Theories of Gravitation

D. Anselmi, 2019

http://renormalization.com

Damiano Anselmi

damiano.anselmi@unipi.it

54 hours	9 credits
36 hours	6 credits

MON	F1	14.00 - 16.00
TUE	V1	9.00 - 11.00
THU	G1	14.00 - 16.00

www.df.unipi.it/~anselmi

- Differential Geometry
- Classical gravity
- General Relativity
- Quantum Gravity

Textbooks R. Wald, General Relativity

S. Carroll, Lecture Notes on General Relativity

arXiv: gr-qc/9712019

P. Menotti, Lectures on gravitation, arXiv: 1703.05155 [gr-qc]

Lecture notes on differential geometry G. P. Pirola

www-dimat.unipv.it/~pirola/corso-intero.pdf

Differential forms, Weintraub

Natural operations in differential geometry, Kolar, Michor, Slóvak, Topological space $\{X, \mathcal{V}\}$ pair

- X is a nonempty set, $X \neq \phi$
- $\mathcal{V}$ is a family of subsets of X, called open sets, such that
  - $X \in \mathcal{V}$, $\phi \in \mathcal{V}$
  - $A \in \mathcal{V}$, $B \in \mathcal{V} \Rightarrow A \cup B \in \mathcal{V}$, $A \cap B \in \mathcal{V}$
  - the union of an infinite number of subets $A_i \in \mathcal{V}$ also belongs to $\mathcal{V}$

A closed set is the complement of an open set

A topological space is separated (or Hausdorff space) if

$\forall\ p, q \quad p \neq q \quad p \in X \quad q \in X$

$\exists$ open sets U_p and U_q belonging to $\mathcal{V}$

such that $p \in U_p$, $q \in U_q$, $U_p \cap U_q = \phi$

A neighborhood of $p \in X$ is any open set that contains p.

A **cover** of X is a family $\mathcal{U}$ of open sets $\mathcal{U}_i \in \mathcal{V}$ such that

$$\bigcup_i \mathcal{U}_i = X$$

A topological manifold is a separated topological space that admits a cover $\mathcal{U}$ such that every open set $\mathcal{U}_i$ is homeomorphic to $\mathbb{R}^n$.

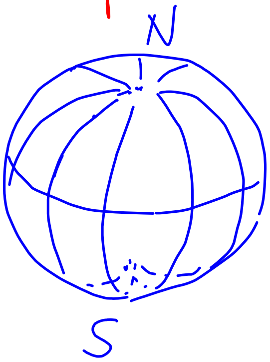
A homeomorphism is a continuous one-to-one map between topological spaces such that its inverse is also continuous

A function $f : X \to Y$ between two topological spaces is continuous if

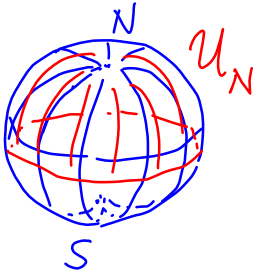
$f^{-1}(A)$ is an open set of X whenever

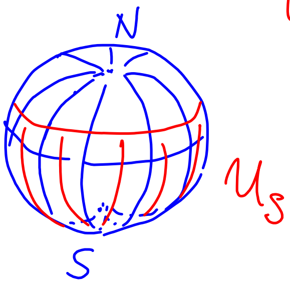
A is an open set of Y

Example : the sphere



We choose a cover made of two segments of the sphere



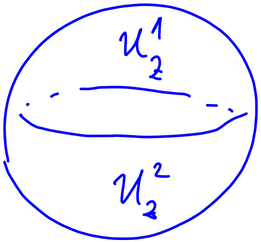


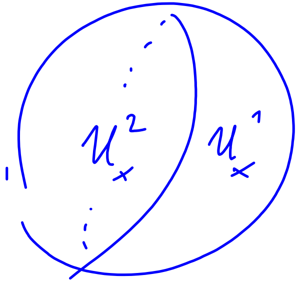
$\mathcal{U}_N \cup \mathcal{U}_S =$ sphere

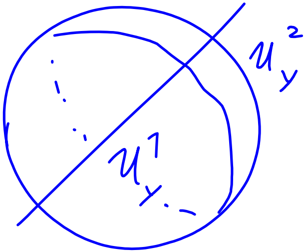
$$S^n = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}, \; x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1 \}$$

$$\mathcal{U}_i^\pm = \{ (x_1, \dots, x_{n+1}) \in S^n : \; x_i \gtrless 0 \}$$

$$S^n = \Big(\bigcup_i \mathcal{U}_i^+ \Big) \cup \Big(\bigcup_i \mathcal{U}_i^- \Big)$$







Homeomorphisms: $\varphi_i^\pm : \mathcal{U}_i^\pm \longrightarrow \mathbb{R}^n$

$$\varphi_i^\pm (x_1, \dots x_i, \dots x_{n+1}) = (x_1, \dots x_{i-1}, x_{i+1}, \dots x_{n+1})$$

Every pair $(\mathcal{U}_i^{\pm}, \varphi_i^{\pm})$ is called a chart

The family of charts is called atlas

Other examples:

- $\mathbb{R}^n$ is a topological manifold, every open subset of $\mathbb{R}^n$ is a topological manifold
- S^n
- If X and Y are topological manifolds $X \times Y$ is a topological manifold

Changes of coordinates

$\mathcal{U}_i \qquad \varphi_i : \mathcal{U}_i \longrightarrow B_i \subset \mathbb{R}^n \qquad B_i = \varphi_i(\mathcal{U}_i)$

is an open set

$\mathcal{U}_j \qquad \varphi_j : \mathcal{U}_j \longrightarrow B_j \subset \mathbb{R}^n \qquad B_j = \varphi_j(\mathcal{U}_j)$

If $\mathcal{U}_{ij} \equiv \mathcal{U}_i \cap \mathcal{U}_j \neq \phi$

$$\varphi_i : \mathcal{U}_{ij} \longrightarrow B_{ij} = \varphi_i(\mathcal{U}_{ij}) \subset \mathbb{R}^n$$

$$\varphi_j : \mathcal{U}_{ij} \longrightarrow C_{ij} = \varphi_j(\mathcal{U}_{ij}) \subset \mathbb{R}^n$$

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : C_{ij} \longrightarrow \mathcal{U}_{ij} \longrightarrow B_{ij}$$

$$\begin{tikzcd} \varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : & C_{ij} \arrow[r] \arrow[d] & \mathcal{U}_{ij} \arrow[r] & B_{ij} \arrow[d] \\ & \text{open set of } \mathbb{R}^n & & \text{open set of } \mathbb{R}^n \\ & (y_1, \dots y_n) & & (x_1, \dots x_n) \end{tikzcd}$$

$$\varphi_{ij} : \quad y(x) \qquad y^i = y^i(x_1, \dots x_n)$$

$$i = 1, \dots n$$

$$\varphi_{ij}(x_1, \dots x_n) = (y_1, \dots y_n)$$

The manifold is differentiable of class $\mathcal{C}^k$ if every φ_{ij} is k times differentiable

A topological manifold is a manifold of class $\mathcal{C}^0$

If $k>0$ the maps φ_{ij} are called diffeomorphisms

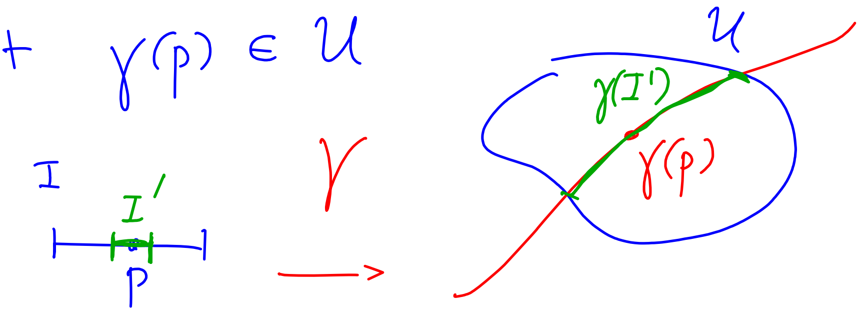
Curve

Let M denote a differential manifold of class $\mathcal{C}^k$ $k \geq 1$ and I denote an interval of the real line

A curve is a map $\gamma : I \longrightarrow M$ of class $\mathcal{C}^k$

Let $(\mathcal{U}, \varphi)$ denote a chart and

$p \in I$ such that $\gamma(p) \in \mathcal{U}$



$\exists\, I'$ neighborhood of p that is mapped in $\mathcal{U}$

$\gamma : I' \longrightarrow \mathcal{U}$ $\qquad$ $\varphi : \mathcal{U} \longrightarrow \mathbb{R}^n$

$\varphi \circ \gamma : I' \longrightarrow \gamma(I') \longrightarrow \mathbb{R}^n$

$t \in \mathbb{R}$ $\qquad$ $(x_1, \dots x_n) \in \mathbb{R}^n$

$\gamma: \quad (x_1(t), \ldots, x_n(t))$

$x_i(t)$ is a k-times differentiable function for every i and every chart

The tangent vector to γ in p is

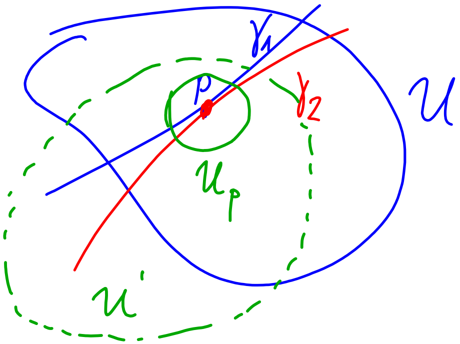
$(\dot{x}_1(p), \ldots \dot{x}_n(p))$ This is a vector of $\mathbb{R}^n$

Two curves $\gamma_1: (x_1(t), \ldots x_n(t))$ ← x & y refer to the same coordinate system (chart)

$\gamma_2: (y_1(t), \ldots y_n(t))$ ←

are tangent to each other in p if

$$(\dot{x}_1(p), \dots \dot{x}_n(p)) = (\dot{y}_1(p), \dots \dot{y}_n(p))$$



This property does $\underline{\text{not}}$ depend on the coordinates

$$\left.\begin{array}{lll} \varphi \circ \gamma_1 & : & (x_1(t), \dots x_n(t)) \\ \varphi \circ \gamma_2 & : & (y_1(t), \dots y_n(t)) \end{array}\right\} \text{using } \mathcal{U} \text{ coordinates } \varphi$$

Consider another chart $(\mathcal{U}', \varphi')$ $\quad p \in \mathcal{U}'$

$$\left.\begin{array}{lll} \varphi'\circ\gamma_1 & : & (x_1'(t), \dots\ x_n'(t)) \\ \varphi'\circ\gamma_2 & : & (y_1'(t), \dots\ y_n'(t)) \end{array}\right\}$$ using $\mathcal{U}'$ coordinates φ'

We want to show that

$$\dot{x}_i(p) = \dot{y}_i(p) \implies \dot{x}_i'(p) = \dot{y}_i'(p)$$

$\exists\ \mathcal{U}_p$ neighborhood of p that is contained both in $\mathcal{U}$ and $\mathcal{U}'$

There, we can use both coordinate systems

$$\underbrace{\varphi' \circ \gamma_1}_{\text{1st curve in the 2nd coordinate system}} = \underbrace{\varphi' \circ \varphi^{-1}}_{\text{change of coordinates}} \circ \underbrace{\varphi \circ \gamma_1}_{\text{1st curve in the 1st coordinate system}}$$

$$x'(x) \qquad\qquad (x_1(t), \dots x_n(t))$$

$$(x_1'(t), \dots \; x_n'(t))$$

$$(x_1'(t), \dots, x_n'(t)) \dashrightarrow (x_1'(x(t)), \dots \; x_n'(x(t)))$$

$$\frac{dx_i'}{dt}(p) = \frac{\partial x_i'}{\partial x_j}\bigg|_p \frac{dx^j}{dt}\bigg|_p$$

Similar formulas for γ_2 $(x \to y)$

$$\frac{dy_i'}{dt}(p) = \left.\frac{\partial y_i'}{\partial y_j}\right|_p \left.\frac{dy_j}{dt}\right|_p$$

Let us take the difference of the two equations

$$\underbrace{\frac{dx_i'}{dt}(p) - \frac{dy_i'}{dt}(p)}_{} = \left.\frac{\partial x_i'}{\partial x_j}\right|_p \underbrace{\left.\frac{dx^j}{dt}\right|_p}_{} - \left.\frac{\partial y_i'}{\partial y_j}\right|_p \underbrace{\left.\frac{dy_j}{dt}\right|_p}_{}$$

we want to show that this difference is zero

these should also be equal : indeed they are $\boxed{\varphi' \circ \varphi^{-1}}$

are equal by assumption

We can define an equivalence relation between curves through the same point p:

$\gamma_1 \sim \gamma_2$ if they are tangent to each other in p

The quotient between the set $\Omega(p)$ of all curves through p and the equivalence relation just defined is called tangent space to the manifold M in p and denoted by T_p or $T_{M,p}$

In other words, if we consider a $v \in T_p$, i.e. an equivalence class of curves through p and associate it with the vector $(\dot{x}_1(p), \dots \dot{x}_n(p)) \in \mathbb{R}^n$ (common to all representatives of the equivalence class and independent of the coordinates and chart we use), then we obtain an isomorphism between T_p and $\mathbb{R}^n$ which endows T_p with a structure of vector space

Let M denote a smooth manifold

Smooth = differentiable of class $\mathcal{C}^\infty$

A smooth function is a $\mathbb{C}^\infty$ function

Let $p \in M$

A smooth function in a neighborhood U of p is the pair (f, U) where $f : U \to \mathbb{R}$ is $\mathcal{C}^\infty(U)$

Two smooth functions f and g are equivalent, $f \sim g$, if there exists a neighborhood W of p

such that $f|_W = g|_W$.

The quotient $\mathcal{G}_p$ between the set of smooth functions in p and the equivalence relation is called space of the germs of the smooth functions in p

A derivation in p is a map

$X : \mathcal{G}_p \longrightarrow \mathbb{R}$ such that

$$X(f+g) = X(f) + X(g)$$

$X(\lambda) = 0$ if λ = real constant

$X(fg) = f\,X(g) + X(f) \cdot g$

in p

In particular, $X(\lambda f) = \lambda X(f) + \cancel{f\,X(\lambda)} =$

$= \lambda X(f)$

The space of derivations in p is denoted by D_p and is a vector space

Note. Let $\mathcal{U}$ denote an open set of $\mathbb{R}^n$, $p \in \mathcal{U}$ and $f \in \mathcal{C}^\infty(\mathcal{U})$. Then there exist smooth functions $g_i \in \mathcal{C}^\infty(\mathcal{U})$ such that $\forall x \in \mathcal{U}$

$$f(x) - f(p) = \sum_i (x_i - x_i(p))\, g_i(x) \quad \text{and}$$

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

Proof. Let us assume $x_i(p) = 0$

$$f(x) - f(0) = \int_0^1 dt \, \frac{df}{dt}(t x_1, \dots, t x_n) =$$

$$= \int_0^1 dt \, \frac{\partial f}{\partial x_i}(tx)\, x_i = x_i\, g_i(x) \quad \text{where}$$

$$g_i(x) = \int_0^1 dt \, \frac{\partial f}{\partial x^i}(tx) \quad : \text{ they are } \mathcal{C}^\infty$$

$$g_i(0) = \frac{\partial f}{\partial x_i}(0)$$

Again, let $\mathcal{U}$ denote an open set of $\mathbb{R}^n$, $p \in \mathcal{U}$

The $X_i = \left.\frac{\partial}{\partial x_i}\right|_p$ is a basis of the space D_p of derivations in p

If $f : \mathcal{U} \longrightarrow \mathbb{R}$ $\quad X_i(f) = \left.\frac{\partial f}{\partial x^i}\right|_p$

$f \in \mathcal{C}^\infty(\mathcal{U})$

If $f = x_j$ $\quad X_i(x_j) = \delta_{ij}$

The X_i's generate D_p :

let $X \in D_p$, let us define $X(x_i) = a_i$

Then we want to show that $X = a_i X_i$

Consider $Y = X - a_i X_i$

We want to show $Y(f) = 0$ $\forall f \in \mathcal{G}_p$ (in p)

$$Y(x_i) = X(x_i) - a_j X_j(x_i) = a_i - a_j \delta_{ij} = 0$$

$Y(f)$: $f(x) = f(p) + \sum_i (x_i - x_i(p)) g_i(x)$

$$Y(f) = \cancel{Y(f(p))} + \sum_i \cancel{Y(x_i - x_i(p))} g_i(x) +$$

$$+ \sum_i (x_i - x_i(p))\, Y(g_i(x))$$

$Y(f) = 0$ in p

The tangent space T_p in p is isomorphic to the space D_p of derivations in p

$$\zeta : T_p \longrightarrow D_p$$

Let $v \in T_p$ $\quad v = [\gamma(t)] \quad \gamma(0) = p$

Isomorphism : $\quad \zeta(v)(f) = \frac{d}{dt} f(\gamma(t)) \Big|_{t=0}$

Let us consider $\quad v_i = [t e_i] \quad e_i = (0, \dots, 0, \underset{\text{ith place}}{1}, \dots 0)$

$$j(v_i)(f) = \frac{d}{dt} f(0, \dots 0, \underset{ith}{t}, 0 \dots 0)\Big|_{t=0} =$$

$$= \frac{\partial f}{\partial x_i}\Big|_{t=0} = X_i(f)\Big|_p$$

$$j(v_i) = X_i = \frac{\partial}{\partial x_i}\Big|_p$$ isomorphism

A **vector field** is a linear function

$X : \mathcal{C}^{k}(M) \longrightarrow \mathcal{C}^{k-1}(M)$ such that

$$X(fg) = f\,X(g) + g\,X(f)$$

Locally (i.e. in a specific chart) $X_i = \frac{\partial}{\partial x_i}$

and $X = a_i(x) \dfrac{\partial}{\partial x_i}$

In the intersection between two charts

let us call x and y the two coordinate systems

$$X = b^i(y) \frac{\partial}{\partial y^i} = a^i(x) \frac{\partial}{\partial x^i} =$$

$$= b^{i\,j}(y(x)) \frac{\partial x^{j\,i}}{\partial y^{i\,j}} \frac{\partial}{\partial x^{j\,i}}$$

$$a^i(x) = b^j(y(x)) \frac{\partial x^i}{\partial y^j} \qquad b^i(y(x)) = \frac{\partial y^i}{\partial x^j} a^j(x)$$

$\mathcal{X}^{\infty}(M)$ = space of smooth ($\mathcal{C}^{\infty}$) vector fields of a smooth manifold M

If X and Y are vector fields XY is not a vector field in general, because it does not obey the Leibniz rule :

$$X(Y(fg)) = X(fY(g) + gY(f)) =$$

$$= \underbrace{X(f)\,Y(g)}_{} + \underbrace{f\,\underbrace{XY}_{}(g)}_{} + \underbrace{X(g)\,Y(f)}_{} +$$

$$+ \underbrace{g\,\underbrace{XY}_{}(f)}_{}$$

$[X, Y] \equiv XY - YX$ is a vector field

$$[X, Y](fg) = f\,[X, Y](g) + g\,[X, Y](f)$$

Locally $X = a_i(x)\dfrac{\partial}{\partial x^i}$ $Y = b^i(x)\dfrac{\partial}{\partial x^i}$

$$Z = [X, Y] = c_i(x)\frac{\partial}{\partial x^i}$$

$$Z(f) = c_i\frac{\partial f}{\partial x^i} = [X, Y](f) = X(Y(f)) - Y(X(f))$$

$$= a_i\frac{\partial}{\partial x^i}\left(b_j\frac{\partial f}{\partial x^j}\right) - b_i\frac{\partial}{\partial x^i}\left(a_j\frac{\partial f}{\partial x^j}\right) =$$

$$= a_i \frac{\partial b_j}{\partial x^i} \frac{\partial f}{\partial x^j} - b_i \frac{\partial a_j}{\partial x^i} \frac{\partial f}{\partial x^j}$$

$$c_i = a_j \frac{\partial b_i}{\partial x^j} - b_j \frac{\partial a_i}{\partial x^j}$$

Properties:

- $[X, Y] = -[Y, X]$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

(Jacobi identity)

$$[fX, Y] = f[X, Y] - Y(f) X(g)$$

$$[fX, Y](g) = fXY(g) - Y(fX(g)) =$$

$$= fXY(g) - Y(f)X(g) - fYX(g) =$$

$$= f[X, Y](g) - Y(f)X(g)$$

If $\mathcal{A}$ is an algebra, a derivation D of $\mathcal{A}$ is a linear map $D : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\mathcal{D}(ab) = a\,\mathcal{D}(b) + b\,\mathcal{D}(a) \quad \forall\, a, b \in \mathcal{A}$$

Note: smooth function f on M:

$f(x)$ in a chart $\mathcal{U}$

in the intersection of two charts $x'(x)$

$f'(x') = f(x)$ (a "scalar")

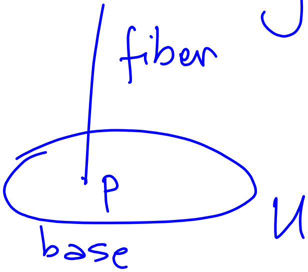
We can take $\mathcal{A} = \mathcal{C}^\infty(M)$. Let X denote a vector field of $\mathcal{X}^\infty(M)$.

Then $f \longrightarrow X(f)$ is a derivation of $\mathcal{A}$

Conversely, every derivation of $\mathcal{C}^\infty(M)$ is associated with one and only one vector field of $\mathcal{X}^\infty(M)$

Tangent bundle $T_M = \bigcup_{p\in M} T_p$

Let U denote an open set of $\mathbb{R}^n$ with coordinate system $(x_1, \dots x_n)$



$$T_U = U \times \mathbb{R}^n$$

$$\dim T_U = 2n$$

$$(\underbrace{x_1, \dots, x_n,}_{U} \underbrace{a_1, \dots a_n}_{\mathbb{R}^n})$$

tangent

$$X = a_i \frac{\partial}{\partial x^i}$$

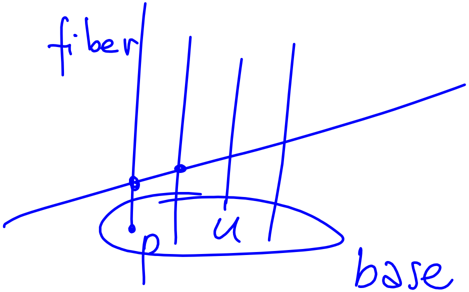
In the intersection of two charts we have coordinates

$$(x_1, \dots x_n, a_1, \dots a_n)$$

$$(y_1, \dots y_n, b_1, \dots b_n)$$

We also a relation $y = y(x)$

We add $$b^i = \frac{\partial y^i}{\partial x^j} a^j$$



Section (or cross section):

$$(x_1, \dots x_n, a_1(x), \dots a_n(x))$$

$$X = a_i(x) \frac{\partial}{\partial x_i}$$

A vector field is a section of the tangent bundle

For a section, $b^{i} = \dfrac{\partial y^{i}}{\partial x^{j}} a^{j}$ becomes the correct transformation rule for vector fields

Differential of a function

$f : M \to \mathbb{R}$ $\quad f \in \mathcal{C}^{k}(M)$ $\quad k \geq 1$

Let $p \in M$ $\quad df(p) : T_p \longrightarrow \mathbb{R}$ $\quad df(p) \in T_p^{*}$

Let $X \in T_p$ We interpret it as a derivation

Then df is defined by $df(X) = X(f)$

obs.: $df(gX) = gX(f) = g\, df(X)$

In local coordinates $X = a^i(x) \frac{\partial}{\partial x^i}$

$$X(f) = a^i(x)\Big|_p \frac{\partial f}{\partial x_i}\Big|_p = df(X)$$

Let us take $f = x^i$

$$dx^i(X) = X(x_i) = a^i$$

$$df(X) = \frac{\partial f}{\partial x^i} dx^i(X) \qquad dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^{ij}$$

$$df = \frac{\partial f}{\partial x^i} dx^i$$

This formula shows that the dx^i are a basis of T_p^* (cotangent space)

Change of coordinates

$$x^i \longrightarrow y^i(x) \qquad\qquad \delta^{ij} = dx^i\left(\frac{\partial}{\partial x_j}\right) = dy^i\left(\frac{\partial}{\partial y_j}\right)$$

$$\delta^{ij} = \frac{\partial y^k}{\partial x^j}\,\frac{\partial x^i}{\partial y^k} = \frac{\partial x^i}{\partial y^k}\, dy^k\left(\frac{\partial}{\partial x^j}\right)$$

$$dy^k\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i}\, dx^i(X) = \frac{\partial y^k}{\partial x^i}\, dx^i\left(\frac{\partial}{\partial x^j}\right) =$$

$$\overset{\shortparallel}{f} \qquad \overset{\shortparallel}{X} \qquad\qquad = \frac{\partial y^k}{\partial x^j}$$

$$dx^i = \frac{\partial x^i}{\partial y^j}\, dy^j \qquad\qquad dy^i = \frac{\partial y^i}{\partial x^j}\, dx^j$$

Cotangent bundle

$$T^*_M = \bigcup_{p\in M} T^*_p$$

M = base manifold

Locally:

$$(x_1, \dots x_n, \underbrace{\omega_1, \dots \omega_n}_{\text{in the basis } dx^i})$$

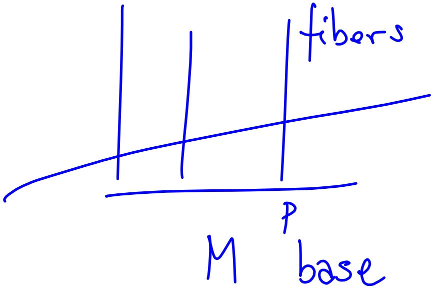
Element of the cotangent space $\omega_i \, dx^i$

(tangent space) $a_i \frac{\partial}{\partial x^i}$

Change of coordinates:

$$\omega_i \, dx^i = \omega_i \frac{\partial x^i}{\partial y^j} dy^j = \omega_i' \, dy^i$$

$$\omega_i' = \omega_j \frac{\partial x^j}{\partial y^i}$$



T_p^*

Section = differential form of degree 1

1- form

Locally: $\omega = \omega_i(x)\, dx^i$

Action of a differential form on a vector field

$$\omega = \omega_i(x)\, dx^i \qquad X = a^i(x)\frac{\partial}{\partial x^i}$$

$$\omega(X) = \omega_i(x)\, dx^i\left(a^j(x)\frac{\partial}{\partial x^j}\right) =$$

out

$$= \omega_i(x)\, a^j(x)\, dx^i\left(\frac{\partial}{\partial x^j}\right) = \omega_i(x)\, a^i(x)$$

We define $\Lambda^s T^*_M$ as the space of the antisymmetric forms of degree s (Λ = wedge)

$s=2$ locally : $\omega_2 = \frac{\omega_{ij}(x)}{2} dx^i \wedge dx^j$

$\omega_{ij} = -\omega_{ji}$

$\omega_2 : T_p \times T_p \longrightarrow \mathbb{R}$

$X = a^i \frac{\partial}{\partial x^i}$

$X, Y \in T_M \qquad \omega_2(X, Y) \in \mathbb{R}$

$Y = b^i \frac{\partial}{\partial x^i}$

$$dx^i \wedge dx^j (X, Y) = \det \begin{pmatrix} dx^i(X) & dx^i(Y) \\ dx^j(X) & dx^j(Y) \end{pmatrix} =$$

$$= \det \begin{pmatrix} a^i & b^i \\ a^j & b^j \end{pmatrix} = a^i b^j - a^j b^i$$

$$\omega_2(X,Y) = \frac{\omega_{ij}}{2}\left(a^i b^j - b^i a^j\right) = \omega_{ij}\, a^i b^j$$

arbitrary degree s:

$$\omega_s = \frac{1}{s!}\, \underbrace{\omega_{i_1 \dots i_s}}_{\text{completely antisymmetric}}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

where

$$dx^{i_1} \wedge \dots \, dx^{i_s}\left(X_1, \dots X_s\right) = \det \begin{pmatrix} dx^{i_1}(X_1) & \dots & dx^{i_s}(X_1) \\ \vdots & & \vdots \\ dx^{i_s}(X_1) & \dots & dx^{i_s}(X_s) \end{pmatrix}$$

$(T^*_M)^{\otimes s}$: space of symmetric forms of degree s

Locally: $\omega_s = \underbrace{\omega_{i_1 \dots i_s}}_{\text{completely symmetric}}(x)\, dx^{i_1} \otimes \dots \otimes dx^{i_s}$

$s=2$ $\quad dx^i \otimes dx^j (X, Y) = a^i b^j + a^j b^i$

$X = a^i \dfrac{\partial}{\partial x^i} \quad Y = b^i \dfrac{\partial}{\partial x^i}$

$$g = \underbrace{g_{ij}}\, \frac{dx^i \otimes dx^j}{2} = g_{ij}\, a^i b^j$$

$$g_{ij} = g_{ji}$$

Exterior derivative $\quad d : \Lambda^m T^*_M \longrightarrow \Lambda^{m+1} T^*_M$

locally $\quad \omega = \omega_{i_1 \dots i_m}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_m}$

$$d\omega = \partial_j \omega_{i_1 \dots i_m}(x)\, dx^j \wedge dx^{i_1} \wedge \dots dx^{i_m}$$

Exercise: transformations under a change of coordinates

Nilpotency $\quad d^2 = 0$

$$d\omega = \partial_i \omega_{i_1 \dots i_m} dx^i dx^{i_1} \dots dx^{i_m}$$

$$d^2\omega = \underbrace{\partial_j \partial_i}_{\text{symmetric}} \omega_{i_1 \dots i_m} \underbrace{dx^j dx^i}_{\text{antisymmetric}} dx^{i_1} \dots dx^{i_m} \implies d^2 \equiv 0$$

A differential form ω is closed if $d\omega = 0$

Locally: $\quad \partial_{[i} \omega_{i_1 \dots i_m]} = 0$

Example $\quad \omega = \omega_i dx^i \qquad d\omega = \partial_j \omega_i \, dx^j \wedge dx^i$

$d\omega = 0 \implies \quad \partial_i \omega_j = \partial_j \omega_i$

If $\omega_i = E_i$ (electric field) $\quad \underline{\underline{d\omega = 0}} \iff$

curl $\vec{E} = 0 \qquad \dfrac{\partial E_i}{\partial x^j} = \dfrac{\partial E_j}{\partial x^i}$

$$\frac{\partial}{\partial x^i}$$

[$\overset{?}{\Longrightarrow} \quad \vec{E} = -\vec{\nabla} V \qquad E_i = -\partial_i V$

$$\omega = \omega_i \, dx^i = E_i \, dx^i = -\partial_i V \, dx^i = -dV$$

exact]

A differential form ω of degree $k \geq 1$ is **exact** if $\exists\, \alpha$, differential form of degree $k-1$, such that $\omega = d\alpha$.

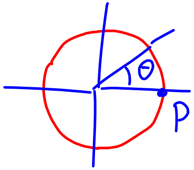
An exact form is always closed:

$$\omega = d\alpha \implies d\omega = dd\alpha = 0$$

(exact) (closed)

A closed form is NOT always exact, but it is always exact in R^n or an open subset of $\mathbb{R}^n$

Example $M = S^1$



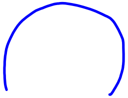


chart 1

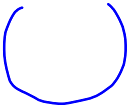


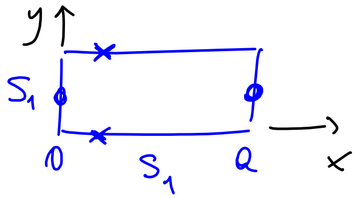
chart 2

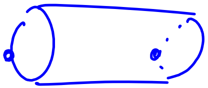
$\underline{\omega = d\theta}$

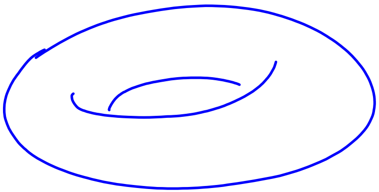
ω is closed : $d\omega = 0$

But it is NOT exact, because θ is not a zero form (a function on S^1)

Let us consider the torus $T = S^1 \times S^1$







Differential forms that are closed but not exact

$1 \qquad dx \qquad dy \qquad dx \wedge dy$

$M = S^2$

$\omega_0 = 1$

$\omega_2 = \text{sen}\,\theta \, d\theta \wedge d\varphi = d\Omega$

Stoke's theorem

Let V denote an open set, ∂V denote the boundary of V. Let Ω denote a differential form

$$\int_{\underset{\uparrow}{V}} d\Omega = \int_{\partial V} \Omega$$

If $\omega = \operatorname{sen}\theta \, d\theta \wedge d\varphi$ were exact, $\omega = d\sigma$

$$4\pi = \int_{\underset{V}{S^2}} \omega = \int_{S^2} d\overset{\Omega}{\sigma} = \int_{\underset{=0}{\partial S^2}} \sigma = 0 \qquad \text{absurd}$$

Electromagnetism $\quad A = A_\mu dx^\mu$

$$F = dA = \partial_\nu A_\mu \, dx^\nu \wedge dx^\mu = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$dF = 0$ (Bianchi identity)

Two closed differential forms ω_1 and ω_2 are equivalent if their difference $\omega_1 - \omega_2$ is exact:

$\exists\, \alpha$ such that $\omega_1 = \omega_2 + d\alpha$ $\quad \omega_1 \sim \omega_2$

An exact differential form $\omega = d\alpha$ is equivalent to zero

The equivalence classes of the differential forms with respect to this equivalence relation define the cohomology of differential forms

The boundary operator ∂ is also nilpotent : $\partial^2 = 0$

$\gamma = \partial V$:
γ is exact

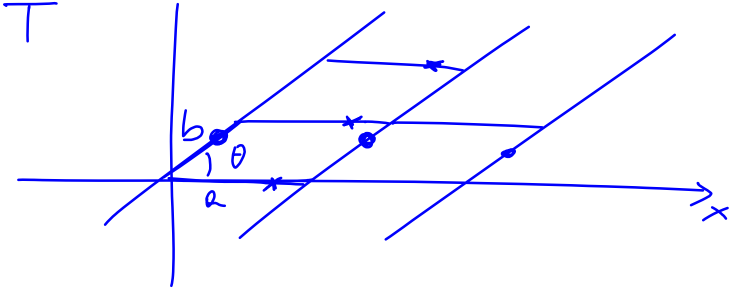
You can define a notion of "homology" from the nilpotency of ∂

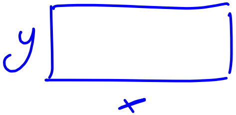
S^2:

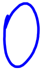
$$\begin{tikzcd}
\omega_0 = 1 \arrow[dr] & \omega_2 = \text{sen}\,\theta\, d\theta \wedge d\varphi \arrow[dl] \\
\text{point} & S^2
\end{tikzcd}$$

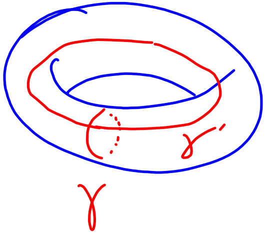
no curve is homologically nontrivial
(every closed curve on S^2 is a boundary)

Torus T

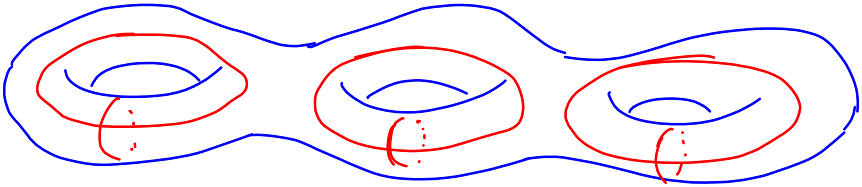




Cohomology:	1	dx	dy	$dx\wedge dy$
Homology:	T torus			• point



Riemann surfaces



g = genus = # of handles

Homology :

$\cdot$ point	$2g$	surface
b_0	b_1	b_2

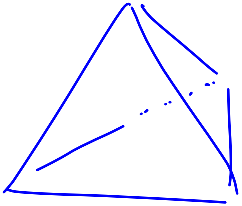
Betti numbers : dimensions of the homology spaces

$b_0 = 1 \qquad b_1 = 2g \qquad b_2 = 1$

$b_0 - b_1 + b_2 = 2 - 2g$ = Euler characteristics of the Riemann surface

Polyhedra

Euler identity



$b_0 = \#$ vertices $= 4$

$b_1 = \#$ edges $= 6$

$b_2 = \#$ faces $= 4$

$4 - 6 + 4 = 2$

Exterior product of differential forms :

$\omega \wedge \Omega$ $\qquad \omega = \omega_{i_1 \dots i_m} dx^{i_1} \wedge \dots dx^{i_m}$

$$\Omega = \Omega_{i_1 \dots i_n} dx^{i_1} \wedge \dots dx^{i_n}$$

$$\omega \wedge \Omega = \omega_{i_1 \dots i_m} \Omega_{i_{m+1} \dots i_{m+n}} dx^{i_1} \wedge \dots\dots \wedge dx^{i_{m+n}}$$

$$\omega \wedge \Omega = (-1)^{\deg \omega \cdot \deg \Omega} \, \Omega \wedge \omega$$

$$dx \wedge dy = - dy \wedge dx$$

$$dx \wedge \underbrace{dy \wedge dz} = dy \wedge dz \wedge dx$$

$$d(\omega \wedge \Omega) = d\omega \wedge \Omega + (-1)^{\text{degree } \omega} \, \omega \wedge d\Omega$$

Exercise

Derivations of vector fields

M = smooth manifold $\quad \mathcal{X}(M)$ = smooth vector fields on M

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

"del" or "nabla"

$$X, Y \in \mathcal{X}(M)$$

$$X, Y \longmapsto \nabla_X Y \equiv Z$$

such that

a) $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$

$\forall f, g \in C^\infty(M)$

$\forall X, Y, Z \in \mathcal{X}(M)$

b) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

c) $\nabla_X (fY) = f \nabla_X Y + X(f) Y$

$\forall f \in C^\infty(M)$

∇ is called (linear) connection

Locally, let us consider the canonical basis $\frac{\partial}{\partial x^i}$ of derivations

$$\nabla_{\partial/\partial x^i} \left(\frac{\partial}{\partial x^j} \right) = \Gamma_{ij}^{k} \frac{\partial}{\partial x^k}$$

Γ_{ij}^{k} are called Christoffel symbols

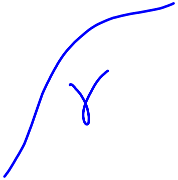
For a generic basis X_i

$$\nabla_{X_i} X_j = b_{ij}^{k} X_k$$

Let $\gamma(t)$ denote a curve $\quad \gamma(t) = (a_1(t), \dots, a_n(t))$

Let us define $\quad \dfrac{d}{dt} = \dot{a}_i(t) \dfrac{\partial}{\partial x^i} \qquad \dot{a}_i(t) = \dfrac{da_i}{dt}$

"the derivative along γ"



Let X denote a vector field $\quad X = u^i(x) \dfrac{\partial}{\partial x^i}$

$$\nabla_{\frac{d}{dt}} X = \nabla_{\frac{d}{dt}} \left(u^i \frac{\partial}{\partial x^i} \right) = u^i \nabla_{\frac{d}{dt}} \left(\frac{\partial}{\partial x^i} \right) + \frac{du^i}{dt} \frac{\partial}{\partial x^i} =$$

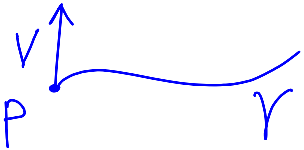
$$= u^i \dot{a}_j \nabla_{\frac{\partial}{\partial x^j}} \left(\frac{\partial}{\partial x^i}\right) + \frac{du^i}{dt}\frac{\partial}{\partial x^i} =$$

$$= \frac{du^i}{dt}\frac{\partial}{\partial x^i} + u^i \dot{a}_j \Gamma^k_{ji}\frac{\partial}{\partial x^k} =$$

$$= \left[\frac{du^i}{dt} + \Gamma^i_{jk}\dot{a}_j u^k\right]\frac{\partial}{\partial x^i}$$

X is parallel to γ if $\nabla_{\frac{d}{dt}} X = 0$

Given a point $p \in M$ and a vector $V \in T_p$ and a curve γ through p, $\gamma(0) = p$



, $\exists$ a unique field X such that

$$X(0) = V \qquad \frac{\nabla_d}{dt} X = 0 \quad \text{along } \gamma$$

This defines a Cauchy problem. It's unique solution is the parallel transport of the vector V along the curve γ

Curvature

Let M denote a smooth manifold, $\mathcal{X}(M)$ = smooth vector fields on M

The curvature R_∇ of the connection ∇ is a map $\mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$X, Y, Z \in \mathfrak{X}(M)$

[Shortcut : $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_Z - \nabla_{[X,Y]}$

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

Properties : $R(X,Y)Z$ is linear in X, Y and Z

$$R(X,Y)Z = -R(Y,X)Z$$

If $f, g, h \in C^\infty(M)$ then

$$R(fX, gY)(hZ) = fgh\, R(X,Y)Z$$

Exercise

Example : $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$ $\qquad \frac{\partial}{\partial x^i} = \partial_i$

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k =$$

$$= \nabla_{\partial_i} \left(\Gamma_{jk}^{\ell} \partial_\ell \right) - \nabla_{\partial_j} \left(\Gamma_{ik}^{\ell} \partial_\ell \right) =$$

$$= \Gamma_{jk}^{\ell} \nabla_{\partial_i} (\partial_\ell) + \partial_i \Gamma_{jk}^{\ell} \partial_\ell - \Gamma_{ik}^{\ell} \nabla_{\partial_j} (\partial_\ell) - \partial_j \Gamma_{ik}^{\ell} \partial_\ell =$$

$$= \Gamma_{jk}^{\ell} \Gamma_{i\ell}^{m} \partial_m + \partial_i \Gamma_{jk}^{m} \partial_m - \Gamma_{ik}^{\ell} \Gamma_{j\ell}^{m} \partial_m + \partial_j \Gamma_{ik}^{m} \partial_m =$$

$$= \left(\partial_i \Gamma_{jk}^{m} - \partial_j \Gamma_{ik}^{m} + \Gamma_{i\ell}^{m} \Gamma_{jk}^{\ell} - \Gamma_{j\ell}^{m} \Gamma_{ik}^{\ell} \right) \partial_m =$$

$$= R_{ij}{}^{m}{}_{k} \partial_m \qquad R_{ij}{}^{m}{}_{k} = \text{Riemann tensor}$$

Torsion

$$T : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Properties : $T(X, Y) = -T(Y, X)$

$$T(fX, gY) = fg\,T(X, Y) \quad \forall f, g \in C^\infty(M)$$

$$\parallel$$

$$\nabla_{fX}(gY) - \nabla_{gY}(fX) - [fX, gY] =$$

$$= f\left(\nabla_X Y\, g + X(g) Y\right) - g\left(\nabla_Y X\, f + Y(f) X\right) +$$

$$- fXgY + gYfX =$$

$$= fg\left(\nabla_X Y - \nabla_Y X\right) + \cancel{f\,X(g)\,Y} - \cancel{g\,Y(f)\,X} +$$

$$- fg\,[X, Y] - \cancel{f\,X(g)\,Y} + \cancel{g\,Y(f)\,X}$$

$$T(\partial_i, \partial_j) = \nabla_{\partial_i}(\partial_j) - \nabla_{\partial_j}(\partial_i) - \cancel{[\partial_i, \partial_j]} =$$

$$= \Gamma_{ij}^{k}\,\partial_k - \Gamma_{ji}^{k}\,\partial_k = \left(\Gamma_{ij}^{k} - \Gamma_{ji}^{k}\right)\partial_k$$

The torsion vanishes if and only if the Christoffel symbols are symmetric in the canonical basis

Bianchi identities

Exercises

1st Bianchi identity

$$R(X,Y)Z + \text{cyclic permutations} =$$

$$= \nabla_X T(Y,Z) + T(X,[Y,Z]) +$$

$$+ \text{cyclic permutations of } X,Y,Z$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$$

2nd Bianchi identity

$$\nabla_X R(Y,Z)W + \text{cyclic perms of } X,Y,Z =$$

$$= R(X,T(Y,Z))W + \text{cyclic perms of } X,Y,Z$$

Riemannian manifold

(M, g) $\quad M =$ manifold

$g =$ metric $=$ positive definite section of $(T^*_M)^{\otimes 2}$

Basis of $(T^*_M)^{\otimes 2}$: $\dfrac{dx^i \otimes dx^j}{2}$

$$g = g_{ij} \frac{dx^i \otimes dx^j}{2} = g_{ij}\, dx^i dx^j \qquad dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$$

$g_{ij} = g_{ji}$ $\qquad g_{ij}$ positive definite

$$V = V^i \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$$

$$g(V, W) = g_{ij} V^i W^j$$

$$W = W^i \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$$

$$g(\partial_i, \partial_j) = g_{ij}$$

Let us orthonormalize g

Let E_i denote vector fields such that

$$g(E_i, E_j) = \delta_{ij} \qquad X = a^i E_i \in \mathcal{X}(M)$$

$$g(X, E_i) = a^j g(E_j, E_i) = a^j \delta_{ji} = a_i$$

$$X = g(X, E_i) E_i$$

Let ∇ denote a linear connection

We define the function

$$\nabla_g : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{C}^\infty(M)$$

$$\nabla_g(X,Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) + - g(Y, \nabla_X Z)$$

Exercise:

$$\nabla_g(fX, hY, kZ) = fhk\,\nabla_g(X,Y,Z)$$

$$\forall f, h, k \in C^\infty(M)$$

The connection ∇ is compatible with the metric g if $\nabla_g \equiv 0$

In a Riemannian manifold there exists one and only one torsionless connection that is compatible with a given metric g

We have to solve $\nabla g = 0$ $T = 0$

We work with ∂_i . $T = 0 \Rightarrow \Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

$$\nabla g(\partial_i, \partial_j, \partial_k) = \partial_i (g(\partial_j, \partial_k)) - g(\nabla_{\partial_i}\partial_j, \partial_k) +$$

$$- g(\partial_j, \nabla_{\partial_i}\partial_k) =$$

$$= \partial_i (g_{jk}) - g(\Gamma_{ij}^{m}\partial_m, \partial_k) +$$

$$- g(\partial_j, \Gamma_{ik}^{m}\partial_m) =$$

$$= \partial_i g_{jk} - \Gamma_{ij}^{m} g_{mk} - \Gamma_{ik}^{m} g_{jm} = 0$$

$$\partial_i g_{jk} - \underline{\Gamma_{ij}^{\ m} g_{mk}} - \cancel{\Gamma_{ik}^{\ m} g_{jm}} = 0 \qquad (1)$$

$(i \leftrightarrow j)$

$$\partial_j g_{ik} - \underline{\Gamma_{ji}^{\ m} g_{mk}} - \cancel{\Gamma_{jk}^{\ m} g_{im}} = 0 \qquad (2)$$

$(k \leftrightarrow j)$

$$\partial_k g_{ij} - \cancel{\Gamma_{ki}^{\ m} g_{mj}} - \cancel{\Gamma_{jk}^{\ m} g_{im}} = 0 \qquad (3)$$

$$(1) + (2) - (3) = 0 = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} +$$

$$- 2 \Gamma_{ij}^{\ m} g_{mk}$$

$$\Gamma_{ij}^{\ m} g_{mk} = \frac{1}{2} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) \quad \vdots \quad \cdot g^{kn}$$

g^{ij} inverse of the metric $\quad g^{ij} g_{jk} = \delta^i_k$

$$\Gamma_{ij}^{\;\;n} = \frac{1}{2} g^{kn} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right)$$

Levi-Civita connection

Basis E_i : $\quad g(E_i, E_j) = \delta_{ij}$

$$\nabla_{E_i} E_j = b^k_{ij} E_k \qquad [E_i, E_j] = a^k_{ij} E_k$$

$T = 0 \qquad \nabla g = 0$

$$T(E_i, E_j) = 0 = \nabla_{E_i} E_j - \nabla_{E_j} E_i - [E_i, E_j] =$$
$$= \left(b^k_{ij} - b^k_{jk} - a^k_{ij} \right) E_k$$

$$a_{ij}^{k} = b_{ij}^{k} - b_{ji}^{k}$$

$$\nabla g(E_i, E_j, E_k) = E_i(\cancel{\overbrace{g(E_j, E_k)}^{\delta_{jk}}}) - g(\nabla_{E_i} E_j, E_k) +$$

$$- g(E_j, \nabla_{E_i} E_k) =$$

$$= -g(b_{ij}^{m} E_m, E_k) - g(E_j, b_{ik}^{m} E_m) =$$

$$= -b_{ij}^{m} \delta_{mk} - b_{ik}^{m} \delta_{mj}$$

Riemann tensor, again

$$R : \mathcal{X}(M)^4 \longrightarrow \mathcal{C}^\infty(M)$$

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

$$\left[R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \right]$$

Properties:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) =$$

$$= -R(X,Y,W,Z) =$$

$$= R(Z,W,X,Y)$$

Exercise

Bianchi identity: $\nabla R = 0$ where

$$\nabla R(X,Y,Z,T,W) = X(R(Y,Z,T,W)) +$$

$$- R(\nabla_X Y, Z, T, W) - R(Y, \nabla_X Z, T, W) +$$

$$- R(Y, Z, \nabla_X T, W) - R(Y, Z, T, \nabla_X W)$$

Exercise (highlight if and where you need to use $T = 0$ and $\nabla g = 0$)

Ricci tensor

$$Ric(X, Y) = \sum_i R(X, E_i, Y, E_i)$$

$$Ric(Y, X) = Ric(X, Y)$$

Scalar curvature

$$R = \sum_j Ric(E_j, E_j)$$

Interior product or contraction

V = vector field ω = k-form

$i_V \omega$ is $k-1$ form

$$i_V \omega (V_1, \dots, V_{k-1}) = \omega (V, V_1, \dots V_{k-1})$$

In local coordinates $\omega = \omega_{i_1 \dots i_k} dx^{i_1} \dots dx^{i_k}$

$$i_V \omega = k \, V^i \omega_{i \, i_1 \dots i_{k-1}} dx^{i_1} \dots dx^{i_{k-1}}$$

Lie derivative

V = vector field, ω = form

$$\mathcal{L}_V \omega = d \, i_V \omega + i_V d \omega$$

$$\mathcal{L}_V = d\, i_V + i_V\, d$$

$\omega =$ 0-form (a function f)

$$\mathcal{L}_V f = i_V\, df = i_V\, dx^i \frac{\partial f}{\partial x^i} = V^i \frac{\partial f}{\partial x^i}$$

$\omega =$ 1-form $\qquad \omega = \omega_i\, dx^i$

$$i_V \omega = V^i \omega_i \qquad d\omega = \partial_j \omega_i\, dx^j\, dx^i$$

$$d\, i_V \omega = \partial_{\underset{i}{j}} V^{\overset{j}{i}}\, \omega_{\underset{j}{i}}\, dx^{\overset{i}{j}} + V^i \partial_j \omega_i\, dx^j$$

$$i_V d\omega = V^j \partial_j \omega_i\, dx^i - V^i \partial_j \omega_i\, dx^j$$

$$\mathcal{L}_V \omega = \left(V^j \partial_j \omega_i + \partial_i V^j \omega_j \right) dx^i$$

Property: $\mathcal{L}_V(\omega_1 \wedge \omega_2) = \mathcal{L}_V \omega_1 \wedge \omega_2 +$

$+ \omega_1 \wedge \mathcal{L}_V \omega_2$

Exercise

$$\omega = \omega_{i_1 \dots i_n} dx^{i_1} \dots dx^{i_n}$$

$$\mathcal{L}_V \omega = \left(V^i \partial_i \omega_{i_1 \dots i_n} + n \, \partial_{i_1} V^i \omega_{i i_2 \dots i_n} \right) dx^{i_1} \dots dx^{i_n}$$

Lie derivative of a vector field:

$X, Y \in \mathcal{X}(M)$ $\quad \mathcal{L}_X Y = [X, Y]$

Flow of a vector field

Let $X = a^i(x) \frac{\partial}{\partial x^i}$ denote a vector field

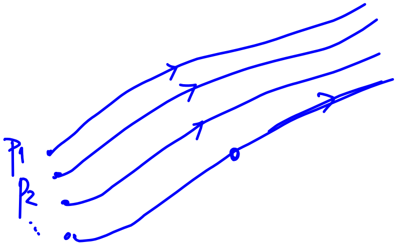
Let $(x_1, \dots x_n)$ denote any point of M

We define a curve $\gamma : [0,1] \longrightarrow M$

$$(\phi^{i_1}(t), \dots, \phi^{i_n}(t))$$

as the solution of the Cauchy problem

$$\begin{cases} \frac{d\phi^i(t)}{dt} = a^i(\phi(t)) \\ \phi^i(0) = x^i \end{cases}$$



Let us rewrite the system and its solution as

$$\phi^i(t,x) \qquad \begin{cases} \dfrac{\partial \phi^i(t,x)}{\partial t} = a^i(\phi(t,x)) \\ \phi^i(0,x) = x^i \end{cases}$$

$$\left.\frac{\partial \phi^i}{\partial x^j}\right|_{t=0} = \delta^i_j$$

Let $f : M \to \mathbb{R}$

$$\mathcal{L}_X f = \lim_{t \to 0} \frac{1}{t}\left[f(\phi(t,x)) - f(x) \right] =$$

$$= \frac{\partial f}{\partial x^i}(\phi(t,x)) \left.\frac{\partial \phi^i(t,x)}{\partial t}\right|_{t=0} =$$

$$= a_i(x) \frac{\partial f}{\partial x^i}(x)$$

Let $Y = b^i(x) \frac{\partial}{\partial x^i}$

$$b^i(\phi(t,x)) \frac{\partial}{\partial \phi^i}$$

$$\mathcal{L}_X Y = \frac{d}{dt}\left[b^i(\phi(t,x)) \frac{\partial x^j}{\partial \phi^i}(\phi(t,x)) \frac{\partial}{\partial x^j} \right]\Bigg|_{t=0} =$$

$$= \frac{\partial \phi^k}{\partial t} \frac{\partial b^i(\phi)}{\partial \phi^k} \frac{\partial x^j}{\partial \phi^i} \frac{\partial}{\partial x^j} \Bigg|_{t=0} +$$

$$+ \ b^i(\phi) \frac{\partial^2 x^j}{\partial t \, \partial \phi^i} \frac{\partial}{\partial x^j} \Bigg|_{t=0} =$$

$$= a^k(x) \, \partial_k b^i(x) \frac{\partial}{\partial x^i} - b^i(x) \, \partial_i a^j(x) \frac{\partial}{\partial x^j} = [X, Y]$$

$$\left.\frac{\partial^2 x^j}{\partial t\, \partial \phi^i}\right|_{t=0} = - \underline{\partial_i a^j}(x)$$ Indeed

$$M^i{}_j = \frac{\partial \phi^i}{\partial x^j} \qquad M\big|_{t=0} = \mathbb{1}$$

$$\dot{M}^i{}_j = \frac{\partial^2 \phi^i}{\partial t\, \partial x^j} \qquad \dot{M}^{-1} = - M^{-1} \dot{M} M^{-1}$$

$$\dot{M}^{-1}\big|_{t=0} = - \dot{M}\big|_{t=0}$$

$$\left.(\dot{M}^{-1})^i{}_j\right|_{t=0} = - \left.\frac{\partial^2 \phi^i}{\partial x^j\, \partial t}\right|_{t=0} = - \partial_j a^i$$ <u>ok</u>

Let ω denote a 1 form

$$\omega = \omega_i(x)\, dx^i$$

Consider $\omega_i(\phi(t,x))\, d\phi^i = \omega_i(\phi(t,x)) \frac{\partial \phi^i}{\partial x^j} dx^j$

$$\mathcal{L}_v \omega = \frac{\partial}{\partial t}\left[\omega_i(\phi(t,x)) \frac{\partial \phi^i}{\partial x^j} dx^j\right]\Bigg|_{t=0} =$$

$$= \frac{\partial \phi^j}{\partial t}\Bigg|_{t=0} \partial_j \omega_i(x)\, dx^i + \omega_i(x) \frac{\partial}{\partial x^j} \frac{\partial \phi^i}{\partial t}\Bigg|_{t=0} dx^j =$$

$$= \left(a^j \partial_j \omega_i + \omega_j \partial_i a^j\right) dx^i$$

<u>ok</u>

Lie derivative : infinitesimal diffeomorphisms

Metric $g_{ij}\, dx^i dx^j$

$$\frac{\partial}{\partial t}\left[g_{ij}(\phi(t,x)) \frac{\partial \phi^i}{\partial x^k} \frac{\partial \phi^j}{\partial x^m}\, dx^k dx^m \right]\Bigg|_{t=0} =$$

$$= \left(a^l \partial_l g_{ij}\, \delta^i_k \delta^j_m + g_{ij}\, \partial_k a^i\, \delta^j_m + g_{ij}\, \delta^i_k\, \partial_m a^j \right) dx^k dx^m =$$

$$= \left(a^l \partial_l g_{ij} + g_{lj}\, \partial_i a^l + g_{il}\, \partial_j a^l \right) dx^i dx^j =$$

$$= \mathcal{L}_X g \qquad \mathcal{L}_X g_{ij} = a^m \partial_m g_{ij} + g_{mj} \partial_i a^m + g_{im} \partial_j a^m$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} = g'_{\mu\nu}(x+\xi) \simeq$$

$$\simeq g'_{\mu\nu}(x) + \xi^{\rho} \partial_{\rho} g'_{\mu\nu}(x)$$

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} \simeq \delta^{\mu}_{\rho} + \partial_{\rho} \xi^{\mu} , \quad \frac{\partial x^{\rho}}{\partial x'^{\nu}} \simeq \delta^{\rho}_{\nu} - \partial_{\nu} \xi^{\rho}$$

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) - \partial_{\mu} \xi^{\rho} g_{\rho\nu} - \partial_{\nu} \xi^{\rho} g_{\rho\mu}$$

$$- \xi^{\rho} \partial_{\rho} g_{\mu\nu} = g_{\mu\nu}(x) - \mathcal{L}_{\xi} g_{\mu\nu}$$

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x) = - \mathcal{L}_{\xi} g_{\mu\nu}$$

Scalar $\varphi'(x') = \varphi(x) = \varphi'(x+\xi) \simeq$

$$\simeq \varphi'(x) + \xi^\rho \partial_\rho \varphi'$$

$$\varphi'(x) - \varphi(x) \simeq - \xi^\rho \partial_\rho \varphi = - \mathcal{L}_\xi \varphi$$

Lorentzian manifolds

An invertible metric with signature $(1,-1,-1,-1)$ is assumed $\quad \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$$x^\mu \qquad \frac{\partial}{\partial x^\mu} = \partial_\mu \qquad dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$$

$$\nabla_{\partial_\mu}(\partial_\nu) = \Gamma^\rho_{\mu\nu}\,\partial_\rho$$

$E_i \qquad g(E_i, E_j) = \delta_{ij} \qquad \longrightarrow$

$e_a \qquad g(e_a, e_b) = \eta_{ab}$

e_a are vector fields $\qquad e_a = e_a^{\mu} \partial_\mu$

$$\nabla_{e_a}(e_b) = \gamma_{ab}^{c}\, e_c$$

$a, b, c, \ldots$ flat-space indices

$\mu, \nu, \rho, \ldots$ spacetime indices

We want to introduce differential forms e^a

dual to e_a : $\qquad e^a = H_\mu^a \, dx^\mu \qquad e^a(e_b) = \delta_b^a$

$$\delta^a_b = e^a(e_b) = H^a_\mu dx^\mu (e^\nu_b \partial_\nu) =$$

$$= H^a_\mu e^\nu_b \, dx^\mu(\partial_\nu) = \underline{\underline{H^a_\nu e^\nu_b}}$$

$H^a{}_\mu$ is the inverse matrix of $e^\mu{}_a$

$\hookrightarrow \quad e^a_\mu$

$$\boxed{e^a_\mu e^\mu_b = \delta^a_b}$$

$A \cdot B = I$

$B \cdot A = I$:

$$e^a = e^a_\mu dx^\mu$$

$$\begin{cases} e^a{}_\mu e^\mu{}_b = \delta^a{}_b \\ e^\mu{}_a e^a{}_\nu = \delta^\mu{}_\nu \end{cases}$$

$$e^\mu_a e^a_\nu = \delta^\mu_\nu$$

$$g(e_a, e_b) = \eta_{ab} = g_{\mu\nu}\, e^\mu_a\, e^\nu_b$$

$$e^a_\rho\, \eta_{ab}\, e^b_\sigma = \underbrace{e^a_\rho\, \overbrace{e^b_\sigma\, g_{\mu\nu}\, e^\mu_a\, e^\nu_b}^{\delta^\nu_\sigma}}_{\delta^\mu_\rho} = g_{\rho\sigma}$$

$$g_{\mu\nu} = e^a_\mu\, \eta_{ab}\, e^b_\nu$$

e^a_μ tetrad

vierbein

$$g_{\mu\nu}\, e^\nu_c = e^\nu_c\, e^a_\mu\, \eta_{ab}\, e^b_\nu = e^a_\mu\, \eta_{ac}$$

$g_{\mu\nu}$ lowers spacetime indices

η_{ab} lowers flat-space indices

$$\nabla_{e_a} e_b = \gamma_{ab}^{\ \ c}\, e_c \qquad\qquad g(e_a, e_b) = \eta_{ab}$$

Metric compatibility condition

$$\nabla_g (X, Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) +$$

$$- g(Y, \nabla_X Z) = 0$$

$$0 = \nabla_g (e_a, e_b, e_c) = e_a\big(\overset{/\!/\ \eta_{bc}}{\cancel{g(e_b, e_c)}}\big) +$$

$$- g(\nabla_{e_a} e_b, e_c) - g(e_b, \nabla_{e_a} e_c) =$$

$$= - \gamma_{ab}^{\ \ d}\, g(e_d, e_c) - g(e_b, e_d)\, \gamma_{ac}^{\ \ d} =$$

$$= - \gamma_{ab}^{\ \ d}\, \eta_{dc} - \gamma_{ac}^{\ \ d}\, \eta_{bd}$$

$\gamma^a_{cb} e^c = \omega^a{}_b = \omega_\mu{}^a{}_b \, dx^\mu$ 1-form

spin connection

$\nabla g = 0 \iff \omega^d{}_b \eta_{dc} + \omega^d{}_c \eta_{db} = 0$

$\omega^{ab} = -\omega^{ba}$

Torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

$T(e_a, e_b) = \nabla_{e_a} e_b - \nabla_{e_b} e_a - [e_a, e_b] =$

$= \gamma_{ab}^c e_c - \gamma_{ba}^c e_c - [e_a, e_b]$

Multiply by $\dfrac{e^a \wedge e^b}{2}$

$$\frac{e^a \wedge e^b}{\cancel{2}} \left[\cancel{2}\, \gamma^c_{ab}\, e_c - e_a\, e_b\, \cancel{2} \right] = \omega^c{}_b \wedge e^b\, e_c +$$

$$- e^a \wedge e^b\, e_a\, e_b = \omega^a{}_b \wedge e^b\, e_a +$$

$$- e^a_\mu\, dx^\mu \wedge e^b_\nu\, dx^\nu\, e^\rho_a \frac{\partial}{\partial x^\rho}\, e^\sigma_b \frac{\partial}{\partial x^\sigma} =$$

$$= \omega^a{}_b\, e^b\, e_a - dx^\mu \wedge e^b \frac{\partial}{\partial x^\mu}\, e_b =$$

$$= \omega^a{}_b\, e^b\, e_a - dx^\mu\, e^b_\nu\, dx^\nu\, \partial_\mu\, e^\sigma_b\, \partial_\sigma =$$

$$= \omega^a{}_b\, e^b\, e_a - dx^\mu\, \cancel{e^b_\nu\, dx^\nu\, e^\sigma_b}\, \partial_\mu\, \partial_\sigma +$$

$$- dx^\mu\, e^b_\nu\, dx^\nu \left(\partial_\mu\, e^\sigma_b \right) \partial_\sigma$$

$$e^b_\nu\, \partial_\mu\, e^\sigma_b = - \partial_\mu\, e^b_\nu\, e^\sigma_b \qquad \partial_\mu \left(e^b_\nu\, e^\sigma_b \right) = 0$$

$$\frac{e^a \wedge e^b}{2} T(e_a, e_b) = \omega^a{}_b e^b e_a +$$

$$+ dx^\mu \partial_\mu e^b_\nu dx^\nu e^\sigma_b \partial_\sigma =$$

$$= \omega^a{}_b e^b e_a + de^a e_a = (de^a + \omega^a{}_b e^b) e_a$$

$$\partial_\mu e^b_\nu dx^\mu dx^\nu = de^b$$

$$e^b = e^b_\nu dx^\nu$$

$$T^a = de^a + \omega^a{}_b e^b = \nabla e^a$$ Torsion

Curvature $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

$$R(e_a, e_b) e_c = \nabla_{e_a} \nabla_{e_b} e_c - \nabla_{e_b} \nabla_{e_a} e_c - \nabla_{[e_a, e_b]} e_c =$$

$$[e_a, e_b] = a^c_{ab} e_c$$

$$= \nabla_{e_a}\left(\gamma^d_{bc}\, e_d\right) - \nabla_{e_b}\left(\gamma^d_{ac}\, e_d\right) - Q^d_{ab}\, \gamma^f_{dc}\, e_f =$$

$$= \gamma^d_{bc}\, \gamma^f_{ad}\, e_f + e_a\left(\gamma^d_{bc}\right) e_d +$$

$$- \gamma^d_{ac}\, \gamma^f_{bd}\, e_f + e_b\left(\gamma^d_{ac}\right) e_d - Q^d_{ab}\, \gamma^f_{dc}\, e_f$$

Let us multiply by $\frac{e^a \wedge e^b}{2}$ $\qquad dx^\mu \partial_\mu = d$

$$\frac{1}{2} e^a \wedge e^b\, R(e_a, e_b)\, e_c = \omega^f{}_d\, \omega^d{}_c\, e_f +$$

$$+ e^a \wedge e^b\, e_a\left(\gamma^d_{bc}\right) e_d - \frac{1}{2} e^a \wedge e^b\, Q^d_{ab}\, \gamma^f_{dc}\, e_f$$

$$e^a \wedge e^b\, e_a\left(\gamma^d_{bc}\right) e_d = \underline{e^a_\mu}\, \underset{\cdots}{e^b_\nu}\, \underline{dx^\mu}\, \underset{\cdots}{dx^\nu}\, e^\rho_a\, \partial_\rho\left(\gamma^d_{bc}\right) e_d =$$

$$= d\gamma^d_{bc}\, e^b\, e_d = d\omega^d{}_c\, e_d - \gamma^d_{bc}\, de^b\, e_d$$

$$-\frac{1}{2}\, e^a \wedge e^b \,[e_a, e_b] = - e^a_\mu \, e^b_\nu \, dx^\mu dx^\nu \, e_a^\rho \left(\partial_\rho e_b^\sigma\right) \partial_\sigma =$$

$$= - de_b^\sigma \; e^b_\nu \, dx^\nu \, \partial_\sigma = d e^b_\nu \, dx^\nu \, e_b =$$

$$= \partial_\mu e_\nu^b \, dx^\mu dx^\nu \, e_b = de^b \, e_b$$

$$\longrightarrow \; -\frac{1}{2}\, e^a \wedge e^b \; a_{ab}^c \, e_c$$

$$d\left(e_b^\sigma e^b_\nu\right) = d\,\delta^\sigma_\nu = 0 = de_b^\sigma \; e^b_\nu + e_b^\sigma \, de^b_\nu$$

$$-\frac{1}{2}\, e^a \wedge e^b \; a_{ab}^c = de^c$$

Summarizing, we have

$$\frac{1}{2} e^a \wedge e^b \; R(e_a, e_b) e_c = \omega^e{}_d \wedge \omega^d{}_c \, e_e + e^a \wedge e^b \; e_a(\gamma^d_{bc}) e_d - \frac{1}{2} e^a \wedge e^b \, a^d_{ab} \gamma^e_{dc} \, e_e$$

$$e^a \wedge e^b \; e_a(\gamma^d_{bc}) e_d = d\omega^d{}_c \, e_d - \gamma^d_{bc} \, de^b \, e_d$$

$$-\frac{1}{2} e^a \wedge e^b \, a^d_{ab} = de^d$$

$$\frac{1}{2} e^a \wedge e^b \; R(e_a, e_b) e_c = \omega^e{}_d \wedge \omega^d{}_c \, e_e + d\omega^d{}_c \, e_d +$$

$$- \cancel{\gamma^d_{bc} \, de^b \, e_d} + \cancel{de^d \, \gamma^e_{dc} \, e_e} =$$

$$= \left(d\omega^d{}_c + \omega^d{}_b \, \omega^b{}_c \right) e_d \equiv R^d{}_c \, e_d$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c\,\omega^c{}_b$$

$$R = d\omega + \omega\omega \qquad T = \nabla e = de + \omega e$$

$\Lambda^k_{(m,n)}$ = space of k-forms with m upper flat-space indices and n lower flat-space indices

$$e^a \in \Lambda^1_{(1,0)} \qquad T^a \in \Lambda^2_{(1,0)} \qquad R^a{}_b \in \Lambda^2_{(1,1)}$$

Let $T^{a_1\dots a_m}{}_{b_1\dots b_n} \in \Lambda^k_{(m,n)}$

Covariant derivative

$$\nabla T^{a_1\dots a_m}{}_{b_1\dots b_n} = d\,T^{a_1\dots a_m}{}_{b_1\dots b_n} + \sum_{i=1}^{m} \omega^{a_i}{}_c\, T^{a_1\dots \overset{c}{\cancel{a_i}}\dots a_m}{}_{b_1\dots b_n} +$$

$$- (-1)^k \sum_{i=1}^{n} T^{a_1 \dots a_m}{}_{b_1 \dots \underset{c}{\cancel{b_i}} \dots b_n} \; \omega^c{}_{b_i}$$

$$T^a = \nabla e^a = de^a + \omega^a{}_b e^b$$

$k=1$

Let $W^a \in \Lambda^0_{(1,0)} \quad V_a \in \Lambda^0_{(0,1)}$

$$\nabla W^a = dW^a + \omega^a{}_b W^b$$

$$\nabla V_a = dV_a - V_c \omega^c{}_a$$

$$\nabla (W^a V_a) = \nabla W^a V_a + W^a \nabla V_a =$$

$$= dW^a V_a + \cancel{\omega^a{}_b W^b V_a} + W^a dV_a - \cancel{W^a V_b \omega^b{}_a} =$$

$$= d(W^a V_a)$$

In general (exercise)

$$\nabla \left(T_k{}^{a_1 \cdots a_m}{}_{b_1 \cdots b_n} \wedge S_h{}^{c_1 \cdots c_r}{}_{d_1 \cdots d_p} \right) = \nabla T_k{}^{a_1 \cdots a_m}{}_{b_1 \cdots b_n} \wedge S_h{}^{c_1 \cdots c_r}{}_{d_1 \cdots d_p} +$$

$$+ (-1)^k \, T_k{}^{a_1 \cdots a_m}{}_{b_1 \cdots b_n} \, \nabla S_h{}^{c_1 \cdots c_r}{}_{d_1 \cdots d_p}$$

Bianchi identities

$$T = \nabla e = de + \omega e \qquad\qquad T^a$$

$$\nabla T = dT + \omega T = d(\cancel{de} + \omega e) + \omega (de + \omega e) =$$

$$= d\omega\, e - \omega \cancel{de} + \omega \cancel{de} + \omega \omega e = R \wedge e$$

$$R = d\omega + \omega \omega$$

$$T = 0 \quad \Rightarrow \quad R \wedge e = 0$$

$$R(X,Y)Z + \text{cyd.} = \nabla_X T(Y,Z) + T(X,[Y,Z]) + \text{cyd.}$$

$$T=0 \quad \Rightarrow \quad R^a{}_b \, e^b = 0$$

$$R^a{}_b = R_{cd}{}^a{}_b \, \frac{e^c \wedge e^d}{2} \qquad R_{cb}{}^a{}_b = \text{Riemann tensor}$$

$$(\text{also}) = R^a{}_{bcd}$$

$$0 = R^a{}_b e^b = \frac{1}{2} R_{cd}{}^a{}_b \, e^c e^d e^b$$

$$0 = R_{ab}{}^c{}_d + R_{da}{}^c{}_b + R_{bd}{}^c{}_a$$

$$R = d\omega + \omega\omega \qquad 2^{nd} \text{ Bianchi identity: } \nabla R = 0$$

$$\nabla R^a{}_b = dR^a{}_b + \omega^a{}_c R^c{}_b - R^a{}_c \omega^c{}_b$$

$$\nabla R = dR + \omega R - R\omega =$$

$$= d(\cancel{d\omega} + \omega\omega) + \omega(d\omega + \omega\omega) +$$

$$-(d\omega + \omega\omega)\omega =$$

$$= \cancel{d\omega\,\omega} - \cancel{\omega\, d\omega} + \cancel{\omega\, d\omega} + \cancel{\omega\omega\omega} +$$

$$- \cancel{d\omega\,\omega} - \cancel{\omega\omega\omega} = 0$$

Change of basis for flat-space indices

$$\begin{cases} e^{a'} = \Omega^a{}_b\, e^b & \Omega = \Omega(x) \quad \Omega \in GL(4,\mathbb{R}) \\ e_a{}' = (\Omega^{-1})^b{}_a\, e_b & \text{Indeed, } \quad e^a(e_b) = \delta^a_b \end{cases}$$

$$e^{a'}(e_b{}') = \delta^a_b = \Omega^a{}_c\, e^c\left((\Omega^{-1})^d{}_b\, e_d\right) =$$

$$= \Omega^a{}_c (\Omega^{-1})^d{}_b\, \delta^c_d = \delta^a_b$$

$$\nabla_{e_a} e_b = \gamma_{ab}^{\ \ c}\, e_c \qquad \nabla_{e'_a} e'_b = \gamma_{ab}^{\ \ c'}\, e'_c =$$

$$= \nabla_{(\Omega^{-1})^d{}_a e_d} \left((\Omega^{-1})^e{}_b\, e_e \right) =$$

$$= (\Omega^{-1})^d{}_a\, \nabla_{e_d}\left((\Omega^{-1})^e{}_b\, e_e \right) =$$

$$= (\Omega^{-1})^d{}_a (\Omega^{-1})^e{}_b\, \nabla_{e_d} e_e +$$

$$+ (\Omega^{-1})^d{}_a\, e_d\left((\Omega^{-1})^e{}_b \right) e_e =$$

$$= (\Omega^{-1})^d{}_a (\Omega^{-1})^e{}_b\, \gamma_{de}^{\ \ f}\, \Omega^g{}_f\, e'_g +$$

$$+ (\Omega^{-1})^d{}_a\, e_d\left((\Omega^{-1})^e{}_b \right) \Omega^g{}_e\, e'_g$$

$$\gamma_{ab}{}^{g\,\prime} = (\Omega^{-1})^d{}_a (\Omega^{-1})^e{}_b \, \gamma_{de}{}^f \, \Omega^g{}_f \; +$$

$$+ (\Omega^{-1})^d{}_a \, e_d((\Omega^{-1})^e{}_b) \, \Omega^g{}_e \qquad \omega^f{}_e$$

$$\omega^a{}_b{}' = \gamma_{cb}{}^{a\,\prime} \, e^{c\,\prime} = (\Omega^{1})^e{}_b \, \boxed{\gamma_{ce}{}^f} \, \Omega^a{}_f \, \boxed{e^c} +$$

$$+ \; e_c((\Omega^{-1})^e{}_b) \, \Omega^a{}_e \, e^c \; =$$

$$= \Omega^a{}_f \, \omega^f{}_e \, (\Omega^{-1})^e{}_b \; +$$

$$+ \; \Omega^a{}_e \;\; e^c \, e_c((\Omega^{-1})^e{}_b)$$

$$e^c \, e_c(f) = \boxed{e^c_\mu} \, dx^\mu \, \boxed{e^\nu_c} \frac{\partial}{\partial x^\nu} f = df$$

$$\omega' = \Omega \omega \Omega^{-1} + \Omega d\Omega^{-1}$$

$$e' = \Omega e$$

$$T = \nabla e = de + \omega e$$

$$T' = \nabla' e' = de' + \omega' e' =$$

$$= d(\Omega e) + \left(\Omega \omega \Omega^{-1} + \Omega d \Omega^{-1}\right) \Omega e =$$

$$= \cancel{d\Omega e} + \underbrace{\Omega de} + \underbrace{\Omega \omega e} +$$

$$+ \Omega \cancel{d\Omega^{-1} \Omega} e = \Omega \nabla e = \Omega T$$

$$1 = \Omega \Omega^{-1} \qquad d1 = 0 = d\Omega\, \Omega^{-1} + \Omega d \Omega^{-1}$$

$$0 = d\Omega + \Omega d\Omega^{-1} \Omega$$

$$R = d\omega + \omega\omega$$

$$R' = d\omega' + \omega'\omega' = d\left(\Omega\omega\Omega^{-1} + \Omega d\Omega^{-1}\right) +$$

$$+ \left(\Omega\omega\Omega^{-1} + \Omega d\Omega^{-1}\right)\left(\Omega\omega\Omega^{-1} + \Omega d\Omega^{-1}\right) =$$

$$= d\Omega\,\underline{\omega\Omega^{-1}} + \underline{\Omega\, d\omega\, \Omega^{-1}} - \cancel{\Omega\omega d\Omega^{-1}} +$$

$$+ \underline{\cancel{d\Omega\, d\Omega^{-1}}} + \underline{\Omega\omega\omega\Omega^{-1}} + \Omega d\Omega^{-1}\Omega\,\underline{\omega\Omega^{-1}} +$$

$$+ \cancel{\Omega\omega d\Omega^{-1}} + \underline{\cancel{\Omega d\Omega^{-1}}\,\Omega d\Omega^{-1}} =$$

$$= \Omega\left(d\omega + \omega\omega\right)\Omega^{-1} = \Omega R \Omega^{-1}$$

$$R^{a}{}_{b}$$

$$T = \nabla e \qquad e' = \Omega e \qquad \Omega^{-1} e' = e$$

$$T' = \Omega\, T = \Omega\, \nabla e' = \Omega\, \nabla \Omega^{-1} e' =$$

$$= \nabla' e' \qquad \nabla' = \Omega \nabla \Omega^{-1}$$

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu \qquad \underline{g(e_a, e_b) = \eta_{ab}}$$

$$e^{a\,\prime}_\mu = \Omega^a{}_b\, e^b_\mu$$

Do we have $g'_{\mu\nu} = e^{a\,\prime}_\mu \eta_{ab} e^{b\,\prime}_\nu = g_{\mu\nu}$?

$$g'_{\mu\nu} = \Omega^a{}_c\, e^c_\mu\, \eta_{ab}\, \Omega^b{}_d\, e^d_\nu = g_{\mu\nu}$$

only if $\Omega^a{}_c\, \eta_{ab}\, \Omega^b{}_d = \eta_{cd}$ $\qquad \cancel{GL(4,\mathbb{R})}$

Local Lorentz transformation :

change of basis $e^{a'}(x) = \Lambda^a{}_b(x)\, e^b$

Λ : Lorentz transformation

$$\eta_{ab} = \Lambda^c{}_a(x)\, \eta_{cd}\, \Lambda^d{}_b(x) \qquad \forall$$

What we can do with a more general change of basis

$$e^a = e^a_\mu\, dx^\mu \qquad \nabla_{\partial_\mu}(\partial_\nu) = \Gamma^\rho_{\mu\nu}\, \partial_\rho \qquad \Gamma^\mu{}_\nu = \Gamma^\mu_{\rho\nu}\, dx^\rho$$

$$e^{a'} = \Omega^a{}_b\, e^b \qquad \nabla_{e_a}(e_b) = \gamma^c_{ab}\, e_c \qquad \omega^a{}_b = \gamma^a_{cb}\, e^c$$

$$\omega' = \Omega\, \omega\, \Omega^{-1} + \Omega\, d\Omega^{-1}$$

$$" \omega = e\, \Gamma\, e^{-1} + e\, d\, e^{-1} "$$

$$\omega^a{}_b = e^a_\mu \Gamma^\mu{}_\nu e^\nu_b + e^a_\mu\, d e^\mu_b$$

$$dx^\mu \omega_\mu{}^a{}_b = e^a_\rho \Gamma^\rho_{\mu\nu}\, dx^\mu e^\nu_b + e^a_\nu\, dx^\mu \partial_\mu e^\nu_b$$

$$\omega_\mu{}^a{}_b = e^a_\rho \Gamma^\rho_{\mu\nu} e^\nu_b + e^a_\nu \partial_\mu e^\nu_b$$

completely general

(no assumptions $T = 0$, $\nabla g = 0$)

We can define

$\nabla T^{a_1 \dots a_m \nu_1 \dots \nu_s}{}_{b_1 \dots b_n \mu_1 \dots \mu_r}$ by replacing $\omega^a{}_b$ with $\Gamma^\mu{}_\nu$

$$\nabla e^a_\mu = d e^a_\mu + \omega^a{}_b \, e^b_\mu - e^a_\nu \, \Gamma^\nu{}_\mu$$

This is always zero $\qquad \nabla = dx^\mu \nabla_\mu$

$$dx^\rho \nabla_\rho e^a_\mu = \nabla e^a_\mu = dx^\rho \partial_\rho e^a_\mu + dx^\rho \omega_\rho{}^a{}_b \, e^b_\mu +$$

$$- dx^\rho e^a_\nu \Gamma^\nu_{\rho\mu} \overset{?}{=} 0$$

$$\partial_\rho e^a_\mu + \omega_\rho{}^a{}_b \, e^b_\mu - e^a_\nu \Gamma^\nu_{\rho\mu} \overset{?}{=} 0$$

$$\partial_\rho e^a_\mu + \left(e^a_\sigma \Gamma^\sigma_{\rho\nu} e^\nu_b + e^a_\sigma \partial_\rho e^\sigma\right) e^b_\mu - e^a_\nu \Gamma^\nu_{\rho\mu} =$$

$$= \partial_\rho e^a_\mu + \cancel{e^a_\sigma \Gamma^\sigma_{\rho\mu}} + e^a_\sigma \partial_\rho e^\sigma_b e^b_\mu - \cancel{e^a_\nu \Gamma^\nu_{\rho\mu}} =$$

$$- \partial_\rho e^a_\sigma e^\sigma_b e^b_\mu$$

$$= \partial_\rho e^a_\mu - \partial_\rho e^a_\mu = 0$$ ok!

We can consider

$$\nabla g_{\mu\nu} = d g_{\mu\nu} - g_{\rho\nu} \Gamma^\rho{}_\mu - g_{\mu\rho} \Gamma^\rho{}_\nu$$

$$\|$$

$$dx^\rho \nabla_\rho g_{\mu\nu} = dx^\rho \partial_\rho g_{\mu\nu} - dx^\rho g_{\sigma\nu} \Gamma^\sigma_{\rho\mu} - dx^\rho g_{\mu\sigma} \Gamma^\sigma_{\rho\nu}$$

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - g_{\sigma\nu} \Gamma^\sigma_{\rho\mu} - g_{\mu\sigma} \Gamma^\sigma_{\rho\nu}$$

If $\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$ and $\nabla_\mu g_{\nu\rho} = 0$, then

$\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho}(g)$, which implies $\left[g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b \right]$

$$\omega_\mu{}^a{}_b = e_\rho^a \, \Gamma_{\mu\nu}{}^{\rho}(g) \, e_b^\nu + e_\nu^a \partial_\mu e_b^\nu = \omega_\mu{}^a{}_b(e)$$

$$\nabla e_\mu^a = d e_\mu^a + \omega^a{}_b \, e_\mu^b - e_\nu^a \, \Gamma^\nu{}_\mu = 0$$

$$dx^\mu = \delta_\nu^\mu \, dx^\nu$$

$$\nabla \delta_\mu^\nu = d \delta_\mu^\nu + \Gamma^\nu{}_\rho \, \delta_\mu^\rho - \delta_\rho^\nu \, \Gamma^\rho{}_\mu = 0$$

$T^a = \nabla e^a$ Basis dx^μ

$$T^\mu = \nabla dx^\mu = \cancel{d\,dx^\mu} + \Gamma^\mu{}_\nu \, dx^\nu$$

$$T = 0 \iff \Gamma^\mu{}_\nu \, dx^\nu = dx^\rho dx^\nu \, \Gamma^\mu{}_{\rho\nu}$$

$$\iff \Gamma^\mu{}_{[\rho\nu]} = 0$$

$$T^a = \nabla e^a = \nabla\left(e^a{}_\mu \, dx^\mu\right) = \cancel{\left(\nabla e^a{}_\mu\right) dx^\mu} + e^a{}_\mu \left(\nabla dx^\mu\right) = e^a{}_\mu \, \nabla dx^\mu$$

$$\nabla e^a = 0 \iff \nabla dx^\mu = 0$$

2^{nd} Bianchi identity again

$$\nabla R = 0 \qquad \nabla R^a{}_b = 0 \qquad R^a{}_b = R^a{}_{bcd} \frac{e^c \wedge e^d}{2} =$$

$$= R^a{}_{b\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2}$$

$$0 = \nabla \left(R^a{}_{bcd} \frac{e^c e^d}{2} \right)$$

If $T = \nabla e = 0$ $\qquad 0 = (\nabla R^a{}_{bcd}) \frac{e^c \wedge e^d}{2} =$

$$= e^f \, \nabla_f R^a{}_{bcd} \frac{e^c \wedge e^d}{2} \qquad \text{3-form}$$

$$0 = \nabla_c R^a{}_{bde} + \nabla_e R^a{}_{bcd} + \nabla_d R^a{}_{bec}$$

Contracted Bianchi identities

(under the metric compatibolity assumption $\nabla\eta_{ab} = 0$)

$$R^{a}{}_{bcd}\,\delta^{c}_{a} = R_{bd} \qquad R_{bd}\,\eta^{bd} = R$$

$$0 = \nabla_c R_{be} - \nabla_e R_{bc} + \nabla_a R^{a}{}_{bec}$$

$$\times\ \eta^{be}$$

$$0 = \nabla_c R - \nabla_b Ric^{b}{}_{c} - \nabla_a Ric^{a}{}_{c}$$

$$\nabla_a Ric^{a}{}_{b} = \frac{1}{2}\nabla_b R \qquad \nabla_\mu R^{\mu}_{\nu} = \frac{1}{2}\nabla_\nu R$$

Scalar fields $\varphi : M \longrightarrow \mathbb{R}$

$$\varphi'(x') = \varphi(x) \qquad \underset{\displaystyle \overset{\shortparallel}{dx^\mu \nabla_\mu \varphi}}{\nabla\varphi} = d\varphi = dx^\mu \partial_\mu \varphi$$

$$\nabla_\mu \varphi = \partial_\mu \varphi$$

Action

$$S = \frac{1}{2}\int_M d^4x \sqrt{-g}\left[g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + \underbrace{\xi R\varphi^2}_{\substack{\uparrow \\ \text{non minimal term}}} - m^2\varphi^2\right]$$

$$= \frac{1}{2}\int_M d^4x \sqrt{-g}\, \mathcal{L}(x)$$

Flat space $g_{\mu\nu}(x) = \eta_{\mu\nu}$ $\qquad S = \frac{1}{2}\int d^4x \left(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right)$

$$\varphi'(x') = \varphi(x)$$

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x^{\mu'}} g_{\rho\sigma}(x) \frac{\partial x^{\sigma}}{\partial x^{\nu'}}$$

$$g = \det(g_{\mu\nu}) \qquad d^4x' \sqrt{-g(x')} = d^4x \sqrt{-g(x)}$$

$$\nabla'_{\mu}\varphi'(x') = \partial'_{\mu}\varphi'(x') = \frac{\partial x^{\nu}}{\partial x^{\mu'}} \partial_{\nu}\varphi(x)$$

$$g^{\mu\nu'}(x') = \frac{\partial x^{\mu'}}{\partial x^{\rho}} \frac{\partial x^{\nu'}}{\partial x^{\sigma}} g^{\rho\sigma}(x)$$

$$R'(x') = R(x) \qquad \mathcal{L}'(x') = \mathcal{L}(x)$$

Infinitely many terms can be added

$$\int d^4x \sqrt{-g}\, R^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi, \quad \int d^4x \sqrt{-g} \left(g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi\right)^2,$$

....

Gauge fields

QED $\quad A_\mu \qquad A = A_\mu dx^\mu$

$$F = F_{\mu\nu} \frac{dx^\mu dx^\nu}{2} = \partial_\mu A_\nu \, dx^\mu dx^\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \nabla_\mu A_\nu - \nabla_\nu A_\mu = ?$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^{\mu'}} F_{\rho\sigma}(x) \frac{\partial x^\sigma}{\partial x^{\nu'}}$$

$$S = -\frac{1}{4}\int_M d^4x \sqrt{-g}\, F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$$

What about $F_{\mu\nu} \longrightarrow \nabla_\mu A_\nu - \nabla_\nu A_\mu =$

$$= \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu A_\mu + \Gamma^\rho_{\nu\mu} A_\rho$$

$$= F_{\mu\nu} - \Big(\underbrace{\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}}_{\text{tensor!}}\Big) A_\rho$$

$$\nabla A_\nu = dA_\nu - A_\rho \Gamma^\rho{}_\nu = dx^\mu \partial_\mu A_\nu - A_\rho\, dx^\mu\, \Gamma^\rho_{\mu\nu}$$

$$\|$$

$$dx^\mu \nabla_\mu A_\nu \qquad\qquad \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho$$

Transformation of $\Gamma^{\rho}_{\mu\nu}$

$$\omega' = \Omega \omega \Omega^{-1} + \Omega \, d\Omega^{-1}$$

$$e^{a} \longrightarrow dx^{\mu}$$

$$\omega^{a}{}_{b} \longrightarrow \Gamma^{\mu}{}_{\nu} = \Gamma^{\mu}_{\rho\nu} \, dx^{\rho}$$

$$e^{a'} = \Omega^{a}{}_{b} \, e^{b} \longrightarrow dx^{\mu'} = \Omega^{\mu}{}_{\nu} \, dx^{\nu} =$$

$$\Omega^{\mu}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \qquad\qquad = \frac{\partial x^{\mu'}}{\partial x^{\nu}} dx^{\nu}$$

$$(\Omega^{-1})^{\nu}{}_{\rho} = \frac{\partial x^{\nu}}{\partial x^{\rho'}}$$

$$dx^{\rho'}\Gamma^{\mu'}_{\rho\nu} = \Omega^{\mu}{}_{\alpha}\, dx^{\rho}\,\Gamma^{\alpha}{}_{\rho\beta}\,(\Omega^{-1})^{\beta}{}_{\nu} \;+$$

$$+\;\Omega^{\mu}{}_{\alpha}\, dx^{\rho}\,\partial_{\rho}(\Omega^{-1})^{\alpha}{}_{\nu}$$

$$\Big\downarrow$$

$$dx^{\rho}\,\Omega^{\sigma}{}_{\rho}\,\Gamma^{\mu'}_{\sigma\nu}$$

$$\Omega^{\sigma}{}_{\rho}\,\Gamma^{\mu'}_{\sigma\nu} = \Omega^{\mu}{}_{\alpha}\,\Gamma^{\alpha}{}_{\rho\beta}\,(\Omega^{-1})^{\beta}{}_{\nu} + \Omega^{\mu}{}_{\alpha}\,\partial_{\rho}(\Omega^{-1})^{\alpha}{}_{\nu}$$

$$\Gamma^{\mu'}_{\sigma\nu} = \Omega^{\mu}{}_{\alpha}\,\Gamma^{\alpha}{}_{\rho\beta}\,(\Omega^{-1})^{\rho}{}_{\sigma}\,(\Omega^{-1})^{\beta}{}_{\nu} \;+$$

$$+\;\Omega^{\mu}{}_{\alpha}\,(\Omega^{-1})^{\rho}{}_{\sigma}\,\partial_{\rho}(\Omega^{-1})^{\alpha}{}_{\nu}$$

$$\Gamma^{\mu'}_{\sigma\nu} - \Gamma^{\mu'}_{\nu\sigma} = \Omega^{\mu}{}_{\alpha}\left(\Gamma^{\alpha}_{\rho\beta} - \Gamma^{\alpha}_{\rho\rho}\right)(\Omega^{-1})^{\rho}{}_{\sigma}(\Omega^{-1})^{\beta}{}_{\nu} +$$

$$\underbrace{+\ \Omega^{\mu}{}_{\alpha}\left[(\Omega^{-1})^{\rho}{}_{\sigma}\,\partial_{\rho}(\Omega^{-1})^{\alpha}{}_{\nu} - (\Omega^{-1})^{\rho}{}_{\nu}\,\partial_{\rho}(\Omega^{-1})^{\alpha}{}_{\sigma}\right]}$$

$$= -\underbrace{\partial_{\rho}\Omega^{\mu}{}_{\alpha}}_{\text{symmetric}}\underbrace{\left[(\Omega^{-1})^{\rho}{}_{\sigma}(\Omega^{-1})^{\alpha}{}_{\nu} - (\Omega^{-1})^{\rho}{}_{\nu}(\Omega^{-1})^{\alpha}{}_{\sigma}\right]}_{\text{antisymmetric}}$$

$\rho <\!\!> \alpha$

$$\partial_{\rho}\Omega^{\mu}{}_{\alpha} = \underbrace{\partial_{\rho}\partial_{\alpha}}\, x^{\mu'}$$

At the end:

$$\Gamma^{\mu'}_{\sigma\nu} - \Gamma^{\mu'}_{\nu\sigma} = \Omega^{\mu}{}_{\alpha}\left(\Gamma^{\alpha}_{\rho\beta} - \Gamma^{\alpha}_{\rho\rho}\right)(\Omega^{-1})^{\rho}{}_{\sigma}(\Omega^{-1})^{\beta}{}_{\nu}$$

$$\int_M F \wedge F \qquad\qquad F = F_{\mu\nu} \frac{dx^\mu dx^\nu}{2}$$

$$= \int_M F_{\mu\nu} F_{\rho\sigma} \frac{1}{4} \underbrace{dx^\mu dx^\nu dx^\rho dx^\sigma}_{= \varepsilon^{\mu\nu\rho\sigma} d^4x} \qquad\qquad \varepsilon^{0123} = 1$$

$$= \frac{1}{4} \int_M F_{\mu\nu} F_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} d^4x \qquad \sim \qquad \vec{E} \cdot \vec{H}$$

$$\downarrow \qquad\qquad \int F_{\mu\nu} F^{\mu\nu} \quad \sim \quad \vec{E}^2 - \vec{H}^2$$

$$= \frac{1}{4} \int_M \partial_\mu \left(A_\nu \partial_\rho A_\sigma \, \varepsilon^{\mu\nu\rho\sigma} \right) d^4x$$

$$\int_M F\wedge F = \int_M dA\wedge dA = \int_M d(A\,dA) =$$

$$= \int_{\partial M} A\,dA$$

Non-Abelian Yang-Mills theories

$$F^{\dot a}_{\mu\nu} = \partial_\mu A^{\dot a}_\nu - \partial_\nu A^{\dot a}_\mu + f^{\dot a}{}_{\dot b\dot c}\, A^{\dot b}_\mu A^{\dot c}_\nu \qquad f^{\dot a}{}_{\dot b\dot c} = - f^{\dot a}{}_{\dot c b}$$

gauge group G [Ex.: $SU(N)$]

$A^{\dot a}_\mu$: adjoint representation of G $\qquad U \in G$

$$\omega^a{}_b \qquad A' = UAU^{-1} + U\,dU^{-1} \qquad F' = UFU^{-1}$$

$$\Gamma^\mu{}_\nu \qquad \omega' = \Omega\,\omega(\Omega^{-1}) + \Omega\,d\Omega^{-1} \qquad R' = \Omega R \Omega^{-1}$$

$$-\frac{1}{4}\int_M d^4x \sqrt{-g}\, F^{\dot a}_{\mu\nu} F^{\dot a}_{\rho\sigma}\, g^{\mu\nu} g^{\rho\sigma}$$

Abelian case (electrodynamics) $G = U(1)$

$$U(x) = e^{i\Lambda(x)}$$

$$A' = UAU^{-1} + UdU^{-1} = e^{i\Lambda} A e^{-i\Lambda} +$$

$$+ e^{i\Lambda}(-id\Lambda)e^{-i\Lambda} = A - id\Lambda$$

($\cancel{A} = i$ "usual A") $\qquad A'_\mu = A_\mu + \partial_\mu \Lambda$

Fermions

Flat space

$$\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

χ, ϕ doublets

Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\sigma_0, \vec{\sigma}) \qquad \sigma^{\mu\,+} = \sigma^\mu$$

$$\tilde{\sigma}^\mu = (\sigma_0, -\vec{\sigma}) \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\tilde{\sigma}^\mu = \sigma_2 \, \sigma^{\mu\,*} \sigma_2$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (\gamma^0)^2 = 1$$

Lorentz transformations

$$x^{\mu'} = \Lambda^\mu{}_\nu x^\nu \qquad \eta_{\mu\nu} = \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu$$

$$x'^2 = x'^\mu \eta_{\mu\nu} x'^\nu = x^2 = x^\mu \eta_{\mu\nu} x^\nu$$

$$\hookrightarrow = \underbrace{\Lambda^\mu{}_\rho x^\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma}_{= \eta_{\rho\sigma}} x^\sigma$$

$$\eta_{\mu\nu} \eta^{\nu\alpha} = \delta_\mu^\alpha = \Lambda^\rho{}_\mu \underbrace{\eta_{\rho\sigma} \Lambda^\sigma{}_\nu \eta^{\nu\alpha}} =$$

$$= \Lambda^\rho{}_\mu \Lambda_\rho{}^\alpha \qquad \Lambda_\mu{}^\rho \Lambda^\nu{}_\rho = \delta_\mu^\nu$$

$$\psi' = \mathcal{A}\,\psi \qquad \mathcal{A} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & A \end{pmatrix}$$

$$\bar{\psi} = \psi^{+}\gamma^{0}$$

$$\bar{\psi}' = \psi^{+}A^{+}\gamma^{0} = \bar{\psi}\gamma^{0}A^{+}\gamma^{0}$$

$$\bar{\psi}'\psi' = \bar{\psi}\gamma^{0}\mathcal{A}^{+}\gamma^{0}\mathcal{A}\,\psi = \bar{\psi}\psi$$

$$\gamma^{0}\mathcal{A}^{+}\gamma^{0}\mathcal{A} = 1 \qquad \boxed{\gamma^{0}\mathcal{A}^{+}\gamma^{0} = \mathcal{A}^{-1}} \qquad \bar{\psi}' = \bar{\psi}\mathcal{A}^{-1}$$

We also want $\bar{\psi}'\gamma^{\mu}\partial_{\mu}'\psi' = \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi =$

$$= \bar{\psi}\mathcal{A}^{-1}\gamma^{\mu}\frac{\partial x^{\nu}}{\partial x^{\mu\prime}}\frac{\partial}{\partial x^{\nu}}\mathcal{A}\,\psi$$

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad \frac{\partial x^{\mu\prime}}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\nu}$$

$$\frac{\partial x^{\nu}}{\partial x^{\mu\prime}} = \Lambda_{\mu}{}^{\nu}$$

$$\bar{\psi}'\gamma^{\mu}\partial'_{\mu}\psi' = \bar{\psi}A^{-1}\gamma^{\mu}\Lambda_{\mu}{}^{\nu}A\,\partial_{\nu}\psi = \bar{\psi}\gamma^{\nu}\partial_{\nu}\psi$$

$$\boxed{A^{-1}\gamma^{\mu}A\,\Lambda_{\mu}{}^{\nu} = \gamma^{\nu}}$$

γ^{μ} are invariant

$\gamma^{\mu}_{\alpha\beta}$

$$A^{-1}\gamma^{\mu}A = \Lambda^{\mu}{}_{\nu}\,\gamma^{\nu}$$

We have to consider $SL(2,\mathbb{C})$

complex 2×2 matrices with unit determinant

$A \in SL(2,\mathbb{C}) \qquad A = a\sigma_0 + \vec{b}\cdot\vec{\sigma}$

where $a, \vec{b}$ are complex and $a^2 - \vec{b}^2 = 1$

$A^{-1} = a\sigma_0 - \vec{b}\cdot\vec{\sigma}$

Property

$$A^+ \sigma^\mu A = \Lambda^\mu{}_\nu \sigma^\nu \qquad \Lambda^\mu{}_\nu = \text{Lorentz transformation}$$

$SL(2,\mathbb{C})$ = double cover of $SO^+(1,3)$

$$\gamma^0 A^+ \gamma^0 = A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{A}^+ & 0 \\ 0 & A^+ \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} A^+ & 0 \\ 0 & \tilde{A}^+ \end{pmatrix} = \begin{pmatrix} \tilde{A}^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} \Rightarrow A^+ = \tilde{A}^{-1}$$

$$\tilde{A} = (A^{+})^{-1} \qquad A = a\sigma_0 + \vec{b}\cdot\vec{\sigma}$$

$$A^{+} = a^{*}\sigma_0 + \vec{b}^{*}\cdot\vec{\sigma}$$

$$\tilde{A} = (A^{+})^{-1} = a^{*}\sigma_0 - \vec{b}^{*}\cdot\vec{\sigma}$$

Property : $\tilde{A}^{+}\tilde{\sigma}^{\mu}\tilde{A} = \Lambda^{\mu}{}_{\nu}\tilde{\sigma}^{\nu}$

$A^{+}\sigma^{\mu}A = \Lambda^{\mu}{}_{\nu}\sigma^{\nu}$ complex conjugate :

$A^{T}\sigma^{\mu *}A^{*} = \Lambda^{\mu}{}_{\nu}\sigma^{*\nu}$ multiply by $\sigma_2 \dots \sigma_2$

$$\underbrace{\sigma_2 A^{T}\sigma_2}\;\underbrace{\sigma_2\sigma^{\mu *}\sigma_2}\;\underbrace{\sigma_2 A^{*}\sigma_2} = \Lambda^{\mu}{}_{\nu}\sigma_2\sigma^{*\nu}\sigma_2$$

$$\tilde{A}^{+}\tilde{\sigma}^{\mu}\tilde{A} = \Lambda^{\mu}{}_{\nu}\tilde{\sigma}^{\nu}$$

$$\tilde{A} = \sigma_2 A^{*} \sigma_2 = \sigma_2 \left(a^{*} \sigma_0 + \vec{b}^{*} \cdot \vec{\sigma}^{*} \right) \sigma_2 =$$

$$= a^{*} \sigma_0 + \vec{b}^{*} \cdot \tilde{\vec{\sigma}} = a^{*} \sigma_0 - \vec{b}^{*} \cdot \vec{\sigma} \quad \underline{\text{ok}}$$

We also have $A^{-1} \gamma^{\mu} A = \Lambda^{\mu}{}_{\nu} \gamma^{\nu}$

$$\begin{pmatrix} A^{+} & 0 \\ 0 & \tilde{A}^{+} \end{pmatrix} \begin{pmatrix} 0 & \sigma^{\mu} \\ \tilde{\sigma}^{\mu} & 0 \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & A \end{pmatrix} =$$

$$= \begin{pmatrix} A^{+} & 0 \\ 0 & \tilde{A}^{+} \end{pmatrix} \begin{pmatrix} 0 & \sigma^{\mu} A \\ \tilde{\sigma}^{\mu} \tilde{A} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^{+} \sigma^{\mu} A \\ \tilde{A}^{+} \tilde{\sigma}^{\mu} \tilde{A} & 0 \end{pmatrix} =$$

$$= \Lambda^{\mu}{}_{\nu} \begin{pmatrix} 0 & \sigma^{\nu} \\ \tilde{\sigma}^{\nu} & 0 \end{pmatrix} = \Lambda^{\mu}{}_{\nu} \gamma^{\nu}$$

The relation between Λ and A is as follows:

Let us write $\Lambda = e^{T}$ $\quad (T_{\mu\nu} = -T_{\nu\mu})$

Then $A = e^{\Sigma^{\mu}{}_{\nu} T^{\nu}{}_{\mu}}$, where

$$\Sigma^{\mu}{}_{\nu} = -\frac{1}{8}[\gamma^{\mu}, \gamma_{\nu}] \qquad \gamma_{\nu} = \gamma^{\alpha}\eta_{\alpha\nu}$$

$$A = e^{\mathrm{tr}[\Sigma T]}$$

To prove $A^{-1}\gamma^{\mu}A = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$ we need the Campbell-Baker-Hausdorff formula

For every matrices A and B

$$e^{A} B e^{-A} = e^{ad_A} B = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \frac{1}{n!}\underbrace{[A, \cdots [A, B]]}_{n \text{ commutators}}$$

$$ad_A B = [A, B]$$

$$B = \gamma^\mu \qquad A = -\mathrm{tr}[\Sigma T]$$

$$e^{A} = \mathcal{A}^{-1} \qquad e^{-A} = \mathcal{A}$$

$$\mathcal{A}^{-1}\gamma^\mu \mathcal{A} = \gamma^\mu - [\mathrm{tr}[\Sigma T], \gamma^\mu] + \cdots$$

$$-[\Sigma^\rho{}_\nu T^\nu{}_\rho, \gamma^\mu] = -T^\nu{}_\rho [\Sigma^\rho{}_\nu, \gamma^\mu]$$

$$[\Sigma^{\rho\sigma}, \gamma^{\mu}] = \frac{1}{2}\left(\eta^{\mu\rho}\gamma^{\sigma} - \eta^{\mu\sigma}\gamma^{\rho}\right)$$

$$\|$$

$$-\frac{1}{8}\left[[\gamma^{\rho}, \gamma^{\sigma}], \gamma^{\mu}\right]$$

$\searrow$ $\mu=\rho$:

$\frac{1}{2} 3 \gamma^{\sigma}$

$\mu=\rho$: $\quad -\frac{1}{8}\left[\gamma^{\mu}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\mu}, \gamma_{\mu}\right] =$

$$= -\frac{1}{8}\left\{-2\gamma^{\sigma} - 4\gamma^{\sigma} - 4\gamma^{\sigma} - 2\gamma^{\sigma}\right\} = \frac{3}{2}\gamma^{\sigma} \qquad \underline{\text{ok}}$$

$$\gamma^{\mu}\gamma^{\sigma}\gamma_{\mu} = \gamma^{\mu}\left(-\gamma_{\mu}\gamma^{\sigma} + 2\delta^{\sigma}_{\mu}\right) = -2\gamma^{\sigma}$$

$$\gamma^{\mu}\gamma_{\mu} = 4$$

$$\Lambda^{-1}\gamma^\mu\Lambda = \gamma^\mu - T^\nu{}_\rho\frac{1}{2}(\eta^{\mu\rho}\gamma_\nu - \delta^\mu_\nu\gamma^\rho) + \dots$$

$$= \gamma^\mu - T^{\nu\mu}\gamma_\nu\frac{1}{2} + \frac{1}{2}T^\mu{}_\rho\gamma^\rho + \dots$$

$$= \gamma^\mu + T^\mu{}_\rho\gamma^\rho + \dots = (e^T\gamma)_\mu$$

Coupling to gravity

$$\Lambda^a{}_b \qquad \gamma^a \qquad \sigma^a \qquad \tilde\sigma^a \qquad \gamma_a = \eta_{ab}\gamma^b$$

$$\Sigma^a{}_b = -\frac{1}{8}[\gamma^a, \gamma^b]$$

$$e^{a\prime} = \Omega^a{}_b(x)\, e^b \qquad \Omega^a{}_b(x) \longrightarrow \Lambda^a{}_b(x)$$

What does $\bar{\psi} \partial\!\!\!/ \psi$ turn into ? $\quad \partial\!\!\!/ = \gamma^\mu \partial_\mu$

$$S = i \int_M e \, d^4x \, e^\mu_a \bar{\psi} \gamma^a (\partial_\mu + \tilde{\omega}_\mu) \psi +$$

$$- \int_M e \, d^4x \, m \bar{\psi} \psi$$

$$e = \sqrt{-g} = \det(e^a_\mu)$$

$$g = \det g_{\mu\nu} = \det(e^a_\mu \eta_{ab} e^b_\nu) =$$

$$= e(-1)e \Rightarrow -g = e^2$$

$$\tilde{\omega}_\mu \equiv \Sigma^a{}_b \, \omega_\mu{}^b{}_a \qquad \Sigma^a{}_b = -\frac{1}{8}[\gamma^a, \gamma_b]$$

Diffeomorphisms :

$$e'(x')\, d^4x' = e(x)\, d^4x$$

$$\psi'(x') = \psi(x) \qquad \bar{\psi} = \psi^\dagger \gamma^0 \qquad \bar{\psi}'(x') = \psi(x)$$

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu \qquad {e^{\mu}_{a}}'(x') = \frac{\partial x'^{\mu}}{\partial x^\nu} e^\nu_a(x)$$

$$\omega'_\mu{}^a{}_b(x') = \frac{\partial x^\nu}{\partial x^{\mu'}} \omega_\nu{}^a{}_b(x)$$

$$\tilde{\omega}'_\mu(x') = \frac{\partial x^\nu}{\partial x^{\mu'}} \tilde{\omega}_\nu(x)$$

$$S' = i \int_M e'(x') d^4x' \, e_a^{\mu}{}'(x') \bar{\psi}'(x') \gamma^a (\partial_\mu' + \tilde{\omega}_\mu'(x')) \psi'(x') +$$

$$- m \int_M e'(x') d^4x' \, \bar{\psi}'(x') \psi'(x) = S$$

Local Lorentz transformations

$$e_\mu^a{}'(x') = \Lambda^a{}_b \, e_\mu^b(x) \qquad x^\mu{}' = x^\mu$$

$$\psi'(x) = A \psi(x) \qquad e'(x) = e(x)$$

$$\bar{\psi}' = \bar{\psi} A^{-1}$$

$$S_K = i\int e\, d^4x\; e_a^{\mu}\, \bar{\psi}\, \gamma^a (\partial_\mu + \tilde{\omega}_\mu)\psi$$

$$S_K' = i\int e\, d^4x\; e_b^{\mu}\, \Lambda_a{}^b\, \bar{\psi} A^{-1} \gamma^a \cdot$$

$$\cdot\, (\partial_\mu + \tilde{\omega}_\mu')\, A\psi =$$

$$= i\int e\, d^4x\; e_b^{\mu}\, \Lambda_a{}^b\, \bar{\psi} A^{-1} \gamma^a (\partial_\mu A\, \psi +$$

$$+ A\, \partial_\mu \psi + \tilde{\omega}_\mu' A \psi)$$

$$\boxed{\tilde{\omega}' = A\omega A^{-1} + A\, dA^{-1}}$$

Then $S_K' = i\int e\, d^4x\; e_b^{\mu}\, \Lambda_a{}^b\, \bar{\psi} A^{-1} \gamma^a \cancel{(\underbrace{\partial_\mu A}_{}\, \psi} +$

$$+\mathcal{A}\partial_\mu\psi + \mathcal{A}\tilde{\omega}_\mu\psi + \cancel{\mathcal{A}\partial_\mu\mathcal{A}^{-1}}\mathcal{A}\psi\Big) =$$

$$= i\int e\,d^4x\, e^\mu_b \boxed{\Lambda_a{}^b}\,\bar{\psi}\,\boxed{\mathcal{A}^{-1}\gamma^a\mathcal{A}}\,(\partial_\mu + \tilde{\omega}_\mu)\psi = S_K$$

$$\partial_\mu\mathcal{A} = -\mathcal{A}\,\partial_\mu\mathcal{A}^{-1}\,\mathcal{A}$$

$$\Lambda_a{}^b\mathcal{A}^{-1}\gamma^a\mathcal{A} = \gamma^b \qquad \gamma^a\Lambda_a{}^b = \mathcal{A}\gamma^b\mathcal{A}^{-1}$$

We need to prove $\tilde{\omega}' = \mathcal{A}\,\omega\,\mathcal{A}^{-1} + \mathcal{A}\,d\,\mathcal{A}^{-1}$

we Know $\omega' = \Lambda\,\omega\,\Lambda^{-1} + \Lambda\,d\,\Lambda^{-1}$

$$\tilde{\omega} = \mathrm{tr}\left[\Sigma\,\omega\right] = \Sigma^a{}_b\,\omega^b{}_a$$

$$\tilde{\omega}' = \mathrm{tr}\left[\Sigma\,\omega'\right] = \mathrm{tr}\left[\Sigma\Lambda\omega\Lambda^{-1}\right] + \mathrm{tr}\left[\Sigma\Lambda d\Lambda^{-1}\right] =$$

$$= \mathcal{A}\,\omega\,\mathcal{A}^{-1} + \mathcal{A}\,d\,\mathcal{A}^{-1}$$

We want to prove $\mathrm{tr}\left[\Sigma\Lambda\omega\Lambda^{-1}\right] = \mathcal{A}\,\omega\,\mathcal{A}^{-1}$

and $\mathrm{tr}\left[\Sigma\Lambda d\Lambda^{-1}\right] = \mathcal{A}\,d\,\mathcal{A}^{-1}$

$$\mathrm{tr}\left[\Sigma\Lambda\omega\Lambda^{-1}\right] = \mathrm{tr}\left[\Lambda^{-1}\Sigma\Lambda\,\omega\right] = \mathcal{A}\,\tilde{\omega}\,\mathcal{A}^{-1}$$

$$(\Lambda^{-1}\Sigma\Lambda)^{a}{}_{b} = (\Lambda^{-1})^{a}{}_{c}\,\Sigma^{c}{}_{d}\,\Lambda^{d}{}_{b} =$$

$$= -\frac{1}{8}(\Lambda^{-1})^{a}{}_{c}\left[\gamma^{c},\gamma_{d}\right]\Lambda^{d}{}_{b} =$$

$$= -\frac{1}{8}\left[\mathcal{A}\gamma^{a}\mathcal{A}^{-1},\mathcal{A}\gamma_{b}\mathcal{A}^{-1}\right] = \mathcal{A}\,\Sigma^{a}{}_{b}\,\mathcal{A}^{-1}$$

$$\mathrm{tr}\left[\Sigma \wedge d\Lambda^{-1}\right] = \mathcal{A}\, d\mathcal{A}^{-1} \qquad : \qquad \gamma_a \Lambda^a{}_b = \mathcal{A} \gamma_b \mathcal{A}^{-1}$$

$$\parallel$$

$$= -\mathrm{tr}\left[\Sigma\, d\Lambda\, \Lambda^{-1}\right] = \frac{1}{8}\left(\gamma^a \gamma_b - \gamma_b \gamma^a\right) d\Lambda^b{}_c \left(\Lambda^{-1}\right)^c{}_a =$$

$$= \frac{1}{8}\left[\left(\Lambda^{-1}\right)^c{}_a \gamma^a, d\left(\gamma_b \Lambda^b{}_c\right)\right] = \qquad\qquad \gamma^a \gamma_a = 4$$

$$= \frac{1}{8}\left[\mathcal{A} \gamma^a \mathcal{A}^{-1}, d\left(\mathcal{A} \gamma_a \mathcal{A}^{-1}\right)\right] =$$

$$= \frac{1}{8}\Big(\mathcal{A}\gamma^a\mathcal{A}^{-1}\big(d\underset{\sim}{\mathcal{A}\gamma_a}\mathcal{A}^{-1} + \mathcal{A}\underline{\gamma_a\, d\mathcal{A}}^{-1}\big) +$$

$$- \big(d\underline{\mathcal{A}\gamma_a}\mathcal{A}^{-1} + \mathcal{A}\underset{\sim}{\gamma_a\, d\mathcal{A}}^{-1}\big)\mathcal{A}\gamma^a\mathcal{A}^{-1}\Big) =$$

$$= \frac{1}{2}\mathcal{A}\, d\mathcal{A}^{-1} - \frac{1}{2} d\mathcal{A}\, \mathcal{A}^{-1} +$$

$$+\ \frac{1}{8}\left(A\gamma^a A^{-1}\underset{\overset{\shortparallel}{0}}{d}A\gamma_a A^{-1} - A\gamma_a \underset{\overset{\shortparallel}{0}}{d}A^{-1}A\gamma^a A^{-1}\right) =$$

$$= A\, dA^{-1} \qquad \underline{\text{ok}}$$

$$\gamma^a A^{-1} dA\, \gamma_a = 0 \qquad !$$

$$\gamma^a \Sigma^b{}_c\, \gamma_a = 0 \qquad\qquad \gamma^a\gamma^b\gamma^c\gamma_a = 4\eta^{bc}$$

$$A = e^{tr[\Sigma T]}$$

$$e^{-M} d e^{M} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \underbrace{[M,[M, \dots [M, dM] \dots]}_{n} =$$

$$= dM - \frac{1}{2!}[M, dM] + \frac{1}{3!}[M,[M,dM]] + \dots$$

$$\Lambda = e^{-M} \qquad \Lambda^{-1} = e^{M}$$

$$e^{''tr[\Sigma T]} \qquad \Lambda = e^{T} \qquad M = -tr[\Sigma T]$$

$$[\Sigma^{ab}, \Sigma^{cd}] = \frac{1}{2}\left(\eta^{ac}\Sigma^{bd} - \eta^{ad}\Sigma^{bc} - \eta^{bc}\Sigma^{ad} + \eta^{bd}\Sigma^{ac}\right)$$

$$[M, dM] = [\Sigma^{a}{}_{b} T^{b}{}_{a}, \Sigma^{c}{}_{d}\, dT^{d}{}_{c}] =$$

$$= [\Sigma^{a}{}_{b}, \Sigma^{c}{}_{d}]\, T^{b}{}_{a}\, dT^{d}{}_{c}$$

$$M^{ab} = -2i\,\Sigma^{ab} = \text{generators of the Lorentz group}$$

$$[M^{ab}, M^{cd}] = i\left(\eta^{bc} M^{ad} - \eta^{ac} M^{bd} - \eta^{bd} M^{ac} + \eta^{ad} M^{bc}\right)$$

$$\mathcal{A} = e^{\Sigma^a{}_b T^b{}_a} = e^{-\frac{i}{2} M^{ab} T_{ab}}$$

$$\Lambda = e^{-\frac{i}{2} \bar{M}^{ab} T_{ab}} = e^{T}$$

$$(\bar{M}^{ab})_{cd} = i\left(\delta^a_c \delta^b_d - \delta^a_d \delta^b_c\right)$$

$$S_K = i\int e\, d^4x\ \bar{\psi}\, e^\mu_a \gamma^a (\partial_\mu + \tilde{\omega}_\mu)\psi =$$

$$= i\int e\, d^4x\ \bar{\psi}\, e^\mu_a \gamma^a \nabla_\mu \psi$$

$dx^\mu \quad \nabla_\mu \psi = \partial_\mu \psi + \tilde{\omega}_\mu \psi$

$$\nabla \psi = d\psi + \tilde{\omega} \psi = \left(d - \frac{1}{8} [\gamma^a, \gamma_b] \omega^b{}_a \right) \psi$$

$\nabla = d + \tilde{\omega}$ on spinors

Bianchi identity $\quad \nabla^2 \psi = \tilde{R} \psi \qquad \tilde{R} = \mathrm{tr}[\Sigma R] =$

$= R^a{}_b \Sigma^b{}_a$

$$\nabla^2 \psi = (d + \tilde{\omega})(d + \tilde{\omega}) \psi =$$

$$= d\tilde{\omega} \psi - \cancel{\tilde{\omega} d\psi} + \cancel{\tilde{\omega} d\psi} + \tilde{\omega}\tilde{\omega} \psi =$$

$$= (d\tilde{\omega} + \tilde{\omega}\tilde{\omega}) \psi \qquad \tilde{\omega} = \mathrm{tr}[\Sigma \omega]$$

$$\tilde{R} \overset{?}{=} d\tilde{\omega} + \tilde{\omega}\tilde{\omega} = \Sigma^a{}_b \, d\omega^b{}_a +$$

$$+ \Sigma^a{}_b\, \omega^b{}_a\, \Sigma^c{}_d\, \omega^d{}_c =$$

$$= \Sigma^a{}_b\, d\omega^b{}_a + \frac{1}{2}\left[\Sigma^a{}_b, \Sigma^c{}_d\right] \omega^b{}_a\, \omega^d{}_c =$$

$$= \Sigma^a{}_b\, d\omega^b{}_a + \frac{1}{\cancel{4}}\cancel{4}\, \omega_{\underset{a}{\overset{\curvearrowleft}{ba}}}\, \omega^{\overset{b}{d}}{}_{\underset{\underset{c}{\downarrow}}{a}}\, \Sigma^{\overset{a}{b}}{}_{\underset{b}{d}}$$

$$\left(\nabla \eta^{ab} = 0\right)$$

we assume metric compatibility

$$\omega^{ab} = -\omega^{ba}$$

$$= \Sigma^a{}_b \left(d\omega^b{}_a + \omega^b{}_c\, \omega^c{}_a \right)$$

$$= \Sigma^a{}_b\, R^b{}_a$$

ok!

$$\left[\Sigma^{ab}, \Sigma^{cd}\right] = \frac{1}{2}\left(\eta^{ac}\Sigma^{bd} - \eta^{ad}\Sigma^{bc} - \eta^{bc}\Sigma^{ad} + \eta^{bd}\Sigma^{ac}\right)$$

Infinitesimal transformations

φ , ψ , A_μ , $e_\mu^{\ a}$, $g_{\mu\nu}$, $T_{\mu\nu}^{\ \ \rho}$, $\omega_\mu^{\ a}{}_b$

Diffeomorphisms $\quad x^{\mu\,\prime} = x^\mu - \xi^\mu(x) \qquad \xi^\mu$ small

Local Lorentz transformations $\quad e_\mu^{\ a\,\prime} = \Lambda^a{}_b e_\mu^{\ b} = e_\mu^{\ a} + \theta^a{}_b e_\mu^{\ b}$

$$\Lambda^a{}_b = \delta^a_b + \theta^a{}_b \qquad \theta_{ab} = -\theta_{ba} \qquad \theta \text{ small}$$

Scalar:

$$\varphi'(x') = \varphi(x) = \varphi'(x-\xi) = \varphi'(x) - \xi^\mu \partial_\mu \varphi + O(\xi^2)$$

$$\varphi' = \varphi + \xi^\mu \partial_\mu \varphi + O(\xi^2)$$

Vector: $\quad A'_\mu(x') = A_\nu(x) \dfrac{\partial x^\nu}{\partial x^{\mu\,\prime}} = A'_\mu(x) - \xi^\rho \partial_\rho A_\mu +$

$+ O(\xi^2)$

$$\frac{\partial x^{\mu'}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} - \partial_{\nu} \xi^{\mu} + O(\xi^2) \qquad \frac{\partial x^{\nu}}{\partial x^{\rho'}} = \delta^{\nu}_{\rho} + \partial_{\rho} \xi^{\nu} + O(\xi^2)$$

$$A'_{\mu} = \xi^{\rho} \partial_{\rho} A_{\mu} + A_{\mu} + \partial_{\mu} \xi^{\rho} A_{\rho} + O(\xi^2) =$$

$$= A_{\mu} + \xi^{\rho} \partial_{\rho} A_{\mu} + \partial_{\mu} \xi^{\rho} A_{\rho} + O(\xi^2)$$

$$e^{a'}_{\mu} = e^{a}_{\mu} + \xi^{\nu} \partial_{\nu} e^{a}_{\mu} + \partial_{\mu} \xi^{\nu} e^{a}_{\nu} + \theta^{a}{}_{b} e^{b}{}_{\mu} + \text{higher orders}$$

$$\psi' = \psi + \xi^{\nu} \partial_{\nu} \psi + \Sigma^{a}{}_{b} \theta^{b}{}_{a} \psi + \text{higher orders}$$

Lorentz

$$\psi' = A \psi = e^{\Sigma^{a}{}_{b} \theta^{b}{}_{a}} \psi = \psi + \Sigma^{a}{}_{b} \theta^{b}{}_{a} \psi$$

$$\Lambda^{a}{}_{b} \simeq \delta^{a}_{b} + \theta^{a}{}_{b} \simeq e^{\theta}$$

$$\omega' = \Lambda \omega \Lambda^{-1} + \Lambda d \Lambda^{-1}$$

$$\Lambda \simeq 1 + \theta$$

$$\omega' \simeq \omega + \boxed{\theta \omega - \omega \theta - d\theta} + O(\theta^2) =$$

$$= \omega - \nabla \theta + O(\theta^2)$$

$$\nabla \theta^a{}_b = d\theta^a{}_b + \omega^a{}_c \theta^c{}_b - \theta^a{}_c \omega^c{}_b$$

$$\delta \omega_\mu{}^a{}_b = \xi^\rho \partial_\rho \omega_\mu{}^a{}_b + \partial_\mu \xi^\rho \omega_\rho{}^a{}_b - \nabla_\mu \theta^a{}_b$$

$$\delta e_\mu^a = \xi^\nu \partial_\nu e_\mu^a + \partial_\mu \xi^\nu e_\nu^a + \theta^a{}_b e^b{}_\mu$$

$$\delta g_{\mu\nu} = \delta (e_\mu^a \eta_{ab} e_\mu^b) = e_\mu^a \eta_{ab} \delta e_\nu^b + (\mu \leftrightarrow \nu) =$$

$$= e_\mu^{\ a} \underline{\underline{\eta_{ab}}} \left(\xi^\rho \partial_\rho e^b_{\ \nu} + \partial_\nu \xi^\rho e^b_{\ \rho} + \underline{\underline{\theta^b_{\ c}}}\, e^c_{\ \nu} \right)$$

$$\theta_{ac} = -\theta_{ca}$$

$$+ (\mu \leftrightarrow \nu) = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\nu \xi^\rho \underbrace{e_\mu^{\ a} \eta_{ab} e_\rho^{\ b}}_{g_{\mu\rho}} +$$

$$+ \partial_\mu \xi^\rho e_\nu^{\ a} \eta_{ab} e_\rho^{\ b} =$$

$$= \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\nu \xi^\rho g_{\mu\rho} + \partial_\mu \xi^\rho g_{\nu\rho} =$$

$$(\text{if } T = 0 = \nabla g)$$

$$= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \qquad \xi_\mu = \xi^\rho g_{\rho\mu}$$

$$= \partial_\mu (g_{\nu\rho} \xi^\rho) - T_{\mu\nu}^{\rho} \xi_\rho + (\mu \leftrightarrow \nu) =$$

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

$$= \partial_\mu g_{\nu\rho}\, \xi^\rho + g_{\nu\rho}\, \partial_\mu \xi^\rho - \frac{1}{2} \underline{\xi^\sigma (\partial_\mu g_{\nu\sigma}} + + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) + (\mu \leftrightarrow \nu) =$$

$$= \partial_\mu \cancel{g_{\nu\rho}\, \xi^\rho} + \underline{g_{\nu\rho}\, \partial_\mu \xi^\rho} + + \partial_\nu \cancel{g_{\mu\rho}\, \xi^\rho} + \underline{g_{\mu\rho}\, \partial_\nu \xi^\rho} - \xi^\rho \cancel{\partial_\mu g_{\nu\rho}} + - \cancel{\xi^\rho \partial_\nu g_{\mu\rho}} + \underline{\xi^\rho\, \partial_\rho}\, g_{\mu\nu} \qquad \underline{\text{ok}}$$

A vector ξ_μ such that $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ is called Killing vector and describes an (infinitesimal) isometry : $\delta g_{\mu\nu}$

Gravitational action: Palatini 1st-order formalism

$$S_{Pal} = c \int_M \underbrace{R^{ab} \wedge e^c \wedge e^d}_{4\text{-form}} \, \varepsilon_{abcd}$$

$$\varepsilon^{0123} = 1 \qquad \varepsilon_{0123} = -1$$

$$S_{Pal} \propto \int_M \sqrt{-g}\, R \, d^4x$$

$$R^a{}_b = (d\omega + \omega \wedge \omega)^a{}_b$$

$$= R^a{}_{b\,mn} \frac{e^m \wedge e^n}{2} = R^a{}_{b\,\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2}$$

$$S_{Pal} = \frac{c}{2} \int_M R^{ab}{}_{mn} \underbrace{e^m e^n e^c e^d}_{} \, \varepsilon_{abcd}$$

$$e^m e^n e^c e^d = A\ \varepsilon^{mncd} \qquad \times\ \varepsilon_{mncd}$$

$$[\varepsilon^{mncd}\varepsilon_{mncd} = -24]$$

$$\parallel$$

$$e^m_\mu e^n_\nu e^c_\rho e^d_\sigma \underbrace{dx^\mu dx^\nu dx^\rho dx^\sigma}_{= \varepsilon^{\mu\nu\rho\sigma} d^4x} =$$

$$= d^4x \quad e^m_\mu e^n_\nu e^c_\rho e^d_\sigma\ \varepsilon^{\mu\nu\rho\sigma}$$

$$-24 A = d^4x \quad \underbrace{\varepsilon_{nncd}\ e^m_\mu e^n_\nu e^c_\rho e^d_\sigma\ \varepsilon^{\mu\nu\rho\sigma}}_{= -24e} =$$

$$= -24\ e\ d^4x$$

$$A = \sqrt{-g}\, d^4x = e\, d^4x$$

$$S_{Pal} = \frac{C}{2} \int_M e \, R^{ab}{}_{mn} \, \varepsilon^{mncd} \varepsilon_{abcd} \, d^4x =$$

$$= \frac{C}{\cancel{2}} \int_M e \, R^{ab}{}_{mn} (-\cancel{2}) \left(\delta^m_a \delta^n_b - \delta^m_b \delta^n_a \right) d^4x =$$

$$= -2C \int_M e R \, d^4x = -2C(-2k^2) S_H = S_H$$

Hilbert action : $S_H = -\frac{1}{2k^2} \int_M \sqrt{-g} \, R \, d^4x$

$$C = \frac{1}{4k^2} \qquad S_{Pal} = \frac{1}{4k^2} \int_{Pal} R^{ab} \wedge e^c \wedge e^d \, \varepsilon_{abcd}$$

Cosmological term :

$$\int_M e^a e^b e^c e^d \mathcal{E}_{abcd} = \int_M e\, d^4x \ \mathcal{E}^{abcd} \mathcal{E}_{abcd} =$$

$$= -24 \int_M d^4x \sqrt{-g}$$

$$S_{HE} = -\frac{1}{2k^2} \int_M \sqrt{-g}\, d^4x \left(R + 2\Lambda \right) =$$

$$= \frac{1}{4k^2} \int_M \left(R^{ab} + \frac{\Lambda}{6} e^a e^b \right) e^c e^d \ \mathcal{E}_{abcd}$$

1st order formalism: e^a $\omega^a{}_b$ are independent variables

We assume metric compatibility ($\nabla \eta_{ab} = 0$, $\omega^{ab} = -\omega^{ba}$),

but NOT vanishing torsion

The field eq. obtained by varying w.r.t. $\omega^a{}_b$ gives vanishing torsion.

$$S_{Pal} = \frac{1}{4\kappa^2} \int_M (d\omega + \omega\omega)^{ab} e^c e^d \varepsilon_{abcd}$$

Variation w.r.t. ω: $R = d\omega + \omega\omega$

$$\delta R = d\delta\omega + \delta\omega\,\omega + \omega\,\delta\omega = \nabla\delta\omega$$

$$d(\omega + \delta\omega) = d\omega + d\delta\omega = d\omega + \delta(d\omega)$$

$$\delta\omega^a{}_b \qquad \nabla\delta\omega = d\delta\omega + \omega\,\delta\omega - \underset{k=1}{(-1)^k}\,\delta\omega\,\omega$$

$$\delta S_{Pd} = \frac{1}{4k^2}\int_M \nabla\delta\omega^a{}_b\; e^c\, e^d\; \varepsilon_a{}^b{}_{cd}$$

$$\nabla\eta^{ab} = 0 \quad\Rightarrow\quad \nabla\varepsilon_{abcd} = 0 = \nabla\varepsilon^{abcd}$$

Indeed, $\nabla\varepsilon^{abcd} = \cancel{d\varepsilon^{abcd}} + \omega^{\underset{0}{\overset{0}{a}}}{}_{\underset{3}{\overset{0}{m}}}\,\varepsilon^{\underset{3}{\overset{0}{m}}\underset{0}{\overset{1}{b}}\underset{1}{\overset{2}{c}}\underset{2}{\overset{3}{d}}} +$

$$+\,\omega^{\underset{0}{\overset{1}{b}}}{}_{\underset{3}{\overset{1}{m}}}\,\varepsilon^{\underset{0}{\overset{0}{a}}\underset{3}{\overset{1}{m}}\underset{1}{\overset{2}{c}}\underset{2}{\overset{3}{d}}} + \omega^{\underset{1}{\overset{2}{c}}}{}_{\overset{2}{m}}\,\varepsilon^{\underset{0}{\overset{0}{a}}\underset{0}{\overset{1}{b}}\overset{2}{m}\underset{2}{\overset{3}{d}}} + \omega^{\underset{2}{\overset{3}{d}}}{}_{\overset{3}{m}}\,\varepsilon^{\underset{0}{\overset{0}{a}}\underset{0}{\overset{1}{b}}\underset{1}{\overset{2}{c}}\overset{3}{m}}$$

$abcd = 0123$ OK

$abcd = 0012$ OK

$$\delta S_{Pal} = \frac{1}{4\kappa^2}\int_M \nabla\cancel{\left(\delta\omega^{ab}e^c e^d\right.}\left.\varepsilon_{abcd}\right) + \;= d\left(\delta\omega^{ab}e^c e^d \varepsilon_{abcd}\right)$$

$$+ \frac{2}{4\kappa^2}\int_M \delta\omega^{ab}\left(\nabla e^c\right)e^d\,\varepsilon_{abcd}$$

$$\nabla(e^c e^d) = \nabla e^c\, e^d - e^c\,\underbrace{\nabla e^d}_{2\,\text{form}} =$$

$$= \nabla e^c\, e^d \underset{\uparrow}{-} \nabla e^d\, e^c$$

Stokes theorem :

$$\int_M d\Omega = \int_{\partial M}\Omega = 0 \quad \left(\delta\omega^a{}_b \text{ on } \partial M\right)$$

$$\delta S_{Pal} = \frac{1}{2k^2} \int_M \delta\omega^{ab} T^c e^d \varepsilon_{abcd} =$$

$$= \frac{1}{2k^2} \int_M \delta\omega_\mu^{ab} T^c_{\nu\rho} e^d_\sigma \varepsilon^{\mu\nu\rho\sigma} d^4x \, \varepsilon_{abcd} = 0$$

$$\forall \, \delta\omega_\mu^{ab}$$

$$\Longrightarrow \quad T^c_{\nu\rho} e^d_\sigma \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} = 0 \qquad \times \, e^f_\mu$$

$$0 = T^c_{mn} \varepsilon^{fmnd} \varepsilon_{abcd} = -T^c_{mn} \left| \Big\downarrow \begin{matrix} \delta^f_a & \delta^m_a & \delta^n_a \\ \delta^f_b & \delta^m_b & \delta^n_b \\ \delta^f_c & \delta^m_c & \delta^n_c \end{matrix} \right| =$$

$$= -2T^c_{bc} \delta^f_a + 2 \delta^f_b T^c_{ac} - 2T^f_{ab}$$

$$b=f: \quad 0 = -2T^c_{ac} + 2 \cdot 4 \, T^c_{ac} - 2 T^c_{ac} = 4 T^c_{ac} \Rightarrow T^f_{ab} = 0$$

Variation w.r.t. e_μ^a :

$$\delta S_{Pal} = \frac{1}{4k^2} 2 \int_M R^{ab}\, \delta e^c\, e^d\, \varepsilon_{abcd} = 0 \qquad \forall A_m^c$$

$$\underbrace{e^c}\;\underbrace{\delta e^d}$$

$$\delta e^c = A_m^c\, e^m$$

$$0 = \delta S_{Pal} = \frac{1}{2k^2} \int_M R^{ab}{}_{mn}\, \frac{e^m e^n}{2}\, A_p^c\, e^p\, e^d\, \varepsilon_{abcd} =$$

$$= \frac{1}{4k^2} \int_M R^{ab}{}_{mn}\, A_p^c\, e\, d^4x\, \varepsilon^{mnpd}\, \varepsilon_{abcd} =$$

$$= \frac{1}{4k^2} \int_M R^{ab}{}_{mn}\, A_p^c\, e\, d^4x\, (-1) \begin{vmatrix} \delta_a^m & \delta_a^n & \delta_a^p \\ \delta_b^m & \delta_b^n & \delta_b^p \\ \delta_c^m & \delta_c^n & \boxed{\delta_c^p} \end{vmatrix} =$$

$$= -\frac{1}{4k^2}\int_M e\, d^4x \left(-A_a{}^c \underset{\uparrow\ \text{Ricci}}{R^a{}_c}\, 2 - \underline{2A_b{}^c R^b{}_c} + 2A_c{}^c R\right) =$$

$$= \frac{1}{k^2}\int_M e\, d^4x\, A_a{}^b \left(R^a{}_b - \frac{1}{2}\delta^a_b R\right) = 0 \qquad \forall A_b^a$$

$$\Rightarrow \quad R^a{}_b - \frac{1}{2}\delta^a_b R = 0 \quad \Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$R_{\mu\nu} = e_\mu{}^c R^a{}_b\, e^b{}_\nu\, \eta_{ac}\,, \quad \text{etc.}$$

$S_{Pal} = S_{Pal}(e, \omega)$: 1st order formalism

The ω equation can be solved : $\omega = \omega(e)$ $(T = 0)$

$\tilde{S}_{Pal}(e) = S_{Pal}(e, \omega(e))$: 2nd order formalism

$$\frac{\delta \tilde{S}_{Pal}(e)}{\delta e^a_\mu} = \frac{\delta S_{Pal}(e,\omega)}{\delta e^a_\mu}\Bigg|_{\omega}, \quad \omega \to \omega(e) \quad +$$

$$+ \cancel{\frac{\delta \omega^c_{\nu\,d}(e)}{\delta e^a_\mu} \frac{\delta S_{Pal}(e,\omega)}{\delta \omega^c_{\nu\,d}}}\Bigg|_{e,\ \omega \to \omega(e)}$$

The two formalisms give the same equation

$$S_\Lambda = \frac{\Lambda}{24\kappa^2}\int_M e^a e^b e^c e^d \, \varepsilon_{abcd}$$

$$\delta S_\Lambda = \frac{\Lambda}{6\kappa^2}\int_M \delta e^a e^b e^c e^d \, \varepsilon_{abcd} =$$

$$= \frac{\Lambda}{6\kappa^2}\int_M A^a_m \, e^m e^b e^c e^d \, \varepsilon_{abcd} =$$

$$= \frac{\Lambda}{6\kappa^2} \int_M e\, d^4x\; A^a_m\; \varepsilon^{mbcd}\; \varepsilon_{abcd} =$$

$$= - \frac{\Lambda}{\kappa^2} \int_M e\, d^4x\; A^a_a$$

$$\Rightarrow \quad R^a_b - \frac{1}{2} \delta^a_b R - \delta^a_b \Lambda = 0$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + 2\Lambda) = 0$$

Quadratic terms

$$\int d^4x \sqrt{-g} \left\{ a R^2 + b\; R_{\mu\nu} R^{\mu\nu} + c\; R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\}$$

$[R_{\mu\nu\rho\sigma}] = 2$ $\qquad$ Riemann $\simeq \partial\Gamma + \Gamma\Gamma$

$$(T=0\ ,\ \nabla_g = 0) \qquad \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

$$[g_{\mu\nu}]=0 \qquad [\partial]=1 \qquad [\Gamma]=1 \qquad [R]=2$$

$$-\frac{1}{2\kappa^2}\int \sqrt{-g}\ (R + 2\Lambda)$$

$$[R]=2 \qquad [\Lambda]=0 \qquad\qquad [R^2]=4$$

$$\int \sqrt{-g}\ \{ R^3 + R\Box R + \cdots \}$$

$$[R\Box R]=6 \qquad\qquad \int \partial_\mu J^\mu\, d^4x$$

$$\int_M \sqrt{-g}\ \Box R = \int_M \sqrt{-g}\ \nabla_\mu(\nabla^\mu R) = \int_M \partial_\mu \underbrace{(\sqrt{-g}\,\nabla^\mu R)} d^4x$$

$$\int d^4x \sqrt{-g} \; \nabla^\mu \nabla^\nu R_{\alpha\beta\gamma\delta} \longrightarrow$$

$$\int d^4x \sqrt{-g} \; \nabla^\mu \nabla^\nu R_{\alpha\beta} \longrightarrow \int d^4x \sqrt{-g} \; \nabla^\mu \nabla^\nu R_{\mu\nu}$$

Contracted Bianchi identity: $\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R$

Then $\longrightarrow \frac{1}{2} \int \sqrt{-g} \; \Box R$

$$\int d^4x \sqrt{-g} \; R^{\mu\nu\rho\sigma} R_{\alpha\beta\gamma\delta}$$

$$R^{\mu\nu\rho\sigma} R_{\alpha\beta} \qquad\qquad R^{\mu\nu} R_{\alpha\beta}$$

$$\int d^4x \sqrt{-g} \left\{ R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \right.$$

$$\left. + R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma} \right\}$$

1st Bianchi identity: $R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$

$$R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma} = - R^{\mu\nu\rho\sigma} \left(R_{\mu\sigma\rho\nu} + R_{\mu\nu\sigma\rho} \right) =$$

$$= - R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$$

$$R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma} = \frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

Euler characteristics

$$\chi(M) = -\frac{1}{32\pi^2}\int_M R^{ab}\wedge R^{cd}\,\varepsilon_{abcd} =$$

$$= -\frac{1}{32\pi^2}\int_M R^{ab}{}_{mn}\,\frac{e^m e^n}{2}\,R^{cd}{}_{fg}\,\frac{e^f e^g}{2}\,\varepsilon_{abcd} =$$

$$= -\frac{1}{128\pi^2}\int_M R^{ab}{}_{mn}\,R^{cd}{}_{fg}\,\varepsilon^{mnfg}\,\varepsilon_{abcd}\,e\,d^4x =$$

$$= \frac{1}{128\pi^2}\int_M R^{ab}{}_{mn}\,R^{cd}{}_{fg}\left|\Big\downarrow\begin{matrix}\delta_a^m & \delta_a^n & \delta_a^f & \delta_a^g\\ \delta_b^m & \delta_b^n & \delta_b^f & \delta_b^g\\ \delta_c^m & \delta_c^n & \delta_c^f & \delta_c^g\\ \delta_d^m & \delta_d^n & \delta_d^f & \delta_d^g\end{matrix}\right| e\,d^4x =$$

$$= \frac{1}{128\pi^2} 2 \int_M e\, d^4x \left\{ R^b_n \left(2 \delta^u_b R - R^{ug} g_{bg}\, 2\, 2 \right) + \right.$$

$$\left. + R^{ab}{}_{mn} R^{md}{}_{fg} \begin{vmatrix} \boxed{\delta^n_a} & \delta^f_a & \delta^g_a \\ \delta^n_b & \delta^f_b & \delta^g_b \\ \delta^u_d & \delta^f_d & \delta^g_d \end{vmatrix} \right\} =$$

$$= \frac{1}{\underset{32}{\cancel{64}}\pi^2} \int_M e\, d^4x \left\{ \cancel{2} R^2 - \overset{2}{\cancel{4}} R_{\mu\nu} R^{\mu\nu} + \right.$$

$$- R^b_m R^{md}{}_{bd} \cancel{2} - \cancel{2} R^a_m R^{md}{}_{ad} +$$

$$\left. + R^{ab}{}_{mn} R^{mn}{}_{ab} \cancel{2} \right\} =$$

$$= \frac{1}{32\pi^2} \int_M e\, d^4x \left\{ R^2 - 2 R_{\mu\nu} R^{\mu\nu} - 2 R_{\mu\nu} R^{\mu\nu} + \right.$$

$$\left. + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\} =$$

$$= \frac{1}{32\pi^2} \int_M e\, d^4x \left\{ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right\}$$

$$\chi(M) = -\frac{1}{32\pi^2} \int_M R^{ab} R^{cd} \varepsilon_{abcd} = -\frac{1}{32\pi^2} \int_M dC$$

C = Chern Simons form

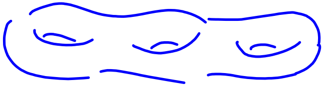
$$R^{ab} R^{cd} \varepsilon_{abcd} = dC$$

This does Not mean that the form is exact

$$C = \varepsilon_{abcd}\, \omega^{ab}\left(d\omega^{cd} + \frac{2}{3}\omega^{c}{}_{g}\, \omega^{gd} \right)$$

$\delta C \neq 0$ under local Lorentz transformations

$$\delta\omega = -\nabla\theta$$



$\chi(M) = 2-2g$

d= 2 first (Riemann surfaces)

$$e^{m} = e^{m}_{\mu}\, dx^{\mu}$$

$$\chi(M) = \frac{1}{4\pi}\int_{M} R^{ab}\, \varepsilon_{ab} =$$

$$= \frac{1}{4\pi}\int_{M} R^{ab}{}_{mn}\, \frac{e^{m} e^{n}}{2}\, \varepsilon_{ab} =$$

$$= \frac{1}{8\pi}\int_{M} R^{ab}{}_{mn}\, e\, d^{2}x\, \varepsilon^{mn}\varepsilon_{ab} =$$

$$= \frac{1}{8\pi} \int_M e \, R^{ab}{}_{mn} \left(\delta^m_a \delta^n_b - \delta^m_b \delta^n_a \right) d^2x =$$

$$= \frac{1}{4\pi} \int_M d^2x \sqrt{-g} \, R =$$

$$= \frac{1}{4\pi} \int_M \left(d\omega^{ab} + \omega^{ac} \omega_c{}^b \right) \varepsilon_{ab} =$$

$$= \frac{1}{4\pi} \int_M d\left(\omega^{ab} \varepsilon_{ab} \right)$$

$$\omega^{ac} \omega_c{}^b \varepsilon_{ab} = 0 \qquad \Big(\nabla \eta_{ab} = 0$$

$$\cancel{\omega^{1\overset{2}{c}} \omega_{\underset{1}{c}}{}^2} - \cancel{\omega^{2\overset{1}{c}} \omega_{\underset{2}{c}}{}^1} = 0 \qquad \omega^{ab} = -\omega^{ba} \Big)$$

$d = 4$ $\qquad \left(\nabla \eta^{ab} = 0 , \quad \omega^{ab} = - \omega^{ba} \right)$

$$\int_M R^{ab} \wedge R^{cd} \varepsilon_{abcd} = \int_M (d\omega + \omega\omega)^{ab} (d\omega + \omega\omega)^{cd} \varepsilon_{abcd} =$$

$$= \int_M d\left(\omega^{ab} d\omega^{cd} \varepsilon_{abcd}\right) + 2 \int_M d\omega^{ab} \omega^c{}_f \omega^{fd} \varepsilon_{abcd} +$$

$$+ \int_M \cancel{\omega^a{}_f \omega^{fb} \omega^c{}_g \omega^{gd} \varepsilon_{abcd}}$$

$$\omega^{af} \omega_f{}^b \omega^c{}_g \omega^{gd} \varepsilon_{abcd} = - \frac{1}{4} \overbrace{\varepsilon^{afmn} \varepsilon_{mnpq}}^{-2\left(\delta^a_p \delta^f_q - \delta^a_q \delta^f_p\right)} \omega^{pq} .$$

$$\cdot \; \omega_f{}^b \omega^c{}_g \omega^{gd} \varepsilon_{abcd} = + \frac{1}{4} \varepsilon_{mnpq} \omega^{pq} \omega_f{}^b \omega^c{}_g$$

$$\omega^{gd} \cdot \left| \downarrow \begin{matrix} \cancel{\delta^f_b} & \delta^m_b & \delta^n_b \\ \delta^f_c & \delta^m_c & \delta^n_c \\ \delta^f_d & \delta^m_d & \delta^n_d \end{matrix} \right| =$$

$$= -\frac{1}{2}\,\varepsilon_{mnpq}\,\omega^{pq}\,\omega_c{}^m\,\omega^c{}_g\,\omega^{gn} +$$

$$+\frac{1}{2}\,\varepsilon_{mnpq}\,\omega^{pq}\,\boxed{\omega_f{}^m}\,\omega^n{}_g\,\omega^{gf} =$$

$$= \frac{1}{2}\,\varepsilon_{mnpq}\,\omega^{pq}\,(\omega\omega\omega)^{mn} +$$

$$+\frac{1}{2}\,\varepsilon_{mnpq}\,\omega^{pq}\,(\omega\omega\omega)^{nm} = 0$$

$$\omega_f{}^m\,\omega^n{}_g\,\omega^{gf} = (\omega\omega\omega)^{nm} =$$

$$= -\,\omega^m{}_f\,\omega_g{}^n\,\omega^{fg} = \omega^m{}_f\,\omega^{fg}\,\omega^{gn} =$$

$$= (\omega\omega\omega)^{mn}$$

$$2\, d\omega^{ab}\,\omega^{c}{}_{f}\,\omega^{fd}\,\varepsilon_{abcd} \overset{\text{(to be proved)}}{=} \frac{2}{3}\, d\left[\omega^{ab}\,\omega^{c}{}_{g}\,\omega^{gd}\,\varepsilon_{abcd}\right] =$$

$$= \frac{2}{3}\, d\omega^{ab}\,\omega^{c}{}_{g}\,\omega^{cd}\,\varepsilon_{abcd} - \frac{4}{3}\,\omega^{ab}\, d\omega^{c}{}_{g}\,\omega^{gd}\,\varepsilon_{abcd}$$

$$\omega^{ab}\, d\omega^{c}{}_{g}\,\omega^{gd}\,\varepsilon_{abcd} = -\frac{1}{4}\,\omega^{ab}\, d\omega^{pq}\,\varepsilon_{pqmn}\,\varepsilon^{mncg}$$

$$\cdot\ \omega_{g}{}^{d}\,\varepsilon_{abcd} =$$

$$= +\frac{1}{4}\,\omega^{ab}\, d\omega^{pq}\,\omega_{g}{}^{d}\,\varepsilon_{pqmn} \begin{vmatrix} \delta_{a}^{m} & \delta_{a}^{n} & \delta_{a}^{g} \\ \delta_{b}^{m} & \delta_{b}^{n} & \delta_{b}^{g} \\ \delta_{d}^{m} & \delta_{d}^{n} & \cancel{\delta_{d}^{g}} \end{vmatrix} \downarrow =$$

$$= \frac{\cancel{4}}{\cancel{4}}\,\omega^{ab}\, d\omega^{pq}\,\omega_{a}{}^{d}\,\varepsilon_{pqbd} = -\, d\omega^{pq}\,\omega^{ba}\,\omega_{a}{}^{d}\,\varepsilon_{pqbd}$$

$d=3$ $\int_M C$ is an invariant

Under local Lorentz transformations $\delta_\theta C = d\,\Omega_\theta$

$\delta_\theta \int_M C = 0$: the integral is invariant

Chern-Simons QED $\vec{E}\cdot\vec{H}$

$$d=4 \quad \int_M F\wedge F = \frac{1}{4}\int_M F_{\mu\nu}F_{\rho\sigma}\,\varepsilon^{\mu\nu\rho\sigma}d^4x =$$

$$F = dA = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu \qquad = \int_M \partial_\mu A_\nu\,\partial_\rho A_\sigma\,\varepsilon^{\mu\nu\rho\sigma}d^4x =$$

$$dx^\mu dx^\nu dx^\rho dx^\sigma = \varepsilon^{\mu\nu\rho\sigma}d^4x \qquad = \int_M \partial_\mu C^\mu$$

$$C^{\mu} = \varepsilon^{\mu\nu\rho\sigma} A_{\nu} \partial_{\rho} A^{\sigma}$$

$d = 4 \longrightarrow d = 3$ $\quad C^{0} = \varepsilon^{0\nu\rho\sigma} A_{\nu} \partial_{\rho} A^{\sigma}$

$$\longrightarrow \quad C = \varepsilon^{ijk} A_{i} \partial_{j} A_{k}$$

$$\int_{M} C \, d^{3}x = \int_{M} \varepsilon^{ijk} A_{i} \partial_{j} A_{k} \, d^{3}x$$

gauge invariance : $\delta A_{i} = \partial_{i} \Lambda$

$$\delta \left[\varepsilon^{ijk} A_{i} \partial_{j} A_{k} \right] = \varepsilon^{ijk} \partial_{i} \Lambda \, \partial_{j} A_{k} =$$

$$= \frac{1}{2} \partial_{i} \left(\varepsilon^{ijk} \Lambda F_{jk} \right) \neq 0 \qquad \delta \int_{M} C d^{3}x = 0$$

Weyl tensor " = Riemann tensor to which we subtract all the traces "

n = spacetime dimension

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{n-2}\left(R_{\mu\sigma} g_{\nu\rho} - R_{\nu\sigma} g_{\mu\rho} - R_{\mu\rho} g_{\nu\sigma} + R_{\nu\rho} g_{\mu\sigma}\right) + \frac{1}{(n-1)(n-2)} R \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$

$$W^{\mu}{}_{\nu\mu\sigma} = \cancel{R_{\nu\sigma}} + \frac{1}{n-2}\left(2\cancel{R}_{\nu\sigma} - n \cancel{R}_{\nu\sigma} - R \cancel{g_{\nu\sigma}}\right) + \frac{1}{(n-1)(n-2)} R \left(n \cancel{g_{\nu\sigma}} - \cancel{g_{\nu\sigma}}\right) = 0$$

Symmetry properties :

$$W_{\mu\nu\rho\sigma} = -W_{\nu\mu\rho\sigma} = W_{\rho\sigma\mu\nu}$$

$$W_{\mu\nu\rho\sigma} + W_{\mu\sigma\nu\rho} + W_{\mu\rho\sigma\nu} = 0$$

$$W_{abcd} = -\frac{1}{4}\,\varepsilon_{abmn}\,\varepsilon_{cdpq}\,W^{mnpq}$$

ε_{abcd} is a tensor, ε^{abcd} is a tensor

$\varepsilon_{\mu\nu\rho\sigma}$ is not a tensor

$$\int\sqrt{-g}\,d^4x = -\frac{1}{24}\int\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}\,dx^\mu dx^\nu dx^\rho dx^\sigma$$

$$dx^\mu dx^\nu dx^\rho dx^\sigma = \varepsilon^{\mu\nu\rho\sigma}\,d^4x \qquad \cdot\ \varepsilon_{\mu\nu\rho\sigma}$$

$$\varepsilon_{\mu\nu\rho\sigma}\, dx^{\mu} dx^{\nu} dx^{\rho} dx^{\sigma} = -24\, d^4x$$

$\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$ is a tensor, because it collects the components of the volume form

$\varepsilon^{\mu\nu\rho\sigma}$ is a tensor (it is a number)

$$\varepsilon^{0123} = 1$$

$$\varepsilon_{\mu\nu\rho\sigma} = g_{\mu\alpha}\, g_{\nu\beta}\, g_{\rho\gamma}\, g_{\sigma\delta}\, \varepsilon^{\alpha\beta\gamma\delta}$$

Weyl transformations (local conformal transformations)

$$g_{\mu\nu}(x) \longrightarrow e^{-2\Omega(x)}\, g_{\mu\nu}(x)$$

$W^{\mu}{}_{\nu\rho\sigma}$ is invariant (in all dimensions), which $\Rightarrow$ (in four dimensions only) the invariance of the Weyl action

$$S_W = \int_M \sqrt{-g}\; W^{\mu}{}_{\nu\rho\sigma}\, W^{\alpha}{}_{\beta\gamma\delta}\, \underset{e^{-2\Omega}}{g_{\mu\alpha}}\, \underset{e^{2\Omega}}{g^{\nu\beta}}\, \underset{e^{2\Omega}}{g^{\rho\gamma}}\, \underset{e^{2\Omega}}{g^{\sigma\delta}}\, d^4x$$

$$g = \det g_{\mu\nu} \longrightarrow e^{-8\Omega} g$$

Similarly,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\int \underset{e^{-4\Omega}}{\sqrt{-g}}\; F_{\mu\nu} F_{\rho\sigma}\, \underset{e^{2\Omega}}{g^{\mu\rho}}\, \underset{e^{2\Omega}}{g^{\nu\sigma}}\, d^4x$$ is invariant

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} - \frac{1}{2}\left(R_{\mu\sigma} g_{\nu\rho} - R_{\nu\sigma} g_{\mu\rho} - R_{\mu\rho} g_{\nu\sigma} + R_{\nu\rho} g_{\mu\sigma} \right) - \frac{1}{6} R \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \mathrm{Riem}^2 = W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} + \qquad 1 + \frac{2}{3}$$

$$+ \frac{1}{4}\left(\mathrm{Ric}^2\, 4 \cdot 4 - 2\, \mathrm{Ric}^2 (1+1+1+1) + 2 R^2 (1+1) \right) + \frac{1}{36} R \left(16 \cdot 2 - 2 \cdot 4 \right) +$$

$$+ \cancel{2} \left(\cancel{+} \frac{1}{\cancel{6}} \right) \left(-\frac{1}{\cancel{2}} \right) R \cdot 2 \left(R - \cancel{4} R - \cancel{4} R \cancel{+} R \right) =$$

$$= W^2 + 2\, \mathrm{Ric}^2 + R^2 \left(1 + \frac{2}{3} - 2 \right) =$$

$$= W^2 + 2\, \mathrm{Ric}^2 - \frac{1}{3} R^2$$

$$\int_M \sqrt{-g}\, |W|^2 d^4x = \int_M \sqrt{-g}\, d^4x \left(Riem^2 - 2\, Ric^2 + \frac{1}{3} R^2 \right)$$

In three dimension the Weyl tensor vanishes identically

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

If $d > 3$ and $W_{\mu\nu\rho\sigma} = 0$, then the metric $g_{\mu\nu}$ is (locally) conformally flat, i.e. there locally exists a $\phi(x)$ such that

$$g_{\mu\nu}(x) = e^{2\phi(x)} \eta_{\mu\nu}$$

Codazzi tensor: any tensor $T_{\mu\nu}$ that is symmetric and such that $\nabla_\mu T_{\nu\rho} = \nabla_\nu T_{\mu\rho}$

$$\left[\nabla_X T(Y,Z) = \nabla_Y T(X,Z) \right]$$

Schouten tensor $P_{\mu\nu} = \frac{1}{n-2} R_{\mu\nu} - \frac{R}{2(n-1)} g_{\mu\nu}$

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + g_{\mu\rho} P_{\nu\sigma} - g_{\nu\rho} P_{\mu\sigma} - g_{\mu\sigma} P_{\nu\rho} + g_{\nu\sigma} P_{\mu\rho}$$

Cotton tensor $C_{\mu\nu\rho} = \nabla_\mu P_{\nu\rho} - \nabla_\nu P_{\mu\rho}$

In three dimensions a metric is locally conformally flat if and only if the Schouten tensor is a Codazzi tensor, i.e. the Cotton tensor vanishes identically.

$$\nabla_\mu P_{\nu\rho} = \nabla_\nu P_{\mu\rho}$$

Hilbert action with the Palatini (1^{st} order) formalism

$$S_H = -\frac{1}{2k^2}\int_M \sqrt{-g}\, d^4x\, R$$

$g_{\mu\nu}$ and $\Gamma_{\mu\nu}^{\rho}$ are treated as independent fields

If we assume that the torsion vanishes ($\Gamma_{\mu\nu}^{\rho} = \Gamma_{\nu\mu}^{\rho}$), then the Γ-field equation implies metric compatibility ($\nabla_\mu g_{\nu\rho} = 0$)

and viceversa

We assume $\Gamma_{\mu\nu}^{\rho} = \Gamma_{\nu\mu}^{\rho}$

$$S(g, \Gamma) = -\frac{1}{2k^2} \int_M \sqrt{-g}\, d^4x \; g^{\mu\nu} R_{\mu\nu}$$

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma_{\nu\sigma}^{\mu} - \partial_\sigma \Gamma_{\nu\rho}^{\mu} + \Gamma_{\sigma\nu}^{\alpha} \Gamma_{\rho\alpha}^{\mu} - \Gamma_{\rho\nu}^{\alpha} \Gamma_{\sigma\alpha}^{\mu}$$

$$R_{\nu\sigma} = \partial_\mu \Gamma_{\nu\sigma}^{\mu} - \partial_\sigma \Gamma_{\nu\mu}^{\mu} + \Gamma_{\sigma\nu}^{\alpha} \Gamma_{\mu\alpha}^{\mu} - \Gamma_{\mu\nu}^{\alpha} \Gamma_{\sigma\alpha}^{\mu}$$

$$\frac{\delta S}{\delta g^{\mu\nu}}$$

$$\delta g = g\, g^{\alpha\beta} \delta g_{\alpha\beta} = -g\, g_{\mu\nu} \delta g^{\mu\nu}$$

$$\delta \sqrt{-g} = \frac{1}{2\sqrt{-g}} (-1)(-g)\, g_{\mu\nu} \delta g^{\mu\nu} = -\frac{1}{2} \sqrt{-g}\, \delta g^{\mu\nu} g_{\mu\nu}$$

$$\delta S = -\frac{1}{2k^2}\int_M \sqrt{-g}\, d^4x\, \delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Longrightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$S(g,\Gamma) = -\frac{1}{2k^2}\int_M \sqrt{-g}\, d^4x\, g^{\mu\nu}\left(\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\lambda\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\nu\alpha}\right) =$$

$$= -\frac{1}{2k^2}\int_M d^4x \Big[\partial_\lambda\left(\sqrt{-g}\, g^{\mu\nu}\, \Gamma^\lambda_{\mu\nu}\right) - \underset{\lambda}{\partial_\nu}\left(\sqrt{-g}\, g^{\mu\overset{\lambda}{\nu}}\, \Gamma^{\overset{\alpha}{\lambda}}_{\mu\underset{\alpha}{\lambda}}\right) +$$

$$-\partial_\lambda\left(\sqrt{-g}\, g^{\mu\nu}\right)\Gamma^\lambda_{\mu\nu} + \partial_\nu\left(\sqrt{-g}\, g^{\mu\nu}\right)\Gamma^\lambda_{\mu\lambda} +$$

$$+\left(\Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\lambda\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\nu\alpha}\right) g^{\mu\nu}\Big]$$

Exercise

$$\frac{\delta S}{\delta \Gamma^{\rho}_{\mu\nu}} = 0 \implies \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

Then we can switch to the 2nd order formalism

$$S_H(g) = S(g, \Gamma(g))$$

$$\frac{\delta S_H(g)}{\delta g_{\mu\nu}} = \left.\frac{\delta S(g,\Gamma)}{\delta g_{\mu\nu}}\right|_{\substack{\Gamma, \\ \Gamma \to \Gamma(g)}} + \cancel{\left.\frac{\delta S(g,\Gamma)}{\delta \Gamma^{\gamma}_{\alpha\beta}}\right|_{\substack{g, \\ \Gamma \to \Gamma(g)}}} \cdot \frac{\delta \Gamma^{\gamma}_{\alpha\beta}(g)}{\delta g_{\mu\nu}}$$

$$= \left.\frac{\delta S(g,\Gamma)}{\delta g_{\mu\nu}}\right|_{\substack{\Gamma, \\ \Gamma \to \Gamma(g)}}$$

Let us consider a generic quadratic action

$$S(x) = \int \left(\frac{1}{2} x^t M x + x^t A + B \right)$$

$$\frac{\delta S}{\delta x} = M x + A = 0 \qquad x = - M^{-1} A$$

$$S(x)\Big|_{x=-M^{-1}A} = \int \left(\frac{1}{2} \left(+ A^t \cancel{M^{-1}} \cancel{M} (+1) M^{-1} A \right) + \right.$$

$$\left. - A^t M^{-1} A + B \right) = \int \left(-\frac{1}{2} A^t M^{-1} A + B \right) =$$

$$= \int \left(- \frac{1}{2} x^t M x + B \right) \Big|_{x = -M^{-1}A}$$

$$S_H(g) = \frac{1}{2k^2}\int_M d^4x \sqrt{-g}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} - \Gamma^{\lambda}_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}\right)g^{\mu\nu} +$$

$$- \frac{1}{2k^2}\int_M d^4x\, \partial_\lambda w^\lambda$$

$$w^\lambda = \sqrt{-g}\left(g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\Gamma^{\alpha}_{\mu\alpha}\right)$$

$$S_{\Gamma\Gamma}(g) = \frac{1}{2k^2}\int_M d^4x \sqrt{-g}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} - \Gamma^{\lambda}_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}\right)g^{\mu\nu}$$

$\downarrow$

it depends on $g_{\mu\nu}$ and $\partial_\lambda g_{\mu}$

$$\int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t), t)\, dt$$

$$q(t_1) = q_1$$
$$q(t_2) \equiv q_2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\frac{\dot{q}^2}{2} \qquad \sqrt{1-\dot{q}^2} \qquad F_{\mu\nu}F^{\mu\nu} \qquad \sqrt{\det(\eta_{\mu\nu}-F_\mu{}^\rho F_{\rho\nu})}$$

Born Infeld

$$f(\dot{q}^2) \qquad f(F_{\mu\nu}F^{\mu\nu})$$

In gravity there are <u>no</u> local invariants

Ex. scalar $\varphi'(x') = \varphi(x)$

$$\delta\varphi = \xi^\rho\partial_\rho\varphi$$

$F_{\mu\nu}(x)F^{\mu\nu}(x)$ is a local invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu\Lambda$

S_H : unique action depending only on $g_{\mu\nu}$ & $\partial g_{\mu\nu}$ up to boundary terms

To build invariants in gravity we need to integrate scalar densities on the manifold

$$\int_M d^4x \sqrt{-g}\, \varphi(x)$$

where $\varphi(x)$ is a scalar
$(\varphi = R,\ 1,\ R^2,\ R_{\mu\nu}R^{\mu\nu},\ g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi, \dots)$

$$\mathcal{L} = \sqrt{-g}\,\varphi \qquad \delta\varphi = \xi^\rho\partial_\rho\varphi$$

$$\delta g_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu\xi^\rho + g_{\nu\rho}\partial_\mu\xi^\rho =$$

$$= \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$$

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-g)\,g^{\mu\nu}\,\delta g_{\mu\nu} = \frac{\sqrt{-g}}{\cancel{2}}\,g^{\mu\nu}\nabla_\mu\xi_\nu \cdot \cancel{2}$$

$$\delta \mathcal{L} = \sqrt{-g}\, \xi^{\rho} \nabla_{\rho} \varphi + \sqrt{-g}\, g^{\mu\nu} \nabla_{\mu} \xi_{\nu} \varphi =$$

$$= \sqrt{-g}\, \nabla_{\mu} \left(g^{\mu\nu} \xi_{\nu} \varphi \right) = \sqrt{-g}\, \nabla_{\mu} \left(\xi^{\mu} \varphi \right) =$$

$$= \partial_{\mu} \left(\sqrt{-g}\, \xi^{\mu} \varphi \right)$$

Energy momentum tensor

$$T = T(g)$$

$$S = - \frac{1}{2\kappa^2} \int \sqrt{-g} \left(\overset{g^{\mu\nu} R_{\mu\nu}}{R} + 2\Lambda \right) + \underset{\nwarrow \text{ matter part}}{S_m}$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \qquad T^{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

$$\delta S = \int d^4x \, \delta g^{\mu\nu} \underbrace{\frac{\delta S}{\delta g^{\mu\nu}}}_{\text{functional derivative}} = -\frac{1}{2\kappa^2} \int_M d^4x \sqrt{-g} \, \delta g^{\mu\nu} \Big(R_{\mu\nu}$$

$$- \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} \Big) + \int_M d^4x \sqrt{-g} \, \delta g^{\mu\nu} \frac{\delta S_m}{\delta g^{\mu\nu}} =$$

$$= -\frac{1}{2\kappa^2} \int_M d^4x \sqrt{-g} \, \delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} - \frac{\cancel{2}\kappa^2}{\cancel{2}} T_{\mu\nu} \right)$$

$$\delta S = 0 \Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

$T_{\mu\nu}$ is covariantly constant : $\nabla^\mu T_{\mu\nu} = 0$

because $\nabla_\mu g_{\nu\rho} = 0$ and because of the contracted Bianchi identity

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R$$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is called Einstein tensor

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^\mu R = 0$$

With fermions, we define

$$T_a^{\ \mu} \equiv -\frac{1}{e}\frac{\delta S_m}{\delta e_\mu^a}$$

If S_m just depends on $g_{\mu\nu}$:

$$T_a^{\ \mu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\rho\sigma}}\frac{\delta g_{\rho\sigma}}{\delta e_\mu^a} =$$

$$= \frac{1}{2}T^{\rho\sigma}\frac{\delta(e_\rho^b\eta_{bc}e_\sigma^c)}{\delta e_\mu^a} =$$

$$= T^{\rho\sigma}e_\rho^b\eta_{bc}\delta_a^c\delta_\sigma^\mu = T^{\rho\mu}e_\rho^b\eta_{ab}$$

$$T_a^{\ \mu}e_\nu^a = T^{\rho\mu}g_{\rho\nu}$$

$$S_{Pal} = \frac{1}{4\kappa^2} \int_M \left(R^{ab} + \frac{1}{6} e^a e^b \right) e^c e^d \varepsilon_{abcd} + S_m$$

$$\delta e^a_\mu = A^a_b \, e^b_\mu$$

$$\delta S = \frac{1}{k^2} \int_M e \, d^4x \, A^b_a \left(R^a_b - \frac{1}{2} \delta^a_b (R + 2\Lambda) \right) +$$

$$+ \int d^4x \, \delta e^a_\mu \frac{\delta S_m}{\delta e^a_\mu} = \int d^4x \, (-e) A^a_b e^b_\mu T^\mu_a =$$

$$= \frac{1}{k^2} \int_M e \, d^4x \, A^b_a \left[R^a_b - \frac{1}{2} \delta^a_b (R + 2\Lambda) - k^2 e^a_\mu T^\mu_b \right] = 0$$

$$R_{ab} - \frac{1}{2}\eta_{ab} R - \eta_{ab}\Lambda = \kappa^2 T_{ab}$$

$$T_{ab} = T^{\mu}_{\ a}\, e_{\mu b} = -\frac{1}{e}\, e_{\mu b} \frac{\delta S_m}{\delta e_{\mu}^{\ a}}$$

T_{ab} is <u>not</u> symmetric off-shell, but it is on-shell

Theorem

The theory admits a symmetric energy-momentum tensor (on the solutions of the field equations) if and only if it is (globally) Lorentz invariant

S_m is invariant under local Lorentz transformations

$$\delta_\theta e^a_\mu = \theta^a{}_b e^b_\mu \qquad \theta^{ab} = -\theta^{ba}$$

χ = generic matter field $\qquad \delta_\theta \chi = A^\chi_{ab}\,\theta^{ab}$

$$\delta_\theta S_m = 0 = \int d^4x \left(\delta_\theta e^a_\mu \frac{\delta S_m}{\delta e^a_\mu} + \sum_\chi \frac{\delta S_m}{\delta \chi} \delta_\theta \chi \right) =$$

$$= \int d^4x \left(-e\, \theta^{ab} \underbrace{e_{\mu b} T^\mu_a}_{T_{[ab]}} + \sum_\chi \frac{\delta S_m}{\delta \chi} A^\chi_{ab}\, \theta^{ab} \right)$$

$\forall\, \theta^{ab}$ arbitrary functions

$$\Longrightarrow \quad T_{[ab]} = \frac{1}{e} \sum_\chi \frac{\delta S_m}{\delta \chi} A^\chi_{[ab]} \quad \left[= 0 \text{ on shell} \right]$$

Perturbative expansion around flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\,\phi_{\mu\nu} \qquad g^{\mu\nu} = \eta^{\mu\nu} - 2\kappa\,\phi^{\mu\nu} + O(\phi^2)$$

Indices are raised and lowered with $\eta_{\mu\nu}$

$$\Gamma_{\mu\nu}^{\ \ \rho} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) =$$

$$= \frac{1}{\cancel{2}}\cancel{2}\kappa\left(\partial_\mu\phi_\nu^{\ \rho} + \partial_\nu\phi_\mu^{\ \rho} - \partial^\rho\phi_{\mu\nu}\right) + O(\phi^2)$$

$$S_{\Gamma\Gamma}(g) = \frac{1}{2k^2}\int_M d^4x\,\sqrt{-g}\,\left(\Gamma_{\mu\nu}^{\ \ \alpha}\,\Gamma_{\alpha\lambda}^{\ \ \lambda} - \Gamma_{\mu\alpha}^{\ \ \lambda}\,\Gamma_{\nu\lambda}^{\ \ \alpha}\right)g^{\mu\nu}$$

$$\Gamma_{\mu\lambda}^{\ \ \lambda} = \kappa\,\partial_\mu\phi + O(\phi^2) \qquad \phi = \phi_{\mu\nu}\,\eta^{\mu\nu}$$

$$\Gamma_{\mu\nu}{}^{\rho}\,\eta^{\mu\nu} = k\left(2\,\partial^{\mu}\phi_{\mu}{}^{\rho} - \partial^{\rho}\phi\right)$$

$$S_{\Gamma\Gamma} = \frac{1}{2}\int d^4x\; \partial_\mu\phi\left(2\partial^\nu\phi_\nu{}^\mu - \partial^\mu\phi\right) +$$

$$-\frac{1}{2}\int d^4x\left(\partial_\mu\phi_\alpha{}^\lambda + \partial_\alpha\phi_\mu{}^\lambda - \partial^\lambda\phi_{\mu\alpha}\right)\left(\partial^\mu\phi_\lambda{}^\alpha + \partial_\lambda\phi^{\alpha\mu} - \partial^\alpha\phi^\mu{}_\lambda\right) =$$

$$= \frac{1}{2}\int d^4x\Big[\;\underline{2\,\partial_\mu\phi\,\partial^\nu\phi_\nu{}^\mu} - \underline{\partial_\mu\phi\,\partial^\mu\phi} +$$

$$-\underline{\partial_\mu\phi_\alpha{}^\lambda\,\partial^\mu\phi_\lambda{}^\alpha} - \underline{\cancel{\partial_\mu\phi_\alpha{}^\lambda\,\partial_\lambda\phi^{\alpha\mu}}} + \underline{\cancel{\partial_\mu\phi_\alpha{}^\lambda\,\partial^\alpha\phi^\mu{}_\lambda}}$$

$$-\underline{\partial_\alpha\phi_\mu{}^\lambda\,\partial^\mu\phi_\lambda{}^\alpha} - \underline{\partial_\alpha\phi_\mu{}^\lambda\,\partial_\lambda\phi^{\alpha\mu}} + \underline{\partial_\alpha\phi_\mu{}^\lambda\,\partial^\alpha\phi^\mu{}_\lambda} +$$

$$+\underline{\cancel{\partial^\lambda\phi_{\mu\alpha}\,\partial^\mu\phi_\lambda{}^\alpha}} + \underline{\partial^\lambda\phi_{\mu\alpha}\,\partial_\lambda\phi^{\mu\alpha}} - \underline{\cancel{\partial^\lambda\phi_{\mu\alpha}\,\partial^\alpha\phi^\mu{}_\lambda}}\;\Big] =$$

$$= \frac{1}{2}\int d^4x \left[\partial_\mu \phi_{\nu\rho}\, \partial^\mu \phi^{\nu\rho} - \partial_\mu \phi\, \partial^\mu \phi + 2\, \partial_\mu \phi\, \partial^\nu \phi^\mu_\nu + \right.$$

$$\left. - 2\, \partial_\mu \phi^\mu_\rho\, \partial_\nu \phi^{\nu\rho} \right] + O(\phi^3)$$

Lowest order interaction ("$J_\mu A^\mu$") $T_{\mu\nu}\phi^{\mu\nu}$

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}} \qquad g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}$$

$$T^{\mu\nu} = -\cancel{2}\frac{1}{\cancel{2}\kappa}\frac{\delta S_m}{\delta \phi_{\mu\nu}} + \cdots \qquad \frac{\delta S_m}{\delta \phi_{\mu\nu}} = -\kappa T^{\mu\nu} + \cdots$$

$$S_m = S_m(\phi) = S_m(0) + \int \phi_{\mu\nu} \left.\frac{\delta S_m}{\delta \phi_{\mu\nu}}\right|_{\phi=0} + \cdots$$

$$= S_m(0) - \kappa \int T^{\mu\nu}\phi_{\mu\nu} + \cdots$$

↖ flat-space matter action

Note : we work at $\Lambda = 0$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R + 2\Lambda \right) = 0$$

$g_{\mu\nu} = \eta_{\mu\nu}$ is a solution <u>only</u> if $\Lambda = 0$

$$S_A + S_m = \frac{1}{2} \int d^4x \Big[\partial_\mu \phi_{\nu\rho} \partial^\mu \phi^{\nu\rho} - \partial_\mu \phi \partial^\mu \phi$$

$$+ 2 \partial_\mu \phi \partial^\nu \phi^\mu_\nu - 2 \partial_\mu \phi^\mu_\rho \partial_\nu \phi^{\nu\rho} - 2\kappa T^{\mu\nu} \phi_{\mu\nu} \Big] + \dots$$

Field equations : $-\Box \phi_{\nu\rho} + \eta_{\nu\rho} \Box \phi - \eta_{\nu\rho} \partial^\mu \partial^\nu \phi_{\mu\nu} +$

$$- \partial_\nu \partial_\rho \phi + \partial_\nu \partial_\mu \phi^\mu_\rho + \partial_\rho \partial_\mu \phi^\mu_\nu = \kappa T_{\nu\rho}$$

$$\delta \phi = \eta^{\mu\nu} \delta \phi_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}$$

$$R_{\nu\sigma} = \partial_\mu \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\mu} + \Gamma^\alpha_{\sigma\nu} \Gamma^\mu_{\mu\alpha} - \Gamma^\alpha_{\mu\nu} \Gamma^\mu_{\sigma\alpha} =$$

$$= \kappa \Big[\partial_\mu \left(\partial_\nu \phi^\mu_\sigma + \partial_\sigma \phi^\mu_\nu - \partial^\mu \phi_{\nu\sigma} \right) +$$

$$- \partial_\sigma \partial_\nu \phi \Big] + O(\phi^2) =$$

$$= \kappa \left[- \Box \phi_{\nu\sigma} - \partial_\nu \partial_\sigma \phi + \partial_\nu \partial_\mu \phi^\mu_\sigma + \partial_\sigma \partial_\mu \phi^\mu_\nu \right] + O(\phi^2)$$

$$R = g^{\mu\nu} R_{\mu\nu} = 2\kappa \left[- \Box \phi + \partial_\mu \partial_\nu \phi^{\mu\nu} \right] + O(\phi^2)$$

$$\cancel{\kappa} \Big[- \Box \phi_{\nu\sigma} - \partial_\nu \partial_\sigma \phi + \partial_\nu \partial_\mu \phi^\mu_\sigma + \partial_\sigma \partial_\mu \phi^\mu_\nu + \eta_{\nu\sigma} \Box \phi +$$

$$- \eta_{\nu\sigma} \partial_\alpha \partial_\beta \phi^{\alpha\beta} \Big] = \kappa^{\cancel{2}} T_{\nu\sigma}$$

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho$$

$$\parallel$$

$$\delta(\eta_{\mu\nu} + 2K \phi_{\mu\nu}) = 2k\, \delta\phi_{\mu\nu} = 2k\, \xi^\rho \partial_\rho \phi_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + O(\phi)$$

$$\delta\phi_{\mu\nu} = \partial_\mu C_\nu + \partial_\nu C_\mu + O(\phi) \qquad \delta\phi = 2\, \partial \cdot C + O(\phi)$$

QED $\quad \Box A_\mu - \cancel{\partial_\mu(\partial \cdot A)} \sim J_\mu \qquad \delta A_\mu = \partial_\mu \Lambda \qquad \partial \cdot A = 0$

$$\Box A_\mu \sim J_\mu \qquad \delta(\partial \cdot A) = \Box \Lambda = 0$$

Harmonic or De Donder gauge:

$$\partial^\mu \phi_{\mu\nu} - \frac{1}{2} \partial_\nu \phi = 0$$

$$\delta\left(\partial^\mu\phi_{\mu\nu} - \frac{1}{2}\partial_\nu\phi\right) = \partial^\mu\left(\partial_\mu C_\nu \cancel{+\partial_\nu C_\mu}\right) - \frac{1}{\cancel{2}}\partial_\nu \cancel{2}(\partial C) =$$

$$= \Box C_\nu = 0 \quad : \text{residual gauge}$$

$$-\Box\phi_{\nu\rho} + \eta_{\nu\rho}\Box\phi - \eta_{\nu\rho}\partial^\mu\partial^\nu\phi_{\mu\nu} - \partial_\nu\partial_\rho\phi + \partial_\nu\partial_\mu\phi^\mu_\rho + \partial_\rho\partial_\mu\phi^\mu_\nu = 0$$

$$\partial^\mu\phi_{\mu\nu} - \frac{1}{2}\partial_\nu\phi = 0$$

$$-\Box\phi_{\nu\rho} + \eta_{\nu\rho}\Box\phi - \eta_{\nu\rho}\frac{1}{2}\Box\phi - \cancel{\partial_\nu\partial_\rho\phi} + \cancel{\frac{1}{2}\partial_\nu\partial_\rho\phi} + \cancel{\frac{1}{2}\partial_\rho\partial_\nu\phi} =$$

$$-\Box\phi_{\nu\rho} + \frac{1}{2}\eta_{\rho\nu}\Box\phi = 0$$

Trace : $-\Box\phi + \frac{1}{2}4\,\Box\phi = \Box\phi = 0$

$$\Rightarrow \quad \Box\phi_{\mu\nu} = 0$$

We move to momentum space

$$\Box \phi_{\mu\nu} = 0 \quad \Rightarrow \quad k^2 \tilde{\phi}_{\mu\nu}(k) = 0 \quad \Rightarrow k^2 = 0$$

We can choose $k^\mu = (k, 0, 0, k)$

$$\partial^\mu \phi_{\mu\nu} - \frac{1}{2} \partial_\nu \phi = 0 \qquad \partial_\mu \longrightarrow -i k_\mu$$

$$k^\mu \tilde{\phi}_{\mu\nu}(k) - \frac{1}{2} k_\nu \tilde{\phi}(k) = 0 = k\left(\tilde{\phi}_{0\nu} + \tilde{\phi}_{3\nu}\right) - \frac{1}{2} k_\nu \tilde{\phi}$$

$\nu = 0$ $$\cancel{k}\left(\cancel{\tilde{\phi}_{00}} + \tilde{\phi}_{03}\right) - \frac{1}{2} \cancel{k} \tilde{\phi} = 0$$

$\nu = 1$ $$\cancel{\tilde{\phi}_{01}} + \tilde{\phi}_{31} = 0$$

$\nu = 2$ $$\cancel{\tilde{\phi}_{02}} + \tilde{\phi}_{32} = 0$$

$\nu = 3$ $$\tilde{\phi}_{03} + \cancel{\tilde{\phi}_{33}} + \frac{1}{2} \tilde{\phi} = 0$$

Residual gauge freedom $\delta \phi_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu$ $\Box c_\mu = 0$

$$\delta \tilde{\phi}_{\mu\nu} = -i k_\mu \tilde{c}_\nu - i k_\nu \tilde{c}_\mu \quad \text{with} \quad k^2 \tilde{c}_\mu(k) = 0$$

$$\delta\left(\tilde{\phi}_{02} + \tilde{\phi}_{32}\right) = -i k \tilde{c}_2 - i k_3 \tilde{c}_2 = 0$$

We can impose $\tilde{\phi}_{02} = 0$ using $\tilde{c}_2$

$\tilde{\phi}_{01} = 0$ using $\tilde{c}_1$

$\tilde{\phi}_{00} = 0$ using $\tilde{c}_0$

$\tilde{\phi}_{33} = 0$ using $\tilde{c}_3$

$$\delta \tilde{\phi}_{00} = -2 \kappa \tilde{c}_0 \qquad \delta \tilde{\phi}_{33} = 2 i k \tilde{c}_3$$

$$\tilde{\phi}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Polarizations of the graviton

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \alpha\, \epsilon_+ + \beta\, \epsilon_- = \begin{pmatrix} \alpha+\beta & i(\alpha-\beta) \\ i(\alpha-\beta) & -\alpha-\beta \end{pmatrix}$$

$$\epsilon_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad \epsilon_- = \epsilon_+^* \qquad \beta = \alpha^*$$

Rotation: $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}$

$$\epsilon_\pm \to R_\theta \, \epsilon_\pm \, R_\theta^{-1} = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix} \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} =$$

$$= \begin{pmatrix} e^{\pm i\theta} & \pm i e^{\pm i\theta} \\ \pm i e^{\pm i\theta} & -e^{\pm i\theta} \end{pmatrix} \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} =$$

$$= e^{\pm i\theta} \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} = e^{\pm 2i\theta} \epsilon_\pm$$

QED $\quad \varepsilon_\pm = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \longrightarrow R_\theta \, \varepsilon_\pm = e^{\pm i\theta} \varepsilon_\pm$

In d dimensions

A_μ : $d-2$ elicities $(d \geq 2)$

$A_\mu : d$ $\quad \Lambda : 1$ $\quad 1$

gauge $\quad$ residual gauge

$g_{\mu\nu}$: $\dfrac{d(d-3)}{2}$ elicities $(d \geq 3)$

$$g_{\mu\nu} : \frac{d(d+1)}{2} - 2d = \frac{d^2+d-4d}{2} = \frac{d(d-3)}{2}$$

$C_\mu : d$

residual : d

Quantum field theory

$$S_H \longrightarrow S_{\text{gauge-fixed}}$$

$$g_\mu = \partial^\nu \phi_{\mu\nu} - \frac{\omega}{2} \partial_\mu \phi \qquad \omega = \text{gauge-fixing parameter}$$

$$S_{gf} = S_H + \frac{1}{2\lambda} \int d^4x \, g_\mu \eta^{\mu\nu} g_\nu + S_{\text{ghost}}$$

λ = arbitrary parameter (gauge-fixing parameter)

$\bar{c}^\mu$: antighosts $\dashrightarrow$ residual gauge freedom

c^μ : Faddeev-Popov ghosts $\dashrightarrow$ gauge freedom

$\bar{c}^\mu$, c^μ have the wrong statistics

They expressed in terms of Grassmann variables θ_i ,
which are assumed to anticommute :

$$\{\theta_i , \theta_j\} \equiv \theta_i \theta_j - \theta_j \theta_i = 0$$

$$\theta_i \longrightarrow C(x)$$

$$S_{ghost} = \int d^4x \ \bar{C}^{\mu}(x) \overbrace{\frac{\delta g_{\mu}(x)}{\delta \phi_{\rho\sigma}(y)} \delta_c \phi_{\rho\sigma}(y)}^{\delta_c g_{\mu}} d^4y =$$

parameters of the infintesimal transformation

$$= \int d^4x \ \bar{C}^{\mu} \left(\partial^{\nu} \delta_c \phi_{\mu\nu} - \frac{\omega}{2} \partial_{\mu} \delta_c \phi \right)$$

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho$$

$$\|$$

$$\delta(\eta_{\mu\nu} + 2K \phi_{\mu\nu}) = 2\kappa\, \delta\phi_{\mu\nu} \qquad \xi^\rho = 2\kappa\, C^\rho$$

$$\delta_C \phi_{\mu\nu} = \partial_\mu C_\nu + \partial_\nu C_\mu + 2\kappa \Big(C^\rho \partial_\rho \phi_{\mu\nu} + \phi_{\mu\rho} \partial_\nu C^\rho + \phi_{\nu\rho} \partial_\mu C^\rho \Big)$$

The physical processes are gauge independent: they do not depend on the gauge-fixing parameters (λ, ω, ...) and they do not depend on the gauge-fixing function G_μ

Quadratic parts ($\omega = 1$, $\lambda = \frac{1}{2}$)

$$\frac{1}{2}\int d^4x \Big[\partial_\mu \phi_{\nu\rho}\, \partial^\mu \phi^{\nu\rho} - \partial_\mu\phi\, \partial^\mu\phi + 2\,\partial_\mu\phi\, \partial^\nu\phi^\mu_\nu +$$

$$-2\, \partial_\mu\phi^\mu_\rho\, \partial_\nu\phi^{\nu\rho} + 2\left(\partial^\nu\phi_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\right)\left(\partial^\rho\phi^\mu_\rho - \frac{1}{2}\partial^\mu\phi\right)\Big] =$$

$$= \frac{1}{2}\int d^4x \Big[\partial_\mu \phi_{\nu\rho}\, \partial^\mu \phi^{\nu\rho} - \partial_\mu\phi\, \partial^\mu\phi + 2\,\cancel{\partial_\mu\phi\, \partial^\nu\phi^\mu_\nu} +$$

$$-2\, \cancel{\partial_\mu\phi^\mu_\rho\, \partial_\nu\phi^{\nu\rho}} + 2\,\cancel{\partial^\nu\phi_{\mu\nu}\, \partial^\rho\phi^\mu_\rho} - 2\,\cancel{\partial_\mu\phi\, \partial^\rho\phi^\mu_\rho} +$$

$$+ \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi \Big] = \frac{1}{2}\int d^4x \left[\partial_\mu \phi_{\nu\rho}\, \partial^\mu \phi^{\nu\rho} - \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi \right]$$

$$= \frac{1}{2}\int d^4x\, \phi_{\mu\nu}\,(-\Box)\left(\mathbb{1}^{\mu\nu\rho\sigma} - \frac{\eta^{\mu\nu}\eta^{\rho\sigma}}{2} \right)\phi_{\rho\sigma}$$

$$\mathbb{1}^{\mu\nu\rho\sigma} = \frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{2} \qquad \mathbb{1}^{\mu\nu\rho\sigma}\phi_{\rho\sigma} = \phi_{\mu\nu}$$

Field equations:

$$\underbrace{(-\Box)\frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}}{2}}_{\text{Inverse of this} \to \text{Green function}} \phi_{\rho\sigma} = \kappa T_{\mu\nu} + \mathcal{O}(\phi^2)$$

In momentum space $\Box \to -k^2$, we need to invert

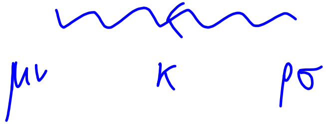
$$Q_{\mu\nu\rho\sigma} = k^2\left(\mathbb{1}_{\mu\nu\rho\sigma} - \frac{\eta_{\mu\nu}\eta_{\rho\sigma}}{2}\right)$$

Inverse: $P_{\mu\nu\rho\sigma} = \frac{1}{k^2}\left(\mathbb{1}_{\mu\nu\rho\sigma} - \frac{\eta_{\mu\nu}\eta_{\rho\sigma}}{2}\right)$

$$P_{\mu\nu\rho\sigma}\, Q^{\rho\sigma}{}_{\alpha\beta} = \left(\mathbb{1}_{\mu\nu\rho\sigma} - \frac{\eta_{\mu\nu}\eta_{\rho\sigma}}{2}\right)\left(\mathbb{1}^{\rho\sigma}{}_{\alpha\beta} - \frac{\eta^{\rho\sigma}\eta_{\alpha\beta}}{2}\right) =$$

$$= \mathbb{1}_{\mu\nu\alpha\beta} - \cancel{\eta_{\mu\nu}\eta_{\alpha\beta}} + \frac{1}{\cancel{4}}\cancel{4}\,\cancel{\eta_{\mu\nu}\eta_{\alpha\beta}} = \mathbb{1}_{\mu\nu\alpha\beta}$$

$iP_{\mu\nu\rho\sigma}(k)$ is the graviton propagator

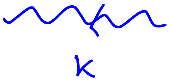


$$= \frac{i}{k^2}\,\frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}}{2}$$

$$\tilde{\Phi}_{\mu\nu}\tilde{\Phi}^{\mu\nu}$$

Graviton polarizations
$$\tilde{\phi}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{C}_{\mu} = 0$$
$$\tilde{C}^{\mu} = 0$$

$\tilde{\phi}_{\mu\nu}$


$$\tilde{\phi}_{\rho\sigma} = \frac{i}{\cancel{2}k^2}(a^2+b^2)\cancel{2} = i\,\frac{a^2+b^2}{k^2}$$

Pole of propagator : $k^2=0$: on-shellness condition

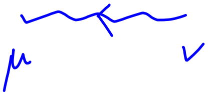
Residue at the pole : $i \times$ something positive

QED

$$\mathcal{L}_{gf} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\lambda}(\partial\cdot A)^2 + \bar{c}\,\frac{\overbrace{\delta(\partial\cdot A)}^{\partial^\mu}}{\delta A_\mu}\partial_\mu c \qquad \lambda=-1$$

$$= -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\mu A_\nu - \frac{1}{2}(\partial A)^2 + \bar{c}\,\Box c =$$

$$= -\frac{1}{2}A_\mu(-\Box)A^\mu + \cancel{\frac{1}{2}(\partial\cdot A)^2} - \cancel{\frac{1}{2}(\partial\cdot A)^2} + \bar{c}\,\Box c$$



$= -\frac{i}{k^2}\eta^{\mu\nu}$

4

$= -\frac{i}{k^2}$

-2

We move to the Coulomb gauge $\quad g = \vec{\nabla}\cdot\vec{A} \qquad \lambda = -1$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\vec{\nabla}\cdot\vec{A})^2 + \bar{c}\,\frac{\delta(\vec{\nabla}\cdot\vec{A})}{\delta A_i}\nabla_i c =$$

$$= -\frac{1}{2}A_\mu(-\Box)A^\mu + \frac{1}{2}(\partial\cdot A)^2 - \frac{1}{2}(\vec{\nabla}\cdot\vec{A})^2 + \bar{c}\,\Delta c$$

$$\cdot\cdot\,\sim\,\cdot\cdot = -\frac{i}{\vec{k}^2}$$ no propagation

$$-\frac{1}{2}A_\mu\Big[\underbrace{k^2\eta^{\mu\nu} - k^\mu k^\nu + \eta^{\mu i}k_i\,\eta^{\nu j}k_j}_{\equiv Q^{\mu\nu}}\Big]A_\nu$$

$$Q^{\mu\nu} = \begin{pmatrix} k^2 - k^0k^0 & -k^0k^i \\ -k^0k_j & -k^2\delta_{ij}\cancel{-k^ik^j}\cancel{+k^ik^j} \end{pmatrix}$$

$$P_{\mu\nu} = \begin{pmatrix} a & d\,k_i \\ d\,k_j & b\delta_{ij} + c k_i k_j \end{pmatrix}$$

$$P_{\mu\nu} Q^{\nu\rho} = \begin{pmatrix} a & d k_i \\ d k_j & b\,\delta_{ij} + c k_i k_j \end{pmatrix} \begin{pmatrix} -\vec{k}^2 & k^0 k_m \\ k^0 k_i & -k^2 \delta_{im} \end{pmatrix} =$$

$$= \begin{pmatrix} (-a + d k^0)\vec{k}^2 & k_m(a k^0 - d k^2) \\ k_j(-d\vec{k}^2 + b k^0 + c\vec{k}^2 k^0) & d k^0 k_j k_m - k^2 b\,\delta_{jm} - k^2 c\, k_j k_m \end{pmatrix}$$

$$a = d\,\frac{k^2}{k^0} \qquad b = -\frac{1}{k^2} \qquad d = \frac{k^2}{k^0}\,c$$

$$a = -c \qquad b = -\frac{1}{k^2}$$

$$-a+dk^0=\frac{1}{\vec{k}^2}=-d\frac{k^2}{k^0}+dk^0=\frac{d}{k^0}\left(-k^2+k^{0\,2}\right)=$$

$$=\frac{d}{k^0}\vec{k}^2$$

$$d=\frac{k^0}{(\vec{k}^2)^2}\qquad a=\frac{k^2}{(\vec{k}^2)^2}\qquad c=\frac{(k^0)^2}{(\vec{k}^2)^2k^2}$$

$$b=-\frac{1}{k^2}$$

$$0\overset{?}{=}-d\vec{k}^2+bk^0+c\vec{k}^2k^0=-\frac{k^0}{\vec{k}^2}-\frac{k^0}{k^2}+\frac{(k^0)^2k^0}{k^2\vec{k}^2}=$$

$$=-k^0\frac{k^2+\vec{k}^2-k^{0\,2}}{k^2\vec{k}^2}=0$$

$$P_{\mu\nu} = -\frac{1}{k^2}\begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} - \frac{k^{0^2} k_i k_j}{(\vec{k}^2)^2} \end{pmatrix}$$

on shell

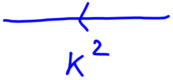
on the pole : $k^2 = 0 \Rightarrow k^{0^2} = \vec{k}^2$ $\quad k^\mu = (k, 0, 0, k)$

$$\begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In gravity you find analogous results in the Prentki gauge : $\mathcal{G}_\mu = \partial^i g_{\mu i} + \omega\, \delta^i_\mu \partial_i g_{\rho\sigma} \eta^{\rho\sigma}$

Quantization

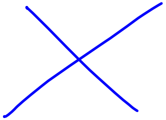
φ^4 - theory
$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2}_{\frac{1}{2}\varphi\left(-\Box - m^2\right)\varphi} - \frac{\lambda}{4!}\varphi^4$$



$$= \frac{i}{k^2-m^2+i\epsilon} \qquad \frac{1}{x-i\epsilon} = \mathcal{P}\frac{1}{x} + i\pi\delta(x)$$

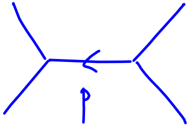
$$\|$$

$$= i\mathcal{P}\frac{1}{k^2-m^2} - i\pi\delta(k^2-m^2)$$



$$= -i\lambda$$

Build Feynman diagrams

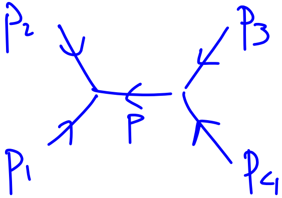


$$= (-i\lambda)\frac{i}{p^2-m^2+i\epsilon}(-i\lambda) =$$

$$= \frac{-i\lambda^2}{p^2-m^2+i\epsilon}$$

Unitarity: $S^\dagger S = 1$

non amputated diagram



$$= \frac{i}{p_1^2 - m_1^2} \frac{i}{p_2^2 - m_2^2} \frac{i}{p_3^2 - m_3^2} \frac{i}{p_4^2 - m_4^2} \frac{-i\lambda^2}{p^2 - m^2 + i\epsilon}$$

$$p = p_3 + p_4 = -p_1 - p_2$$

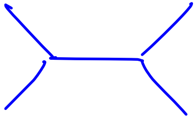
$S =$ scattering matrix $= 1 + iT$

$iT =$ Feynman diagrams

$$S^\dagger S = 1 = (1 - iT^\dagger)(1 + iT) = 1 + iT +$$

$$-iT^\dagger + T^\dagger T$$

$$iT - iT^\dagger = -T^\dagger T$$ optical theorem

$iT =$  $= \dfrac{-i\lambda}{p^2-m^2+i\epsilon} = -i\lambda^2\,\mathcal{P}\,\dfrac{1}{p^2-m^2} +$

$-i\lambda^2\pi\,\delta(p^2-m^2)$

$$\frac{1}{x \mp i\epsilon} = \mathcal{P}\,\frac{1}{x} \pm i\pi\,\delta(x)$$

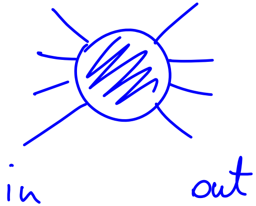
$$2\,\mathrm{Re}\left[iT\right] = -\,T^\dagger T \leq 0$$

$$\parallel$$

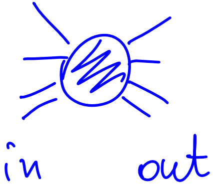
$$-2\lambda^2\pi\,\delta(p^2-m^2) \leq 0 \qquad \text{ok!}$$

$\dfrac{i}{p^2-m^2+i\epsilon}$ $\qquad$ $\dfrac{-i}{p^2-m^2-i\epsilon}$ $\qquad$ are both ok with unitarity

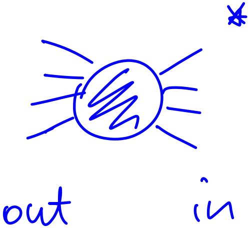
S $\qquad\qquad$ $S^\dagger$



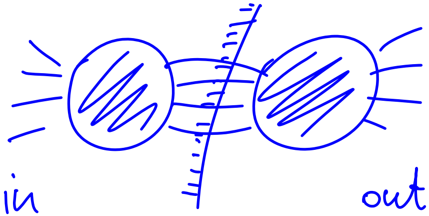
$$= iT$$



$$+$$



$$=$$



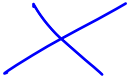
$$iT + (-iT)^\dagger = -T^\dagger T$$

$$\langle\alpha| iT |\beta\rangle + \langle\alpha|(-iT)^\dagger|\beta\rangle =$$

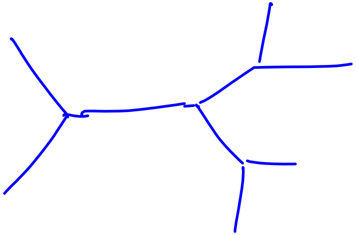
$$= -\sum_{|n\rangle} \langle\alpha| T^\dagger |n\rangle\langle n| T |\beta\rangle$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

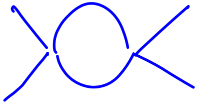
$$= \frac{i}{p^2 - m^2 + i\epsilon}$$



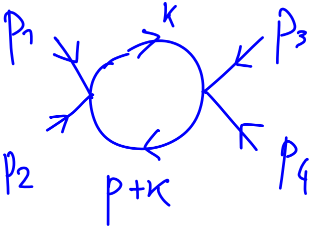
$$= -i\lambda$$



tree diagrams



loop diagrams



$$\propto \int \frac{d^4k}{(2\pi)^4} (-i\lambda)^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p+k)^2 - m^2 + i\epsilon} = \infty$$

$$p = p_3 + p_4 = -p_1 - p_2$$

k large : $\displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} \sim \ln \Lambda$

$|k^2| \gg \Lambda^2$

$$\int^{\Lambda} \frac{d^3k}{k^4} \propto \int^{\Lambda} \frac{dk}{k} \sim \ln \Lambda$$

$$g(p,\Lambda) = \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} \frac{1}{(p+k)^2-m^2+i\epsilon} = g_{fin}(p,\Lambda) + g_{div}(p,\Lambda)$$

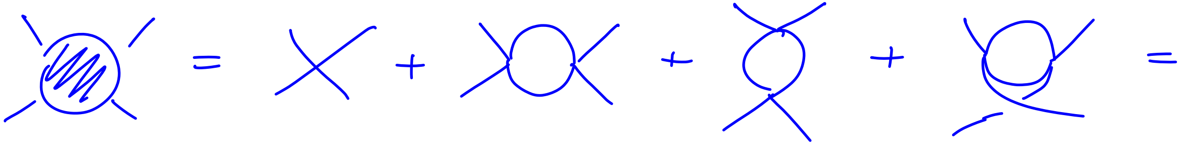
$$\frac{\partial g(p,\Lambda)}{\partial p^\mu} = \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} \frac{-2(p+k)^\mu}{((p+k)^2-m^2+i\epsilon)^2}$$

$|k^2| \sim \infty$: $\displaystyle\int^{\Lambda} \frac{d^4k}{(k^2)^3} k^\mu \sim \int^{\Lambda} \frac{dk}{k^2} < \infty$

$$\frac{\partial}{\partial p^\mu} g_{fin} + \frac{\partial}{\partial p^\mu} g_{div} = \text{finite}$$

If we parametrize $g_{div}(p,\Lambda)$ as $\ln\Lambda \cdot \tilde{g}(p)$

$$\Rightarrow \frac{\partial}{\partial p^\mu} g_{div} = 0 \qquad \Rightarrow \qquad g_{div} = \text{const. in } p$$



$$= -i\lambda + a\lambda^2 g_{fin}(p,\Lambda) + b\lambda^2 \ln\Lambda$$

a = number b = number

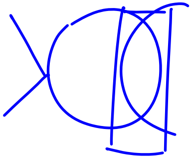
counterterms

locality: every term of the Lagrangian should contain a finite number of derivatives

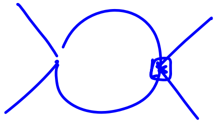
Locality of counterterms: the divergent parts of all diagrams are local, which means that they are polynomial in the external momenta

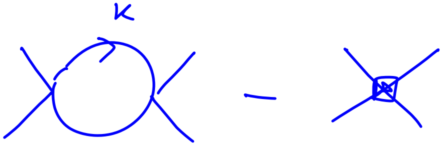
$$g(p) = \int \prod_{i=1}^{L} \frac{d^4 k_i}{(2\pi)^4} \prod_j \frac{1}{k_j^2 - m^2 + i\epsilon} \prod \frac{P(k, p)}{(p + k_m + k_n)^2 - m^2 + i\epsilon}$$

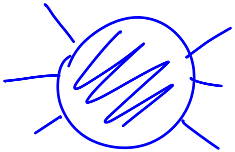
$$\frac{\partial^n}{\partial p^{\mu_1} \cdots \partial p^{\mu_n}} g(p) < \infty \Rightarrow \frac{\partial^n}{\partial p^{\mu_1} \cdots \partial p^{\mu_n}} g_{div}(p) = 0$$



two-loop diagram







The counterterms are local, like the terms of the Lagrangian and can be subtracted by modifying the Lagrangian

The starting Lagrangian may or may not contain all the types of local terms it generates as counterterms by means of Feynman diagrams

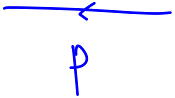
If yes, the theory is renormalizable

If not, new terms must be added to the Lagrangian. If a finite number of additions is sufficient, the resulting theory is renormalizable

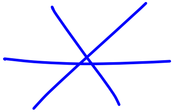
If not, the theory is nonrenormalizable

The φ^4 theory is renormalizable, the φ^6 theory is not

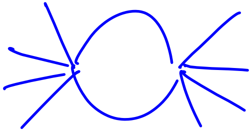
$$\mathcal{L}_6 = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda_6}{6!}\varphi^6 + \Delta\frac{\varphi^8}{8!}$$



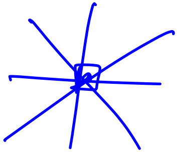
$= \dfrac{i}{p^2-m^2+i\epsilon}$



$= -i\lambda_6$

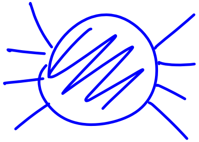


$\Big|_{div} = c \ln\Lambda\, \lambda_6^2$

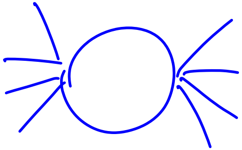


$= -ib\ln\Lambda$

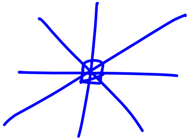
c = number   b = number



$=$



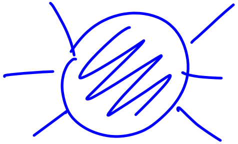
$+$



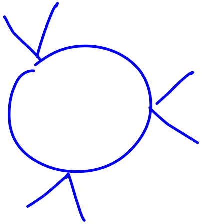
$< \infty$

φ^4 theory

$$\mathcal{L} = \frac{Z_\varphi}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{Z_\varphi Z_{m^2}}{2} m^2\varphi^2 - \frac{\lambda}{4!} Z_\lambda \frac{Z_\varphi^2\varphi^4}{4!}$$



$=$



$$\propto \int \frac{d^4k}{(k^2)^3} < \infty$$

$$|k^2| \gg \Lambda^2$$

Power counting: $\mathcal{L} = \frac{(\partial_\mu\varphi)^2}{2} - \frac{m^2}{2}\varphi^2 - \frac{\lambda\varphi^4}{4!}$

$[\mathcal{L}] = 4$ $\quad [S] = [\int d^4x\, \mathcal{L}] = 0$

$[m] = 1$ $\quad [\varphi] = 1$ $\quad [\partial_\mu] = 1$ $\quad [\lambda] = 0$ $\quad + \lambda_6\varphi^6$

No parameters with negative dimension $\quad [\lambda_6] = -2$

φ^6 theory $\quad \mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda_6}{6!}\varphi^6$

$[\varphi] = 1 \qquad [\lambda_6] = -2$

$$\lambda_6^3 \varphi^{10} \qquad \underbrace{\lambda_6^{\cdots} \varphi^8 \Box \varphi^8}_{\lambda_6^2 \varphi^8}$$

Quantum gravity (Hilbert action) is nonrenormalizable

$$S_H + \frac{1}{2\lambda}\int \ell_\mu \eta^{\mu\nu} \ell_\nu + S_{ghost}$$

$$S_H = -\frac{1}{2k^2}\int d^4x \sqrt{-g}\, R = \int \partial\phi\partial\phi + k\, \partial\phi\partial\phi\phi \cdots$$

$$g_{\mu\nu} = \eta_{\mu\nu} + 2k\phi_{\mu\nu} \qquad [\phi_{\mu\nu}] = 1 \quad [g_{\mu\nu}] = 0 \quad [k] = -1$$

Candidate counterterms :

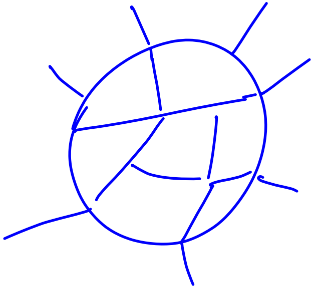
$$\int \sqrt{-g} \left(R^2 + R_{\mu\nu} R^{\mu\nu} + \kappa^2 R \Box R + \kappa^2 R^3 + \kappa^4 R^4 \dots \right)$$

$$\kappa^4 R^4 \rightarrow \frac{1}{M_{Pl}^4} R^4$$

$[R] = [\partial \Gamma] = 2 \qquad [\kappa] = -1$

$[\Gamma] = [g^{-1} \partial g] = 1$

$[\kappa] = -1 \qquad \kappa = \frac{1}{M_{Pl}} \qquad M_{Pl}$ = Planck mass = $= 10^{19}$ GeV



L = # of loops

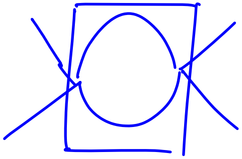
I = # of internal legs

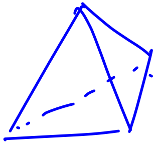
V = # of vertices

$$\underline{L = I - V + 1}$$

Generic vertex of Hilbert action : $\kappa^{n-2} \partial^2 \phi^n$

$L = I - V + 1$ $(1+L) - I + V = 2$



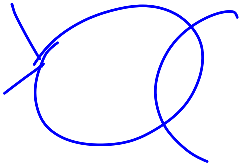


$\# \text{faces} - \# \text{edges} + \# \text{vertices} = 2$

$4 - 6 + 4 = 2$

$L + 1 - I + V$

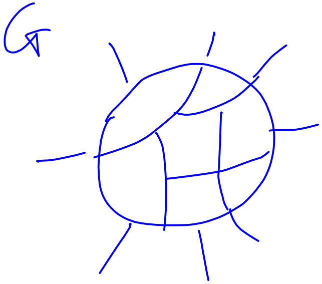
$2 - 2 + 2 = 2$



$L + 1 - 4 + 3 = 2$

3

generic vertex of the Hilbert action: $\kappa^{n-2}\partial^2\phi^n$



power of κ :

$$\prod_{i \text{ vertices}} \kappa^{n_i-2} = \kappa^{\sum_i n_i - 2V} = \kappa^{E+2(I-V)}$$

n_i = # of legs of the i-th vertex

$$\sum_i n_i = \underset{\substack{\uparrow \\ \text{external legs}}}{E} + 2I$$

G: $\kappa^{E+2(I-V)} = \kappa^{E+2L-2}$

$$\kappa^{E+2(L-1)}\phi^E = \underbrace{(\kappa\phi)^E}_{\text{invariant}}(\kappa^2)^{L-1}$$

$(R, R^2, \dots)$

$L=0$: $-\frac{1}{2\kappa^2}\int\sqrt{g}\,R$

one loop: $(\kappa^2)^0 \int \left(\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}\right) \sqrt{-g}$

two loops: $(\kappa^2) \int \Big(a\, R^3 + b\, R_{\mu\nu} R^{\mu\rho} R_{\rho}^{\ \nu} + c\, R \Box R \cdots +$

$[\kappa] = -1$ $\qquad \cdots + R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R^{\mu\nu}_{\ \ \alpha\beta} \Big)$

$\ln \Lambda \quad — \quad (\ln \Lambda + C)$ $\qquad C =$ arbitrary finite (for $\Lambda \to \infty$)

Every counterterm of new type brings a new, independant coupling constant (unless it is proportional to the field equations)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Pure gravity (no cosmological constant): $R_{\mu\nu} = 0$

Every counterterm that vanishes on shell can be absorbed away by redefining $g_{\mu\nu}$

Example: $S_H = -\frac{1}{2\kappa^2}\int\sqrt{-g}\,R$

At one loop: $S_H \to S_H + \int\sqrt{-g}\left(\alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2\right) =$

$$\alpha, \beta \propto \ln\Lambda$$

$$= S_H + \int \alpha' R_{\mu\nu}\frac{\delta S_H}{\delta g_{\mu\nu}} + \beta' R g_{\mu\nu}\frac{\delta S_H}{\delta g_{\mu\nu}} =$$

$$= S_H(g) + \int \Delta g_{\mu\nu}\frac{\delta S_H}{\delta g_{\mu\nu}} \simeq S_H(g+\Delta g)$$

$$\Delta g_{\mu\nu} = \alpha' R_{\mu\nu} + \beta' g_{\mu\nu} R$$

Pure gravity is one-loop finite

It is not finite at two loops :

$$\int \sqrt{-g}\, R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R^{\mu\nu}_{\ \ \alpha\beta}$$ Goroff Sagnotti

The theory is not even finite at one loop in the presence of matter ('t Hooft and Veltman)

$$-\frac{1}{2\kappa^2}\int\sqrt{-g}\, R + \int\sqrt{-g}\,\frac{1}{2}\, g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi$$

One-loop counterterms R^2, $R_{\mu\nu}R^{\mu\nu}$, $R^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi$,

$\ldots \Box\varphi\Box\varphi \quad \ldots \quad (\nabla_\mu\varphi\nabla^\mu\varphi)^2$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa^2 T_{\mu\nu} \sim \nabla\varphi\nabla\varphi$$

All the counterterms that are quadratic in the curvature tensors can be converted into cubic terms up to total derivatives and term that vanish on shell

$$R_{\mu\nu}R^{\mu\nu}\,,\quad R^2\,,\quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,,\quad R\Box R\,,\quad R_{\mu\nu}\Box R^{\mu\nu}\,,\quad R_{\mu\nu\rho\sigma}\Box R^{\mu\nu\rho\sigma}$$

$\Rightarrow$ The propagator is unaffected

$$R_{\mu\nu\rho\sigma}\quad \nabla \cdots \quad \nabla^{\lambda}\cdots R_{\alpha\beta\gamma\delta}$$

We can freely commute derivatives $[\nabla,\nabla]\sim R$

$$\nabla^{\alpha}R_{\alpha\beta\gamma\delta} = -\nabla_{\delta}R_{\beta\gamma} + \nabla_{\gamma}R_{\beta\delta}$$

$$R_{\mu\nu\rho\sigma} \cdots \nabla^{\lambda} \cdots \nabla_{\lambda} R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} \cdots \nabla^{\lambda} \cdots \left(- \nabla_{\nu} R_{\lambda\mu\rho\sigma} - \nabla_{\mu} R_{\nu\lambda\rho\sigma} \right)$$

$\longrightarrow$ back to previous case

The classical action of quantum gravity

$$- \frac{1}{2\kappa^2} \int \sqrt{-g} \left\{ 2\Lambda + R + W^3 + \cdots \right\} + S_m$$

only Weyl and
at least three

Field equations :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(m)} + \kappa^2 T_{\mu\nu}^{(g)}$$

$$T_{\mu\nu}^{(g)} \propto W^2$$

FLRW metric

$$T^{(m)\,\nu}_{\mu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

$\rho = \rho(t)$

$p = p(t)$

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right]$$

This metric has $W_{\mu\nu\rho\sigma} = 0$

There are ways to make the theory renormalizable

$$S_{HD} = -\frac{1}{2k^2}\int \sqrt{-g}\left(2\Lambda + \zeta R + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2\right)$$

higher-derivative quantum gravity

α and β are of order 1

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\, \phi_{\mu\nu}$$

Quadratic part: $\sim \zeta (\partial\phi)^2 + \alpha \underbrace{\phi\, \partial^4 \phi} + \beta \underbrace{\phi\, \partial^4 \phi}$

← from $R_{\mu\nu}R^{\mu\nu}$ and R^2

Propagator $\sim \dfrac{1}{\alpha(p^2)^2 + \beta(p^2) - \zeta p^2} \underset{|p^2| \text{ large}}{\sim} \dfrac{1}{(p^2)^2}$

Let us choose $[\alpha] = [\beta] = 0$ Then $[\phi] = 0$

$[\zeta] = 2 \quad [\Lambda] = 4 \quad [\kappa] = 0$

No parameter of negative dimension $\quad \int R^3 \sqrt{-g}$

Every parameter (but α and β) must appear polynomially in the counterterms

Propagator $P = \dfrac{1}{\alpha(p^2)^2 + \beta(p^2)^2 + \zeta p^2 + \Lambda} \sim \dfrac{1}{(p^2)^2}$

$$\frac{\partial P}{\partial \zeta} = -\frac{p^2}{\left(\alpha(p^2)^2 + \beta(p^2)^2 + \zeta p^2 + \Lambda\right)^2} \sim \frac{1}{(p^2)^3}$$

$$\frac{\partial P}{\partial \alpha} = -\frac{(p^2)^2}{\left(\alpha(p^2)^2 + \beta(p^2)^2 + \zeta p^2 + \Lambda\right)^2} \sim \frac{1}{(p^2)^2}$$

$$S_{HD} = -\frac{1}{2k^2}\int\sqrt{-g}\left(2\Lambda + \zeta R + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2\right)$$

is strictly renormalizable

$$S_{HD} = -\frac{1}{2\kappa^2}\int\sqrt{-g}\,\Big[\,2\Lambda + \zeta R + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2 +$$
$$+\gamma R\Box R + \delta R_{\mu\nu}\Box R^{\mu\nu} + O(R^3)\,\Big]$$

is super-renormalizable

dominant quadratic terms: $\delta\,\phi\,\Box^3\phi + \gamma\,\phi\,\Box^3\phi$

We choose $[\delta] = [\gamma] = 0 \quad [\beta] = [\alpha] = 2 \quad [\zeta] = 4$

$[\Lambda] = 6 \quad [\phi] = -1 \quad [\kappa] = 1$

At L loops the counterterms carry an extra factor $(\kappa^2)^L$

One loop: $\int R^2 + R_{\mu\nu}^2 + R + \Lambda$

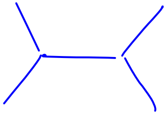
Two loops: $\kappa^2\int R + \Lambda \qquad$ Three loops $(\kappa^2)^2\int 1$

These theories are *not* unitary (if quantized in the ordinary way).

Propagator $\sim \dfrac{1}{(p^2)^2 + p^2}$

$$\frac{-m^2}{p^2(p^2-m^2)} = \frac{1}{p^2} - \frac{1}{p^2-m^2}$$

negative residue

$iT =$ 

$$2\,\mathrm{Re}\left[iT\right] = -\pi\left[\delta(p^2) - \delta(p^2-m^2)\right]$$

This cannot be $-T^\dagger T$!

Higher derivative theories have problems also at the classical level

Instead of having a Lagrangian $\mathcal{L}(q,\dot q)$ we have a Lagrangian $\mathcal{L}(q,\dot q,\ddot q,\dddot q,\dots)$

Adding variables I can always get a $\mathcal{L}(q,\dot q, S, \dot S, J, \dot J \dots)$

but then the energy is <u>not</u> bounded from below

Abraham - Lorentz force

$$m\left(a - \tau\frac{da}{dt}\right) = F_{ext} \qquad\qquad \tau = \frac{2e^2}{3mc^3} > 0$$

Runaway solutions : $F_{ext} = 0$ $\qquad x(0)=0=\dot x(0) \not\Rightarrow x(t)\equiv 0$

$$a = \tau \dot{a} \qquad a = c e^{t/\tau} = \ddot{x}$$

$$\dot{x}(t) = c\tau\left(e^{t/\tau} - 1\right) + \cancel{v_0}$$

$$x(t) = c\tau^2\left(e^{t/\tau} - 1\right) - c\tau t + \cancel{v_0 t} + \cancel{x_0}$$

$$m\left(a - \tau \frac{da}{dt}\right) = F_{ext} = m\left(1 - \tau \frac{d}{dt}\right) a$$

$$m a = \frac{1}{1 - \tau \frac{d}{dt}} F_{ext}(t) \equiv \langle F_{ex}(t) \rangle$$

Correct $\tau \to 0$ limit $\Rightarrow$ unique Green function

Most general solution :

$$ma = -\frac{1}{\tau}\int_{-\infty}^{t} dt'\, e^{(t-t')/\tau}\, F_{ext}(t') + \underbrace{m a_0 e^{t/\tau}}_{\text{runaway solution}}$$

Regular solution for $\tau \to 0$:

$$ma = \frac{1}{\tau}\int_{t}^{+\infty} dt'\, e^{(t-t')/\tau}\, F_{ext}(t')$$

$$\tau \simeq 10^{-23}\,s$$

$$-u = \frac{t-t'}{\tau}$$

$$ma = \int_0^{\infty} du\, e^{-u}\, F_{ext}(t+u\tau)$$

$$du = \frac{dt'}{\tau}$$

$$t' = t + u\tau$$

$\tau \to 0$ $\quad ma = F_{ext}$

Unruh effect

An accelerated detector detects the presence of particles even in vacuum, with a black body spectrum

Detector: pointlike particle with discrete energy levels $E_0, E_1, E_2, \dots$

We couple it to a quantized scalar field $\phi(x)$

τ = proper time of the detector

$x^\mu(\tau)$ = trajectory of the detector

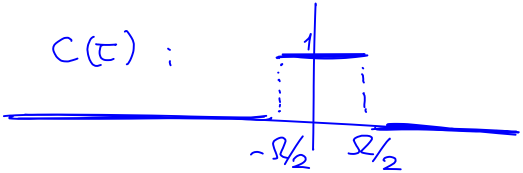
We describe the interaction ("$J_\mu A^\mu$") as

$$c(\tau)\,\chi(\tau)\,\phi(x(\tau))$$

on/off switch ↗

↑ detector

$c(\tau)$:



$$c(\tau) = \begin{cases} 1 & \text{for} \quad |\tau| \leq \frac{\Omega}{2} \\ 0 & \text{for} \quad |\tau| > \frac{\Omega}{2} \end{cases}$$

Later: $\Omega \to \infty$

Initial state of the system: $|E_0\rangle |0\rangle_\phi$

Final state: any $|E_i\rangle |\psi\rangle_\phi$

Transition amplitude in the Bohr approximation:

$$\mathcal{A}_i = i \int_{-\infty}^{+\infty} d\tau \; c(\tau) \; \langle E_i| {}_\phi\langle \psi | \, \chi(\tau) \, \phi(x(\tau)) \, | E_0\rangle |0\rangle_\phi =$$

$$= i \int_{-\infty}^{+\infty} d\tau \; c(\tau) \; \langle E_i | \chi(\tau) | E_0\rangle \; {}_\phi\langle \psi | \phi(x(\tau)) | 0\rangle_\phi$$

$$\chi(\tau) = e^{iH\tau} \chi(0) \, e^{-iH\tau}$$

H = Hamiltonian

$$\langle E_i | \chi(\tau) | E_0 \rangle = \langle E_i | e^{iH\tau} \chi(0) e^{-iH\tau} | E_0 \rangle =$$

$$= e^{i(E_i - E_0)\tau} \langle E_i | \chi(0) | E_0 \rangle$$

$$\mathcal{A}_i = i \langle E_i | \chi(0) | E_0 \rangle \int_{-\infty}^{+\infty} d\tau \, c(\tau) \, e^{i(E_i - E_0)\tau} \, {}_\phi\langle \psi | \phi(x(\tau)) | 0 \rangle_\phi$$

Transition probability :

$$P_i = \sum_{|\psi\rangle_\phi} |\mathcal{A}_i|^2 = | \langle E_i | \chi(0) | E_0 \rangle |^2 \cdot$$

$$\cdot \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \, c(\tau) c(\tau') \, e^{i(E_i - E_0)(\tau - \tau')} \underbrace{\sum_{|\psi\rangle_\phi} {}_\phi\langle 0 | \phi(x(\tau')) | \psi\rangle_\phi \, {}_\phi\langle \psi |}_{= \mathbb{1}} \phi(x(\tau)) | 0 \rangle_\phi =$$

$$= |\langle E_i | \chi(0) | E_0 \rangle|^2 \int_{-\infty}^{+\infty} d\tau \, c(\tau) \int_{-\infty}^{+\infty} d\tau' \, c(\tau') \, e^{i(E_i - E_0)(\tau - \tau')} \underbrace{\langle 0 | \phi(x(\tau')) \phi(x(\tau)) | 0 \rangle}_{\text{retarded potentials}}$$

Free massless scalar field $\phi(x)$:

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = -\frac{1}{4\pi^2} \frac{1}{(x-y)^2}$$

Momentum space : $\frac{1}{p^2}$

Retarded
Advanced potentials : $\frac{1}{(p^0 \pm i\epsilon)^2 - \vec{p}^{\,2}}$ $(y=0)$

$$-\frac{1}{4\pi^2} \frac{1}{(x^0 \mp i\epsilon)^2 - \vec{x}^{\,2}}$$

Feynman : $\frac{1}{p^2 + i\epsilon}$ $-\frac{1}{4\pi^2} \frac{1}{x^2 - i\epsilon}$

$$P_i = \frac{|\langle E_i|\chi(0)|E_0\rangle|^2}{4\pi^2}\int_{-\infty}^{+\infty} d\tau\, c(\tau)\int_{-\infty}^{+\infty} d\tau'\, c(\tau')\, e^{i(E_i-E_0)(\tau-\tau')} .$$

$$\Bigg[\cdot\frac{1}{-(x^0(\tau')-x^0(\tau)-i\epsilon)^2+(\vec{x}(\tau')-\vec{x}(\tau))^2} .$$

Straight uniform motion: $t(\tau) = x^0(\tau) = t$

$$\vec{x}(\tau) = \vec{x}_0 + \vec{v}\cdot t$$

$$= \frac{1}{v^2(\tau-\tau')^2-(\tau-\tau'+i\epsilon)^2}$$

$$P_i = \frac{\Omega}{4\pi^2}|\langle E_i|\chi(0)|E_0\rangle|^2\int_{-\infty}^{\infty} du\; e^{i(E_i-E_0)u}\,\frac{1}{v^2u^2-(u+i\epsilon)^2}$$

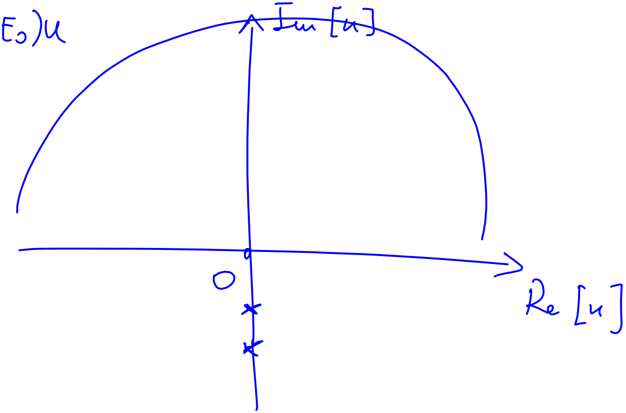
$$\tau-\tau' = u$$

poles : $\pm v u = u + i\epsilon$

$$u \mp v u = -i\epsilon = (1 \mp v)u$$

$$u = -\frac{i\epsilon}{1 \mp v}$$

$e^{i(E_i - E_0)u}$



$E_i > E_0$

$u = x + iy \quad y > 0$

$$\left| e^{i(E_i - E_0)u} \right| = e^{-y(E_i - E_0)}$$

Hyperbolic motion

$x^\mu(\tau)$: $t = \frac{1}{g}\sinh(g\tau)$ $\quad x = \frac{1}{g}\cosh(g\tau)$

g = acceleration $\quad t^2 - x^2 = \frac{-1}{g^2}$ = constant ← $\cancel{2}tdt = \cancel{2}xdx$

$$\frac{dx}{dt} = \frac{t}{x} = \tanh(g\tau) \qquad \frac{dx^\mu}{dt} = (1, \tanh(gt), 0, 0)$$

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\,\frac{dt}{d\tau} = (\cosh(g\tau), \sinh(g\tau), 0, 0) = u^\mu$$

$$u^\mu u_\mu = 1$$

$$\frac{dt}{d\tau} = \cosh(g\tau)$$

$$a^\mu = \frac{dx^\mu}{d\tau} = g\,(\sinh(g\tau), \cosh(g\tau), 0, 0) \Rightarrow a^\mu a_\mu = g$$

We have to compute

$$-\left(x^0(\tau') - x^0(\tau) - i\epsilon\right)^2 + \left(\vec{x}(\tau') - \vec{x}(\tau)\right)^2 = \frac{1}{g^2}\left(\cosh(g\tau') - \cosh(g\tau)\right)^2 +$$

$$-\frac{1}{g^2}\left(\sinh(g\tau') - \sinh(g\tau) - i\bar{\epsilon}\right)^2 = \frac{1}{g^2}\Big[2 - 2\cosh(g\tau')\cosh(g\tau) +$$

$\bar{\epsilon} = g\epsilon$

$$+ 2\sinh(g\tau')\sinh(g\tau) + 2i\bar{\epsilon}\left(\sinh(g\tau') - \sinh(g\tau)\right)\Big] =$$

$$= \frac{2}{g^2}\left[1 - \cosh\left(g(\tau'-\tau)\right) + i\bar{\epsilon}\left(\sinh(g\tau') - \sinh(g\tau)\right)\right] =$$

$$= -\frac{4}{g^2}\sinh^2\left(\frac{g(\tau'-\tau)}{2}\right) + \frac{2i\bar{\epsilon}}{g^2}\boxed{\left(\sinh(g\tau') - \sinh(g\tau)\right)} =$$

$$= -\frac{4}{g^2}\sinh^2\left(\frac{g(\tau'-\tau-i\tilde{\epsilon})}{2}\right) \equiv f(\tilde{\epsilon}) =$$

$$= -\frac{4}{g^2}\sinh^2\left(\frac{g(\tau'-\tau)}{2}\right) + \tilde{\epsilon}\left.\frac{df}{d\tilde{\epsilon}}\right|_{\tilde{\epsilon}=0}$$

$$\left.\frac{df}{d\tilde{\epsilon}}\right|_{\tilde{\epsilon}=0} = +\frac{4}{g^{\cancel{2}}}\,\cancel{2}\,\sinh\left(\frac{g(\tau'-\tau)}{2}\right)\cosh\left(\frac{g(\tau'-\tau)}{2}\right)\frac{+i\cancel{g}}{\cancel{2}} =$$

$$= 2i\,\boxed{\sinh\left(g(\tau'-\tau)\right)}$$

We just need to show that $\quad x = g\tau' \quad y = g\tau$

$$\frac{\sinh(x) - \sinh(y)}{\sinh(x-y)} > 0 \qquad \forall x,y \qquad a = e^{x} > 0$$

$$b = e^{y} > 0$$

$$\frac{a - \frac{1}{a} - b + \frac{1}{b}}{\frac{a}{b} - \frac{b}{a}} = \frac{a^2 b - b - b^2 a + a}{a^2 - b^2} =$$

$$= \frac{\cancel{(a-b)}(1+ab)}{\cancel{(a-b)}(a+b)} = \frac{1+ab}{a+b}$$

$$P_i = \frac{|\langle E_i|\chi(0)|E_0\rangle|^2}{4\pi^2} \int_{-\infty}^{+\infty} d\tau\, c(\tau) \int_{-\infty}^{+\infty} d\tau'\, c(\tau')\, e^{i(E_i - E_0)(\tau-\tau')} \frac{-g^2}{4} \; .$$

$u = \tau - \tau'$

$$\frac{1}{\sinh^2\left(\frac{g(\tau' - \tau - i\hat{\varepsilon})}{2}\right)} =$$

$$= -\frac{g^2\Omega}{(4\pi)^2}\,|\langle E_i|\,\chi(0)\,|E\rangle|^2 \int_{-\infty}^{+\infty} du\, \frac{e^{i(E_i-E_0)u}}{\sinh^2\left(\frac{g(u+i\tilde{\epsilon})}{2}\right)}$$

$$\frac{1}{\sinh^2(\pi y)} = \frac{1}{\pi^2}\sum_{k=-\infty}^{+\infty}\frac{1}{(y+ik)^2}$$

$$y = \frac{g}{2\pi}(u+i\tilde{\epsilon})$$

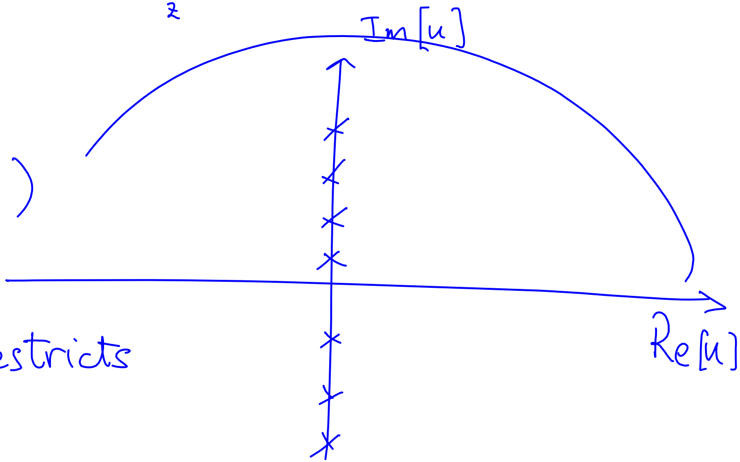
$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{C_z}\frac{f(\zeta)\,d\zeta}{(\zeta-z)^{n+1}}$$

$$u = -i\tilde{\epsilon} + \frac{2\pi}{g}y$$

$$dy = \frac{g}{2\pi}du$$

Poles: $y = -ik = \frac{g}{2\pi}(u+i\tilde{\epsilon})$

$$u = -i\tilde{\epsilon} - ik\frac{2\pi}{g}$$



The sum restricts to $k<0$

$$P_i = - \frac{g^{\cancel{2}}\Omega}{(4\pi)^2} |\langle E_i | X(0) | E_0 \rangle|^2 \int_{-\infty}^{+\infty} \frac{2\cancel{\pi}}{\cancel{g}\pi k} dy \sum_{k=-1}^{-\infty} \frac{e^{i(E_i-E_0)\frac{2\pi}{g}y}}{(y+ik)^2} =$$

$$= - \frac{\cancel{g}\Omega}{\underset{2}{\cancel{8}}\pi^{\cancel{3}}} |\langle E_i | X(0) | E_0 \rangle|^2 \sum_{k=-1}^{-\infty} \cancel{2}\cancel{\pi} i \; e^{i(E_i-E_0)\frac{2\pi}{g}(-ik)} \; i(E_i-E_0)\frac{\cancel{2\pi}}{\cancel{g}}$$

W_i = probability per unit time $= \dfrac{P_i}{\Omega} =$

$k=-n$

$$= \frac{E_i - E_0}{2\pi} |\langle E_i | X(0) | E_0 \rangle|^2 \sum_{n=1}^{\infty} e^{-\frac{2\pi}{g}(E_i-E_0)n}$$

$$a = e^{\frac{2\pi}{g}(E_i-E_0)} \qquad \sum_{n=1}^{\infty} a^{-n} = \frac{1}{1-\frac{1}{a}} - 1 = \frac{\frac{1}{a}}{1-\frac{1}{a}} =$$

$$= \frac{1}{a-1}$$

$$W_i = \frac{E_i - E_0}{2\pi} \left| \langle E_i | \mathcal{X}(0) | E_0 \rangle \right|^2 \frac{1}{e^{\frac{2\pi}{g}(E_i - E_0)} - 1}$$

Black body spectrum with $T = \frac{g}{2\pi}$

— end of 1st part —

Theories of Proca, Pauli-Fierz and Rarita-Schwinger

spin 1	Spin 2	Spin $\frac{3}{2}$
$m \neq 0$	$m \neq 0$	$m = 0$, $m \neq 0$

Proca : $S_p = \int \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + \frac{m^2}{2} A_\mu A_\nu g^{\mu\nu} + \right.$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\left. \underbrace{+ \frac{\eta_p}{2} R^{\mu\nu} A_\mu A_\nu + \frac{\eta_p'}{2} R A_\mu A_\nu g^{\mu\nu}}_{\text{non minimal terms}} \right]$$

Flat space:

$$\mathcal{L}_P = -\frac{1}{2}\partial_\mu A_\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m^2}{2}A_\mu A^\mu$$

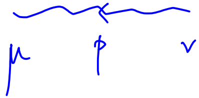
$$\Box A_\mu - \partial_\mu(\partial\cdot A) + m^2 A_\mu = 0 \qquad \leftarrow \partial_\mu$$

$$\cancel{\Box(\partial\cdot A)} - \cancel{\Box(\partial\cdot A)} + m^2\,\partial\cdot A = 0 \qquad \Rightarrow$$

$$\begin{cases} \partial\cdot A = 0 \\ (\Box + m^2)A_\mu = 0 \end{cases}$$

Propagator: $\mathcal{L}_P = \tilde{A}_\mu(-p)\left[-p^2\eta^{\mu\nu} + p^\mu p^\nu + m^2\eta^{\mu\nu}\right]\tilde{A}_\nu(p)$

$$Q_{\mu\nu} = (-p^2+m^2)\eta_{\mu\nu} + p_\mu p_\nu$$



$$= -\frac{i}{p^2-m^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right) \equiv i\, P_{\mu\nu}$$

$$P_{\mu\nu}\, Q^{\nu}{}_{\rho} = -\frac{1}{p^2-m^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)\left((-p^2+m^2)\,\delta^{\nu}_{\rho} + p^\nu p_\rho\right) =$$

$$= \eta_{\mu\rho} - \cancel{\frac{p_\mu p_\rho}{m^2}} + p_\rho \cancel{\frac{-1}{p^2-m^2}\left(p_\mu - \frac{p^2}{m^2}p_\mu\right)} = \eta_{\mu\rho}$$

$$P_{\mu\nu}\, p^\nu = -\frac{1}{p^2-m^2}\, p_\mu \frac{m^2-p^2}{m^2} = \frac{p_\mu}{m}$$

On the pole $p^2 = m^2$ We can choose $p^\mu = (m, 0, 0, 0)$

Residue : $$-\begin{pmatrix} 1-1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Build the most general $P_{\mu\nu}$ with a pole in $p^2 = m^2$, but such that $P_{\mu\nu} p^\nu$ has no pole

$$P_{\mu\nu} = \frac{1}{p^2 - m^2} \left(a \, \eta_{\mu\nu} + b \, p_\mu p_\nu \right)$$

$$P_{\mu\nu} p^\nu = \frac{1}{p^2 - m^2} \left(a + b p^2 \right) p_\mu$$

Residue at $p^2 = m^2$ should be zero : $a + b m^2 = 0$

$$b = - \frac{a}{m^2} \qquad P_{\mu\nu} = \frac{a}{p^2 - m^2} \left(\eta_{\mu\nu} - \underset{\substack{\uparrow \\ \text{nonrenormalizable term}}}{\frac{p_\mu p_\nu}{m^2}} \right)$$

Pauli-Fierz fields $\quad \chi_{\mu\nu} \quad$ symmetric tensor $\quad$ spin 2

10 components

$\partial^\mu \chi_{\mu\nu}$: vector + salar $(\partial^\mu \partial^\nu \chi_{\mu\nu})$

$\chi \equiv \chi_{\mu\nu} \eta^{\mu\nu}$: scalar

Propagator: $\quad P_{\underbrace{\mu\nu}\underbrace{\rho\sigma}}(p) = \frac{1}{p^2-m^2}(\cdots)$

symmetric in $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$, $(\mu\nu \leftrightarrow \rho\sigma)$

uniquely fixed by the requirements that $P_{\mu\nu\rho}{}^{\rho}$ and $P_{\mu\nu\rho\sigma} p^\sigma$ have no poles

$$P_{\mu\nu\rho\sigma} = \frac{i}{2}\frac{1}{p^2-m^2}\left[\pi_{\mu\rho}\pi_{\nu\sigma}+\pi_{\mu\sigma}\pi_{\nu\rho}-\frac{2}{3}\pi_{\mu\nu}\pi_{\rho\sigma}\right]$$

$$\pi_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \qquad \pi_\mu{}^\mu = 4 - \frac{p^2}{m^2} \qquad \pi_{\mu\nu}\pi^\nu{}_\rho = \eta_{\mu\rho} - \frac{2}{m^2}p_\mu p_\rho + \frac{p_\mu p_\rho p^2}{m4} = \pi_{\mu\rho} + O(p^2-m^2)$$

$$\pi_{\mu\nu}p^\nu = p_\mu \frac{m^2-p^2}{m^2} = 0 \quad \text{on shell}$$

$P_{\mu\nu\rho\sigma}p^\sigma$ has no pole : obvious

$$P_{\mu\nu\rho}{}^{\rho} = \frac{i}{2}\frac{1}{p^2-m^2}\left[2\pi_{\mu\nu} + O(p^2-m^2) - \frac{2}{3}\pi_{\mu\nu}\left(4-\frac{p^2}{m^2}\right)\right] =$$

$$= i\frac{\pi_{\mu\nu}}{p^2-m^2}\left(1-\frac{4}{3}+\frac{1}{3}\frac{p^2}{m^2}\right) + O(1) = O(1)$$

$$S_{PF} = \frac{1}{2}\int\sqrt{-g}\Big[\nabla_\rho \chi_{\mu\nu}\nabla^\rho \chi^{\mu\nu} - \nabla_\mu \chi \nabla^\mu \chi + 2\nabla_\mu \chi \nabla_\nu \chi^{\mu\nu} +$$

$$- 2\nabla_\mu \chi_{\nu\rho}\nabla^\rho \chi^{\mu\nu} - m^2\left(\chi_{\mu\nu}\chi^{\mu\nu} - \chi^2\right)\Big] \qquad \chi = \chi_{\mu\nu} g^{\mu\nu}$$

$$S_{PF}^{\text{non minimal}} = \frac{1}{2}\int\sqrt{-g}\Big[a_1 R_{\mu\nu\rho\sigma}\chi^{\mu\rho}\chi^{\nu\sigma} + a_2 R_{\mu\nu}\chi^{\nu\rho}\chi^\mu_\rho +$$

$$+ a_3 R_{\mu\nu}\chi^{\mu\nu}\chi + a_4 R\,\chi_{\mu\nu}\chi^{\mu\nu} + a_5 R\,\chi^2\Big]$$

In flat space : $S_{PF} = \frac{1}{2}\int\Big(\partial_\rho \chi_{\mu\nu}\partial^\rho \chi^{\mu\nu} - \partial_\mu \chi \partial^\mu \chi + 2\partial_\mu \chi \partial_\nu \chi^{\mu\nu} +$

$$- 2\partial_\mu \chi_{\nu\rho}\partial^\rho \chi^{\mu\nu} - m^2 \chi_{\mu\nu}\chi^{\mu\nu} + m^2 \chi^2\Big)$$

Equations of motion:

$$0 = -\Box \chi_{\mu\nu} + \eta_{\mu\nu}\Box \chi - \eta_{\mu\nu}\partial_\rho\partial_\sigma \chi^{\rho\sigma} - \partial_\mu\partial_\nu \chi + \partial_\mu\partial_\rho \chi^\rho_\nu +$$

$$+ \partial_\nu \partial_\rho \chi^\rho_\mu - m^2 \chi_{\mu\nu} + m^2 \eta_{\mu\nu} \chi = 0$$

Divergence ∂_μ :

$$V_\mu = \partial_\nu \chi^\nu_\mu$$

$$0 = -\cancel{\Box V_\nu} + \cancel{\partial_\nu \Box \chi} - \cancel{\partial_\nu (\partial \cdot V)} - \cancel{\partial_\nu \Box \chi} + \cancel{\Box V_\nu} +$$

$$+ \cancel{\partial_\nu (\partial \cdot V)} - m^2 V_\nu + m^2 \partial_\nu \chi = 0$$

$$\Rightarrow V_\mu = \partial_\mu \chi \qquad \Rightarrow \partial \cdot V = \Box \chi$$

Trace :

$$0 = 3 \Box \chi - 4 \partial \cdot V - \Box \chi + 2 \partial \cdot V - m^2 \chi + 4 m^2 \chi =$$

$$= 2 (\cancel{\Box \chi} - \cancel{\partial \cdot V}) + 3 m^2 \chi = 3 m^2 \chi$$

$$\Rightarrow \chi = 0 \qquad V_\mu = 0$$

When $m=0$ we have the quadratic part of the Hilbert action, which has the gauge symmetry $\delta\chi_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$

$$\delta\chi_{\mu_1\dots\mu_s} = \partial_{\{\mu_1}\epsilon_{\mu_2\dots\mu_s\}}$$

Rarita - Schwinger spin $\frac{3}{2}$ in flat space

$$\psi_\mu^\alpha \qquad \mathcal{L} = -\bar{\psi}_\mu \left(\varepsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\nu\partial_\rho + m\,\sigma^{\mu\lambda} \right)\psi_\lambda =$$

$$= -\bar{\psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - m\,\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] \qquad \gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = \frac{1}{3!}\left(\gamma^\mu\gamma^\nu\gamma^\rho + \dots \text{perms}\right)$$

Or $\mathcal{L} = -\bar{\psi}_\mu Q^{\mu\nu} \psi_\nu$

$$Q^{\mu\nu} = (\eta^{\mu\nu} - \gamma^\mu \gamma^\nu)(i\not{\partial} - m) - i\gamma^\nu \partial^\mu + i\gamma^\mu \partial^\nu$$

ψ_μ^α $\quad \partial^\mu \psi_\mu^\alpha$ has spin $\frac{1}{2}$ $\quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$\gamma^\mu \psi_\mu^\alpha$ has spin $\frac{1}{2}$ $\quad \gamma^\mu \psi_\mu = \gamma \cdot \psi$

Equations of motion: $Q^{\mu\nu} \psi_\nu = 0$

We want to show that they imply $\partial \cdot \psi^\alpha = 0 = \gamma^\mu \psi_\mu^\alpha$

$$\partial^\mu Q_{\mu\nu} = (\partial^\nu - \not{\partial} \gamma^\nu)(i\not{\partial} - m) - i\gamma^\nu \Box + i\not{\partial} \partial^\nu =$$

$$= (\partial^\nu + \gamma^\nu \not{\partial} - 2\partial^\nu)(i\not{\partial} - m) - i\gamma^\nu \Box + i\not{\partial} \partial^\nu =$$

$$= -m\left(\gamma^\nu \not\partial - \partial^\nu\right) + i\left[\cancel{\gamma^\nu \Box} - \cancel{\partial^\nu \not\partial} - \cancel{\Box \gamma^\nu} + \cancel{\partial^\nu \not\partial}\right]$$

$$\Rightarrow \quad \gamma^\nu \not\partial \psi_\nu = \partial\cdot\psi = \left(-\not\partial \gamma^\nu + 2\partial^\nu\right)\psi_\nu =$$

$$= -\not\partial(\gamma\cdot\psi) + 2\partial\cdot\psi$$

$$\Rightarrow \not\partial(\gamma\cdot\psi) = \partial\cdot\psi$$

$$\gamma^\mu Q_{\mu\nu} = -3\gamma^\nu(i\not\partial - m) - i\not\partial\gamma^\nu + 4i\partial^\nu =$$

$$= 3m\gamma^\nu + i\left[4\partial^\nu - 3\gamma^\nu\not\partial - \not\partial\gamma^\nu\right] =$$

$$= 3m\gamma^\nu + i\left[4\partial^\nu + 3\not\partial\gamma^\nu - 6\partial^\nu - \not\partial\gamma^\nu\right] =$$

$$= 3m\gamma^\nu + 2i\left(\cancel{\not\partial\gamma^\nu} - \cancel{\partial^\nu}\right)$$

$$\psi_\nu$$

$$\gamma^\mu Q_\mu{}^\nu \psi_\nu = 0 \quad \Rightarrow \quad \gamma\cdot\psi = 0 \quad \Rightarrow \quad \partial\cdot\psi = 0$$

Propagator : $P_{\mu\nu} Q^{\nu\rho} = i\,\delta^{\rho}_{\mu}$

$$P_{\mu\nu} = \frac{i}{p^2-m^2}\left[(\not p + m)\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right) + \frac{1}{3}\left(\gamma_\mu + \frac{p_\mu}{m}\right)(\not p - m)\left(\gamma_\nu + \frac{p_\nu}{m}\right)\right]$$

$P_{\mu\nu} p^\nu$ and $P_{\mu\nu} \gamma^\nu$ have no poles

$$P_{\mu\nu} p^\nu = \frac{i}{p^2-m^2}\left[(\not p + m)\, O(p^2-m^2) + \frac{1}{3}\left(\gamma_\mu + \frac{p_\mu}{m}\right)(\not p - m)\underbrace{\left(\not p + \frac{p^2}{m}\right)}_{= \not p + m \text{ on the pole}}\right] =$$

$$(\not p - m)(\not p + m) = p^2 - m^2$$

$$= O(1)$$

$$P_{\mu\nu}\gamma^{\nu} = \frac{i}{p^2-m^2}\left[(\not p+m)\left(\gamma_\mu - p_\mu\frac{\not p}{m^2}\right) + \frac{1}{3}\left(\gamma_\mu+\frac{p_\mu}{m}\right)(\not p-m)\left(1+\frac{\not p}{m}\right)\right] =$$

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} \qquad \not p = \underbrace{\not p - m}_{drop} + m \qquad \not p = \underbrace{\not p + m}_{drop} - m$$

$$= \frac{i}{p^2-m^2}\left[\underbrace{(\not p+m)\left(\gamma_\mu\right.}\left. - \frac{p_\mu}{m}\right) + \left(\underbrace{\gamma_\mu + \frac{p_\mu}{m}\right)(\not p}-m)\right] + O(1) =$$

$$= \frac{i}{p^2-m^2}\left[\cancel{2p^\mu} - \cancel{\frac{\not p\, p_\mu}{m}} + \cancel{m\gamma_\mu} - \cancel{p^\mu} - \cancel{m\gamma^\mu} + \cancel{\frac{p_\mu}{m}\not p} - \cancel{p_\mu}\right] + O(1) =$$

$$= O(1)$$

Massless limit: $\mathcal{L} = -\bar\psi_\mu \varepsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\nu\partial_\rho\psi_\lambda$

Gauge symmetry: $\delta\psi_\lambda = \partial_\lambda\epsilon$ $\quad\epsilon$ = spinor

Typical gauge-fixing conditions :

$\partial^\mu \psi_\mu = 0$ residual gauge freedom : $\Box \epsilon = 0$

$\gamma^\mu \psi_\mu = 0$ (algebraic) residual gauge freedom : $\not{\partial} \epsilon = 0$

$d=4$ two elicities (gravitino)

Remarks about gauge fixing

synchronous gauge $g_{00} = 1$ $g_{0i} = 0$ $g_{\mu\nu} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & & \\ 0 & & g_{ij} = -\gamma_{ij} & \end{array}\right)$

gauge conditions for the vielbein (for local Lorentz invariance)

symmetric gauge : $e_{\mu a} = \delta_\mu^b e_{\nu b} \delta_a^\nu$

(it also breaks diffeomorphisms) it is algebraic

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\, \phi_{\mu\nu} = e_{\mu a}\, \eta^{ab}\, e_{\nu b} \qquad (*)$$

$$e_{\mu a} = \eta_{\mu a} + \kappa\, \phi_{\mu a} + O(\phi^2)$$

This is the solution of $(*)$ in the symmetric gauge

Alternatively, we can start from

$$e_{\mu a} = \eta_{\mu a} + \kappa\, \phi_{\mu a} \qquad (\text{exact}),$$

impose the symmetric gauge $(\phi_{\mu a} = \phi_{a\mu})$ and

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\, \phi_{\mu\nu} + \kappa^2\, \phi_{\mu a}\, \eta^{ab}\, \phi_{\nu b}$$

local Lorentz transformation $\quad \delta_L e_\mu^{\ a} = \theta^a{}_b\, e_\mu^{\ b}$

gauge fixing $g^A(\phi)$ $\quad \phi^I$: fields $\quad \bar{c}_A \, , \, c^A$

$$\mathcal{L} \longrightarrow \mathcal{L} + \frac{1}{2\lambda} g^A M_{AB} g^B + \underbrace{\bar{c}_A \frac{\delta g^A}{\delta \phi^I} \underbrace{\delta_c \phi^I}}_{\mathcal{L}_{ghost}}$$

M_{AB} = constant matrix

Equivalent way

$$\mathcal{L} \to \mathcal{L} + B_A g^A + B_A N^{AB} B_B + \mathcal{L}_{ghost}$$

B_A equations $\quad 0 = g + 2 N \cdot B \qquad B = -\frac{1}{2} N^{-1} g$

$$B_A g^A + B_A N^{AB} B_B \longrightarrow -\frac{1}{2} g N^{-1} g + \frac{1}{4} g N^{-1} g = -\frac{1}{4} g N^{-1} g$$

$$\frac{M}{\cancel{4}2\lambda} = - \frac{N^{-1}}{\cancel{4}\, 2} \qquad N^{-1} = -\frac{2}{\lambda} M, \qquad N = -\frac{\lambda}{2} M^{-1}$$

$$\mathcal{L} \to \mathcal{L} + B_A g^A - \frac{\lambda}{2} B_A (M^{-1})^{AB} B_B + \bar{c}_A \frac{\delta g^A}{\delta \phi^I} \delta_c \phi^I$$

$\lambda = 0$ Landau gauge, which $\Rightarrow$ $g^A = 0$

QED

$$-\frac{1}{4} \underbrace{F_{\mu\nu} F^{\mu\nu} + B\, \partial \cdot A}_{\text{invertible for the multiplet } \begin{pmatrix} A_\mu \\ B \end{pmatrix}} + \bar{c}\, \Box c$$

Symmetric gauge (at $\lambda = 0$) : $g^A \longrightarrow \phi_{\mu a} - \phi_{a\mu}$

$\bar{c}_{\mu a} = -\bar{c}_{\mu a}$

$c_{\mu a} = -c_{a\mu}$

$$\delta_L e_\mu{}^a = \theta^a{}_b\, e_\mu{}^b \qquad e_{\mu a} = \eta_{\mu a} + \kappa \phi_{\mu a} \qquad \delta_L \phi_{\mu a} = c_{\mu a} + \dots$$

$$\mathcal{L} \to \mathcal{L} + B^{\mu a}(\phi_{\mu a} - \phi_{a\mu}) + \bar{c}^{\mu a}(\delta^\nu_\mu \delta^b_a - \delta^b_\mu \delta^a_\nu)\underbrace{(c_{\nu b} + \dots)}_{= \tilde{c}_{\nu b}}$$

$$\bar{c}^{\mu a} \tilde{c}_{\mu a}$$

Another gauge fixing for the local Lorentz transformations is

$$\nabla^{\mu}\omega_{\mu}{}^{ab} \equiv \mathcal{G}_L^{ab} \qquad\qquad \delta_L\,\omega_{\mu}{}^{ab} = -\nabla_{\mu}\theta^{ab}$$

it does not break diffeomorphisms $\qquad \partial\cdot A \qquad \delta A_\mu = \partial_\mu \Lambda$

Remarks about Weyl invariance

$$g_{\mu\nu} \longrightarrow g_{\mu\nu}\, e^{-2\Omega}$$

$W^{\mu}{}_{\nu\rho\sigma}$ is invariant

$$e^{a}_{\mu} \longrightarrow e^{a}_{\mu}\, e^{-\Omega}$$

All massless actions can be made Weyl invariant for spin $= 0, \frac{1}{2}, 1$

$$-\frac{1}{2}\int\sqrt{-g}\; F_{\mu\nu}\, g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$e^{-4\Omega} \qquad e^{2\Omega} \quad e^{2\Omega} \qquad\qquad A_\mu \longrightarrow A_\mu$$

Fermions $$S_D = i\int \underset{\downarrow}{e}\;\underset{\downarrow}{\bar\psi}\;\underset{\downarrow}{e^\mu_a}\,\gamma^a(\partial_\mu+\tilde\omega_\mu)\,\psi$$

$$e^{-4\Omega} \qquad e^{\frac{3}{2}\Omega} \qquad e^{\Omega} \qquad\qquad e^{\frac{3}{2}\Omega}$$

$$\psi \to e^{\frac{3}{2}\Omega}\,\psi$$

Scalars $$S_s = \frac{1}{2}\int \underset{\downarrow}{\sqrt{-g}}\left[\underset{\downarrow}{\nabla_\mu\varphi}\,\underset{\downarrow}{\nabla_\nu\varphi}\,\underset{\downarrow}{g^{\mu\nu}} + \frac{1}{6}R\varphi^2\right]$$

$$e^{-4\Omega} \qquad e^{\Omega} \qquad e^{\Omega} \qquad e^{2\Omega}$$

$$\varphi \to e^{\Omega}\varphi$$

$$R \longrightarrow e^{2\Omega}\left(R + 6\nabla^2\Omega - 6\nabla_\mu\Omega\,\nabla^\mu\Omega\right)$$ Exercise

$$S_s \longrightarrow \frac{1}{2}\int\sqrt{-g}\,e^{-4\Omega}\Big[e^{2\Omega}g^{\mu\nu}\,\nabla_\mu(e^{\Omega}\varphi)\nabla_\nu(e^{\Omega}\varphi) +$$

$$+\frac{1}{6}\varphi^2 e^{2\Omega}e^{2\Omega}\left(R + 6\nabla^2\Omega - 6\nabla_\mu\Omega\nabla^\mu\Omega\right)\Big] =$$

$$= \frac{1}{2}\int\sqrt{-g}\,\Big[g^{\mu\nu}(\nabla_\mu\varphi + \nabla_\mu\Omega\,\varphi)(\nabla_\nu\varphi + \nabla_\nu\Omega\,\varphi) + \frac{1}{6}R\varphi^2 +$$

$$+ \varphi^2\nabla^2\Omega - \varphi^2\nabla_\mu\Omega\,\nabla^\mu\Omega\Big] = S_s +$$

$$+ \frac{1}{2}\int\sqrt{-g}\,\Big[2\nabla_\mu\Omega\,\varphi\,\nabla^\mu\varphi + \varphi^2\nabla^2\Omega\Big] = S_S$$

Stress tensor of gauge fields

$$-\frac{1}{4}\int\sqrt{-g}\;F_{\mu\nu}\,g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} = S \qquad\qquad T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$$

$$T_{\mu\nu} = -F_{\mu\rho}F_\nu{}^\rho + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \qquad\qquad T = g^{\mu\nu}T_{\mu\nu} = 0$$

Weyl invariance

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{1}{2\sqrt{-g}}(-g)(-g_{\mu\nu}) = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}$$

Stress tensor for a massless scalar field in flat space

$$T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\,\partial_\alpha\varphi\,\partial^\alpha\varphi - \frac{1}{6}\left(\partial_\mu\partial_\nu - \Box\eta_{\mu\nu}\right)(\varphi^2)$$

$$\partial^\mu T_{\mu\nu} = 0 = \underset{\text{on shell}}{\cancel{\Box\varphi\,\partial_\nu\varphi}} + \partial_\mu\varphi\,\cancel{\partial^\mu\partial_\nu\varphi} - \partial_\mu\cancel{\partial_\alpha\varphi}\,\partial^\alpha\varphi$$

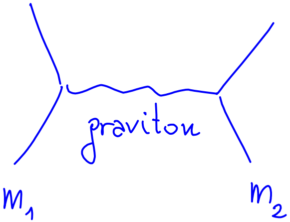
$$T = \eta^{\mu\nu}T_{\mu\nu} = (\partial_\alpha\varphi)(\partial^\alpha\varphi)\,(1-2) - \frac{1}{6}(-3)\,\Box(\varphi^2) =$$

$$= -\cancel{(\partial_\alpha\varphi)(\partial^\alpha\varphi)} + \frac{1}{2}\,2\left(\underset{\text{on shell}}{\cancel{\varphi\Box\varphi}} + \cancel{(\partial_\mu\varphi)(\partial^\mu\varphi)}\right) = 0$$

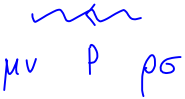
van Dam - Veltman - Zacharov discontinuity

gravitational interaction "$J^\mu A_\mu$" : $-\kappa\,\phi_{\mu\nu}T^{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\,\phi_{\mu\nu}$$



$= -i\kappa T_{\mu\nu}$



$$= \frac{i}{2p^2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)$$

Stress tensor of pointlike particle: $T^{\mu\nu} = m u^\mu u^\nu$

$u^\mu =$ four-velocity At rest, $u^\mu = (1, 0, 0, 0)$

$u^\mu u_\mu = 1$ $T^{00} = m$ $T^{\mu\nu} = \left(\begin{array}{c|c} m & 0 \\ \hline 0 & 0 \end{array}\right)$

Stress tensor of a photon of energy E:

$T^{\mu\nu} = E u^\mu u^\nu$ $T^\mu{}_\mu = 0$ $u^\mu = (1, 0, 0, 1)$

$u^\mu u_\mu = 0$

Static case : no time dependence , $\partial_t = 0$, $p^0 = 0$

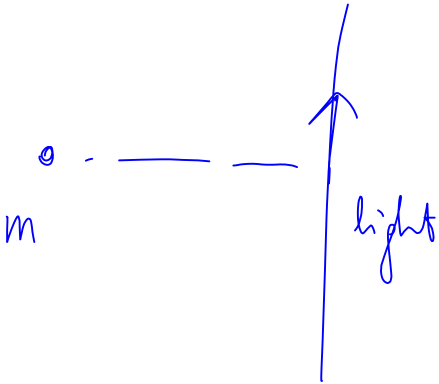
$p^\mu = (0, \vec{p}\,)$

$T^{\mu\nu}$ is conserved : $\partial_\mu T^{\mu\nu} = 0$

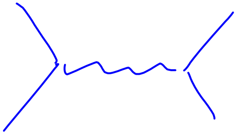
mass m : $p_\mu T^{\mu\nu} = p_0 T^{00} \delta^\nu_0 = 0$

radiation : $0 = p_\mu T^{\mu\nu} = E\, p_\mu u^\mu u^\nu \Rightarrow p_\mu u^\mu = 0$

$\Rightarrow \quad p^\mu = (0, p_x, p_y, 0)$



Potential between two pointlike objects of masses m_1 and m_2



$$= \left(-i\kappa T^{\mu\nu}_{m_1}\right) \frac{i}{2(-\vec{p}^{\,2})} \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)_{0000} \left(-i\kappa T^{\rho\sigma}_{m_2}\right) =$$

$$1 \quad + \quad 1 \quad - \quad 1$$

$$= \frac{i\kappa^2}{2\vec{p}^{\,2}}\, m_1 m_2$$

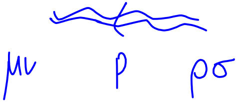
$$\hookrightarrow \qquad \int \frac{d^3\vec{p}}{(2\pi)^3} \; \frac{e^{i\vec{p}\cdot\vec{x}}}{\vec{p}^{\,2}} = \frac{1}{4\pi}\,\frac{1}{|\vec{x}|}$$

$$\kappa^2 \frac{m_1 m_2}{\vec{p}^{\,2}} = G\,\frac{m_1 m_2}{\vec{p}^{\,2}}$$

$G =$ Newton constant

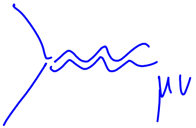
$G = \kappa^2$

What if the objects exchange a massive "graviton"?



$$= \frac{i}{2}\,\frac{1}{p^2 - m_g^2}\left(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\nu\rho}\pi_{\mu\sigma} - \frac{2}{3}\pi_{\mu\nu}\pi_{\rho\sigma}\right)$$

$$\pi_{\mu\nu} = \eta_{\mu\nu} - \xcancel{\frac{p_\mu p_\nu}{m^2}}$$



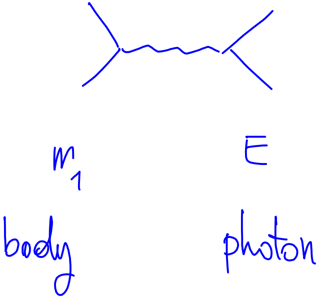
$$= -i\kappa_g T_{\mu\nu}$$

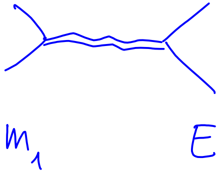
$$= \left(-i\kappa_g T^{00}_{m_1}\right)\frac{i}{2\left(-\vec p^{\,2} - m_g^2\right)}\left(1 + 1 - \frac{2}{3}\right)\left(-i\kappa_g T^{00}_{m_2}\right) =$$

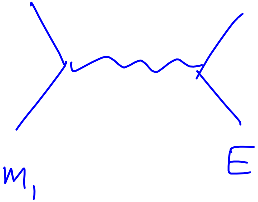
$$= \underbrace{\frac{i}{2}\kappa_g^2 \frac{m_1 m_2}{\vec p^{\,2} + \cancel{m_g^2}}\,\frac{4}{3}} \;—\; G\frac{m_1 m_2}{\vec p^{\,2}}$$

$$\underline{G = \frac{4}{3}\kappa_g^2}$$

Deflection of light :



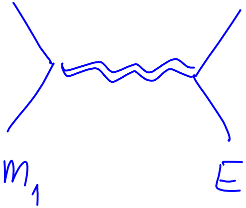




$$= -i\kappa\, m_1\, \overset{0}{u^{\mu}_{m}}\, \overset{0}{u^{\nu}_{m}}\, \frac{i}{2(-\vec{p}^{\,2})} \Big(\underbrace{\underset{0}{\eta_{\mu\rho}}\, \underset{0}{\eta_{\nu\sigma}}}_{1} + \underbrace{\underset{0}{\eta_{\mu\sigma}}\, \underset{0}{\eta_{\nu\rho}}}_{1} - \cancel{\eta_{\mu\nu}\eta_{\rho\sigma}} \Big)(-i\kappa E)\, \overset{0}{u^{\rho}}\, \overset{0}{u^{\sigma}} =$$

$$1 + 1$$

$$= \frac{i\kappa^2}{2\vec{p}^{\,2}}\, m_1 E\, 2 = \frac{i G m_1 E}{\vec{p}^{\,2}}$$



$$= -i\kappa_g\, m_1\, \overset{0}{u^{\mu}}\, \overset{0}{u^{\nu}}\, \frac{i}{2(-\vec{p}^{\,2} - \cancel{m_g^2})} \Big(\overset{\eta}{\pi_{\mu\rho}}\, \overset{\eta}{\pi_{\nu\sigma}} + \overset{\eta}{\pi_{\nu\rho}}\, \overset{\eta}{\pi_{\mu\sigma}} - \cancel{\overset{\eta}{\pi_{\mu\nu}}\, \overset{\eta}{\pi_{\rho\sigma}}} \Big)(-i\kappa_g E)\, u^{\rho} u^{\sigma}$$

$$1 + 1 \qquad\qquad 0000$$

$$= \frac{i\kappa_g^2}{2\vec{p}^{\,2}}\, m_1 E\, 2 = \underbrace{\frac{3}{4}}\, \frac{i G m_1 E}{\vec{p}^{\,2}}$$

van Dam-Veltman-Zacharov discontinuity

Boundary term of the gravitational action

$$\int R\sqrt{-g}\; d^4x = \int \mathcal{L}\left(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho \partial_\sigma g_{\mu\nu}\right)$$

$$\int dt\, \mathcal{L}(q, \dot{q}, \ddot{q})$$

$$\mathcal{L} = \frac{\dot{q}^2}{2} - V(q) \qquad S = \int_{t_0}^{t_1} dt\, \mathcal{L}\left(q(t), \dot{q}(t)\right)$$

$$\delta\dot{q} = \frac{d}{dt}\delta q$$

$$\delta S = \int_{t_0}^{t_1} dt \left[\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q}\right] =$$

$$= \int_{t_0}^{t_1} dt \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right]\delta q + \int_{t_0}^{t_1} dt\, \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right) =$$

$$= \int_{t_0}^{t_1} dt\ \delta q \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right)\right] + \frac{\partial \mathcal{L}}{\partial \dot q}\delta q \Big|_{t_0}^{t_1} \equiv 0 \quad \forall \delta q$$

$$\delta q(t_0) = \delta q(t_1) = 0 \qquad \left\{ \begin{array}{l} q(t_1) \equiv q_1 \\ q(t_0) \equiv q_0 \\ \dfrac{\partial \mathcal{L}}{\partial q} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot q}\right) \end{array} \right.$$

Consider now $\mathcal{L}' = -\frac{q\ddot q}{2} - V(q)$ $\qquad S' = \int_{t_0}^{t_1} dt\ \mathcal{L}'(q(t), \ddot q(t))$

$$\delta S' = \int_{t_0}^{t_1} dt \left(\frac{\partial \mathcal{L}'}{\partial q}\delta q + \frac{\partial \mathcal{L}'}{\partial \ddot q}\delta \ddot q\right) = \int_{t_0}^{t_1} dt \left(\frac{\partial \mathcal{L}'}{\partial q}\delta q - \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \ddot q}\right)\delta \dot q\right) +$$

$$+ \int_{t_0}^{t_1} dt\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \ddot q}\delta \dot q\right) =$$

$$= \int_{t_0}^{t_1} dt \, \delta q \left[\frac{\partial \mathcal{L}'}{\partial q} + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}'}{\partial \ddot{q}} \right) \right] + \frac{\partial \mathcal{L}'}{\partial \ddot{q}} \delta \dot{q} \Big|_{t_0}^{t_1} +$$

$$- \frac{d}{dt} \left(\frac{\partial \mathcal{L}'}{\partial \dot{q}} \right) \delta q \Big|_{t_0}^{t_1} = 0$$

$$\left\{ \begin{array}{ll} q(t_1) = q_1 & q(t_0) = q_0 \\ \dot{q}(t_1) = \dot{q}_1 & \dot{q}(t_0) = \dot{q}_0 \\ \frac{\partial \mathcal{L}'}{\partial q} = - \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}'}{\partial \ddot{q}} \right) & \end{array} \right.$$

$$\delta q(t_1) = \delta q(t_0) = 0$$

$$\delta \dot{q}(t_1) = \delta \dot{q}(t_0) = 0$$

$$- \frac{\ddot{q}}{2} - \frac{\partial V}{\partial q} = - \frac{d^2}{dt^2} \left(- \frac{q}{2} \right) = \frac{\ddot{q}}{2} \Rightarrow \ddot{q} = - \frac{\partial V}{\partial q}$$

$$S(g,\Gamma(g)) = -\frac{1}{2k^2}\int_M \left[\partial_\lambda w^\lambda - \sqrt{-g}\, g^{\mu\nu}\left(\Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\lambda\alpha} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\alpha}\right)\right]$$

$$w^\lambda = \sqrt{-g}\left(g^{\mu\nu}\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\Gamma^\nu_{\mu\nu}\right)$$

$$\int_M \partial_\lambda w^\lambda\, d^4x = \int_{\partial M} w^\lambda \sigma_\lambda \qquad \sigma_\lambda = \sqrt{-g}\,\varepsilon_{\lambda\alpha\beta\gamma}\frac{dx^\alpha dx^\beta dx^\gamma}{3!}$$

If V^λ is a vector then $\int_{\partial M} V^\lambda \sigma_\lambda$ is invariant under diffeomorphisms

$$= \frac{1}{3!}\int_{\partial M}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}V^\mu dx^\nu dx^\rho dx^\sigma$$

J^α_β = Jacobian

$$\varepsilon_{\mu\nu\rho\sigma}J^\mu_\alpha J^\nu_\beta J^\rho_\gamma J^\sigma_\delta = \varepsilon_{\alpha\beta\gamma\delta}\det(J)$$

We look for a different boundary term,

$$\int_{\partial M} \Omega = \int_{\partial M} V^{\lambda} \sigma_{\lambda}$$ V^{λ} = vector, such that

$$\delta \int_{\partial M} w^{\lambda} \sigma_{\lambda} = \delta \int_{\partial M} \Omega$$

and we correct the action accordingly:

$\delta S_{new} = \delta S_{\Gamma\Gamma} \Rightarrow$

S_{new} is covariant and has a welldefined variational problem $\Uparrow$

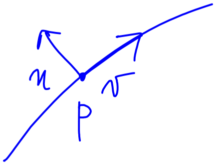
$$S_{H} = -\frac{1}{2k^2} \int_{\partial M} w^{\lambda} \sigma_{\lambda} + S_{\Gamma\Gamma}$$

$$S_{\Gamma\Gamma} = S_{H} + \frac{1}{2k^2} \int_{\partial M} w^{\lambda} \sigma_{\lambda} \longrightarrow$$

$$\longrightarrow S_{new} = S_{H} + \frac{1}{2k^2} \int_{\partial M} \Omega = S_{\Gamma\Gamma} + \frac{1}{2k^2} \int_{\partial M} \left(\Omega - w^{\lambda} \sigma_{\lambda}\right)$$

Submanifolds

M = manifold of dimension 4 with metric $g_{\mu\nu}$



Σ = submanifold of dimension 3

There exists a unique vector n orthogonal to Σ, up to the normalization

Let us consider a basis V_1, V_2, V_3 of vectors that are tangent to Σ in a point p

Let V_0 denote a vector belonging to $T_p(M)$, the tangent space to M in p, that is linearly independent of V_1, V_2, V_3. Then $\{V_0, V_1, V_2, V_3\}$ is a basis of $T_p(M)$.

We can consider the matrix $v_\mu{}^\alpha$, which is invertible

Let $G_{\mu\nu}$ denote $(v_\mu, v_\nu) \equiv v_\mu{}^\alpha g_{\alpha\beta} v_\nu{}^\beta =: G_{\mu\nu}$

Let $G^{\mu\nu}$ denote the inverse of $G_{\mu\nu}$

The normal vector is $\tilde{n} = G^{0\mu} v_\mu$ $\quad \left(\tilde{n}^\nu = G^{0\mu} v_\mu{}^\nu \right)$

$$(\tilde{n}, v_i) = \tilde{n}^\alpha g_{\alpha\beta} v_i{}^\beta = G^{0\mu} \underbrace{v_\mu{}^\alpha g_{\alpha\beta} v_i{}^\beta} = G^{0\mu} G_{\mu i} = \delta^0_i = 0$$

Let $\tilde{n}'$ denote another normal vector

Since v_μ is a basis, there exist coefficients b^μ

such that $\tilde{n}' = b^\mu v_\mu$

We want $(\tilde{n}', v_i) = 0 \qquad i = 1, 2, 3$

$$0 = (\tilde{n}', v_i) = \tilde{n}'^{\alpha} g_{\alpha\beta} v_i^{\beta} = b^{\mu} \underbrace{v_{\mu}^{\alpha} g_{\alpha\beta} v_i^{\beta}}_{G_{\mu i}} = b^{\mu} G_{\mu i}$$

$$\tilde{n}' = b^{\mu} v_{\mu} = b^{\mu} G_{\mu \underset{0}{\nu}} G^{\nu\rho} v_{\rho} =$$

$$= b^{\mu} G_{\mu 0} G^{0\rho} v_{\rho} = \left(b^{\mu} G_{\mu 0} \right) \tilde{n}$$

$\Rightarrow$ The normal vector is unique once it is normalized (if we can)

We will call it n, n^{μ}, $n_{\mu} = n^{\nu} g_{\mu\nu}$

$$n^2 = (n, n) = n^{\mu} g_{\mu\nu} n^{\nu} = n^{\mu} n_{\mu} = \begin{cases} 1 & \text{timelike} & \Sigma \text{ is spacelike} \\ -1 & \text{spacelike} & \Sigma \text{ is timelike} \\ 0 & \text{lightlike} & \Sigma \text{ is lightlike} \end{cases}$$

We focus on $n^2 = \pm 1$

We can define the new basis $(n, v_1, v_2, v_3) = v_\mu$

The metric of Σ is $h_{\mu\nu} = g_{\mu\nu} \mp n_\mu n_\nu$

Property : $h_\mu{}^\nu = \delta_\mu^\nu \mp n_\mu n^\nu$ is a projector on the tangent space to Σ

$$h_\mu{}^\nu h_\nu{}^\rho = \delta_\mu^\rho \mp 2 n_\mu n^\rho + n_\mu \underbrace{n^\nu n_\nu}_{\pm 1} n^\rho = \delta_\mu^\rho \mp n_\mu n^\rho = h_\mu{}^\rho$$

$$h_\mu{}^\nu v_i{}^\mu = v_i{}^\nu \mp \cancel{n_\mu n^\nu v_i{}^\mu} = v_i{}^\nu$$

$$h_\mu{}^\nu n^\mu = n^\nu \mp n_\mu n^\nu n^\mu = n^\nu - n^\nu = 0$$

$$\tilde{h}_{\mu\nu} \equiv w_\mu{}^\alpha h_{\alpha\beta} w_\nu{}^\beta \qquad \tilde{h}_{0\nu} = w_0{}^\alpha h_{\alpha\beta} w_\nu{}^\beta = n^\alpha h_{\alpha\beta} w_\nu{}^\beta = 0$$

$$\tilde{h}_{\mu\nu} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\gamma \end{array}\right) \qquad \gamma = \text{metric of } \Sigma'$$

Let Σ be defined by an equation

$$\chi(t, x, y, z) = 0 \qquad \chi(x) = 0$$

Then $n_\mu \propto \nabla_\mu \chi$

Let γ, $x(\tau) : [0,1] \to \Sigma$ denote a curve on Σ

$$\chi(x(\tau)) = 0 \quad \forall \tau \quad \Rightarrow \quad 0 = \frac{d\chi(x(\tau))}{d\tau} = \underbrace{\frac{dx^\mu(\tau)}{d\tau}}_{\text{tangent to the curve}} \cdot \nabla_\mu \chi$$

$$\Rightarrow \quad n_\mu = \lambda \nabla_\mu \chi$$

From now on we work with $n^2 = 1 = \lambda^2 \nabla_\mu \chi \nabla^\mu \chi$

Extrinsic curvature

We also assume no torsion and metric compatibility

$$K_{\mu\nu} = h_\mu^{\ \rho} h_\nu^{\ \sigma} \nabla_\rho n_\sigma$$

We can also write $K_{\mu\nu} = h_\mu^{\ \rho} \nabla_\rho n_\nu$

Indeed, $K_{\mu\nu} = h_\mu^{\ \rho} \left(\delta_\nu^{\ \sigma} - \cancel{n_\nu n^\sigma} \right) \nabla_\rho n_\sigma = h_\mu^{\ \rho} \nabla_\rho n_\sigma$

$$n^\sigma \nabla_\rho n_\sigma = \frac{1}{2} \nabla_\rho \left(n^2 \right) = 0$$

$K_{\mu\nu}$ is symmetric

$n_\mu = \lambda \nabla_\mu \chi$

$$K_{\mu\nu} - K_{\nu\mu} = h_\mu^{\ \rho} h_\nu^{\ \sigma} \left(\nabla_\rho n_\sigma - \nabla_\sigma n_\rho \right) =$$

$$= h_\mu^{\ \rho} h_\nu^{\ \sigma} \left(\nabla_\rho \lambda \nabla_\sigma \chi + \lambda \cancel{\nabla_\rho \nabla_\sigma \chi} - \nabla_\sigma \lambda \nabla_\rho \chi - \lambda \cancel{\nabla_\sigma \nabla_\rho \chi} \right) =$$

$$= h_\mu^{\ \rho} h_\nu^{\ \sigma} \left(\frac{\nabla_\rho \lambda}{\lambda} n_\sigma - \frac{\nabla_\sigma \lambda}{\lambda} n_\rho \right) = 0$$

$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n h_{\mu\nu}$ $\mathcal{L}_n h_{\mu\nu}$ = Lie derivative of $h_{\mu\nu}$ along n

$$= \frac{1}{2} \left(n^\rho \partial_\rho h_{\mu\nu} + h_{\mu\rho} \partial_\nu n^\rho + h_{\nu\rho} \partial_\mu n^\rho \right) =$$

$$= \frac{1}{2} \left(n^\rho \nabla_\rho h_{\mu\nu} + h_{\mu\rho} \nabla_\nu n^\rho + h_{\nu\rho} \nabla_\mu n^\rho \right) =$$

$$= \frac{1}{2} \Big(n^\rho \partial_\rho h_{\mu\nu} + h_{\mu\rho} \partial_\rho n^\rho + h_{\nu\rho} \partial_\mu n^\rho - \cancel{n^\rho \Gamma^\alpha_{\rho\mu} h_{\alpha\nu}} +$$

$$- \cancel{n^\rho \Gamma^\alpha_{\rho\nu} h_{\alpha\mu}} + \cancel{h_{\mu\rho} \Gamma^\rho_{\nu\alpha}} n^\alpha + \cancel{h_{\nu\rho} \Gamma^\rho_{\mu\alpha}} n^\alpha \Big) =$$

$$= \frac{1}{2} \Big(- n^\rho \nabla_\rho (n_\mu n_\nu) + g_{\mu\rho} \nabla_\nu n^\rho - \cancel{n_\mu n_\rho \nabla_\nu n^\rho} +$$

$$+ g_{\nu\rho} \nabla_\mu n^\rho - \cancel{n_\nu n_\rho \nabla_\mu n^\rho} \Big) =$$

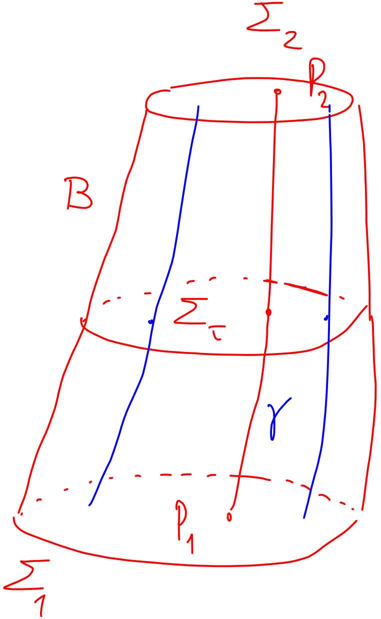
$$= \frac{1}{2}\left(- n^\rho \nabla_\rho n_\mu \, n_\nu - n^\rho \nabla_\rho n_\nu \, n_\mu + \nabla_\nu n_\mu + \nabla_\mu n_\nu \right) =$$

$$= \frac{1}{2}\left(h_\mu^{\ \rho} \nabla_\rho n_\nu + h_\nu^{\ \rho} \nabla_\rho n_\mu \right) =$$

$$= \frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu - n_\mu n^\rho \nabla_\rho n_\nu - n_\nu n^\rho \nabla_\rho n_\mu \right)$$

$$= \frac{1}{2}\left(K_{\mu\nu} + K_{\nu\mu} \right) = K_{\mu\nu}$$

Let Σ_τ denote a family of submanifolds, parametrized by some "time" τ



$M = \bigcup_\tau \Sigma_\tau$ is called foliation of the manifold M

Let Σ_τ be described by an equation $\chi_\tau(x) = 0$

For example we can assume $\chi_\tau(x) = \chi(x) - \tau = 0$

e.g. $t - \tau$ is $\mathbb{R}^3$

$\Sigma_1 = \Sigma_{\tau_1}$ $\tau_1 =$ "initial time"

$\Sigma_2 = \Sigma_{\tau_2}$ $\tau_2 =$ "final time"

Let $\gamma : x(\tau)$ denote a curve such that

$x(\tau_1) = P_1 \in \Sigma_1$ $x(\tau_2) = P_2 \in \Sigma_2$ $x(\tau) \in \Sigma_\tau \; \forall \tau$

We have $\chi(x(\tau)) - \tau = 0$

We obtain a flow of diffeomorphisms that map Σ_τ in $\Sigma_{\tau'}$

Differentiating $\chi(x(\tau))-\tau=0$ w.r.t. τ we get

$$0 = 1 - \frac{dx^\mu}{d\tau} \nabla_\mu \chi \Big|_{x(\tau)} = 1 - \frac{dx^\mu}{d\tau} \frac{1}{\lambda} n_\mu \equiv 1 - \delta^\mu n_\mu$$

$\delta^\mu \equiv \frac{1}{\lambda} \frac{dx^\mu}{d\tau}$ is a vector field that maps Σ_τ in $\Sigma_{\tau+d\tau}$

δ^μ is not uniquely defined: we just need any δ^μ such that $\delta^\mu n_\mu \neq 0$

We decompose δ^μ in its components tangent to Σ and normal to Σ:

$$\begin{cases} N^\mu \equiv \delta^\mu - n^\mu \delta^\nu n_\nu = \delta^\mu - n^\mu N = h^\mu_\nu \delta^\nu \\ N = \delta^\nu n_\nu \end{cases}$$

$$\delta^\mu = N^\mu + n^\mu N \qquad n_\mu N^\mu = n_\mu \delta^\mu - n_\mu n^\mu \delta^\nu n_\nu = 0$$

Let Σ_τ denote the whole space at $t=\tau$ $(\chi(x)=t)$

$\chi_\tau(x) = t - \tau$ $\qquad \nabla_\mu \chi = (1, 0, 0, 0)$

We need δ^μ such that $\delta^\mu \nabla_\mu \chi \neq 0$

We choose $\delta^\mu = (1, 0, 0, 0)$

$$n_\mu = \lambda \nabla_\mu \chi = (\lambda, 0, 0, 0) = (N, 0, 0, 0) \qquad n_0 = N$$

$$N = \delta^\nu n_\nu = \lambda \qquad N^\mu = \delta^\mu - n^\mu N = (1, \vec{0}) - n^\mu N$$

$$0 = n_\mu N^\mu = N N^0 \Rightarrow N^0 = 0 = 1 - n^0 N \qquad n^0 = \frac{1}{N}$$

$$N^\mu = (0, N^i) = (0, -n^i N) \Rightarrow n^i = -\frac{N^i}{N}$$

$$n^\mu = \frac{1}{N}(1, -N^i)$$

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \qquad h_{00} = g_{00} - N^2$$

$$h_{0i} = g_{0i} \qquad h_{ij} = g_{ij} = -k_{ij} \qquad k_{ij} = \text{metric of } \Sigma$$

$$0 = n_i = g_{i\nu} n^\nu = g_{i0} n^0 - g_{ij} n^j = \frac{1}{N}\left(g_{i0} - h_{ij} N^j\right)$$

$$g_{0i} = h_{ij} N^j = -k_{ij} N^j \equiv -N_i$$

$$n^0 = \frac{1}{N} = n_\mu g^{\mu 0} = N g^{00} \qquad \Rightarrow \qquad g^{00} = \frac{1}{N^2}$$

$$1 = n_\mu n^\mu = n^\mu g_{\mu\nu} n^\nu = \frac{1}{N^2}\left(g_{00} - 2 g_{0i} N^i + g_{ij} N^i N^j \right)$$

$$\Rightarrow \quad N^2 = g_{00} + 2 N_i N^i - N_i N^i \quad \Rightarrow \quad g_{00} = N^2 - N_i N^i$$

$$g_{\mu\nu} = \begin{pmatrix} N^2 - N_i N^i & -N_i \\ -N_j & -k_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & \frac{N^i N^j}{N^2} - k^{ij} \end{pmatrix}$$

$$g_{\mu\rho} g^{\rho\nu} = \begin{pmatrix} 1 - \cancel{\frac{N_i N^i}{N^2}} + \cancel{\frac{N^i N_i}{N^2}} & -\cancel{N^i} + \cancel{\frac{N^i N_j N^j}{N^2}} - \cancel{\frac{N^i N_j N^j}{N^2}} + \cancel{N^i} \\ -\cancel{\frac{N_j}{N^2}} + \cancel{\frac{k_{ji} N^i}{N^2}} & \cancel{\frac{N_j N^i}{N^2}} - \cancel{\frac{N_j N^i}{N^2}} + \delta^i_j \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} N^2 - N_i N^i & -N_1 & -N_2 & -N_3 \\ -N_1 & -k_{11} & -k_{12} & -k_{13} \\ -N_2 & -k_{12} & -k_{22} & -k_{23} \\ -N_3 & -k_{13} & -k_{23} & -k_{33} \end{pmatrix}$$

$$g = \det g_{\mu\nu} = (N^2 - N_i N^i)(-k) + N_1 \left[-k N_1 k^{11} - k N_2 k^{12} - k N_3 k^{13} \right] + \dots = (N^2 - \cancel{N_i N^i})(-k) +$$

$$= + N_j \left(\cancel{-k N_i k^{ij}} \right) = -k N^2$$

$$\Rightarrow \sqrt{-g} = \sqrt{k}\, N$$

$$K_{\mu\nu} = h_\mu^\rho h_\nu^\sigma \nabla_\rho n_\sigma = h_\mu^\rho \nabla_\rho n_\nu$$

$$K = g^{\mu\nu} K_{\mu\nu} = g^{\mu\nu} h_\mu^\rho \nabla_\rho n_\nu = h_\mu^\rho \nabla_\rho n^\mu$$

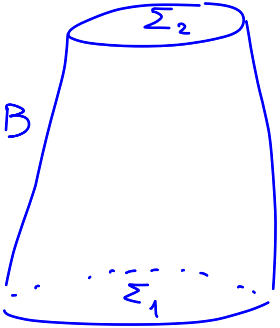
We want to show that

$$\omega^\lambda = g^{\mu\nu} \Gamma_{\mu\nu}^\lambda - g^{\mu\lambda} \Gamma_{\mu\nu}^\nu$$

$$\delta \int_{\Sigma_2} \omega^\lambda \sigma_\lambda = 2 \delta \int_{\Sigma_2} \sqrt{\kappa}\, K\, d^3x$$

$$\sigma_\lambda = \sqrt{-g}\, \epsilon_{\lambda\mu\nu\rho} \frac{dx^\mu dx^\nu dx^\rho}{3!}$$

Variational problem: on



$\partial M = \Sigma_1 \cup B \cup \Sigma_2$ $\quad \delta g_{\mu\nu} = 0 \quad \delta N = 0$

$\delta N^i = 0 \quad \delta \kappa_{ij} = 0 \quad \delta n_\mu = 0 = \delta n^\mu \quad \delta h_{\mu\nu} = 0$

We do not know anything about $\partial_\mu \delta g_{\nu\rho}, \partial_\mu \delta n^\nu \ldots$

Actually, we know that $\partial_\mu \delta n^\nu = c^\nu n_\mu$ for some proportionality factors (functions) c^ν

Indeed, Σ_2 is given by some equation $X(x) = 0$

$n_\mu \propto \partial_\mu X$ $\quad \gamma: x(\tau)$ curve on Σ_2 $\quad X(x(\tau)) = 0 \quad \frac{dx^\mu}{d\tau}\partial_\mu X = 0$

On Σ_2 we also have $\delta n^\nu = 0 \quad \delta n^\nu(x(\tau)) = 0$

$$0 = \frac{dx^\mu}{d\tau}\partial_\mu \delta n^\nu(x(\tau)) \Rightarrow$$ $\partial_\mu \delta n^\nu$ is proportional to n_μ, since the normal is unique

$$I = \delta \int_{\Sigma_2} \sqrt{k}\, \bar{K}\, d^3x = \int_{\Sigma_2} d^3x\ \delta\left(h^\nu_\mu \nabla_\nu n^\mu\right)\sqrt{k} =$$

$$= \int_{\Sigma_2} \sqrt{\kappa}\, d^3x\; h_\mu^\nu\, \delta\left(\cancel{\partial_\nu n^\mu} + \Gamma^\mu_{\nu\rho} n^\rho\right) = \int_{\Sigma_2} d^3x \sqrt{\kappa}\, \underset{\delta_\mu^\nu - u_\mu u^\nu}{\underset{\parallel}{h_\mu^\nu}}\, \delta\Gamma^\mu_{\nu\rho} n^\rho =$$

$$\swarrow$$

$$\delta\partial_\nu n^\mu = c^\mu n_\nu$$

$$= \int_{\Sigma_2} d^3x \sqrt{\kappa} \left(\delta\Gamma^\mu_{\mu\rho} n^\rho - n_\mu n^\nu n^\rho\, \delta\Gamma^\mu_{\nu\rho} \right) \qquad (1)$$

$$I = \delta \int_{\Sigma_2} d^3x \sqrt{\kappa}\; \delta\left(h^{\mu\nu} \nabla_\mu n_\nu\right) = \int_{\Sigma_2} d^3x \sqrt{\kappa}\, h^{\mu\nu}\, \delta\left(\cancel{\partial_\mu n_\nu} - \Gamma^\rho_{\mu\nu} n_\rho\right) =$$

$$\delta\partial_\mu n_\nu = d_\nu n_\mu$$

for some d_ν

$$= \int_{\Sigma_2} d^3x \sqrt{\kappa} \left(-g^{\mu\nu}\, \delta\Gamma^\rho_{\mu\nu} n_\rho + n^\mu n^\nu\, \delta\Gamma^\rho_{\mu\nu} n_\rho \right) \qquad (2)$$

$$I = \frac{1}{2}(1) + \frac{1}{2}(2) = \frac{1}{2} \int_{\Sigma_2} d^3x \sqrt{\kappa} \left(\delta\Gamma^\mu_{\mu\rho} n^\rho - g^{\mu\nu}\, \delta\Gamma^\rho_{\mu\nu} n_\rho \right)$$

$$\delta \int_{\Sigma_2} w^{\lambda} \sigma_{\lambda} = \delta \int_{\Sigma_2} \sqrt{-g}\, \varepsilon_{\lambda\mu\nu\rho} \frac{dx^{\mu} dx^{\nu} dx^{\rho}}{3!} \left(g^{\alpha\beta} \Gamma^{\lambda}_{\alpha\beta} - g^{\alpha\lambda} \Gamma^{\beta}_{\alpha\beta} \right) =$$

Σ_2 is $\mathbb{R}^3$ (space)

$\varepsilon_{0123} = -1$

$\varepsilon^{0123} = 1$

$$= -\delta \int_{\Sigma_2} \sqrt{-g}\, d^3x \left(g^{\alpha\beta} \Gamma^{0}_{\alpha\beta} - g^{\alpha 0} \Gamma^{\beta}_{\alpha\beta} \right) =$$

$$= -\delta \int_{\Sigma_2} \sqrt{\kappa}\, d^3x\, N \left(g^{\mu\nu} \Gamma^{0}_{\mu\nu} - g^{\mu 0} \Gamma^{\nu}_{\mu\nu} \right)$$

$n_{\mu} = (N, 0, 0, 0)$

$$= -\delta \int_{\Sigma_2} d^3x \sqrt{\kappa}\, n_{\rho} \left(g^{\mu\nu} \Gamma^{\rho}_{\mu\nu} - g^{\mu\rho} \Gamma^{\nu}_{\mu\nu} \right) =$$

$$= - \int_{\Sigma_2} d^3x \sqrt{\kappa}\, n_{\rho} \left(g^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} - g^{\mu\rho} \delta\Gamma^{\nu}_{\mu\nu} \right) = 2\,\mathrm{I}$$

$$\Rightarrow \quad \delta\left[\int_{\Sigma_2} \omega^\lambda \sigma_\lambda - 2 \int_{\Sigma_2} \sqrt{k}\, K\, d^3x \right] = 0$$

Trace K action

$$S_K = S_H + \frac{1}{\kappa^2}\left[\int_{\Sigma_2} \sqrt{k}\, K\, d^3x - \int_{\Sigma_1} \sqrt{k}\, K\, d^3x + \right.$$

$$\left. + \int_B d^3x \sqrt{\gamma}\, \textcircled{H} \right]$$

$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$

$u^\mu =$ nomal vector

$\gamma = \det \gamma_{ij}$

$$\gamma_{\mu\nu} = \left(\begin{array}{c|ccc} x & x & x & x \\ \hline x & & & \\ x & & -\gamma_{ij} & \\ x & & & \end{array} \right)$$

$u^\mu u_\mu = -1$

$\textcircled{H} = \gamma_\mu^{\ \nu} \nabla_\nu u^\mu$

Caveat: one must require $n^\mu u_\mu = 0$ on the intersections $\Sigma_2 \cap B$, $B \cap \Sigma_1$. Otherwise, there are extra contributions from $\Sigma_2 \cap B$ and $\Sigma_1 \cap B$

In the end $\delta S_{K} = \delta S_{\Pi}$: well-defined variational problem

Energy of the gravitational field

QED $\quad \partial_\mu F^{\mu\nu} = J^\nu \qquad Q(t) = \int_{\mathbb{R}^3} d^3\vec{x}\, J^0(t, \vec{x}) =$

$$= \int_{\mathbb{R}^3} d^3\vec{x}\, \partial_\mu F^{\mu 0} = \int_{\mathbb{R}^3} d^3x\, \partial_i F^{i0} = \int_{S^2(\infty)} d\sigma\, F^{i0} n_i$$

$$\partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0 = \partial_0 J^0 + \partial_i J^i$$

$$\frac{dQ(t)}{dt} = \int_{\mathbb{R}^3} d^3\vec{x}\ \partial_0 J^0 = -\int_{\mathbb{R}^3} d^3\vec{x}\ \partial_i J^i = 0$$

$$T^{(m)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \qquad S_{tot} = S_H + S_m$$

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu} = \kappa^2 T^{\mu\nu}_m \quad \Longrightarrow \quad \nabla_\mu T^{\mu\nu}_m = 0$$

$$\left[\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R , \quad \nabla_\rho g^{\mu\nu} = 0 \right]$$

$$P^\mu = \int_{\mathbb{R}^3} d^3\vec{x}\ T^{0\mu}(t, \vec{x}) = P^\mu(t)$$

$\frac{dP^\mu}{dt} = 0$ requires $\partial_\nu T^{\nu\mu} = 0$, not $\nabla_\nu T^{\nu\mu} = 0$

$$\underbrace{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu}}_{\equiv E^{\mu\nu}(g)} = \kappa^2 T_m^{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa \phi_{\mu\nu}$$

$$E^{\mu\nu}(g) = E^{\mu\nu}(\eta) + 2\kappa \int \phi_{\rho\sigma}(y) \left. \frac{\delta E^{\mu\nu}(g)}{\delta \phi_{\rho\sigma}(y)} \right|_{g=\eta} d^4y + \kappa^2 X^{\mu\nu}$$

$$X^{\mu\nu} = O(\phi^2)$$

$$E^{\mu\nu}(\eta) + 2\kappa \int \phi_{\rho\sigma} \left. \frac{\delta E^{\mu\nu}(g)}{\delta \phi_{\rho\sigma}} \right|_{g=\eta} = \kappa^2 \underbrace{\left(T_m^{\mu\nu} - X^{\mu\nu} \right)}_{\partial_\mu \left(T_m^{\mu\nu} - X^{\mu\nu} \right) = 0}$$

Let $\bar{g}_{\mu\nu}$ be a generic background such that

$E^{\mu\nu}(\bar{g}) = 0$ We expand $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$

ϵ small

$$E^{\mu\nu}(g) = 0 + \epsilon Y^{\mu\nu} + O(\epsilon^2) = k^2 T^{\mu\nu}_m$$

$$\nabla_\mu = \bar{\nabla}_\mu + O(\epsilon) \qquad \text{"} \nabla_\mu = \partial_\mu + \Gamma_\mu \text{"}$$

$$\nabla_\mu E^{\mu\nu}(g) = \epsilon \bar{\nabla}_\mu Y^{\mu\nu} + O(\epsilon^2) = 0$$

$$\Rightarrow \quad \bar{\nabla}_\mu Y^{\mu\nu} = 0$$

In particular, let us take $\Lambda = 0$, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, $\epsilon = 1$

$$\partial_\mu Y^{\mu\nu} = 0$$

$$Y^{\mu\nu} = \kappa^2 \left(T_m^{\mu\nu} - X^{\mu\nu} \right) \qquad \partial_\mu Y^{\mu\nu} = 0 \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

$$\equiv \kappa^2 T^{\mu\nu} \qquad T^{\mu\nu} = T_m^{\mu\nu} - X^{\mu\nu}$$

$$P^\mu = \int_{\mathbb{R}^3} d^3\vec{x}\, T^{\mu 0}(t, \vec{x}) \qquad \frac{dP^\mu}{dt} = 0$$

$$R^\alpha{}_{\lambda\rho\sigma} = \partial_\rho \Gamma^\alpha_{\lambda\sigma} - \partial_\sigma \Gamma^\alpha_{\lambda\rho} + O(\Gamma^2)$$

$$R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R = -\frac{1}{4} \delta^{\nu\rho\sigma}_{\mu\alpha\beta} R^{\alpha\beta}{}_{\rho\sigma} = -\frac{1}{4} \begin{vmatrix} \delta_\mu^\nu & \delta_\mu^\rho & \delta_\mu^\sigma \\ \delta_\alpha^\nu & \delta_\alpha^\rho & \delta_\alpha^\sigma \\ \delta_\beta^\nu & \delta_\beta^\rho & \delta_\beta^\sigma \end{vmatrix} R^{\alpha\beta}{}_{\rho\sigma} =$$

$$= -\frac{1}{4} \left[2 \delta_\mu^\nu R - 2 R_\mu^\nu \cdot 2 \right]$$

$$R_\mu^{\ \nu} - \frac{1}{2}\delta_\mu^{\ \nu} R = -\cancel{2}\,\frac{1}{\underset{2}{\cancel{4}}}\,\delta^{\nu\rho\sigma}_{\mu\alpha\beta}\,\eta^{\beta\lambda}\,\partial_\rho \Gamma_{\lambda\sigma}^{\ \ \alpha} + O(\phi^2)$$

$$= \frac{1}{2}\partial_\rho Q_\mu^{\ \rho\nu} + O(\phi^2)$$

$$\parallel$$

$$Y_\mu^{\ \nu}$$

$$Q_\mu^{\ \rho\nu} = -\,\delta^{\nu\rho\sigma}_{\mu\alpha\beta}\,\eta^{\beta\lambda}\,\Gamma_{\lambda\sigma}^{\ \ \alpha} \qquad Q_\mu^{\ \rho\nu} = -\,Q_\mu^{\ \nu\rho}$$

$$\partial_\nu Y_\mu^{\ \nu} = \frac{1}{2}\partial_\rho\partial_\nu Q_\mu^{\ \rho\nu} = 0$$

$$P_\mu = \int_{\mathbb{R}^3} d^3\vec{x}\; T_\mu^{\ 0} = \frac{1}{2\kappa^2}\int_{\mathbb{R}^3} \partial_\rho Q_\mu^{\ \rho 0}\; d^3\vec{x} =$$

$$= -\frac{1}{2\kappa^2}\int_{\mathbb{R}^3} \partial_\rho Q_\mu^{\ \rho\nu}\,\sigma_\nu =$$

$$\sigma_\nu = \varepsilon_{\nu\alpha\beta\gamma}\frac{dx^\alpha dx^\beta dx^\gamma}{3!}$$

$$\varepsilon_{0123} = -1$$

$$= \frac{1}{8\kappa^2}\int_{\mathbb{R}^3} \partial_\rho\, Q_\mu^{\ \xi\zeta}\; \varepsilon^{\rho\nu\lambda\tau}\;\varepsilon_{\lambda\tau\xi\zeta}\;\varepsilon_{\nu\alpha\beta\gamma}\frac{dx^\alpha dx^\beta dx^\gamma}{3!} =$$

$$= \frac{1}{8\kappa^2}\int_{\mathbb{R}^3} \partial_\rho\, Q_\mu^{\ \xi\zeta}\;\varepsilon_{\lambda\tau\xi\zeta}\begin{vmatrix} \delta_\alpha^\rho & \delta_\alpha^\lambda & \delta_\alpha^\tau \\ \delta_\beta^\rho & \delta_\beta^\lambda & \delta_\beta^\tau \\ \delta_\gamma^\rho & \delta_\gamma^\lambda & \delta_\gamma^\tau \end{vmatrix}\frac{dx^\alpha dx^\beta dx^\gamma}{3!} =$$

$$= \frac{1}{8\kappa^2}\int_{\mathbb{R}^3} \partial_\rho Q_\mu^{\xi\zeta}\, dx^\rho\, dx^\lambda\, dx^\tau\, \varepsilon_{\lambda\tau\xi\zeta} =$$

$$= \frac{1}{8\kappa^2}\int_{\mathbb{R}^3} d\left[Q_\mu^{\xi\zeta}\, \varepsilon_{\lambda\tau\xi\zeta}\, dx^\lambda dx^\tau \right] \equiv$$

$$= \frac{1}{\kappa^2}\int_{\mathbb{R}^3} dQ_\mu \qquad Q_\mu \equiv \frac{1}{8} Q_\mu^{\xi\zeta}\, \varepsilon_{\lambda\tau\xi\zeta}\, dx^\lambda dx^\tau$$

$$= \frac{1}{\kappa^2}\int_{S^2(\infty)} Q_\mu$$

<u>Theorem</u>: $P_0 \geq 0$ in the synchronous gauge

$$g_{\mu\nu} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & g_{ij} & \\ 0 & & & \end{array}\right) \qquad e_\mu^a = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & e_j^i & \\ 0 & & & \end{array}\right)$$

Nester

Scalar $\Phi(x)$ $\Phi'(x') = \Phi(x)$

$\int d^4x \, \Phi(x)$ $\varphi^2(x)$ $\nabla_\mu \varphi \nabla_\nu \varphi \, g^{\mu\nu}$

$\bar{\psi}\psi$ $\bar{\psi} \not{D} \psi$

$$g_{\mu\nu} = \eta_{\mu\nu} + 2k \, \phi_{\mu\nu}$$

Quantum gravity

Unitarity

$S^\dagger S = 1$ $S = 1 + iT$ iT = Feynman diagrams

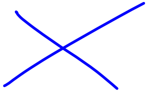
$$(1 - iT^\dagger)(1 + iT) = 1 + iT - iT^\dagger + T^\dagger T = 1$$

$$iT - iT^\dagger = -T^\dagger T$$

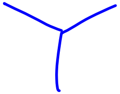
$$-iT + iT^{\dagger} = T^{\dagger}T = 2\,\frac{T - T^{\dagger}}{2i} = 2\,\mathrm{Im}\,T$$

$$2\,\mathrm{Im}\,[\,T\,] = T^{\dagger}T \geqslant 0$$ optical theorem

$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

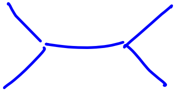


$$= -i\lambda$$



$$= -i$$

$$2\,\mathrm{Im}\Big[(-i)$$



$$\Big] = \pi\,\delta(p^2 - m^2) \geqslant 0$$

The nonrenormalizable theory of quantum gravity

$$\mathcal{L} = -\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\big[\,2\Lambda + R + O(W^3)\,\big]$$

is unitary (It contains infinitely many independent coupling constants)

Propagator $\sim \dfrac{1}{p^2}$ $|p^2|$ large

Power counting : $[\kappa] = -1$

The higher-derivative theory

$$S = -\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\Big[\,2\Lambda + \zeta R + \overbrace{\alpha\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big)}^{=\,\text{Weyl}^2\,+\,\text{total der.}} + $$

$$- \frac{\xi}{6}R^2\Big] \qquad \zeta, \alpha, \xi > 0$$

is renormalizable. Propagator $\sim \dfrac{1}{(p^2)^2}$ $|p^2|$ large

$$[\kappa] = 0$$

What about unitarity?

Higher derivatives have problems with unitarity

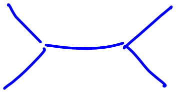
$$\frac{1}{(p^2-m_1^2)(p^2-m_2^2)} = \frac{1}{m_1^2-m_2^2}\left[\frac{1}{p^2-m_1^2} - \frac{1}{p^2-m_2^2}\right]$$

One pole has a negative residue. If we use the Feynman prescription

Propagator:

$$\frac{1}{(p^2-m_1^2+i\epsilon)(p^2-m_2^2+i\epsilon)} = \frac{1}{m_1^2-m_2^2}\left[\frac{1}{p^2-m_1^2+i\epsilon} - \frac{1}{p^2-m_2^2+i\epsilon}\right]$$

$$2\,\mathrm{Im}\Big[(-i)$$



$$\Big] = \frac{\pi}{m_1^2-m_2^2}\left[\delta(p^2-m_1^2) - \delta(p^2-m_2^2)\right]:$$

it is not nonnegative

$$-iT + iT^+ = T^+T$$

Fock space $\mathcal{W}$

$|\alpha\rangle, |\beta\rangle \in \mathcal{W}$

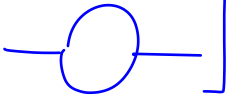
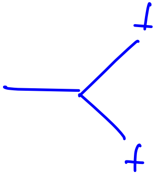
$$\langle\alpha|(-i)T|\beta\rangle + \langle\alpha|iT^+|\beta\rangle = \langle\alpha|T^+T|\beta\rangle =$$

$$= \sum_{|n\rangle\in\mathcal{W}} \langle\alpha|T^+|n\rangle\langle n|T|\beta\rangle$$

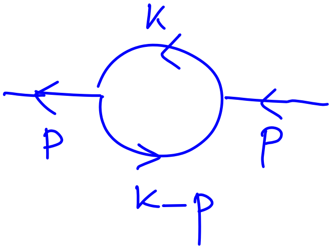
In higher derivative theories (with the Feynman prescription) you can prove the pseudounitarity equation

$$\langle\alpha|(-i)T|\beta\rangle + \langle\alpha|iT^+|\beta\rangle = \sum_{|n\rangle\in\mathcal{W}} \langle\alpha|T^+|n\rangle(-1)^{\sigma_n}\langle n|T|\beta\rangle$$

$$\sigma_n = 0, 1$$

$2\,\mathrm{Im}\Big[(-i)$  $\Big] = \int d\Pi_f \Big|$ 

$\Big|$



$$= \int \frac{d^4k}{(2\pi)^4} S(k)\, S(k-p) \equiv \mathcal{M}(p)$$

$$S(p) = \frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - \vec{p}^{\,2} - m^2} = \frac{1}{(p^0)^2 - \omega_{\vec{p}}^2} =$$

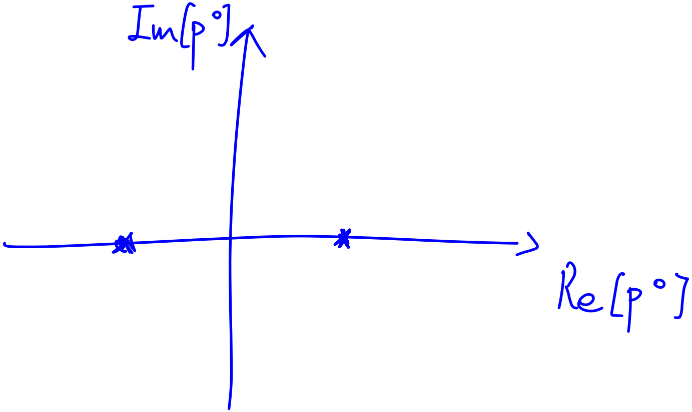
$$= \frac{1}{2\omega}\left[\frac{1}{p^0 - \omega_{\vec{p}}} - \frac{1}{p^0 + \omega_{\vec{p}}}\right] \qquad \omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$$

$$p^0 \in \mathbb{C} \qquad \vec{p} \in \mathbb{R}^3 \qquad k \in \mathbb{C}^4$$

Propagator:

$$S(p) = \frac{1}{p^2 - m^2}$$

$p^0 = \pm \omega_{\vec{p}}$



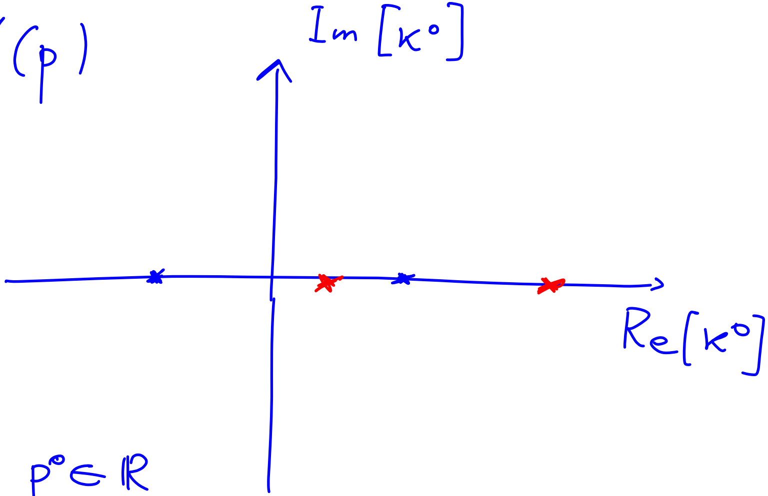
Bubble diagram:

$$\int \frac{d^4k}{(2\pi)^4} S(k)\, S(k-p) \equiv \mathcal{M}(p)$$

Poles: $k^0 = \pm \omega_{\vec{k}}$

$k^0 - p^0 = \pm \omega_{\vec{k}-\vec{p}}$

↪ $k^0 = p^0 \pm \omega_{\vec{k}-\vec{p}}$

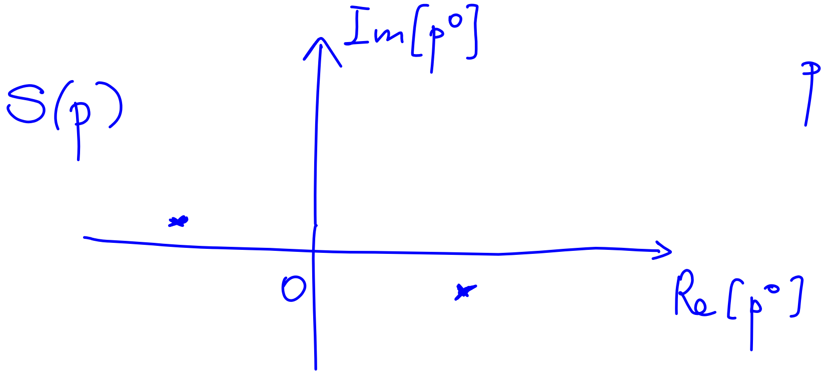


$p^0 \in \mathbb{R}$

$\vec{p} \in \mathbb{R}^3$

Feynman prescription : $\frac{1}{p^2 - m^2 + i\epsilon}$ $\qquad m^2 \longrightarrow m^2 - i\epsilon$

$$\omega_{\vec p} = \sqrt{\vec p^{\,2} + m^2} \longrightarrow \omega_{\vec p} - i\epsilon'$$

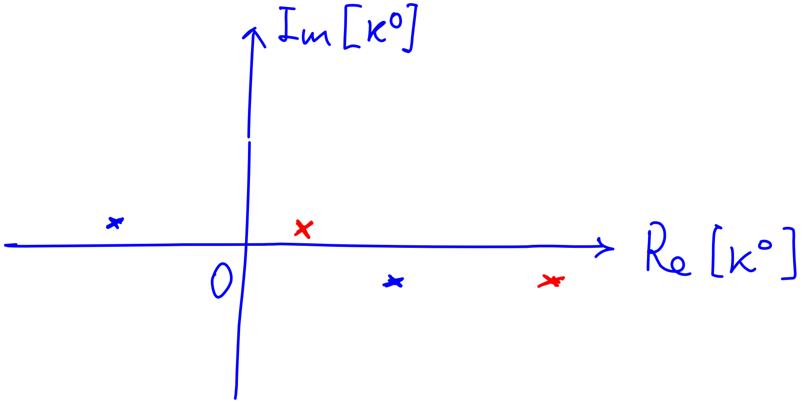


$$p^0 = \pm\omega_{\vec p} \mp i\epsilon'$$

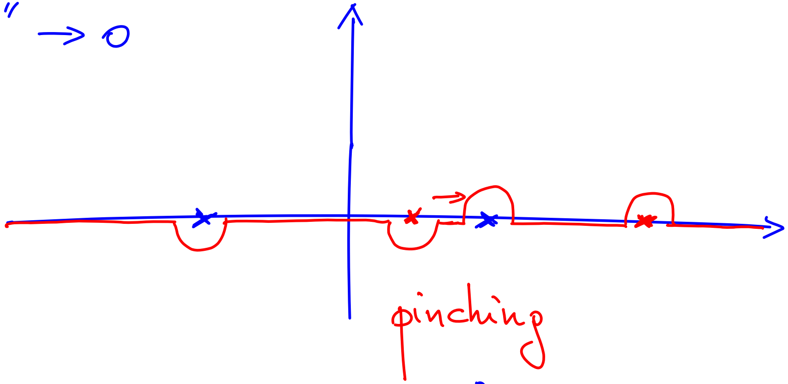
Bubble diagram :

$$k^0 = \pm\omega_{\vec k} \mp i\epsilon'$$

$$k^0 = p^0 \pm \omega_{\vec k - \vec p} \mp i\epsilon''$$

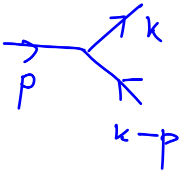


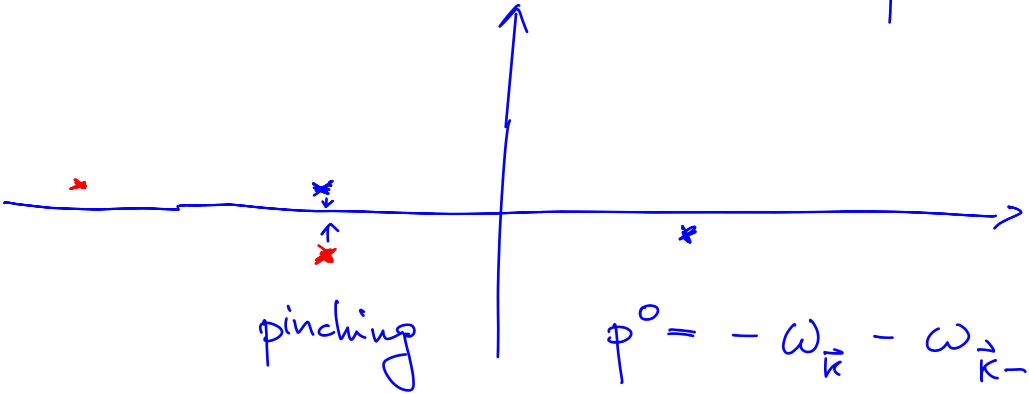
$\epsilon, \epsilon', \epsilon'' \to 0$



$k^0 = \omega_{\vec{k}}$ $\qquad$ $k^0 = p^0 - \omega_{\vec{k}-\vec{p}}$

$\to \quad p^0 = \omega_{\vec{k}} + \omega_{\vec{k}-\vec{p}}$





$p^0 = -\omega_{\vec{k}} - \omega_{\vec{k}-\vec{p}}$

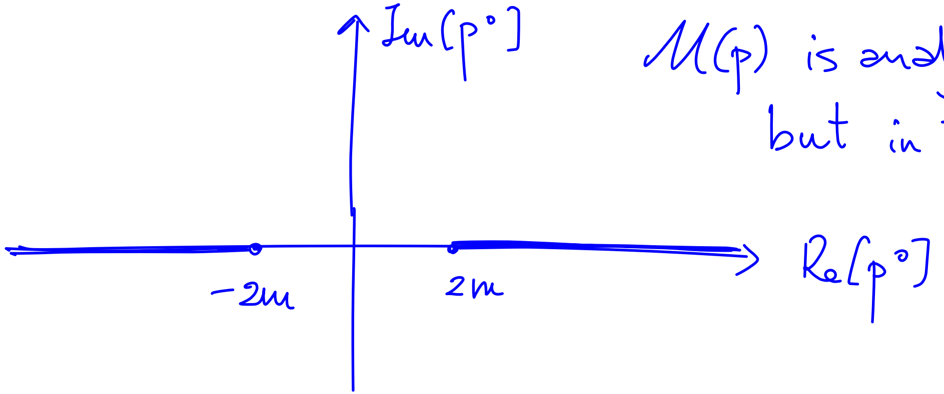
$$\mathcal{M}(p) = \int_{\mathbb{R}} \frac{dk^0}{2\pi} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}\, 2\omega_{\vec{k}-\vec{p}}} \left[\frac{1}{k^0-\omega_{\vec{k}}} - \frac{1}{k^0+\omega_{\vec{k}}} \right] \cdot$$

$$\cdot \left[\frac{1}{k^0-p^0-\omega_{\vec{k}-\vec{p}}} - \frac{1}{k^0-p^0+\omega_{\vec{k}-\vec{p}}} \right] \propto$$

$$\propto \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}\, 2\omega_{\vec{k}-\vec{p}}} \left[\cancel{\frac{1}{\omega_{\vec{k}}-p^0-\omega_{\vec{k}-\vec{p}}}} - \frac{1}{\omega_{\vec{k}}-p^0-\omega_{\vec{k}-\vec{p}}} + \right.$$

$$\left. + \cancel{\frac{1}{p^0+\omega_{\vec{k}-\vec{p}}-\omega_{\vec{k}}}} - \frac{1}{p^0+\omega_{\vec{k}-\vec{p}}+\omega_{\vec{k}}} \right] \propto$$

$$\propto \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\omega_{\vec{k}}\,\omega_{\vec{k}-\vec{p}}} \left(\frac{1}{p^0-\omega_{\vec{k}}-\omega_{\vec{k}-\vec{p}}} - \frac{1}{p^0+\omega_{\vec{k}}+\omega_{\vec{k}-\vec{p}}} \right)$$



$\mathcal{M}(p)$ is analytic everywhere but in two branch cuts

$$|p^0| = \sqrt{\vec{k}^2 + m^2} + \sqrt{(\vec{p}-\vec{k})^2 + m^2} \qquad \underline{\vec{k} \in \mathbb{R}^3}$$

$$\Rightarrow p^2 \geq 4m^2 \quad (\text{minimum for } \vec{k} = \vec{p}/2)$$

$m_1 \neq m_2$ $\qquad \vec{p} = (p_x, 0, 0)$

$$|p^0| - \sqrt{k_x^2 + k_\perp^2 + m_1^2} = \sqrt{p_x^2 + k_x^2 + k_\perp^2 - 2p_x k_x + m_2^2}$$

Square both sides:

$$\underline{(p^0)^2} + \cancel{k_x^2} + \cancel{k_\perp^2} + \underline{m_1^2} - 2|p^0|\sqrt{k_x^2 + k_\perp^2 + m_1^2} = \underline{p_x^2} + \cancel{k_x^2} + \cancel{k_\perp^2} \underline{- 2p_x k_x} + \underline{m_2^2}$$

$$p^2 + 2p_x k_x + m_1^2 - m_2^2 = 2|p_0|\sqrt{k_x^2 + k_\perp^2 + m_1^2}$$

$$\Delta = p^2 + m_1^2 - m_2^2$$

$$\Delta + 2p_x k_x = 2|p^0|\sqrt{k_x^2 + k_\perp^2 + m_1^2}$$

Square again:

$$\Delta^2 + \underline{4p_x^2 k_x^2} + 4p_x k_x \Delta = 4(p^0)^2(\underline{k_x^2} + k_\perp^2 + m_1^2)$$

Solve for k_x:

$$4p^2 k_x^2 - 4p_x k_x \Delta + \underbrace{4(p^0)^2(k_\perp^2 + m_1^2)} - \Delta^2 = 0$$

$$k_x = \frac{1}{\underset{2}{\cancel{4}}p^2}\left[\cancel{2}p_x\Delta \pm \sqrt{\cancel{4}p_x^2\Delta^2 - \underset{4}{\cancel{16}}(p^0)^2p^2(k_\perp^2 + m_1^2) + \cancel{4}\underset{(p^0)^2 - p_x^2}{p^2}\Delta^2}\right] =$$

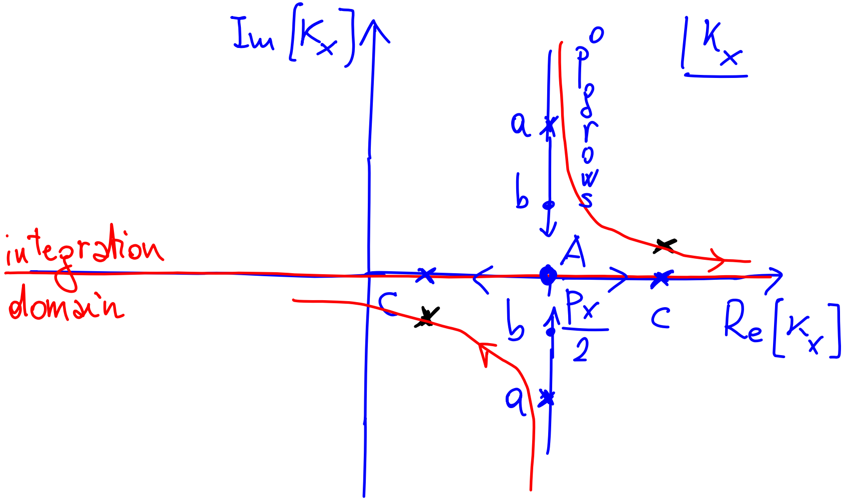
$$= \frac{p_x \Delta}{2p^2} \pm \frac{p^0}{2p^2}\sqrt{\Delta^2 - 4p^2(m_1^2+k_\perp^2)}$$

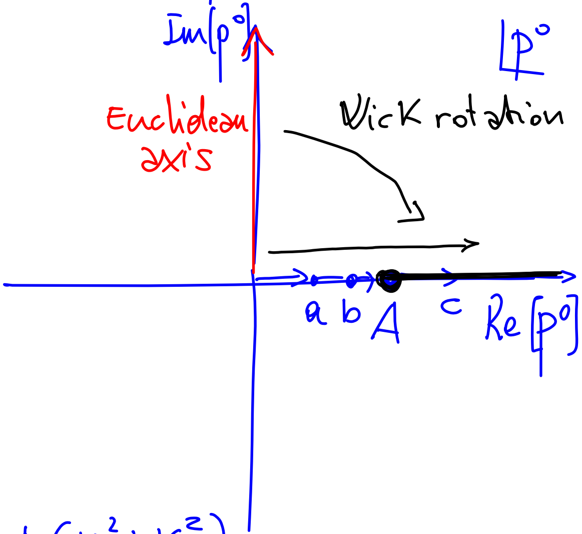
$m_1 = m_2 = m$: $\Delta = p^2 \quad (p^2 > 0)$

$$K_x = \frac{p_x}{2} \pm \frac{p^0}{2p^2}\sqrt{p^2(p^2 - 4(m^2+k_\perp^2))}$$

$m^2 \to m^2 - i\epsilon$

Real K_x as integration domain is ok if $p^2 < 0$ or $0 \leq p^2 \leq 4(m^2 + k_\perp^2$





P_x = fixed $\quad p^0$ grows

A: $p^2 = 4(m^2 + k_\perp^2)$

Higher derivative theories

propagators $\frac{1}{a(p^2)^2+b\,p^2+c}$

$$\frac{1}{p^2-m^2+i\epsilon} \quad \dots \quad \pm\frac{1}{p^2-m^2} = \pm\frac{p^2-m^2}{(p^2-m^2)^2} \longrightarrow$$

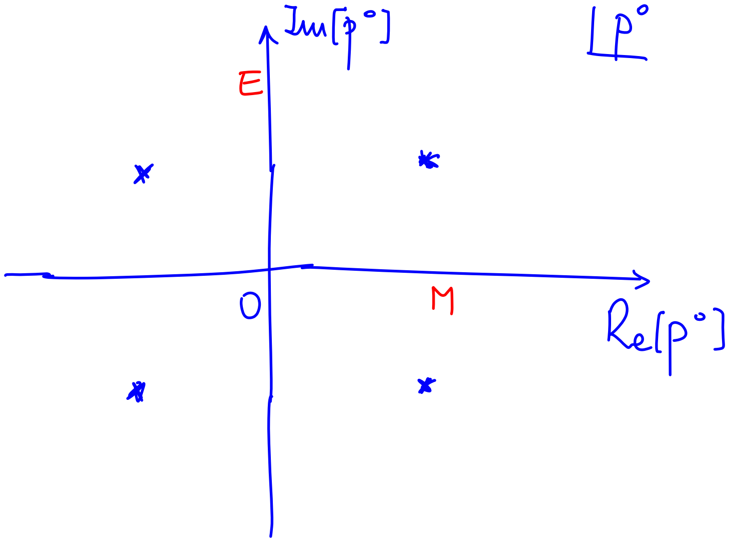
$$\longrightarrow \pm\frac{p^2-m^2}{(p^2-m^2)^2+\varepsilon^4} \qquad \varepsilon\to 0$$

Let us focus on a propagator

$$\frac{1}{(p^2-\mu^2)^2+M^4} = \frac{1}{(p^2-\mu^2-iM^2)(p^2-\mu^2+iM^2)} \equiv S(p)$$

$$m_1^2 = \mu^2 + iM^2 \qquad m_2^2 = \mu^2 - iM^2$$

$$p^0 = \pm\sqrt{\vec{p}^{\,2}+\mu^2 \pm i M^2}$$ (all 4 possibilities)



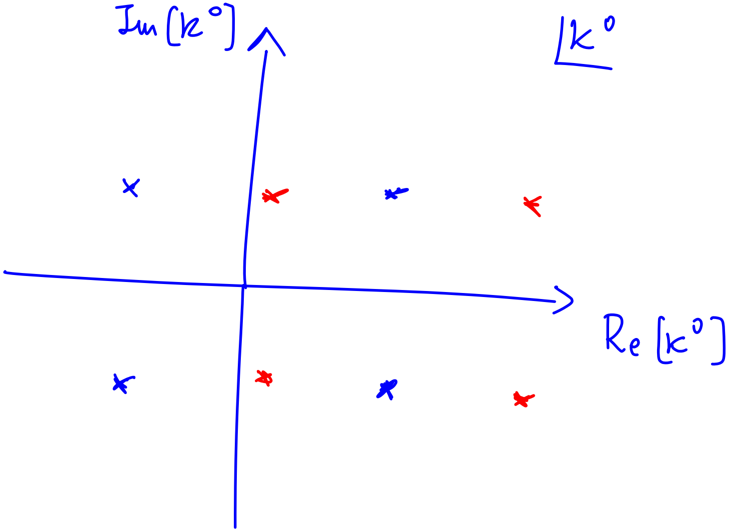
Integrating on Minkowski spacetime generates non local divergences

$$\frac{1}{k^2-m^2}\sim\frac{1}{k^2}$$

$$|k^2| \gg m^2$$

Power counting <u>only</u> works for Euclidean theories

Bubble: $\displaystyle\int_M \frac{d^4k}{(2\pi)^4}\, S(k)\, S(k-p)$ $\displaystyle\left[\underset{E}{\sim}\int\frac{d^4k}{(k^2)^2(k^2)^2}<\infty\right]_{|k^2|\text{ large}}$



pick a pole of $S(k)$: $k^2 = \mu^2 + i M^2$

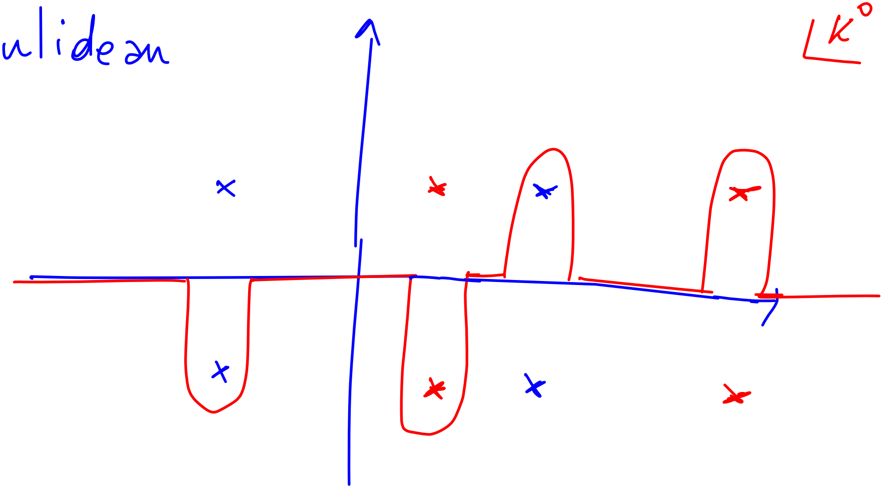
$$S(k) \sim \frac{1}{\omega_{\vec{k}}} \sim \frac{1}{|\vec{k}|} \qquad |\vec{k}| \gg \mu, M$$

$$S(k-p) = \frac{1}{\left((k-p)^2-\mu^2\right)^2+M^4} = \frac{1}{\left(k^2+p^2-2p\cdot k-\mu^2\right)^2+M^4} =$$

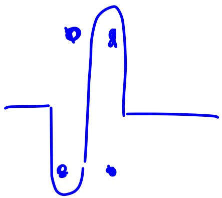
$$\longrightarrow \frac{1}{\left(iM^2+p^2-2p\cdot k\right)^2+M^4} \sim \frac{1}{(p\cdot k)^2} \longleftarrow$$

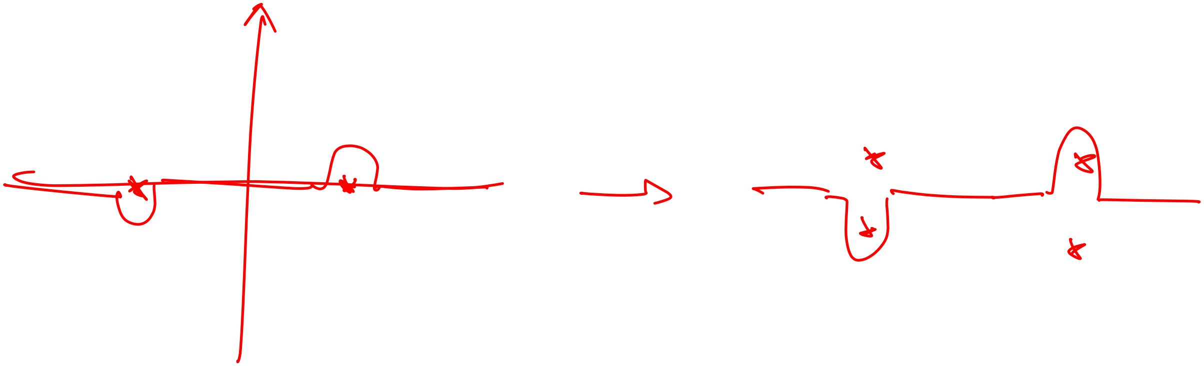
The divergence is nonlocal : $\frac{\ln \Lambda}{p^2}$

Eulidean



pinching :





We now know how to perform the integral on k^0

$$\mathcal{M}(p) = \int d^4k \; S(k)\, S(k-p) = \int_{\mathbb{R}^3(?)} d^3\vec{k} \; f(\vec{k}, p)$$

$\mathcal{M}$ is analytic every time $f(\vec{k}, p)$ is nonsingular for every $\vec{k} \in \mathbb{R}^3$

Singularities : $k_x = \dfrac{p_x \Delta}{2p^2} \pm \dfrac{p^0}{2p^2} \sqrt{\Delta^2 - 4p^2(m_1^2 + k_\perp^2)}$

$\Delta = p^2 + m_1^2 - m_2^2$

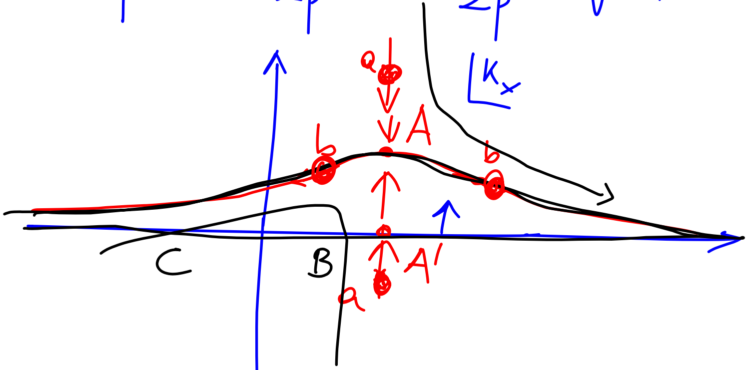
$m_1^2 = \mu^2 \pm i M^2$ $\quad m_2^2 = \mu^2 \pm i M^2$ (all 4 possibilities)

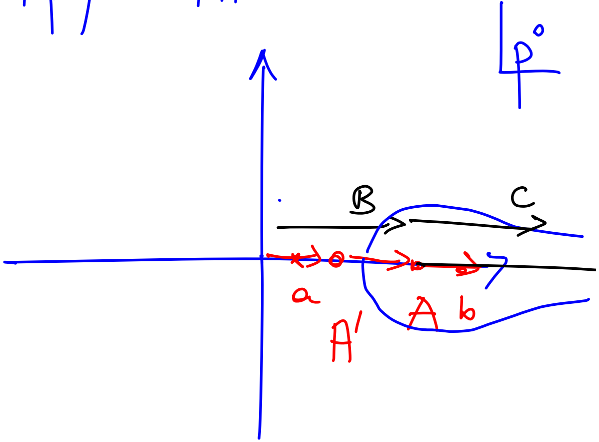
We take $m_1^2 = \mu^2 + iM^2$ and $m_2^2 = \mu^2 - iM^2$ $\quad (k_\perp^2 = 0)$

$$\Delta = p^2 + 2iM^2$$

$$k_x = \frac{p_x(p^2+2iM^2)}{2p^2} \pm \frac{p^0}{2p^2}\sqrt{(p^2)^2 - 4M^4 + \cancel{4ip^2M^2} - 4p^2\mu^2 - \cancel{4p^2 iM^2}} =$$

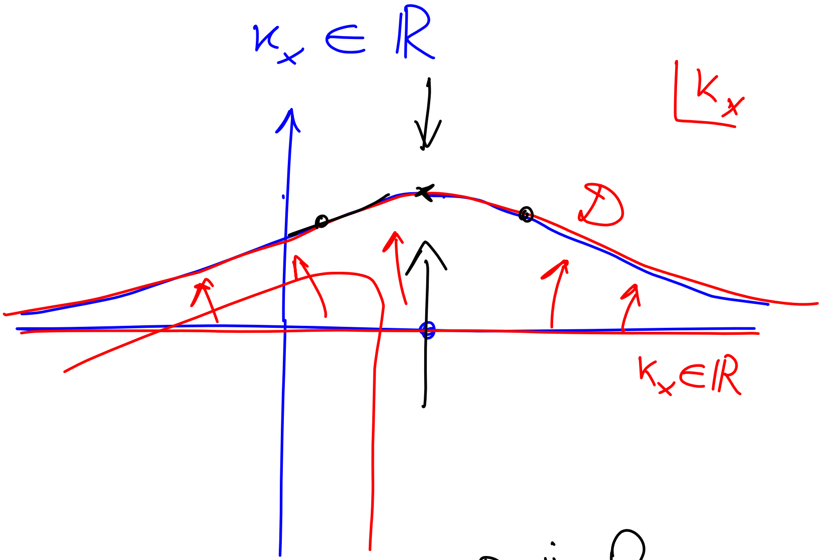
$$= \frac{p_x}{2p^2} + \frac{ip_xM^2}{2p^2} \pm \frac{p^0}{2p^2}\sqrt{(p^2)^2 - 4p^2\mu^2 - 4M^2}$$

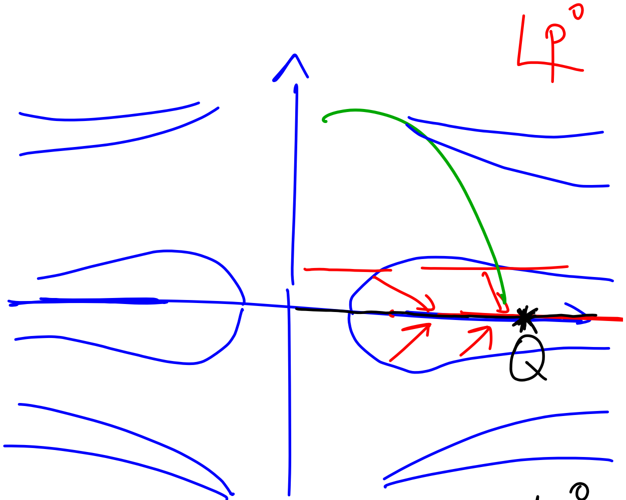




p_x fixed, p_0 growing

$$|p^0| = \sqrt{\vec{k}^2+\mu^2+iM^2} + \sqrt{(\vec{k}-\vec{p})^2+\mu^2-iM^2} \qquad (k_\perp = 0)$$

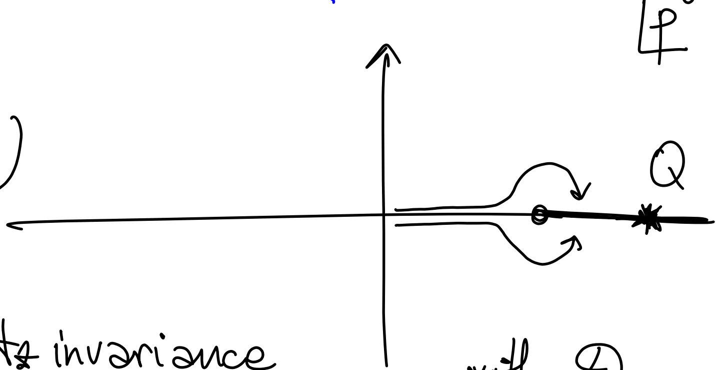




p in Q

$$\int_{\mathbb{R}^3} d^3\vec{k}\, f(\vec{k},p) \longrightarrow \int_{\mathcal{D}_3} d^3\vec{k}\, f(\vec{k},p)$$

You recover analyticity and Lorentz invariance



with $\mathcal{D}$

Let us check how the singularity integrates out ($\mu^2=0$)

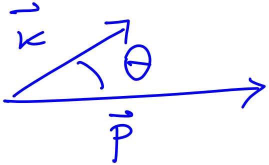
($p^0>0$)

$$\frac{d^3\vec{k}}{D_\varphi} \longrightarrow -\frac{2\pi^2 k_s\, dk_s\, du}{\tau - i(\varphi p^0 + p_s\eta)}$$

$$D_\varphi = p^0 e^{i\varphi} - \sqrt{\vec{k}^2+iM^2} - \sqrt{(\vec{k}-\vec{p})^2-iM^2}$$

Change of variables

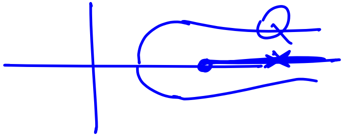
$|\vec{k}| = k_s$ $\quad |\vec{p}| = p_s$ $\quad u = \cos\theta$



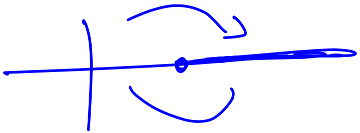
$$k_s = \frac{\sigma_-}{2p^0} + \tau\frac{\sigma_+^2}{2\sigma_-(p^2)^2} + \eta\frac{p_s\sigma_+^2}{4\sigma_- M^2} \qquad \sigma_\pm = \sqrt{(p^0)^4 \pm 4M^4}$$

$$u = \frac{p_s}{2k_s} + \eta\frac{\sigma_+^2}{2\sigma_- M^2}$$

2 cases : a) $\varphi \to 0$ then $p_s \to 0$



b) $p_s \to 0$ then $\varphi \to 0^{\pm}$



a) $\varphi = 0$ $\qquad -2\pi^2 \dfrac{k_s\, dk_s\, du}{\tau - i p_s \eta} \qquad \dfrac{1}{\tau - i\boxed{\epsilon}}$

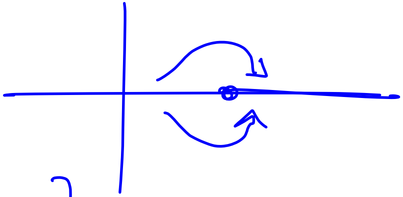
$p_s \to 0$: $\qquad -2\pi^2 k_s\, dk_s\, du \left[\mathcal{P}\left(\dfrac{1}{\tau}\right) + i\pi\, \mathrm{sgn}(\eta)\, \delta(\tau) \right] \propto$

$$\propto d\tau\, du \left[\mathcal{P}\left(\frac{1}{\tau}\right) + i\pi\, \cancel{\mathrm{sgn}(u)}\, \delta(\tau) \right] \to$$

$$\to \propto d\tau\, \mathcal{P}\left(\frac{1}{\tau}\right) \quad : \underline{\text{real !}}$$

[$p_s \to 0$ stands for $\mathbb{R}^3 \to D_3$ deformation in the bubble diagram]

b) $$-\frac{2\pi^2\kappa_s d\kappa_s du}{\tau - i\varphi p^0} \qquad \varphi \to 0^\pm$$



$$\to \propto d\tau\, du \left[\mathcal{P}\left(\frac{1}{\tau}\right) \pm i\pi\,\delta(\tau)\right] \to$$

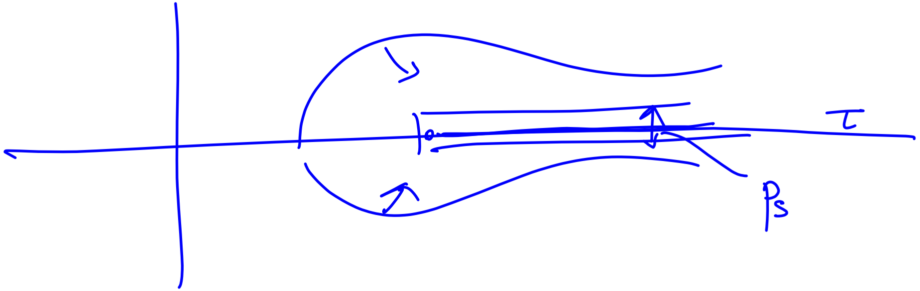
$$\to d\tau \left[\mathcal{P}\left(\frac{1}{\tau}\right) \pm i\pi\,\delta(\tau)\right]$$

Arithmetic average: $d\tau\, \mathcal{P}\left(\frac{1}{\tau}\right)$, as in a)

In general, use the domain deformation

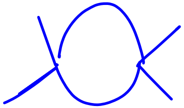
$\mathbb{R}^3 \to \mathcal{D}_3$ as a change of variables

$$\int_{\mathbb{R}^3} \frac{d^3\vec{k}}{\mathcal{D}_\varphi} \longrightarrow \int_{\text{strip}} \frac{d\tau\, d\eta}{\tau - i(\varphi p^0 + \tilde{p}_s \eta)} \qquad -1<\eta<1$$



These unphysical thresholds are bypassed by means of the "average continuation", i.e. the arithmetic average of the two analytic continuations

Example :



$$= \int_E^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 k^2} = \frac{1}{(4\pi)^2} \ln \frac{\Lambda^2}{p^2}$$

p small $\sim \int_E^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \sim \int^{\Lambda} \frac{\cancel{k^3} dk \; 2\pi^2}{16\pi^4 \; k^{\cancel{4}}} =$

$$= \frac{1}{8\pi^2} \ln \Lambda = \frac{1}{(4\pi)^2} \ln \Lambda^2$$

Minkowsky: $\frac{1}{(4\pi^2)} \ln \frac{\Lambda^2}{-p^2 \mp i\epsilon}$

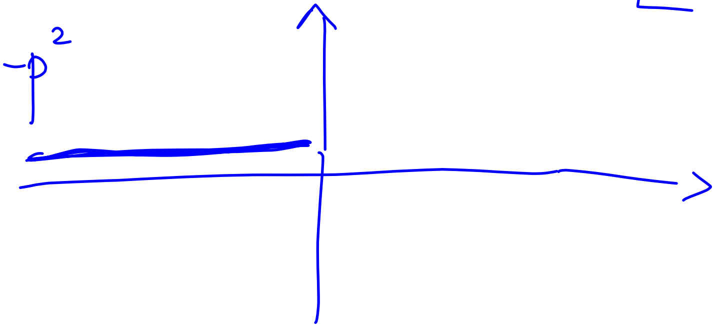
(Feynman from above or below)

Arithmetic average :

$$\frac{1}{2}\frac{1}{(4\pi)^2}\ln\frac{\Lambda^4}{(-p^2)^2+\epsilon^2} \longrightarrow \frac{1}{2}\ln(z)^2$$

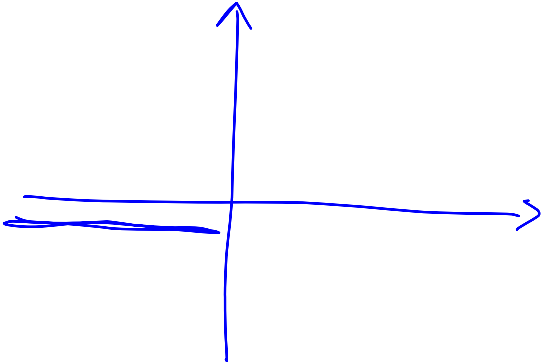
$\angle z$

$z = -p^2$



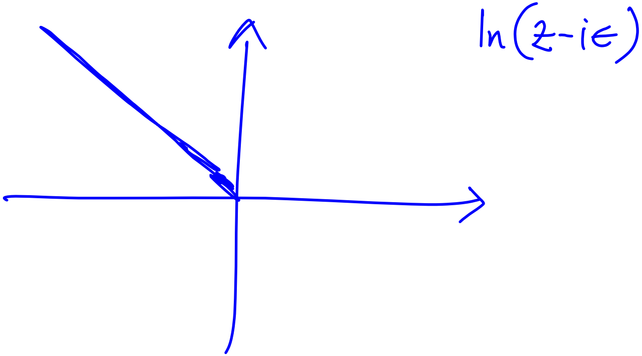
$\ln(z-i\epsilon)$

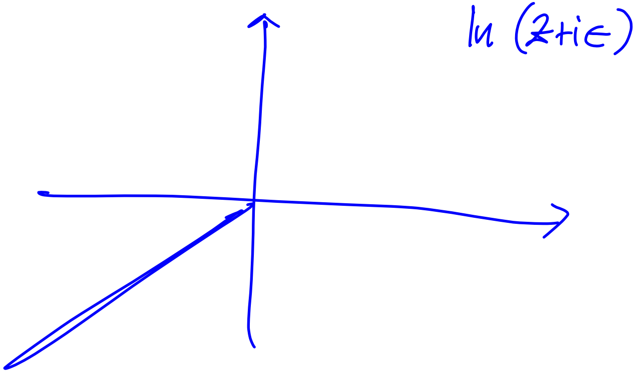
Feynman

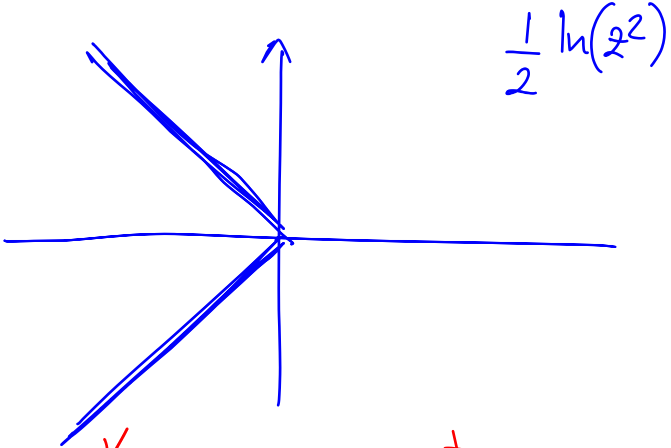


$\ln(z+i\epsilon)$

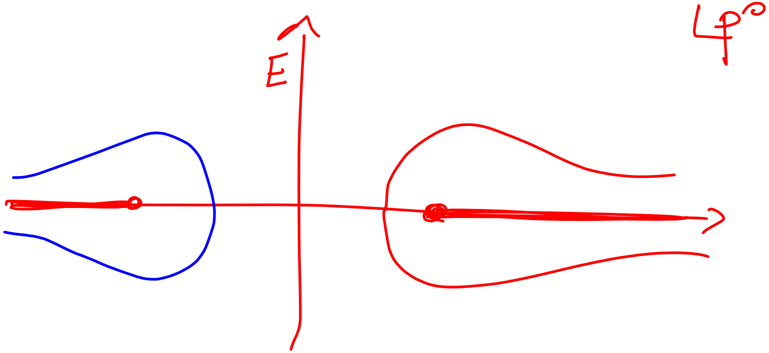
anti Feynman





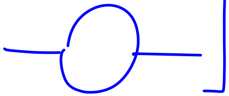
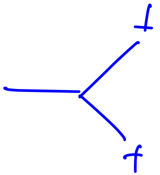


You can compute everything in the Euclidean region and reach every other region by means of the average continuation



Renormalizability is still ok, because it is so in Euclidean space

Unitarity ?

$2\,\mathrm{Im}\Big[(-i)$  $\Big] = \sum_f \int d\Pi_f \Big|$  $\Big|$

||

$0 = 0$

$f \in V$

V physical space

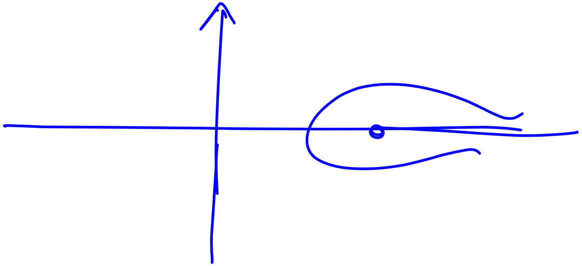
You get the optical theorem in the physical subspace V

$$\langle\alpha|(-i)T|\beta\rangle + \langle\alpha|iT^{+}|\beta\rangle = \langle\alpha|T^{+}T|\beta\rangle =$$

$$= \sum_{|n\rangle\in V} \langle\alpha|T^{+}|n\rangle\langle n|T|\beta\rangle$$

$$V \in W$$

$$\pm\frac{1}{p^2-m^2} \longrightarrow \pm\frac{p^2-m^2}{(p^2-m^2)^2+\varepsilon^4} \qquad \varepsilon\to 0$$



fakeon $\,{}^{+}_{-}$

Quantum Gravity

$$S = -\frac{1}{2\kappa^2}\int\sqrt{-g}\Big[2\Lambda_c + \zeta R + \alpha\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big) +$$

$$-\frac{\xi}{3}R^2\Big] + S_{SM} \qquad \zeta, \alpha, \xi > 0$$

Non-higher-derivative equivalent action ($\Lambda_c = 0$)

$$-\frac{\zeta}{2\kappa^2}\int\sqrt{-g}\,R + \frac{3\zeta}{4}\int\sqrt{-g}\Big[\nabla_\mu\phi\nabla^\mu\phi - \frac{m_\phi^2}{\kappa^2}\big(1-e^{\kappa\phi}\big)^2\Big] +$$

$$+ S_\chi + S'_{SM} \qquad m_\phi^2 = \frac{\zeta}{\xi} \qquad m_\chi^2 = \frac{\zeta}{\alpha}$$

$$S_\chi(g,\chi) = -\frac{\zeta}{\kappa^2}S_{\rm PF}(g,\chi,m_\chi^2) - \frac{\zeta}{2\kappa^2}\int\sqrt{-g}R^{\mu\nu}(\chi\chi_{\mu\nu} - 2\chi_{\mu\rho}\chi_\nu^\rho) + S_\chi^{(>2)}(g,\chi)$$

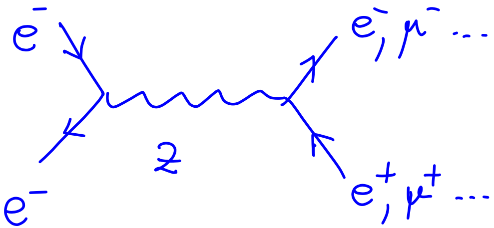
↑ Pauli-Fierz action with the wrong sign

You have to quantize $\chi_{\mu\nu}$ as a fakeon

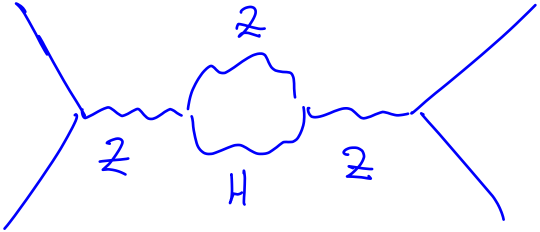
ϕ could be real or fake (two possibilities)

Does the Standard Model contain fakeons ?

Lep 2

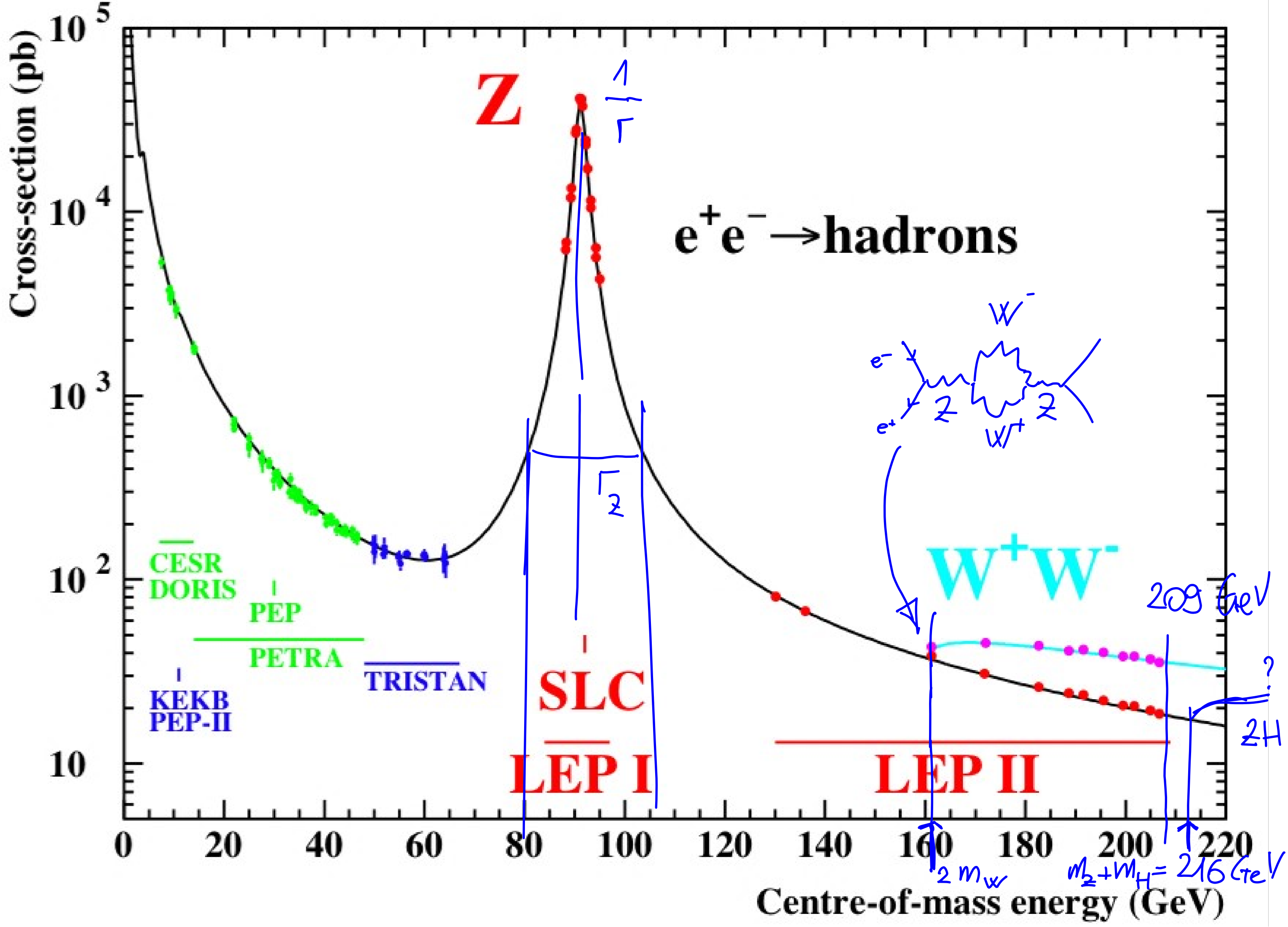


Maybe the Higgs field

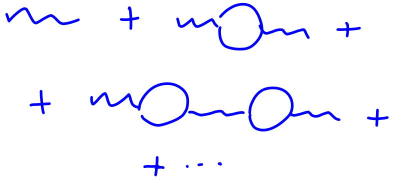


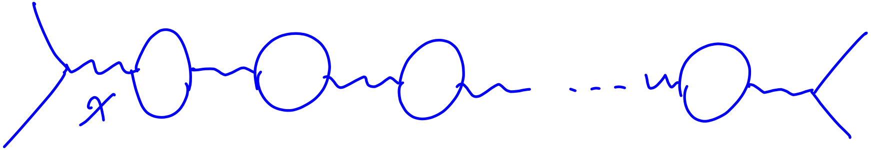
$$m_Z + m_H \sim 216 \text{ GeV}$$

$$91 + 125$$

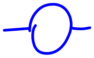

Cross-section (pb)
10 5
10 4
10 3
10 2
10
Z
e+e−→hadrons
W+W-
CESR
DORIS
PEP
PETRA
TRISTAN
KEKB
PEP-II
SLC
LEP I
LEP II
0
20
40
60
80
100
120
140
160
180
200
220
Centre-of-mass energy (GeV)
1/Γ
Γ_Z
e-
e+
Z
W-
W+
Z
209 GeV
?
2H
2 m_W
m_Z+m_H = 216 GeV

Gravity : triplet $g_{\mu\nu}, \phi, \chi_{\mu\nu}$





$$\frac{1}{p^2-m^2} \longrightarrow \frac{1}{p^2-m^2+}$$



$$\longrightarrow \sim \frac{Z}{p^2-\bar{m}^2+i\bar{m}\Gamma}$$

on the (new) pole

dressed propagator

$\Gamma_\chi < 0$ violation of microcausality

$$\frac{i}{E-\bar{m}+i\Gamma} \xrightarrow{\text{Fourier}} \operatorname{sgn}(t)\,\Theta(\Gamma t)\,e^{-i\bar{m}t-\Gamma t/2} = G_{BW}(t)$$

Breit-Wigner

$$\frac{Z}{p^2-\bar{m}^2+i\bar{m}\Gamma}$$

Particle at rest : $p=(E,0)$

$(|\Gamma|\ll\bar{m})$

$$\frac{Z}{E^2-\bar{m}^2+i\bar{m}\Gamma}\sim\frac{Z}{E^2-\left(m-\frac{i\Gamma}{2}\right)^2}=$$

$$=\frac{Z}{2m-i\Gamma}\left[\frac{1}{E-m+\frac{i\Gamma}{2}}-\frac{1}{E+m-\frac{i\Gamma}{2}}\right]$$

$$\int_{-\infty}^{+\infty}dt'\,G_{BW}(t-t')J(t')=\int_{-\infty}^{+\infty}dt'\,\mathrm{sgn}(t-t')\,\theta(\Gamma(t-t'))\,.$$

$$\cdot\, e^{-i\left(\bar{m}-\frac{i\Gamma}{2}\right)(t-t')}J(t')$$

$\Gamma > 0$: $$\int_{-\infty}^{t} dt' \, e^{-i\left(\bar{m} - \frac{i\Gamma}{2}\right)(t-t')} J(t')$$

$$e^{-\frac{\Gamma}{2}(t-t')}$$

$\Gamma < 0$: $$-\int_{t}^{+\infty} dt' \, e^{-i\bar{m}(t-t') - \frac{\Gamma}{2}(t-t')} J(t')$$

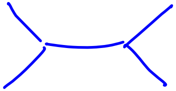
If $\bar{m} \sim 10^{12}$ GeV then you need

$|t-t'| \sim \frac{1}{\bar{m}} \sim 10^{-36}$ s $\qquad \frac{1}{M_{Pl}} \simeq 10^{-43}$ s $\qquad M_{Pl} \sim 10^{19}$ GeV

to appreciate the violation of causality

1 attosecond $\sim 10^{-18}$ s : shortest measured time

Optical theorem for the dressed propagator

$2\,\mathrm{Im}\Big[(-i)$  $\Big] = 2\,\mathrm{Im}\left[(-i)\,i\,(-i)^2\,\frac{-z}{p^2-\bar{m}^2+i\bar{m}\Gamma_\pm}\right] =$

$$= \mp\cancel{2}\,z\,\frac{1}{\cancel{2}i}\left[\frac{1}{p^2-\bar{m}^2+i\bar{m}\Gamma_\pm}-\frac{1}{p^2-\bar{m}^2-i\bar{m}\Gamma_\pm}\right] =$$

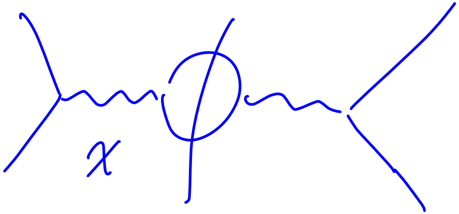
$$= \pm i z\,\frac{-2i\bar{m}\Gamma_\pm}{(p^2-\bar{m}^2)^2+\bar{m}^2\Gamma_\pm^2} = \frac{\pm 2\bar{m}\Gamma_\pm z}{(p^2-\bar{m}^2)^2+\bar{m}^2\Gamma_\pm^2} \geq 0$$

$\Rightarrow \Gamma_+ > 0$, $\Gamma_- < 0$ That is why $\Gamma_x < 0$

$\Gamma_\phi > 0$

Let us take the limit $T_\pm \to 0^\pm$

we get $2 Z \, \delta(p^2 - \bar{m}^2)$



Many effects do survive the classical limit

Classical limit : only tree diagrams without fakeons on the external legs

This is equivalent to "integrate out" the fakeons;
at the classical level it is equivalent to solving their field equations

$$\pm \frac{p^2 - m^2}{(p^2-m^2)^2 + \mathcal{E}^4} \qquad \mathcal{E} = \epsilon^2 \quad (\epsilon > 0)$$

$$\mathcal{E} \to 0$$

$$\frac{1}{2}\left[\frac{1}{p^2 - m^2 + i\epsilon} + \frac{1}{p^2 - m^2 - i\epsilon}\right] \qquad \frac{p^2 - m^2}{(p^2-m^2)^2 + \epsilon^2}$$

Feynman antiFeynman

$$\frac{1}{2}\left[\frac{1}{(p^0+i\epsilon)^2 - \vec{p}^{\,2} - m^2} + \frac{1}{(p^0 - i\epsilon)^2 - \vec{p}^{\,2} - m^2}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{p^2 - m^2 + 2i\epsilon p^0} + \frac{1}{p^2 - m^2 - 2i\epsilon p^0}\right] = \frac{p^2 - m^2}{(p^2-m^2)^2 + 4\epsilon^2 p^{0\,2}}$$

Toy model

$\dot{x} = v \qquad a = \ddot{x}$

$$\mathcal{L} = \frac{m}{2} v^2 - V(x,t)$$

$\tau = \text{constant}$

$$\mathcal{L}_{HD} = \frac{m}{2}\left(v^2 - \tau^2 a^2\right) - V(x,t)$$

Let us take $V(x,t) = -x\, F_{ext}(t)$

unprojected Equations of motion: $m\left(a + \tau^2 \ddot{a}\right) = F_{ext}(t)$

[Remember the Abraham-Lorentz force:

$$m\left(a - \tau \dot{a}\right) = F_{ext} \longrightarrow m a = \int_t^\infty dt'\, e^{(t-t')/\tau} F_{ext}(t') =$$

$$\shortparallel$$

$$m\left(1 - \tau \frac{d}{dt}\right) a$$

Smooth

$\tau \to 0$ limit

$$= \int_0^\infty du\, e^{-u} F_{ext}(t + u\tau)$$

$$m\left(1+\tau^2\frac{d^2}{dt^2}\right)a = F_{ext}(t) \to$$

$$ma = \underbrace{\frac{1}{1+\tau^2\frac{d^2}{dt^2}}} F_{ext}(t)$$

$$\xrightarrow{\text{Fourier}} \frac{1}{1-\tau^2 {p^0}^2} = \frac{-\frac{1}{\tau^2}}{(p^0)^2-\frac{1}{\tau^2}} \to$$

→ half sum of the retarded and advanced potentials

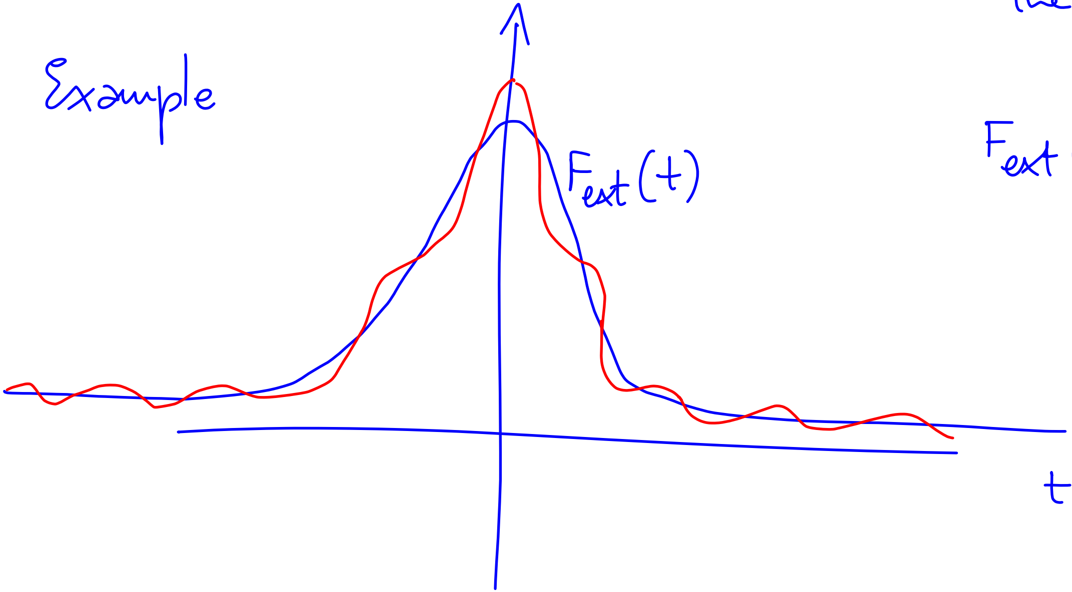
$$-\frac{1}{2\tau^2}\left[\frac{1}{(p^0+i\epsilon)^2-\frac{1}{\tau^2}}+\frac{1}{(p^0-i\epsilon)^2-\frac{1}{\tau^2}}\right]$$

$$ma(t) = \int_{-\infty}^{+\infty} du \, \frac{\sin\left(\frac{|u|}{\tau}\right)}{2\tau} F_{ext}(t-u)$$

$$\lim_{\tau \to 0} \frac{\sin\left(\frac{|u|}{\tau}\right)}{2\tau} = \delta(u)$$

Hint: $\text{sgn}(u) \cos\left(\frac{u}{\tau}\right) \to 0$ by the Riemann-Lebesgue theorem. Then, take the derivative

Example



$$F_{ext}(t) = e^{-\frac{\gamma t^2}{2}}$$

$\gamma =$ constant

$$ma = \langle F_{ext}(t) \rangle = \int_{-\infty}^{+\infty} du \frac{\sin\left(\frac{|u|}{\tau}\right)}{2\tau} F_{ext}(t-u)$$

Classical limit of Quantum Gravity

$$-\frac{\varsigma}{2\kappa^2}\int\sqrt{-g}\,R+\frac{3\varsigma}{4}\int\sqrt{-g}\left[\nabla_\mu\phi\,\nabla^\mu\phi-\frac{m_\phi^2}{\kappa^2}\left(1-e^{\kappa\phi}\right)^2\right]+$$

$$+\,S_\chi+S'_{SM}$$

- derive the field equations of $g_{\mu\nu}$, ϕ, $\chi_{\mu\nu}$
- solve those of χ by means of the fakeon Green function
- insert the solution into the other equations
- repeat the same for ϕ if ϕ is also fake

FLRW metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - a^2(t)\, d\sigma^2 \qquad \text{(ansatz)}$$

$$d\sigma^2 = \frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$\kappa = 0, 1, -1$$

The equations of

$$S = -\frac{1}{2\kappa^2}\int \sqrt{-g}\left[2\Lambda_c + \zeta R + \alpha\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + \right.$$

$$\left. - \frac{\xi}{3} R^2 \right] + S_{SM}$$

$$\begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix}$$

give (at $\Lambda_c = 0$ and $T_{\text{matter}\,\mu}^{\ \ \nu} = \rho(t)\,\delta_0^\nu \delta_\mu^0 - p(t)\,\delta_i^\nu \delta_\mu^i$)

$$i = 1, 2, 3$$

Unprojected equations

$$\Sigma\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^{2}}{a^{2}}+\frac{k}{a^{2}}\right)=\frac{4\pi G}{3}\left(\rho-3p\right)$$

$$\Upsilon\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^{2}}{a}-\frac{k}{a^{2}}\right)=-4\pi\left(\rho+p\right)$$

$$M_{Pl}=\frac{1}{\sqrt{G}}=\frac{\sqrt{8\pi J}}{\kappa}$$

$$\Sigma=1+\frac{1}{m_{\phi}^{2}}\left(3\frac{\dot{a}}{a}+\frac{d}{dt}\right)\frac{d}{dt}$$

$$\Upsilon=\Sigma+\frac{2}{m_{\phi}^{2}}\left[\frac{k}{a^{2}}+3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)\right]$$

Projected equations:

$$\frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{4\pi G}{3}\left(\langle\rho\rangle_\Sigma - 3\langle p\rangle_\Sigma\right)$$

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a} - \frac{k}{a^2} = -4\pi\left(\langle\rho\rangle_\gamma + \langle p\rangle_\gamma\right)$$

$$\langle X\rangle_A = \frac{1}{2}\left[\frac{1}{A}\Big|_{ret} + \frac{1}{A}\Big|_{adv}\right]$$

Continuity equation: $\frac{d\rho}{dt} + 3(\rho + p)\frac{\dot{a}}{a} = 0$

They can be solved exacty for radiation combined with vacuum energy density

$$p = \frac{\rho}{3} + p_0 \qquad p_0 = \text{constant}$$

$$a(t) = \sqrt{\frac{\sinh(\sigma t)}{\sigma}\left(\sigma' \cosh(\sigma t) - \frac{k}{\sigma}\sinh(\sigma t)\right)}$$

$$\sigma, \sigma' = \text{constants}$$