b_2) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\theta(\omega_{\mathbf{p}} - q^0)\hat{\mathcal{Q}}^{12}\hat{\mathcal{Q}}^{13} + \theta(k^0 - q^0 + \omega_{\mathbf{p}-\mathbf{k}})\hat{\mathcal{Q}}^{23}\hat{\mathcal{Q}}^{21} + \hat{\mathcal{Q}}^{32}\hat{\mathcal{Q}}^{31}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}, \tag{4.26}$$

where $\hat{\mathcal{Q}}$ is the "unprescribed $\mathcal{Q}$", that is to say, the same as $\mathcal{Q}$ without the principal-value prescription.

Among the other things, the integrand of (4.26) involves an expression like

$$\frac{1}{xy}-\frac{1}{x(x+y)}-\frac{1}{y(x+y)}, \tag{4.27}$$

with $x=\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{q}}-q^0$ and $y=k^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}$. We focus on the region around $x=y=0$, where the θ functions appearing in some numerators leading to (4.27) are equal to one. Since the sum (4.27) is unprescribed, we do not know whether it is zero or not. For example, it is zero if we slightly move x and y to complex values, but it is nonzero if we take the principal value, due to the identity [11]

$$\mathcal{P}\left(\frac{1}{xy}-\frac{1}{x(x+y)}-\frac{1}{y(x+y)}\right)=-\pi^2\delta(x)\delta(y). \tag{4.28}$$

Note that the double delta function on the right-hand side does not appear in Feynman diagrams (check [11] for details).

The problem just outlined indicates that we have jumped to (4.25) too quickly, since the integral of (4.24) is well defined before taking the limit. To avoid the trouble, we can evaluate (4.24) as follows. First, we move the external energies slightly away from the real axis: $k^0\rightarrow k^0+i\epsilon$, $q^0\rightarrow q^0+i\epsilon'$, with ϵ and ϵ' real (not necessarily positive) and small. Then we use (A.12) to evaluate the limit $\lambda\rightarrow+\infty$. When we apply the residue theorem, we find expressions like (4.27) with $x\rightarrow x-i\epsilon'$ and $y\rightarrow y+i\epsilon$, which do vanish.

Using the principal value and subtracting the right-hand side of (4.28), we can write

$$\begin{aligned}(b_2)=&-i\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}V\frac{\theta(\omega_{\mathbf{p}}-q^0)\mathcal{Q}^{12}\mathcal{Q}^{13}+\theta(k^0-q^0+\omega_{\mathbf{p}-\mathbf{k}})\mathcal{Q}^{23}\mathcal{Q}^{21}+\mathcal{Q}^{32}\mathcal{Q}^{31}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}\\&+i\pi^2\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}V\frac{\delta(q^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{q}})\delta(k^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}.\end{aligned}$$

The total gives

$$\begin{aligned}\mathcal{T}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}},q_{\mathrm{M}})=(a)+(b)=&-i\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}V\frac{\mathcal{Q}^{12}\mathcal{Q}^{13}+\mathcal{Q}^{23}\mathcal{Q}^{21}+\mathcal{Q}^{32}\mathcal{Q}^{31}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}\\&+i\pi^2\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}V\frac{\delta(q^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{q}})\delta(k^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}.\end{aligned}$$

Rearranging it by means of (4.28), we obtain

$$\begin{aligned}\mathcal{T}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}},q_{\mathrm{M}})=&-i\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}V\frac{\mathcal{P}^{12}\mathcal{P}^{13}+\mathcal{P}^{21}\mathcal{P}^{31}+\mathrm{cycl}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}\\&+i\pi^2\int\frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}V\frac{\delta(q^0-\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{q}})\delta(q^0-k^0-\omega_{\mathbf{p}-\mathbf{k}}-\omega_{\mathbf{p}-\mathbf{q}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}.\end{aligned} \tag{4.29}$$

The expected result for a triangle with circulating PVPs is just the first line [11]. The second line is not correct.

The original integral (4.22) is symmetric (up to V) under exchanges of the three energies and the reflection $e \to -e$, but the second line of (4.29) is not. Indeed, the symmetric expression

$$\sum_{a\neq b\neq c\neq a} \delta(e_a - e_b - \omega_a - \omega_b)\delta(e_a - e_c - \omega_a - \omega_c) + (e \to -e),$$

specialized to our case, gives

$$\delta(q^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{q}})\delta(q^0 - k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}-\mathbf{q}}) + \delta(k^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{k}})\delta(q^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{q}}), \quad (4.30)$$

but the second term is missing in (4.29). This means that the calculation fails in the multi-threshold sector, which is the one made by the double deltas. Splitting the limit $\lambda \to \infty$ into three distinct limits is only accurate up to those terms.

The first line of (4.29) is enough to show that there are no propagating degrees of freedom. Indeed, the outcome is purely imaginary. In particular, the single-delta contributions (those which contribute to the optical theorem and highlight the degrees of freedom propagating on-shell inside the diagram) are completely missing.

One may object that corrections proportional to (4.30) are also not acceptable, because they are on-shell. Currently, we lack computational tools that are powerful enough to determine whether they are actually present or not. At any rate, we can explain how we should proceed if they were. Basically, we would be forced to advocate the inherent arbitrariness of the nonlocal deformation to compensate for the extra contributions. The Lagrangian that gives the correct local limit, including the multi-threshold interaction sector, would have to be adjusted along the way by including suitable nonlocal, finite counter-vertices.

5 Asymptotically local quantum gravity

In this section we explain how to deform the (local) theory of quantum gravity with purely virtual particles (PVP-QG) [13] into a unitary, nonlocal theory that tends to it in the local limit. We call the latter "asymptotically local quantum gravity" (AL-QG). We work in Minkowski spacetime.

5.1 PVP-QG

The PVP-QG theory coupled to matter is described by the higher-derivative action [23]

$$S_{\rm QG}(g,\Phi)=-\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(2\Lambda+R-\frac{R^2}{6m_\phi^2}+\frac{\eta}{2m_\chi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)+S_m(g,\Phi), \quad (5.1)$$

where m_ϕ is the mass of the inflaton ϕ (introduced below), m_χ is the mass of the spin-2 massive mode $\chi_{\mu\nu}$ (due to the square of the Weyl tensor $C_{\mu\nu\rho\sigma}$), which must be treated as a PVP, Φ denotes the matter fields, with action $S_m(g,\Phi)$, and $\eta=m_\chi^2(3m_\phi^2+4\Lambda)/(m_\phi^2(3m_\chi^2-2\Lambda))$ is a parameter very close to one. The simplest option is to take S_m equal to the covariantized action of the standard model, equipped with the nonminimal couplings allowed by power counting. The resulting theory is renormalizable[7] and unitary.

The crucial point is the treatment of $\chi_{\mu\nu}$ as a PVP. To clarify how this works it is convenient to introduce ϕ and $\chi_{\mu\nu}$ explicitly as extra fields by eliminating the higher derivatives. The result is [23]

$$S_{\rm QG}(\tilde g,\phi,\chi,\Phi)=\tilde S_{\rm HE}(\tilde g)+S_\phi(\tilde g+\psi,\phi)+S_\chi(\tilde g,\chi)+S_m(\tilde g\mathrm{e}^{\kappa_\phi\phi}+\psi\mathrm{e}^{\kappa_\phi\phi},\Phi). \quad (5.2)$$

The relation between the old metric $g_{\mu\nu}$ and the new metric $\tilde g_{\mu\nu}$ reads

$$g_{\mu\nu}=(\tilde g_{\mu\nu}+\psi_{\mu\nu})\mathrm{e}^{\kappa_\phi\phi},\qquad \psi_{\mu\nu}\equiv 2\kappa_\chi\chi_{\mu\nu}+\kappa_\chi^2\left(\chi_{\mu\nu}\chi_{\rho\sigma}\tilde g^{\rho\sigma}-2\chi_{\mu\rho}\chi_{\nu\sigma}\tilde g^{\rho\sigma}\right), \quad (5.3)$$

and the constants are

$$\kappa_\phi=\frac{m_\phi\sqrt{16\pi G}}{\sqrt{4\Lambda+3m_\phi^2}},\qquad \kappa_\chi=\sqrt{8\pi\tilde G},\qquad \tilde G=\frac{G}{\eta}.$$

Moreover,

$$\tilde S_{\rm HE}(g)=-\frac{1}{16\pi\tilde G}\int \mathrm{d}^4x\sqrt{-g}\,(2\Lambda+R)\,,$$

is the Einstein-Hilbert action with the redefined Newton constant,

$$S_\phi(g,\phi)=\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[D_\mu\phi D^\mu\phi-\frac{m_\phi^2}{\kappa_\phi^2}\left(1-\mathrm{e}^{\kappa_\phi\phi}\right)^2\right], \quad (5.4)$$

is the inflaton action, and

$$S_\chi(\tilde g,\chi)=\tilde S_{\rm HE}(\tilde g+\psi)-\tilde S_{\rm HE}(\tilde g)+\int \mathrm{d}^4x\left[\frac{m_\chi^2}{2}\sqrt{-g}(\chi_{\mu\nu}\chi^{\mu\nu}-\chi^2)-2\kappa_\chi\chi_{\mu\nu}\frac{\delta\tilde S_{\rm HE}(g)}{\delta g_{\mu\nu}}\right]_{g\rightarrow\tilde g+\psi} \quad (5.5)$$

[7] Renormalization works exactly as in the Stelle theory [24], where the spin-2 massive field is quantized in a conventional way and propagates a ghost.

is the $\chi_{\mu\nu}$ action. Specifically, one finds

$$S_{\chi}(g,\chi)=-S_{\mathrm{PF}}(g,\chi,m_{\chi}^{2})+S_{\chi}^{(>2)}(g,\chi), \tag{5.6}$$

where

$$\begin{aligned}S_{\mathrm{PF}}(g,\chi,m_{\chi}^{2})=&\frac{1}{2}\int\mathrm{d}^{4}x\sqrt{-g}\,[D_{\rho}\chi_{\mu\nu}D^{\rho}\chi^{\mu\nu}-D_{\rho}\chi D^{\rho}\chi+2D_{\mu}\chi^{\mu\nu}D_{\nu}\chi-2D_{\mu}\chi^{\rho\nu}D_{\rho}\chi_{\nu}^{\mu}\\&-m_{\chi}^{2}(\chi_{\mu\nu}\chi^{\mu\nu}-\chi^{2})+R^{\mu\nu}(\chi\chi_{\mu\nu}-2\chi_{\mu\rho}\chi_{\nu}^{\rho})]\end{aligned} \tag{5.7}$$

is the covariantized Pauli-Fierz action with a nonminimal term, and $S_{\chi}^{(>2)}(g,\chi)$ are corrections at least cubic in χ.

The theory (5.2) is renormalizable, but not manifestly. This means that, when we calculate its Feynman diagrams, "miraculous" cancellations make it possible to subtract the divergences by means of field redefinitions and renormalizations of the parameters already contained in (5.2).

The crucial problem is the minus sign in front of S_{PF}. If $\chi_{\mu\nu}$ is treated conventionally, it propagates a ghost, and unitarity is violated. For analogous reasons, the theory (5.1)-(5.2) is not acceptable as a classical theory.

The situation changes radically when $\chi_{\mu\nu}$ is understood as a PVP. The field $\chi_{\mu\nu}$ is projected away by integrating it out according to the diagrammatic rules of PVPs, briefly recalled in the appendix. No $\chi_{\mu\nu}$ external legs are considered, and the modified diagrams guarantee that $\chi_{\mu\nu}$ does not give on-shell contributions to the radiative corrections.

Expanding around the flat-space metric $\eta_{\mu\nu}=$diag$(1,-1,-1,-1)$, the $\chi_{\mu\nu}$ propagator of (5.2) is

$$\langle\chi_{\mu\nu}(p)\,\chi_{\rho\sigma}(-p)\rangle_{0}=-\left.\frac{i}{p^{2}-m_{\chi}^{2}}\right|_{\mathrm{PVP}}\left(\frac{\pi_{\mu\rho}\pi_{\nu\sigma}+\pi_{\mu\sigma}\pi_{\nu\rho}}{2}-\frac{1}{3}\pi_{\mu\nu}\pi_{\rho\sigma}\right),\qquad\pi_{\mu\nu}\equiv\eta_{\mu\nu}-\frac{p_{\mu}p_{\nu}}{m_{\chi}^{2}}, \tag{5.8}$$

where the subscript "PVP" is there to remind us that $\chi_{\mu\nu}$ must be treated as a PVP inside diagrams.

The projection applies at the classical level as well. The true, classical theory is neither (5.1) nor (5.2). It is obtained by collecting the tree diagrams with no $\chi_{\mu\nu}$ external legs and only $\chi_{\mu\nu}$ internal legs [18]. The result is

$$S_{\mathrm{cl}}(g,\Phi)=-\frac{1}{16\pi G}\int\mathrm{d}^{4}x\sqrt{-g}\left(2\Lambda+R-\frac{R^{2}}{6m_{\phi}^{2}}\right)+S_{m}(g,\Phi)+\Delta S_{\mathrm{nl}}(g,\Phi),$$

where ΔS_{nl} collects nonlocal vertices that are negligible at energies lower than m_{χ}.

The prices to pay to have renormalizability and unitarity at the same time in quantum gravity are the impossibility to distinguish past, present and future at distances smaller, or intervals shorter, than $1/m_\chi$, as well as a certain "peak uncertainty": in the processes where the PVP is supposed to be "detected", significant complications arise due to the inherent impossibility of its detection [19].

5.2 Kinetic Lagrangians of Proca and Pauli-Fierz AL-QFTs

The residue of the propagator (5.8) at $p^2 = m_\chi^2$ has the wrong sign. If treated conventionally, it gives a ghost. We know that the solution, in the realm of local quantum field theory, is to treat it as a PVP. An alternative option is to alter the propagator (5.8) by means of entire functions, so as to eliminate the zero in the denominator.

The simplest deformation amounts to turning (5.8) into

$$\langle \chi_{\mu\nu}(p)\,\chi_{\rho\sigma}(-p)\rangle_0 = \frac{i}{Q(-p^2+m_\chi^2)}\left(\frac{\tilde{\pi}_{\mu\rho}\tilde{\pi}_{\nu\sigma}+\tilde{\pi}_{\mu\sigma}\tilde{\pi}_{\nu\rho}}{2}-\frac{1}{3}\tilde{\pi}_{\mu\nu}\tilde{\pi}_{\rho\sigma}\right), \tag{5.9}$$

where Q is the second option of formula (2.6) and

$$\tilde{\pi}_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m_\chi^2}\sigma^2(-p^2+m_\chi^2).$$

As anticipated, the new "propagator" (5.9) has no pole, hence it does not actually propagate degrees of freedom. We can view it as the propagator of a NL-PVP. In the local limit, the function σ^2 tends to one, so (5.9) tends to (5.8) (at the tree level), by formula (A.12).

The choice (5.9) corresponds to the nonlocal kinetic Lagrangian

$$\begin{aligned} S'_{\mathrm{PF}} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,\big[&\chi_{\mu\nu}Q(D_\rho D^\rho + m_\chi^2)\chi^{\mu\nu} - \chi Q(D_\rho D^\rho + m_\chi^2)\chi \\ &+2\chi^{\mu\nu}D_\mu\tilde{Q}(D_\rho D^\rho + m_\chi^2)D_\nu\chi - 2\chi^{\sigma\nu}D_\mu\tilde{Q}(D_\rho D^\rho + m_\chi^2)D_\sigma\chi_\nu^\mu \\ &- R^{\mu\nu}(\chi\chi_{\mu\nu} - 2\chi_{\mu\rho}\chi_\nu^\rho)\big], \end{aligned} \tag{5.10}$$

where

$$\tilde{Q}(x) = \frac{x}{m_\chi^2 + (x-m_\chi^2)\sigma^2(x)}.$$

It is not necessary, at this stage, to modify the nonminimal coupling (last line).

The lagrangian of S'_{PF} is singular for $x=0$ and $(m_\chi^2 - x)\sigma^2(x) = m_\chi^2$, where $x = m_\chi^2 - p^2$. The singularities are simple poles in x, and can be prescribed by means of the Cauchy principal value. This keeps S'_{PF} convergent and real.

For reasons similar to those explained in the case of PVPs, the action S'_{PF} is not the true classical action, but a sort of "interim" action. The field $\chi_{\mu\nu}$ must be integrated out, so the singularity of the Lagrangian is harmless. What is important is that the propagator (5.9) is regular. Below we show that the vertices are regular as well.

If we use $\pi_{\mu\nu}$ in (5.9), instead of $\tilde{\pi}_{\mu\nu}$, we have (5.10) with $\tilde{Q} \rightarrow \sigma^{-2}$. Then the Lagrangian has singularities $\sim 1/x^2$, which are more severe, to the extent that the action S'_{PF} becomes also singular. Again, what is important is that the propagator and the vertices are regular.

For reference, let us consider the action

$$S_{\text{Proca}} = \frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\rho}g^{\nu\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) - 2m^2 g^{\mu\nu}A_\mu A_\nu\right]$$

of a PVP Proca vector A_μ. Using the same notation as above with $m_\chi \rightarrow m$, the propagator

$$\langle A_\mu(p)\, A_\nu(-p)\rangle_0 = \left.\frac{i\pi_{\mu\nu}}{p^2 - m^2}\right|_{\text{PVP}}$$

can be deformed into

$$\langle A_\mu(p)\, A_\nu(-p)\rangle_0 = -\frac{i}{Q(-p^2 + m^2)}\tilde{\pi}_{\mu\nu},$$

which is derived from the modified action

$$S'_{\text{Proca}} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\nu}A_\mu Q(D_\rho D^\rho + m^2)A_\nu + A_\mu D^\nu \tilde{Q}(D_\rho D^\rho + m^2)D^\mu A_\nu\right].$$

5.3 AL-QG

Summarizing, asymptotically local quantum gravity is the theory described by the action

$$S_{\text{QG}}(\tilde{g}, \phi, \chi, \Phi) = \tilde{S}_{\text{HE}}(\tilde{g}) + S'_\chi(\tilde{g}, \chi) + S_\phi(\tilde{g} + \psi, \phi) + S_m(\tilde{g}\mathrm{e}^{\kappa_\phi\phi} + \psi\mathrm{e}^{\kappa_\phi\phi}, \Phi), \qquad (5.11)$$

where

$$S_\chi(g, \chi) = -S'_{\text{PF}}(g, \chi, m_\chi^2) + S_\chi^{(>2)}(g, \chi). \qquad (5.12)$$

It is obtained from (5.2) by replacing the Pauli-Fierz $\chi_{\mu\nu}$ action with (5.10).

In fact, (5.11) is just the starting action, because, as we have shown in the previous section, the multi-threshold sectors of involved diagrams may need to be adjusted along the way, in order to reach the correct local limit. Moreover, the deformed theory is not guaranteed to be renormalizable, so further adjustments may be required. We discuss this issue below.

The field $\chi_{\mu\nu}$ does not need a special prescription in AL-QG, since the nonlocal deformation (5.9) of the propagator is self-sufficient. Modulo the adjustments just mentioned, the local limit of (5.12) is the theory of quantum gravity with purely virtual particles [13], that is to say, (5.1) or (5.2), with $\chi_{\mu\nu}$ treated as a PVP.

5.4 Vertices

Now we show that the vertices obtained by expanding (5.12) around flat space are entire functions (see [25] and [6]). Write

$$D_\mu D^\mu + m^2 = \Box + V_\mu \partial^\mu + W,$$

where $\Box$ is the flat-space D'Alembertian. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ denote a generic entire function. We assume that $f(D_\mu D^\mu + m^2)$ belongs to an expression where the derivatives can be integrated by parts. Expanding, the first order in V and W is

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n-1} \Box^k (V_\mu \partial^\mu + W) \Box^{n-k-1} &= (V_\mu \partial^\mu + W) \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n-1} \overleftarrow{\Box}^k \Box^{n-k-1} \\ &= (V_\mu \partial^\mu + W) \sum_{n=0}^{\infty} a_n \frac{\overleftarrow{\Box}^n - \Box^n}{\overleftarrow{\Box} - \Box} = (V_\mu \partial^\mu + W) f(\overleftarrow{\Box}, \Box),\end{aligned} \tag{5.13}$$

where

$$f(x, y) \equiv \frac{f(x) - f(y)}{x - y} \tag{5.14}$$

denotes the incremental ratio of the function f, and the arrow on $\overleftarrow{\Box}$ means that the box acts to the very left, beyond $V_\mu \partial^\mu + W$. In momentum space, the ratio is calculated with respect to the right and left momenta. Formula (5.14) shows that if $f(x)$ is entire, $f(x, y)$ is also entire, that is to say, the incremental ratio of an entire function is an entire function (of two variables): no singularity appears. Moreover, with $f = 1/Q$ the ratio converges to

$$\frac{Q^{-1}(-p^2 + m_\chi^2) - Q^{-1}(-q^2 + m_\chi^2)}{-p^2 + q^2} \to \mathcal{P} \frac{\frac{1}{p^2 - m_\chi^2} - \frac{1}{q^2 - m_\chi^2}}{p^2 - q^2} = -\mathcal{P} \frac{1}{p^2 - m_\chi^2} \frac{1}{q^2 - m_\chi^2} \tag{5.15}$$

in the cone $\mathcal{C}$ fast enough to validate the arguments of the previous sections. Note that (5.15) is the correct result for PVPs at the tree level[8].

To the second order in some operator δ, we find, with $r = \Box$,

$$f(r + \delta) = \sum_{n=0}^{\infty} a_n (r + \delta)^n \to \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n-2} \sum_{l=0}^{n-k-2} r^k \delta r^l \delta r^{n-k-l-2} = \delta_{12} \delta_{23} \frac{f(r_1, r_3) - f(r_2, r_3)}{r_1 - r_2}, \tag{5.16}$$

where the subscripts have been introduced to keep track of the ordering of the various ingredients, which is $r_1 \delta_{12} r_2 \delta_{23} r_3$. For example, in momentum space the ordering gives

$$r(p + k + q)\delta(k) r(p + q) \delta(q) r(p),$$

[8] It is not correct inside loop diagrams, but we know from section 4 that the loop integrals make the right PVP expressions appear there as well, possibly up to multi-thresholds.

where k and q are the incoming momenta of the insertions δ_{12} and δ_{23}, respectively.

Formula (5.16) shows that the second-order vertex is the incremental ratio of the incremental ratio,

$$f(x,y,z)=\frac{f(x,z)-f(y,z)}{x-y}, \tag{5.17}$$

keeping one variable fixed (which one being immaterial). In particular, the vertex is well defined and tends to its local limit fast enough. One can proceed similarly for the rest of the expansion.

To derive the loop integrals of the theory (5.12), we proceed as follows. First, we integrate out $\chi_{\mu\nu}$. This generates "functional diagrams" built by means of the covariantized versions of the propagators (5.9), where $1/Q$ and σ are (entire) functions of $D_\mu D^\mu + m^2$. Later, we expand the metric tensor $g_{\mu\nu}$ around flat space. The expansion also acts inside the covariantized $\chi_{\mu\nu}$ propagators, and is treated by means of identities like (5.13). By the results of this section, it generates only entire functions, through repeated incremental ratios. Once the expansion is worked out to the order of our interest, we are ready to study the loop integrals. They are well defined, because the vertices are entire functions, the propagators of physical particles are standard ones and the propagators of PVPs are also entire functions.

Summarizing,

1. the covariantized Lagrangians of the theories considered here generate entire functions $1/Q$ and σ of the covariant box, upon integrating out the nonlocal purely virtual fields;

2. the vertices, obtained by expanding the covariant derivatives as simple derivatives plus corrections proportional to fields, involve repeated incremental ratios;

3. the repeated incremental ratios of entire functions, formulas (5.14) and (5.17), are entire functions of more variables, because they can be expanded in power series of all their variables at the same time[9];

4. because of this, the diagrammatic structure is the one exhibited in formula (2.9).

That is to say, the asymptotically local theories we have formulated, such as the nonlocal deformation (5.12) of quantum gravity with purely virtual particles, lead to Feynman diagrams where the vertices and the propagators are made of entire functions, apart from the propagators of the physical particles, which have the usual poles.

[9]Their expressions show that the zeros of the denominators are compensated by the zeros of the numerators, so no singularity occurs.

5.5 Arbitrariness and renormalization

The AL-QG theory (5.12) is unitary, and so is its local limit, which is, by construction, the PVP-QG theory of quantum gravity described by the action (5.2), with $\chi_{\mu\nu}$ treated as a PVP. Given that (5.2) is renormalizable, although not manifestly, it is mandatory to inquire whether AL-QG is also renormalizable or not.

The nonlocal theories of the literature [5, 7, 8] are super-renormalizable. Unfortunately, their renormalization properties do not extend to nonlocal deformations of strictly renormalizable theories, such as (5.1) [6]. In view of this, AL-QG is not expected to be renormalizable either. For the reasons that we explain below, we do not think that this is a serious liability.

Although PVP-QG is unique [13], the nonlocal extension PVP-QG $\rightarrow$ AL-QG is not. Any entire function like those considered in the literature [5, 7, 8] can be used for h inside the second option for the function Q in (2.6). This leads to an infinite arbitrariness. The arbitrariness likely turned on by renormalization, together with the one associated with the adjustments of the multi-threshold contributions mentioned in the previous section, is not worse than that.

Moreover, the problem of arbitrariness is predicated on the assumption that the nonlocal theories are fundamental ones, but this is not what we claim here. We merely view AL-QFTs as tools to move beyond common frameworks in quantum field theory.

The main weakness of nonlocal quantum field theory is the lack of a fundamental principle for selecting the form factors that modify the propagators. In our approach this problem is addressed by the very existence of the local limit in Minkowski spacetime, in the sense that we can view *that* as the missing guiding principle. Then the candidate theories of the universe are the local limits themselves.

The arbitrariness of AL-QFT is reminiscent of the one of off-the-mass shell physics [20]. In the latter, the extra parameters are not properties of the fundamental interactions, but describe the environment where the phenomenon is observed, such as the experimental apparatus and the observer itself. We could say that they account for the quantum/classical interplay between the phenomenon and the rest of the universe. It would be interesting to uncover the map relating the arbitrariness of AL-QFT to the one of the off-shell approaches to QFT of ref.s [20]. We postpone this task, because it is beyond the scope of this paper.

We stress that the non uniqueness of the nonlocal deformation does not imply a lack of predictivity. The number of parameters impacting the phenomenon we want to observe is hopefully finite. Once they are identified, they can be fixed by sacrificing an equal number of initial measurements, after which every other measurement is predicted efficiently.

6 Conclusions

We have investigated the relationship between nonlocal and local quantum field theories, and searched for a practical definition of local limit. On general grounds, it is not obvious that a nonlocal model admits a local limit of some sort.

The nonlocal deformations are encoded into form factors that multiply the propagators and remove poles that otherwise would propagate ghosts. We have shown that the form factors inspired by the models studied in the current literature give theories that have well-defined limits only in Euclidean space. Singular behaviors appear in the Minkowskian correlation functions.

To overcome this difficulty, we have relaxed certain requirements and defined a new class of unitary, asymptotically local theories, which have well-defined local limits in Minkowski spacetime. Unitarity forbids target models with ghosts and privileges models with purely virtual particles.

Inside the bubble diagram, the nonlocal deformation generates PVPs directly in the limit. In the triangle diagram, it does so possibly up to multi-threshold corrections, which can be adjusted by fine tuning the deformation itself.

The asymptotically local deformation of a local theory is not unique, and not renormalizable. This is not a liability, in our approach, because we are not proposing AL-QFTs as candidate fundamental theories of nature, but merely as tools to provide alternative formulations of theories with PVPs, approximations to study the violation of microcausality and the peak uncertainty, or alternative approaches to off-shell physics, more commonly treated by restricting QFT to a finite interval of time and a compact space manifold.

In off-shell physics, the non uniqueness is associated with the possibility of introducing parameters that describe the apparatus, the classical environment surrounding the experiment and the observer itself. At the practical level, the theory is predictive in the following sense. One first identifies the parameters that matter to the phenomenon under consideration. Assuming that they are finitely many– as is reasonable to expect under normal circumstances, or in cleverly arranged setups –, one sacrifices an equal number of initial measurements to fix them. After that, one can verify whether the subsequent measurements are correctly predicted or not.

We believe that the nonlocality-locality relation worked out here can stimulate further investigations and some out-of-the-box thinking.

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sions.

Appendices

A Key functions

In this appendix we give entire functions that tend to the absolute value, the sign function and the Cauchy principal value in the local limit $\lambda \to \infty$. They are used in section 2 to build the models studied in the paper.

A.1 Approximating the absolute value

The absolute value of a complex number z can be approximated by the never vanishing entire function

$$h(z) \equiv \exp\left(\frac{1}{2}\int_0^{z^2} \frac{1-\mathrm{e}^{-w}}{w}\mathrm{d}w - \frac{\gamma_{\mathrm{E}}}{2}\right) = \exp\left(\frac{1}{2}\ln z^2 + \frac{1}{2}\Gamma(0,z^2)\right). \tag{A.1}$$

Precisely, in the double cone $\mathcal{C} = \{z : -\pi/4 < \arg[z] < \pi/4 \text{ or } 3\pi/4 < \arg[z] < 5\pi/4\}$, we have [26]

$$h(z) = \sqrt{z^2}\left[1 + \frac{\mathrm{e}^{-z^2}}{2z^2}\left(1 + \mathcal{O}\left(\frac{\mathrm{e}^{-z^2}}{z^2}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right)\right], \tag{A.2}$$

so

$$\lim_{\lambda\to+\infty} \frac{h(\lambda z)}{\lambda} = \sqrt{z^2} \quad \text{for } z \in \mathcal{C}, \qquad \lim_{\lambda\to+\infty} \frac{h(\lambda x)}{\lambda} = |x| \quad \text{for } x \in \mathbb{R}. \tag{A.3}$$

It is important to stress that $h(z)$ has no zeros. Its reciprocal $1/h(z)$ is useful to build the "propagators" that tend to purely virtual particles in the local limit.

The function $h(z)$ is even,

$$h(z) = h(-z), \tag{A.4}$$

and satisfies

$$h^*(z) = h(z^*). \tag{A.5}$$

Moreover, on the real axis it is positive and bounded from below by the absolute value, as illustrated in fig. 2:

$$h(x) \geqslant |x|, \qquad x \in \mathbb{R}. \tag{A.6}$$

Finally, (A.3) gives

$$\lim_{\lambda\to+\infty} \frac{h^2(\lambda z)}{\lambda^2 z} = z \quad \text{for } z \in \mathcal{C}. \tag{A.7}$$

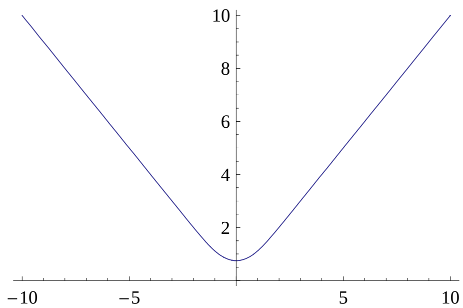

Figure 2: Plot of $h(x)$ for real x

A.2 Approximating the sign function

The sign function $\mathrm{sgn}(x)$ can be approximated on the complex plane by the entire function

$$\sigma(z) = \frac{z}{h(z)}, \tag{A.8}$$

which vanishes at the origin. In $\mathcal{C}$ the rescaled function $\sigma(\lambda z)$ tends to $z/\sqrt{z^2}$ for $\lambda \to \infty$, thus

$$\lim_{\lambda\to+\infty} \sigma(\lambda z) = \mathrm{sgn}(\mathrm{Re}[z]) \quad \text{for } z \in \mathcal{C}, \qquad \lim_{\lambda\to+\infty} \sigma(\lambda x) = \mathrm{sgn}(x) \quad \text{for } x \in \mathbb{R}. \tag{A.9}$$

Note that $\sigma(z)$ is odd in the complex plane:

$$\sigma(z) = -\sigma(-z). \tag{A.10}$$

Moreover, (A.6) gives

$$|\sigma(x)| \leqslant 1, \qquad x \in \mathbb{R}. \tag{A.11}$$

A.3 Approximating the principal value

The combination $\sigma(x)/h(x) = x/h^2(x)$ approximates the Cauchy principal value $\mathcal{P}$ on the real axis. Specifically, we prove that

$$\lim_{\lambda\to+\infty} \frac{\lambda\sigma(\lambda x)}{h(\lambda x)} = \mathcal{P}\frac{1}{x}, \qquad x \in \mathbb{R}, \tag{A.12}$$

in the sense of distributions.

First observe that

$$\lim_{\lambda\to+\infty} \frac{\lambda\sigma(\lambda x)}{h(\lambda x)} = \frac{1}{x} \tag{A.13}$$

for every real $x \neq 0$. This result follows from (A.3) and (A.9).

If $\varphi(x)$ denotes a real test function, consider the integral

$$\mathcal{I} \equiv \lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\lambda \sigma(\lambda x)\varphi(x)}{h(\lambda x)}, \tag{A.14}$$

which can also be written as

$$\mathcal{I} = \lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\lambda \sigma(\lambda x)(\varphi(x) - \varphi(-x))}{2h(\lambda x)},$$

thanks to (A.4) and (A.10). The modulus of the $\mathcal{I}$ integrand is bounded above by a λ-independent function that is integrable on $\mathbb{R}$. Indeed, the inequalities (A.6) and (A.11) give

$$\left| \frac{\lambda \sigma(\lambda x)(\varphi(x) - \varphi(-x))}{2h(\lambda x)} \right| \leqslant \frac{1}{2} \left| \frac{\varphi(x) - \varphi(-x)}{x} \right|, \tag{A.15}$$

for real x. Then the dominated convergence theorem allows us to exchange the limit with the integral. Given that (A.13) holds almost everywhere, we obtain

$$\mathcal{I} = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\varphi(x) - \varphi(-x)}{2x} = \mathcal{P} \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x} \varphi(x), \tag{A.16}$$

as we wished to show.

B Bubble diagram with PVPs

In this appendix we review the calculation of the bubble diagram with one or two circulating PVPs, in the formulation of ref. [11]. We work in Minkowski spacetime, so here p and k denote Minkowskian momenta.

We start from the integral

$$\mathcal{B}_{\mathrm{ph}} \equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon'}$$

of the standard bubble with circulating physical particles. First, we decompose the propagators

$$\frac{1}{q^2 - m^2 + i\epsilon} \to \frac{1}{2\omega_{\mathbf{q}}} \left(\frac{1}{q^0 - \omega_{\mathbf{q}} + i\epsilon} - \frac{1}{q^0 + \omega_{\mathbf{q}} - i\epsilon} \right)$$

by isolating the particle and antiparticle poles, where $\omega_{\mathbf{q}} = \sqrt{\mathbf{q}^2 + m^2}$ is the frequency. Then we expand the integrand and integrate on p^0 by means of the residue theorem. The result is

$$\mathcal{B}_{\mathrm{ph}} = \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{i}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left(\frac{1}{k^0 + \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{p}} - i(\epsilon + \epsilon')} - \frac{1}{k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}} + i(\epsilon + \epsilon')} \right).$$

At this point, we use

$$\frac{1}{x+i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$$

for each term inside the parentheses. We obtain

$$\begin{aligned}\mathcal{B}_{\rm ph} = &\mathcal{P}\int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{i}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left(\frac{1}{k^0+\omega_{\mathbf{p}-\mathbf{k}}+\omega_{\mathbf{p}}} - \frac{1}{k^0-\omega_{\mathbf{p}-\mathbf{k}}-\omega_{\mathbf{p}}}\right) \\ &-\int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{\pi}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left[\delta(k^0+\omega_{\mathbf{p}-\mathbf{k}}+\omega_{\mathbf{p}})+\delta(k^0-\omega_{\mathbf{p}-\mathbf{k}}-\omega_{\mathbf{p}})\right]. \qquad \text{(B.1)}\end{aligned}$$

The bubble diagram $\mathcal{B}_{\rm PVP}$ with one or two circulating PVPs is obtained from $\mathcal{B}_{\rm ph}$ by dropping the delta terms:

$$\mathcal{B}_{\rm PVP} = \mathcal{P}\int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{i}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left(\frac{1}{k^0+\omega_{\mathbf{p}-\mathbf{k}}+\omega_{\mathbf{p}}} - \frac{1}{k^0-\omega_{\mathbf{p}-\mathbf{k}}-\omega_{\mathbf{p}}}\right). \qquad \text{(B.2)}$$

For calculations of more complicated diagrams and the general proof of unitarity, see [11].

It is important to stress that if we adopt the principal value we see in (4.7) as an alternative propagator in Feynman diagrams, we obtain a theory of "Wheelerons" [27], which propagates ghosts. For example, the bubble diagram gives

$$\mathcal{P}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{p^2-m^2}\frac{1}{(p-k)^2-m^2}, \qquad \text{(B.3)}$$

which has a nonvanishing, unphysical absorptive part [28], absent in (B.2). The moral of the story is that in a theory of PVPs the radiative corrections are not given by the usual diagrams with a different propagator, but by new combinations of diagrams [12, 11].

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