# THEORY OF HIGHER SPIN TENSOR CURRENTS AND CENTRAL CHARGES 

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#### Abstract

We study higher spin tensor currents in quantum field theory. Scalar, spinor and vector fields admit unique "improved" currents of arbitrary spin, traceless and conserved. Off-criticality as well as at interacting fixed points conservation is violated and the dimension of the current is anomalous. In particular, currents $\mathcal{J}^{(s, I)}$ with spin $0 \leq s \leq 5$ (and a second label $I$ ) appear in the operator product expansion of the stress tensor. The $T T$ OPE is worked out in detail for free fields; projectors and invariants encoding the space-time structure are classified. The result is used to write and discuss the most general OPE for interacting conformal field theories and off-criticality. Higher spin central charges $c_{s}^{I}$ with arbitrary $s$ are defined by higher spin channels of the many-point $T$-correlators and central functions interpolating between the UV and IR limits are constructed. We compute the one-loop values of all $c_{s}^{I}$ and investigate the RG trajectories of quantum field theories in the conformal window following our approach. In particular, we discuss certain phenomena (perturbative and nonperturbative) that appear to be of interest, like the dynamical removal of the I-degeneracy. Finally, we address the problem of formulating an action principle for the RG trajectory connecting pairs of $\mathrm{CFT}_{4}$ 's as a way to go beyond perturbation theory.


[^0]
## 1 Introduction and motivation.

In quantum field theory, particularly important for a theoretical investigation of phenomena beyond perturbation theory, but not so far from it, is the so-called "conformal window". In QCD with $N_{c}$ colors and $N_{f}$ flavors the theory has a weakly coupled IR fixed point in the neighborhood $N_{f} \lesssim 11 / 2 N_{c}$, where the large distance limit of the theory can be studied to the first orders in the perturbative expansion. By continuity, there is a certain finite interval sharing similar properties as this infinitesimal neighborhood. Its lower bound $N_{f}$ min is not known. $N_{f \min }<N_{f}<11 / 2 N_{c}$ is the conformal window, where the theory has an IR fixed point and, by definition, one can reach the IR fixed point by resumming the perturbative expansion. Below $N_{f \text { min }}$ purely non-perturbative effects become important. QCD does not belong to the conformal window, nevertheless a better understanding of the conformal window is a natural first step to go beyond perturbation theory. In supersymmetric theories, on the other hand, one knows the precise size of the conformal window in various cases and there exist duality arguments in favor of the equivalence of the IR limits of very different theories [1]. Moreover, rigorous computations of IR quantities have been performed [2, 3].

In the conformal window, the renormalization group flow can be thought of as the radiative interpolation between two conformal field theories (CFT), the UV and IR limits of the renormalization group ( RG ) flow trajectory. The purpose of this paper is to make a further step in the program addressed in ref. [4], which indeed amounts to study quantum field theory (QFT) under this point of view. We refine the research purpose itself. The reader interested in the chronological development of this program up to now should consult, in the order, references [5, 6, 4, 2, 3, 3.

In the present paper, we develop techniques similar to those used in the context of deep inelastic scattering [7] and apply them to study the operator product expansion of the stressenergy tensor in the Euclidean framework. We thus clarify the nature of the "secondary" central charges introduced in [5, 6, 4] and study several phenomena in connection with this issue. The stress tensor OPE is worked out in detail for free fields and generalized to interacting conformal field theories and off-criticality. This analysis leads us to consider higher spin currents, which indeed appear as channels of the $T T$ OPE.

We investigate various properties of higher spin tensor currents in the context of our approach to quantum field theory. We can define and study the OPE's of these currents and their anomalies, the RG flow of certain "central functions", defined by the two- and four-point functions of the currents in exam and by their trace anomalies, and compute their UV and IR critical limits (central charges). For spin 1 and 2 this program was successfully realized in [2] for supersymmetric theories in the conformal window. In particular, the exact IR values of the gravitational central charges (called $c$ and $a$ ) were computed for UV free theories using the remarkable properties of supersymmetry. The strategy was applied in 3] to a large amount of models. The results of [2, 3] provided, among the other things, a strong support to the idea of irreversibility of the flux of the RG flow (" $a$-theorem").

The question that we address now is: what can we say in the same spirit using higher
spin currents? First of all, the assumption that a quantum field theory admits a higher spin flavor symmetry is not met by ordinary theories: interactions violate explicitly the higher spin conservation law and higher spin currents acquire an anomalous dimension already at the first orders in the perturbative expansion. For this reason it is not sufficient to study two-point functions of higher spin currents, rather one has to look for higher spin channels inside the correlators of the stress tensor [4]. To begin with, one has to study the TT OPE in detail.
"Improved" higher spin tensor currents provide a basis for the terms appearing in the $T T$ OPE (this and other statements will be made precise in the paper) and provide a simple way to classify the primary operators in the OPE. The construction of certain projectors that encode the space-time structure of the terms allows us to generalize the OPE to interacting conformal field theories and off-criticality. Equipped with this, one can investigate the OPE along the RG trajectory of a quantum field theory.

Along the RG flow trajectory, higher spin currents "move", due to their anomalous dimension. The phenomenology of this moving is an interesting subject to study in this domain. In particular, this phenomenon removes a certain degeneracy that one observes in the free field limit. Examples will be considered in detail, in $\mathrm{N}=4$ supersymmetric Yang-Mills theory as well as in the context of electric-magnetic duality [1].

Before starting the technical analysis, we would like to formulate the idea underlying our approach to QFT in general terms.

One can naturally view perturbation theory as a "Cauchy problem". The starting conformal field theory, say $\mathrm{CFT}_{U V}$ (that we assume free), and the vertices of the classical Lagrangian are, so to speak, the initial conditions ("position" and "velocity", respectively). The functional integral is the step-by-step algorithm to move towards an unknown $\mathrm{CFT}_{I R}$. The problem is then to identify $\mathrm{CFT}_{I R}$ given the initial conditions.


Fig. 1: Is there an action principle for the RG trajectory connecting two conformal field theories?
Alternatively, a trajectory can be identified by its initial and final limits and by the requirement that a certain functional ("action") have its minimum value with these boundary conditions. One is therefore led to ask: is there an action principle identifying the quantum field theory (i.e. the RG trajectory), hopefully unique under suitable assumptions, that radiatively interpolates between two given CFT's? Presumably one will be led to consider a set of theories larger than the set of ordinary renormalizable theories.

We are not ready, yet, to answer this difficult question. The best that we can do is to reconsider ordinary theories, as we know them from perturbation theory, under this alternative point of view (see Fig. 1) and see what we can learn. The properties that we derive might become axioms of the new formulation or maybe stimulate a numerical research.

First we have to fix the set of quantities that should be studied in this approach. A quantum field theory is identified by the set of its correlators, both of elementary fields and composite operators. However, in general there is a undesirable dependence on the description (what are the elementary fields, quarks and gluons or baryons and mesons? and what is composite?), the definition (a composite operator, as well as an elementary field, needs to be normalized at some energy scale, otherwise one can multiply it by an arbitrary function of the running coupling constant), and so on. We look for quantities that are invariant under these details. Introducing an intermediate reference scale does not change the RG trajectory and so it is immaterial to our problem. Moreover, in the reformulation of QFT as we imagine it, it should not be necessary to speak about "elementary" fields, nor "composite" operators, since what is elementary in the UV is not elementary in the IR and vice versa.

It is natural to focus on conserved currents, which do not need any independent normalization, and study their correlators. Conserved currents should be the "elementary" objects of our description, whatever fields are used to represent them explicitly at a given energy scale. Among these, the stress-tensor plays a peculiar role, since it always exists. The idea is to define certain key-quantities (central charges) via correlators of conserved currents, characterizing the theory in the conformal fixed point and to interpolate between the UV and IR values of these quantities by suitable functions. These functions should allow one to study the evolution of the theory along the RG trajectory. Once these functions will be understood it will be possible to make some proposal for the envisaged RG action principle.

We will study, among the other things, the $T T$ OPE and certain structures of the multi- $T$ -point-functions, which encode the central charges just mentioned.

Some interesting phenomena occurring in this approach will be described in the paper. Just to mention one example on which we will not come back again, the irreversibility properties exhibited by some central functions (the " $a$-functions") would imply that, given the two extremal conformal field theories of the RG trajectory, one knows a priori, by comparing the values of the $a$-functions in the two cases, which extremum is the UV limit and which extremum is the IR limit. The $a$-functions should determine a level ("potential") that classifies conformal field theories and puts restrictions on the possible RG trajectories connecting them. For example, two $\mathrm{N}=4$ supersymmetric Yang-Mills theories with the same gauge group and different values of the coupling constant $g$ very presumably will admit no interpolating RG trajectory. The explanation might be: because they have the same $a$-values [2] and theories with the same $a$ values are only marginally connectable. Note that the converse has already been proved [6, 2]: marginal deformations do not change the $a$-values.

The paper is divided into four sections. In the first section (section 2) the general form of improved higher spin currents and their critical two-point functions are studied. In section 3 currents and two-point correlators are extended off-criticality, first under the assumption of
higher spin flavor symmetry and then in presence of an anomalous dimension (explicit violation of the conservation condition). In section 4 the TT OPE is worked out in detail. In section 5 the results of the other sections are combined in order to construct the desired central charges and central functions. Certain phenomena are discussed in supersymmetric theories and some simple remarks in the context of the AdS/CFT correspondence [8] are made. Relationships with more familiar anomalies are pointed out.

For notational convenience, let us write down, before beginning the discussion of higher spin tensors currents, some simple spin-1 and spin-2 formulas that we will generalize. The spin- 1 current and two-point function read in the free field limit

$$
\begin{equation*}
j_{\mu}=\bar{\psi} \gamma_{\mu} \psi, \quad<j_{\mu}(x) j_{\nu}(0)>=\text { const. } \pi_{\mu \nu}\left(\frac{1}{|x|^{4}}\right) \tag{1.1}
\end{equation*}
$$

where $\pi_{\mu \nu}=\partial_{\mu} \partial_{\nu}-\square \delta_{\mu \nu}$ (the precise coefficients will be written in the relevant sections). There is also the axial current

$$
\begin{equation*}
\mathcal{A}_{\mu}^{(1)}=\bar{\psi} \gamma_{5} \gamma_{\mu} \psi=-\frac{1}{6} \varepsilon_{\mu \nu \rho \sigma} \bar{\psi} \gamma_{\nu \rho \sigma} \psi \tag{1.2}
\end{equation*}
$$

where $\gamma_{5}=\gamma_{1} \gamma_{2} \gamma_{3} \gamma_{4}$. For spin- 2 we have

$$
\begin{equation*}
T_{\mu \nu}=\frac{2}{3} \partial_{\mu} \varphi \partial_{\nu} \varphi-\frac{1}{6} \delta_{\mu \nu}\left(\partial_{\alpha} \varphi\right)^{2}-\frac{1}{3} \varphi \partial_{\mu} \partial_{\nu} \varphi, \quad<T_{\mu \nu}(x) T_{\rho \sigma}(0)>=\text { const. } \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x|^{4}}\right), \tag{1.3}
\end{equation*}
$$

where

$$
\prod_{\mu \nu, \rho \sigma}^{(2)}=\frac{1}{2}\left(\pi_{\mu \rho} \pi_{\nu \sigma}+\pi_{\mu \sigma} \pi_{\rho \nu}\right)-\frac{1}{3} \pi_{\mu \nu} \pi_{\rho \sigma}
$$

This stress tensor is known as the "improved" stress tensor. For this notion and references about it we point out the well-known lectures by Jackiw 99 .

A discussion about higher spin currents can be found in ref. [10]. We will use those results and other results of previous work on this subject. The list of references contained here about higher spin fields nevertheless incomplete. We are more concerned with the properties of higher spin currents for lower spin fields (scalar, spinor and vector fields), rather than the dynamical aspects of higher spin fields in themselves, to which a great amount of effort has been nevertheless devoted by physicists so far. In particular, recent developments towards a consistent formulation of dynamical higher spin couplings have shown that the problem of coupling higher spin fields to gravity can be overcome in presence of a cosmological term and that an all-order consistent formulation of higher spin couplings can be achieved at least at the level of equations of motion. For this and related issues, we point out the recent paper by Vasiliev [11], which is also a review of the matter and contains a detailed list of references and an historical survey.

## 2 Higher spin tensor currents and their two-point functions.

A higher spin current $\mathcal{J}_{\mu_{1} \ldots \mu_{s}}^{(s)}(x)$ is a completely symmetric operator in its $s$ indices, satisfying the double trace condition [12, 13]

$$
\begin{equation*}
\mathcal{J}^{(s)}{ }_{\mu}^{\mu}{ }_{\nu}^{\nu} \mu_{5} \cdots \mu_{s}=0 \tag{2.4}
\end{equation*}
$$

and the traceless conservation condition

$$
\begin{equation*}
\partial^{\mu} \mathcal{J}_{\mu \mu_{2} \cdots \mu_{s}}^{(s)}-\frac{1}{2(s-1)} \sum_{i<j=2}^{s} \delta_{\mu_{i} \mu_{j}} \partial^{\mu} \mathcal{J}^{(s)}{ }_{\mu \cdots \mu_{i-1} \alpha \mu_{i+1} \cdots \mu_{j-1} \alpha \mu_{j+1} \cdots \mu_{s}}=0 . \tag{2.5}
\end{equation*}
$$

These properties make the interaction lagrangian

$$
\mathcal{L}_{I}=h_{s}^{\mu_{1} \cdots \mu_{s}} \mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}
$$

invariant under the gauge transformation

$$
\begin{equation*}
\delta h_{s}^{\mu_{1} \cdots \mu_{s}}=\partial_{\left\{\mu_{1}\right.} \xi_{\left.s \mu_{2} \cdots \mu_{s}\right\}}, \quad \xi_{s}{ }_{\mu}{ }^{\mu}{ }_{\mu_{3} \cdots \mu_{s}}=0, \tag{2.6}
\end{equation*}
$$

with $\xi_{s \mu_{2} \cdots \mu_{s}}$ symmetric in its $s-1$ indices. Counting the components and fixing the gauge, one can show that higher spin fields have two elicities. The counting proceeds as follows. A $k$-index symmetric traceless tensor has $(k+1)^{2}$ components and a doubly traceless tensor has $(k+1)^{2}+(k-1)^{2}=2\left(k^{2}+1\right)$ components. (2.5) is an $(s-1)$-indexed such tensor, and therefore contains $s^{2}$ components. Fixing the gauge means to subtract twice as many components ("ghosts" and "antighosts"), therefore giving $2\left(s^{2}+1\right)-2 s^{2}=2$ surviving elicities, the correct number for a massless spin- $s$ propagating field. One can write a kinetic action for $h_{s}^{\mu_{1} \cdots \mu_{s}}$ that reduces to the Klein-Gordon equation for each elicity upon fixing the gauge [12, 13].

Here we consider higher spin tensor currents for scalar, spinor and vector fields. Let us start from free fields. The form of the operator $\mathcal{J}_{\mu_{1} \ldots \mu_{s}}^{(s)}$, as specified by conditions (2.4) and (2.5) and by its minimal dimension $2+s$, is actually not unique. It is unique once one imposes a further condition of simple tracelessness:

$$
\begin{equation*}
\mathcal{J}^{(s)}{ }_{\mu}{ }^{\mu}{ }_{\mu_{3} \cdots \mu_{s}}=0 . \tag{2.7}
\end{equation*}
$$

This condition is the analogue of the tracelessness condition $T_{\mu \mu}=0$ for the stress-energy tensor $T_{\mu \nu}=\mathcal{J}_{\mu \nu}^{(2)}$. As a consequence, it is evident that it is an additional condition and that it does not characterize the higher spin field in itself. It should be related to an extra symmetry of the theory. It is tempting to think that the simple trace condition is the effect of the conformal symmetry, i.e. that condition (2.7) can be imposed only in the critical points (high energy limit or large distance limit) of a quantum field theory. Off-criticality condition (2.7) should not hold. With condition (2.7) the counting is modified as follows. (2.7) leaves the tensor with $(s+1)^{2}$ components and (2.5) subtracts $s^{2}$ components. Thus one remains with $2 s+1$ components, which is correct for a spin- $s$ operator.

Let us assume that $\mathcal{J}^{(s)}$ is conserved and doubly traceless. We are going to show that if the correlators of $\mathcal{J}^{(s)}$ are conformal, then $\mathcal{J}^{(s)}$ is simply traceless and has no anomalous dimension. Vice versa, if $\mathcal{J}^{(s)}$ is simply traceless and has no anomalous dimension, then its correlators are conformal. Some arguments of this subsection are also treated in ref. [14], to which the reader is referred for comparison.

To begin with, let us study the two-point function of $\mathcal{J}^{(s)}$. Conformal invariance assures that the current transforms as [15, 14]

$$
\begin{equation*}
\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}\left(x^{\prime}\right)=|x|^{2 s+4+2 h} \mathcal{I}_{\mu_{1} \mu_{1}^{\prime}}(x) \cdots \mathcal{I}_{\mu_{s} \mu_{s}^{\prime}}(x) \mathcal{J}_{\mu_{1}^{\prime} \cdots \mu_{s}^{\prime}}^{(s)}(x) \tag{2.8}
\end{equation*}
$$

under the coordinate inversion $x_{\mu}^{\prime}=x_{\mu} / x^{2} . h$ is an eventual anomalous dimension of the operator $\mathcal{J}$. Therefore the two-point correlator transforms as

$$
\begin{align*}
<\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}\left(x^{\prime}\right) \mathcal{J}_{\nu_{1} \cdots \nu_{s}}^{(s)}\left(y^{\prime}\right)> & =|x|^{2 s+4+2 h}|y|^{2 s+4+2 h} \mathcal{I}_{\mu_{1} \mu_{1}^{\prime}}(x) \cdots \mathcal{I}_{\mu_{s} \mu_{s}^{\prime}}(x) \mathcal{I}_{\nu_{1} \nu_{1}^{\prime}}(y) \cdots \mathcal{I}_{\nu_{s} \nu_{s}^{\prime}}(y) \times \\
& <\mathcal{J}_{\mu_{1}^{\prime} \cdots \mu_{s}^{\prime}}^{(s)}(x) \mathcal{J}_{\nu_{1}^{\prime} \cdots \nu_{s}^{\prime}}^{(s)}(y)> \tag{2.9}
\end{align*}
$$

To write down the general solution to this condition the basic ingredient is the symmetric orthogonal matrix $\mathcal{I}_{\mu \nu}(x)=\delta_{\mu \nu}-2 x_{\mu} x_{\nu} / x^{2}$, which is proportional to the Jacobian $\partial x_{\mu}^{\prime} / \partial x_{\nu}$ and satisfies the properties

$$
\begin{equation*}
\mathcal{I}_{\mu \nu}\left(x^{\prime}-y^{\prime}\right)=\mathcal{I}_{\mu \mu^{\prime}}(x) \mathcal{I}_{\mu^{\prime} \nu^{\prime}}(x-y) \mathcal{I}_{\nu^{\prime} \nu}(y), \quad \mathcal{I}_{\mu \nu}\left(x^{\prime}\right)=\mathcal{I}_{\mu \nu}(x), \quad \mathcal{I}_{\mu \rho}(x) \mathcal{I}_{\rho \nu}(x)=\delta_{\mu \nu} \tag{2.10}
\end{equation*}
$$

Our correlator can then be written a $\left\{^{2}\right.$

$$
<\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}(x) \mathcal{J}_{\nu_{1} \cdots \nu_{s}}^{(s)}(0)>=\frac{1}{|x|^{4+2 s+2 h}} \sum_{n=0}^{[s / 2]} a_{n} \sum_{\substack{\text { symm }}} \delta_{\mu_{1} \mu_{2}} \delta_{\nu_{1} \nu_{2}} \cdots \delta_{\mu_{n-1} \mu_{n}} \delta_{\nu_{n-1} \nu_{n}} \times
$$

[ $s / 2]$ denotes the integral part. $\sum_{\text {symm }}$ means complete symmetrization in both sets of indices, i.e. the sum over the necessary permutations divided by the number of permutations. $a_{n}$ are arbitrary constants. It is straightforward to verify that the correlator transforms correctly under inversion, since each term in the sum does. The basic rule is that pairs of indices belonging to the same current, like $\mu_{1} \mu_{2}$, are coupled via the identity matrix $\delta_{\mu_{1} \mu_{2}}$, while pairs of indices belonging to different currents, like $\mu_{1} \nu_{1}$, are coupled via the matrix $\mathcal{I}: \mathcal{I}_{\mu_{1} \nu_{1}}$. One then writes down all possible terms. The conclusion is that there are $[s / 2]+1$ free parameters and one anomalous dimension.

We now study the imposition of (2.4), (2.5) and (2.7).
The double trace condition (2.4) imposes $[s / 2]-1$ conditions. First, one notes that doubly tracing a conformal correlator (2.9) one still obtains a conformal correlator. Therefore one can repeat the previous counting of allowed terms with the reduced set of indices.

After imposing the double trace condition, only two free parameters survive (one of which is the overall constant), plus the eventual anomalous dimension $h$. Finally, conservation (2.5) kills another parameter and furthermore imposes $h=0$. One remains just with the overall constant. For the applications it is important to remark that the dimension is fixed to be the canonical one. One can check that the resulting expression automatically satisfies the simple trace condition (2.7).

Now we proceed differently. We start from the conformal correlator (2.9) and impose the double trace condition, therefore remaining with two free parameters and the anomalous dimension $h$. Then, instead of imposing the conservation condition, we just impose the simple trace condition (2.7). The result is that one of the two parameters is killed, but $h$ remains free:

[^1]we remain with the overall constant and the anomalous dimension $h$ : $h$ is not necessarily zero and when it is nonzero the conservation condition (2.5) is violated.

The simple trace condition can always be imposed on the current $\mathcal{J}_{\mu_{1} \ldots \mu_{s}}^{(s)}$, since it is a purely algebraic condition. We will show that in general the conservation condition cannot be satisfied. In particular, it cannot be satisfied in interacting $\mathrm{CFT}_{4}$ 's. We conclude that $h$ measures the violation of the conservation condition at criticality.

Anomalous violations of the conservation conditions can usually be moved away with the socalled finite local counterterms. For example, the violation of the vector current conservation can be moved to the divergence of the axial current [9], the violation of the stress current conservation can be moved to the trace condition.

The $h$-violation of the higher spin conservation condition is not of this type. It is an explicit violation, not an anomalous violation. For this reason it cannot be moved to the simple trace condition by means of finite local counterterms. There is no way to enforce the conservation condition in general, at least for the quantum field theories actually known. It is eventually possible in new theories that admit consistent higher spin couplings, but this is not our main concern here (we plan to develop this issue in a future publication [16]).

Off-criticality there are also anomalous violations and indeed these can be moved to the simple tracelessness condition. We will study some of these aspects in more depth in the next section.

Another way to study the two-point function is to write it as

$$
\begin{equation*}
\left\langle\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}(x) \mathcal{J}_{\nu_{1} \cdots \nu_{s}}^{(s)}(0)>=c_{s} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}\left(\frac{1}{|x|^{4+2 h}}\right),\right. \tag{2.12}
\end{equation*}
$$

where $\prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}$ is a differential operator (a polynomial of degree $2 s$ in derivatives, constructed with $\partial_{\mu}$ and $\delta_{\mu \nu}$ ) fixed by the required symmetries and $c_{s}$ is a numerical factor. This form is more convenient to impose conservation, less convenient to study conformality. It is easy to check that conditions (2.4) and (2.5) do not fix this correlator (i.e. the form of $\Pi^{(s)}$ ) uniquely (see below). It is necessary to impose (2.7).

Now, when (2.7) holds one has also the complete conservation $\partial_{\mu_{1}} \mathcal{J}_{\mu_{1} \ldots \mu_{s}}^{(s)}=0$. A correlator satisfying complete conservation can be easily constructed via the projector $\pi_{\mu \nu}$,

$$
\begin{align*}
& <\underset{\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}(x) \mathcal{J}_{\nu_{1} \cdots \nu_{s}}^{(s)}(0)>=}{ } \quad \sum_{n=0}^{[s / 2]}\left[b_{n} \sum_{\text {symm }} \pi_{\mu_{1} \mu_{2}} \cdots \pi_{\mu_{n-1} \mu_{n}} \pi_{\nu_{1} \nu_{2}} \cdots \pi_{\nu_{n-1} \nu_{n}} \pi_{\mu_{n+1} \nu_{n+1}} \cdots \pi_{\mu_{s} \nu_{s}}\right] \frac{1}{|x|^{4+2 h}} .
\end{align*}
$$

One has $[s / 2]+1$ free parameters $b_{n}$ (different from the $a_{n}$ 's) and the anomalous dimension $h$. We now prove that (2.7) imposes $[s / 2]$ conditions and so leaves with a single overall constant, plus the anomalous dimension $h$.

Indeed, after contraction of, say, $\mu_{s-1}$ and $\mu_{s}$, the result has necessarily the form (using

$$
\begin{align*}
& \pi \pi=-\square \pi) \\
& \square \sum \pi_{\nu_{s-1} \nu_{s}} \sum_{n=0}^{[s / 2]-1}\left[b_{n}^{\prime} \sum_{\text {symm }^{\prime}} \pi_{\mu_{1} \mu_{2}} \cdots \pi_{\mu_{n-1} \mu_{n}} \pi_{\nu_{1} \nu_{2}} \cdots \pi_{\nu_{n-1} \nu_{n}} \pi_{\mu_{n+1} \nu_{n+1}} \cdots \pi_{\mu_{s-2} \nu_{s-2}}\right] . \tag{2.14}
\end{align*}
$$

The internal sum reproduces the projector of a spin $s-2$ current and the external sum takes care of the remaining permutations. The coefficients $b_{n}^{\prime},[s / 2]$ in total, are linear combinations of the coefficients $b_{n}$. Therefore setting (2.14) to zero imposes the desired number of conditions.

In this case, $h$ measures the explicit violation of conformal invariance. Indeed, the final amplitude does not satisfy (2.9) only because of $h$ and enforcing (2.9) sets $h$ to 0 .

This concludes the present subsection. With respect to conformality in its general formulation (2.9), the simple trace condition simplifies enormously the search for the projector $\prod_{\mu_{1} \cdots \mu_{s} \nu_{1} \cdots \nu_{s}}^{(s)}$ and the construction of $\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}$. After presenting other general properties, we proceed to construct the quantities (currents and correlators) for $s=3,4$ and 5 .

### 2.1 Orthonormality.

Some orthonormality properties are noticeable. First we observe that for $s \neq s^{\prime}$ we have the orthogonality relationship

$$
\begin{equation*}
<\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}(x) \mathcal{J}_{\nu_{1} \cdots \nu_{s^{\prime}}}^{\left(s^{\prime}\right)}(0)>=0 \tag{2.15}
\end{equation*}
$$

at criticality. The simple trace condition is here crucial, since otherwise we would get a nonzero result. Indeed, we shall show in the next section that if the simple trace condition is relaxed it is always possible to construct higher spin currents via suitable differentiation of lower spin ones.
(2.15) holds in presence of anomalous dimensions $h_{s}$ and $h_{s^{\prime}}$ and it is proved as follows. Consider the most general conformal structure of the given two-point function, constructed like in (2.11) and let us assume that $s<s^{\prime}$. Then one has $[s / 2]+1$ free parameters before imposing the simple trace condition (the lower spin is the one that dictates the counting). Tracing $\mathcal{J}^{\left(s^{\prime}\right)}$ one imposes precisely $[s / 2]+1$ conditions (this is correct also for $s^{\prime}=s+1$ due to the integral part) therefore proving the statement. We observe that the argument based on the dimensionality of the operators (conformal two-point functions of operators with different dimensions vanish) is weaker than the argument that we have given: the two currents might have precisely the same dimension even if their spins are different.

The second property that we outline has to do with the normalization of the currents. Writing the relevant structure of (2.12) as

$$
\begin{equation*}
\prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}\left(\frac{1}{x^{4}}\right)=\mathcal{I}_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}(x) \frac{1}{|x|^{4+2 s}} \tag{2.16}
\end{equation*}
$$

we have relationships of the form

$$
\begin{equation*}
\frac{1}{p_{s}^{2}} \mathcal{I}_{\mu_{1} \cdots \mu_{s}, \alpha_{1} \cdots \alpha_{s}}^{(s)}(x) \mathcal{I}_{\alpha_{1} \cdots \alpha_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}(x)=\Im_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)} \tag{2.17}
\end{equation*}
$$

$p_{s}$ being normalizations factors and $\Im_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}$ denoting the spin-s identity operators, i.e. the identity operator on the space of symmetric, completely traceless $s$-indexed tensors. For example, $\Im_{\mu \nu, \rho \sigma}^{(2)}=\frac{1}{4}\left[2\left(\delta_{\mu \rho} \delta_{\nu \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}\right)-\delta_{\mu \nu} \delta_{\rho \sigma}\right]$. Property (2.17) means that the symmetric, dimensionless matrices $\mathcal{I}_{\mu_{1} \ldots \mu_{s}, \alpha_{1} \cdots \alpha_{s}}^{(s)}(x)$ are orthogonal. It generalizes the orthogonality property of the matrix $\mathcal{I}_{\mu \nu}(x)$ appearing in the Jacobian $\partial x_{\mu}^{\prime} / \partial x_{\nu}=\mathcal{I}_{\mu \nu}(x) /|x|^{2}=\left(\delta_{\mu \nu}-2 x_{\mu} x_{\nu} /|x|^{2}\right) /|x|^{2}$ of the coordinate inversion $x_{\mu} \rightarrow x_{\mu} /|x|^{2}$. We have indeed already used $\mathcal{I}_{\mu \nu}(x)$ - see formula (2.11) - to build the most general conformal two-point correlator.
$\mathcal{I}_{\mu \nu}(x)$ appears in the numerator of the spin-1 two-point function (1.1). For spin-2, see formula (1.2), definition (2.16) gives $\mathcal{I}_{\mu \nu, \alpha \beta}^{(2)}(x)=80\left(2 \mathcal{I}_{\mu \rho} \mathcal{I}_{\nu \sigma}+2 \mathcal{I}_{\mu \sigma} \mathcal{I}_{\nu \rho}-\delta_{\mu \nu} \delta_{\rho \sigma}\right)$ and (2.17) reads

$$
\begin{equation*}
\frac{1}{\left(2^{6} \cdot 5\right)^{2}} \mathcal{I}_{\mu \nu, \alpha \beta}^{(2)}(x) \mathcal{I}_{\alpha \beta, \rho \sigma}^{(2)}(x)=\Im_{\mu \nu, \rho \sigma}^{(2)} . \tag{2.18}
\end{equation*}
$$

The spin 3,4 and 5 versions of (2.17) will be reported in the respective subsections.
The third observation is that a cross-trace of the spin-s projector $\prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}$ is a spin-$(s-1)$ projector. We fix the overall coefficient of $\prod_{\mu_{1} \cdots \mu_{s}}^{(s)}$ so that the term $\sum_{s} \pi_{\mu_{1} \nu_{1}} \cdots \pi_{\mu_{s} \nu_{s}}$ has unit coefficient. With this convention

$$
\begin{equation*}
\prod_{\mu_{1} \cdots \mu_{s-1} \alpha, \nu_{1} \cdots \nu_{s-1} \alpha}^{(s)}=-\frac{2 s+1}{2 s-1} \square \prod_{\mu_{1} \cdots \mu_{s-1}, \nu_{1} \cdots \nu_{s-1}}^{(s-1)} \tag{2.19}
\end{equation*}
$$

The simplest case is

$$
\prod_{\mu \alpha, \alpha \rho}^{(2)}=-\frac{5}{3} \square \pi_{\mu \rho} .
$$

Finally, it is obvious that the square of the projector $\prod^{(s)}$ is equal to itself. Precisely, the same conventions that give (2.19) give also

$$
\begin{equation*}
\prod_{\mu_{1} \cdots \mu_{s}, \rho_{1} \cdots \rho_{s}}^{(s)} \prod_{\rho_{1} \cdots \rho_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}=(-1)^{s} \square^{s} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)} \tag{2.20}
\end{equation*}
$$

For example,

$$
\pi_{\mu \rho} \pi_{\rho \nu}=-\square \pi_{\mu \nu}, \quad \prod_{\mu \nu, \alpha \beta}^{(2)} \prod_{\alpha \beta, \rho \sigma}^{(2)}=\square^{2} \prod_{\mu \nu, \rho \sigma}^{(2)}
$$

are immediately verifiable.

## $2.2 \quad$ Spin 3.

In this and in the next two subsections we analyze in detail the cases of spin 3, 4 and 5 , that will be relevant for the applications.

For $s=3$ formula (2.11) becomes

$$
\begin{equation*}
<\mathcal{J}_{\mu \nu \rho}^{(3)}(x) \mathcal{J}_{\alpha \beta \gamma}^{(3)}(0)>=\frac{1}{|x|^{10+2 h}} \sum_{\text {symm }}\left[a_{1} \delta_{\mu \nu} \delta_{\alpha \beta} \mathcal{I}_{\rho \gamma}(x)+a_{0} \mathcal{I}_{\mu \alpha}(x) \mathcal{I}_{\nu \beta}(x) \mathcal{I}_{\rho \gamma}(x)\right], \tag{2.21}
\end{equation*}
$$

with $[s / 2]+1=2$ free parameters, as expected. Conservation (2.5) imposes $a_{0}=-2 a_{1}$ and $h=0$. The simple trace condition (2.7) imposes only $a_{0}=-2 a_{1}$.

Similarly, (2.13) becomes (condition (2.4) is empty for $s=3$ )

$$
\begin{equation*}
<\mathcal{J}_{\mu \nu \rho}^{(3)}(x) \mathcal{J}_{\alpha \beta \gamma}^{(3)}(0)>=\sum_{\text {symm }}\left[b_{1} \pi_{\mu \nu} \pi_{\alpha \beta} \pi_{\rho \gamma}+b_{0} \pi_{\mu \alpha} \pi_{\nu \beta} \pi_{\rho \gamma}\right] \frac{1}{|x|^{4+2 h}} . \tag{2.22}
\end{equation*}
$$

One can check that this correlator is conformal if and only if $3 b_{0}+5 b_{1}=0$ and $h=0$. Then the simple trace condition also holds. Vice versa, if the correlator is simply traceless and $h=0$, then it is conserved.

At $h=0$, when $a_{0}=-2 a_{1}$ and $3 b_{0}+5 b_{1}=0$ expressions (2.21) and (2.22) are proportional to each other. We write the final expression of the spin 3 projector as

$$
\prod_{\mu \nu \rho, \alpha \beta \gamma}^{(3)}=\sum_{\operatorname{symm}}\left[\pi_{\mu \alpha} \pi_{\nu \beta} \pi_{\rho \gamma}-\frac{3}{5} \pi_{\mu \nu} \pi_{\alpha \beta} \pi_{\rho \gamma}\right]
$$

The orthonormality property (2.17) reads

$$
\begin{equation*}
\frac{1}{\left(2^{8} \cdot 3^{2} \cdot 7\right)^{2}} \mathcal{I}_{\mu \nu \rho, \tau \zeta \varsigma}^{(3)}(x) \mathcal{I}_{\tau \zeta \varsigma, \alpha \beta \gamma}^{(3)}(x)=\Im_{\mu \nu \rho, \alpha \beta \gamma}^{(3)} \tag{2.23}
\end{equation*}
$$

and formulas (2.19)-(2.20) hold with our conventions.
We now work out the free-field expressions of the spin-3 currents. For a scalar field (which has to be complex, as in the case of the spin- 1 current) one has

$$
\begin{equation*}
\mathcal{J}_{\mu \nu \rho}^{(3)}=C^{i j} \sum_{\text {symm }}\left(3 \partial_{\mu} \varphi^{i} \partial_{\nu} \partial_{\rho} \varphi^{j}+\delta_{\mu \nu} \partial_{\alpha} \partial_{\rho} \varphi^{i} \partial_{\alpha} \varphi^{j}-\frac{1}{3} \varphi^{i} \partial_{\mu} \partial_{\nu} \partial_{\rho} \varphi^{j}\right) . \tag{2.24}
\end{equation*}
$$

This current is uniquely fixed by the additional condition (2.7). $C^{i j}$ is an antisymmetric matrix ( $=\varepsilon^{i j}$ for a complex scalar in real notation). The simple trace

$$
\begin{equation*}
\mathcal{J}_{\mu \mu \rho}^{(3)}=C^{i j}\left[\partial_{\rho} \varphi^{i} \square \varphi^{j}-\frac{1}{3} \varphi^{i} \partial_{\rho} \square \varphi^{j}\right], \tag{2.25}
\end{equation*}
$$

vanishes on shell.
Equivalently, the improved current (2.24) can be fixed by imposing conservation, double tracelessness and the requirement that $\mathcal{J}^{(s)}$ transforms correctly under coordinate inversion, see (2.8). Using $\varphi(x) \rightarrow|x|^{2} \varphi(x), \bar{\varphi}(x) \rightarrow|x|^{2} \bar{\varphi}(x)$ and $\partial_{\mu} \rightarrow|x|^{2} \mathcal{I}_{\mu \nu}(x) \partial_{\nu}$, one can check that, indeed, (2.24) satisfies (2.8) and that this would not be true without the improvement term. Similar checks can be repeated for the other currents that appear in this and in the next subsections, but we will not mention this fact anymore. A good exercise is to perform the check in the simplest cases, namely $s=1,2$, formulas (1.1) and (1.3).

For a spinor we have

$$
\mathcal{J}_{\mu \nu \rho}^{(3)}=\sum_{\operatorname{symm}}\left[\bar{\psi} \gamma_{\mu} \overleftrightarrow{\partial_{\nu}} \overleftrightarrow{\partial_{\rho}} \psi-\frac{1}{5} \pi_{\mu \nu}\left(\bar{\psi} \gamma_{\rho} \psi\right)\right]
$$

The second term is the improvement term and it is constructed via the spin- 1 current. There is also an axial spin-3 current

$$
\mathcal{A}_{\mu \nu \rho}^{(3)}=\sum_{\text {symm }}\left[\bar{\psi} \gamma_{5} \gamma_{\mu} \overleftrightarrow{\partial_{\nu}} \overleftrightarrow{\partial_{\rho}} \psi-\frac{1}{5} \pi_{\mu \nu}\left(\bar{\psi} \gamma_{5} \gamma_{\rho} \psi\right)\right]
$$

that will be useful in the applications.
For free vectors there is no issue of improvement and the spin- 3 current is axial:

$$
\begin{equation*}
\mathcal{A}_{\mu \nu \rho}^{(3)}=\frac{1}{3}\left[F_{\nu \alpha}^{+} \overleftrightarrow{\partial_{\mu}} F_{\alpha \rho}^{-}+F_{\mu \alpha}^{+} \overleftrightarrow{\partial_{\nu}} F_{\alpha \rho}^{-}+F_{\mu \alpha}^{+} \overleftrightarrow{\partial_{\rho}} F_{\alpha \nu}^{-}\right] \tag{2.26}
\end{equation*}
$$

where $F_{\mu \nu}^{ \pm}=\frac{1}{2}\left(F_{\mu \nu} \pm \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} F_{\rho \sigma}\right), F_{\mu \nu}=F_{\mu \nu}^{+}+F_{\mu \nu}^{-}$.

### 2.3 Spin 4.

One can repeat the same steps in the case of spin 4. (2.11) contains $[s / 2]+1=3$ free parameters and the anomalous dimension $h$. After imposing the simple trace condition only $h$ and the overall constant survive. Precisely,

$$
\begin{aligned}
<\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}(x) \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)}(0)>= & \frac{a}{|x|^{14+2 h}} \sum_{\text {symm }}\left[\delta_{\mu \nu} \delta_{\rho \sigma} \delta_{\alpha \beta} \delta_{\gamma \delta}-12 \delta_{\mu \nu} \delta_{\alpha \beta} J_{\rho \gamma}(x) J_{\sigma \delta}(x)\right. \\
& \left.-36 J_{\mu \alpha}(x) J_{\nu \beta}(x) J_{\rho \gamma}(x) J_{\sigma \delta}(x)\right] .
\end{aligned}
$$

Conservation holds for $h=0$. Formula (2.13) produces the desired projector $\prod^{(4)}$, after imposing the simple trace condition:

$$
\begin{equation*}
\prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}=\sum_{\mathrm{symm}}\left[\pi_{\mu \alpha} \pi_{\nu \beta} \pi_{\rho \gamma} \pi_{\sigma \delta}-\frac{6}{7} \pi_{\mu \nu} \pi_{\alpha \beta} \pi_{\rho \gamma} \pi_{\sigma \delta}+\frac{3}{35} \pi_{\mu \nu} \pi_{\rho \sigma} \pi_{\alpha \beta} \pi_{\gamma \delta}\right] . \tag{2.27}
\end{equation*}
$$

The orthonormality property (2.17) reads

$$
\begin{equation*}
\frac{1}{\left(2^{14} \cdot 3^{4}\right)^{2}} \mathcal{I}_{\mu \nu \rho \sigma, \zeta \varsigma \xi \kappa}^{(4)}(x) \mathcal{I}_{\zeta \varsigma \xi \kappa, \alpha \beta \gamma \delta}^{(4)}(x)=\Im_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)} \tag{2.28}
\end{equation*}
$$

and (2.19)-(2.20) hold.
For free scalar fields we have

$$
\begin{aligned}
\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}= & \frac{4}{35} \sum_{\text {symm }} 4 \varphi \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \varphi-64 \partial_{\mu} \varphi \partial_{\nu} \partial_{\rho} \partial_{\sigma} \varphi+72 \partial_{\mu} \partial_{\nu} \varphi \partial_{\rho} \partial_{\sigma} \varphi \\
& +12 \delta_{\mu \nu}\left(2 \partial_{\alpha} \varphi \partial_{\alpha} \partial_{\rho} \partial_{\sigma} \varphi-3 \partial_{\alpha} \partial_{\rho} \varphi \partial_{\alpha} \partial_{\sigma} \varphi\right)+3 \delta_{\mu \nu} \delta_{\rho \sigma} \partial_{\alpha} \partial_{\beta} \varphi \partial_{\alpha} \partial_{\beta} \varphi
\end{aligned}
$$

We can write the current in a more instructive, although less explicit, notation,

$$
\begin{aligned}
\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}= & \varphi \overleftrightarrow{\partial_{\mu}} \overleftrightarrow{\partial_{\nu}} \overleftrightarrow{\partial_{\rho}} \overleftrightarrow{\partial_{\sigma}} \varphi-\frac{1}{24}\left(\delta_{\mu \nu} \delta_{\rho \sigma}+\delta_{\mu \rho} \delta_{\nu \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}\right) \square^{2}\left(\varphi^{2}\right) \\
& -\frac{6}{7} \sum_{\text {symm }} \pi_{\mu \nu}\left[\varphi \overleftrightarrow{\partial_{\rho}} \overleftrightarrow{\partial_{\sigma}} \varphi-\frac{1}{3} \pi_{\rho \sigma}\left(\varphi^{2}\right)\right]-\frac{1}{5} P_{\mu \nu \rho \sigma}^{(4)}\left(\varphi^{2}\right)
\end{aligned}
$$

Let us explain the meaning of the various terms. The first term is the basic term for the construction of the current, since it certainly has a spin-4 content. However, it is not purely spin-4. One has to subtract the double trace (something which is done by the second term of the first line) and, more importantly, one has to subtract spurious lower spin terms. This is the role of the second line, which contains the improvement terms and enforces the simple trace condition. In the rest of the section we shall find several objects like these. In particular, the projector that we called $P_{\mu \nu \rho \sigma}^{(4)}$ is the unique forth-order polynomial in derivatives that satisfies (2.5) and (2.4) identically (but not the simple trace condition). Its explicit expression appears in the next section, formula (3.41), and it subtracts an unwanted spin-0 content. Observe that the first improvement term of the second line contains the stress-tensor $-\frac{1}{4}\left[\varphi \overleftrightarrow{\partial_{\rho}} \overleftrightarrow{\partial_{\sigma}} \varphi-\frac{1}{3} \pi_{\rho \sigma}\left(\varphi^{2}\right)\right]$ and therefore subtracts an undesired spin-2 content.

For a spinor,

$$
\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}=\sum_{\operatorname{symm}}\left[\bar{\psi} \gamma_{\mu} \overleftrightarrow{\partial_{\nu}} \overleftrightarrow{\partial_{\rho}} \overleftrightarrow{\partial_{\sigma}} \psi-\frac{3}{7} \pi_{\mu \nu}\left(\bar{\psi} \gamma_{\rho} \overleftrightarrow{\partial_{\sigma}} \psi\right)\right]
$$

and the improvement term is purely spin- 2 .
For a vector,

$$
\begin{equation*}
\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}=\sum_{\text {symm }}\left[F_{\rho \alpha}^{+} \overleftrightarrow{\partial_{\mu}} \overleftrightarrow{\partial_{\nu}} F_{\alpha \sigma}^{-}-\frac{1}{7} \pi_{\mu \nu}\left(F_{\rho \alpha}^{+} F_{\alpha \sigma}^{-}\right)\right] \tag{2.29}
\end{equation*}
$$

the improvement term is again spin-2.

### 2.4 Spin 5.

For the applications, we need also the expression of the fermionic spin- 5 axial current,

$$
\begin{equation*}
\mathcal{A}_{\mu \nu \rho \sigma \tau}^{(5)}=\sum_{\mathrm{symm}}\left[\bar{\psi} \gamma_{5} \gamma_{\mu} \overleftrightarrow{\partial_{\nu}} \overleftrightarrow{\partial_{\rho}} \overleftrightarrow{\partial_{\sigma}} \overleftrightarrow{\partial_{\tau}} \psi-\frac{2}{3} \pi_{\mu \nu} \mathcal{A}_{\rho \sigma \tau}^{(3)}-\frac{3}{35} \pi_{\mu \nu} \pi_{\rho \sigma} \mathcal{A}_{\tau}^{(1)}\right] \tag{2.30}
\end{equation*}
$$

and the same for the vector field,

$$
\begin{equation*}
\mathcal{A}_{\mu \nu \rho \sigma \tau}^{(5)}=\sum_{\mathrm{symm}}\left[F_{\mu \alpha}^{+} \overleftrightarrow{\partial_{\rho}} \overleftrightarrow{\partial_{\sigma}} \overleftrightarrow{\partial_{\tau}} F_{\alpha \nu}^{-}-\frac{1}{3} \pi_{\mu \nu}\left(F_{\rho \alpha}^{+} \overleftrightarrow{\partial_{\tau}} F_{\alpha \sigma}^{-}\right)\right] \tag{2.31}
\end{equation*}
$$

We see that there are spin- 3 and spin- 1 improvement terms. The spin- 5 projector operator is

$$
\prod_{\mu \nu \rho \sigma \tau, \alpha \beta \gamma \delta \varepsilon}^{(5)}=\sum_{\text {symm }}\left[\pi_{\mu \alpha} \pi_{\nu \beta} \pi_{\rho \gamma} \pi_{\sigma \delta}-\frac{10}{9} \pi_{\mu \nu} \pi_{\alpha \beta} \pi_{\rho \gamma} \pi_{\sigma \delta}+\frac{5}{21} \pi_{\mu \nu} \pi_{\rho \sigma} \pi_{\alpha \beta} \pi_{\gamma \delta}\right] \pi_{\tau \varepsilon}
$$

It satisfies

$$
\frac{1}{\left(2^{16} \cdot 3^{2} \cdot 5^{2} \cdot 11\right)^{2}} \mathcal{I}_{\mu \nu \rho \sigma \tau, \zeta \zeta \xi \kappa \iota}^{(5)}(x) \mathcal{I}_{\zeta \varsigma \xi \kappa \iota, \alpha \beta \gamma \delta \varepsilon}^{(5)}(x)=\Im_{\mu \nu \rho \sigma \tau, \alpha \beta \gamma \delta \varepsilon}^{(5)}
$$

and (2.19)-(2.20) have been verified.

## 3 Critical and off-critical properties.

In this section we study the general structure of higher spin two-point functions. In the first subsections we assume that the higher spin conservation law holds in nontrivially interacting quantum field theories subject to a renormalization group flow. This is equivalent to say that the theory in question has a higher spin flavor symmetry and it is a quite nontrivial requirement. As we have already remarked in section 2 and will discuss in more detail in section 4 this does not happen in ordinary renormalizable quantum field theories, where an explicit $h$-violation of the conservation condition is always present. However, the investigation of the present section is necessary as a preliminary step in order to proceed further.

In the last subsection we include the effect of the explicit $h$-violation. This result will be used in section 5 to define appropriate central charges and central functions. To anticipate the idea, we remark that something similar has been already done in ref. [4]. In absence of explicit violations of the conservation condition (2.5) the current-current two-point functions already define the desired higher spin central charges: this is studied in the present section and corresponds to the work done in section 2 of [4] for spin-2. Instead, in presence of explicit violations one has to construct the higher spin central charges by studying appropriate channels of the stress-tensor four-point function. These channels contain all possible higher spin currents. The construction of higher spin central charges is then similar to the procedure described in section 3 of ref. 4 for the spin- 0 channel.

It is clear therefore, that it is convenient to start the analysis of the current-current twopoint functions under the assumption that conservation and double tracelessness are preserved off-criticality. We also observe, and this should not be underestimated, that if consistent higher spin couplings do exist in some theories [11], this material will apply straightforwardly to those theories. Alternatively, it could constitute the basis for an axiomatic definition of these new theories.

We recall [4] that $<T T>$ contains two terms off-criticality,

$$
\begin{equation*}
<T_{\mu \nu}(x) T_{\rho \sigma}(0)>=\frac{1}{480 \pi^{4}} \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{c(g(x))}{x^{4}}\right)+\pi_{\mu \nu} \pi_{\rho \sigma}\left(\frac{f(\ln x \mu, g))}{x^{4}}\right) \tag{3.32}
\end{equation*}
$$

This formula will be generalized to arbitrary spin. Via a detailed algebraic analysis we classify the projectors that appear in the higher spin two-point functions. We do it explicitly for spin-3 and spin- 4 and then generalize the result to spin $s$. Then, we study the correlators from the point of view of the renormalization group in a generic renormalizable quantum field theory with higher spin flavor symmetries, as well as anomalies and the $h$-violation.

The higher spin flavor symmetry imposes (2.4) and (2.5), while (2.7) holds only at criticality. Therefore, off-criticality the current-current two-point function must contain more invariants. We observe that trivial higher spin currents can be constructs by suitably differentiating lowerspin currents. For example, the operator

$$
\begin{equation*}
\Delta_{\mu \nu \rho}^{(3,1)}=\pi_{\mu \nu} J_{\rho}+\pi_{\mu \rho} J_{\nu}+\pi_{\nu \rho} J_{\mu}-\frac{1}{2}\left(\delta_{\mu \nu} \partial_{\rho}+\delta_{\mu \rho} \partial_{\nu}+\delta_{\nu \rho} \partial_{\mu}\right) \partial_{\alpha} J_{\alpha} \tag{3.33}
\end{equation*}
$$

satisfies (2.5) identically (which means that no use of the equations of motion is necessary), for any spin- 1 current $J_{\mu}$, not necessarily conserved ( $J_{\mu}$ could be anomalous, like the axial current in QED, or even classically non-conserved, like the Konishi current [22] in supersymmetric theories with a superpotential). This property is precisely what defines it to be an "improvement term". Therefore, (3.33) is not a genuine spin-3 current. It is more appropriate to consider it as a higher-spin descendant of a lower-spin current. The simple trace condition (2.7) is not satisfied by $\Delta_{\mu \nu \rho}^{(3,1)}$.

It is natural to expect that there is some sort of conflict between $\mathcal{J}_{\mu \nu \rho}^{(3)}$ and $\Delta_{\mu \nu \rho}^{(3,1)}$. From the point of view of renormalization, this is nothing but the operator mixing. Let us assume, for simplicity, that $J_{\mu}$ is conserved. There is no loss of generality in this, since our purpose is just to classify the possible invariants. We write, from (1.1),

$$
\begin{equation*}
<J_{\mu}(x) J_{\nu}(0)>=\pi_{\mu \nu}\left(\frac{c_{1}[g(t)]}{|x|^{4}}\right) . \tag{3.34}
\end{equation*}
$$

The flavor central charge $c_{1}[g(t)]$ has well-defined non-vanishing UV and IR limits [2] and the correlator is conformal in these two limits. Then we have

$$
\begin{equation*}
<\Delta_{\mu \nu \rho}^{(3,1)}(x) \Delta_{\alpha \beta \gamma}^{(3,1)}(0)>=9 \sum_{\text {symm }} \pi_{\mu \nu} \pi_{\rho \gamma} \pi_{\alpha \beta}\left(\frac{c_{1}[g(t)]}{|x|^{4}}\right) . \tag{3.35}
\end{equation*}
$$

This correlator is non-vanishing and not conformal in the UV and IR limits, but it is related in a simple way to a conformal correlator, via the combined action of a certain number of derivatives on (3.34). As a further check of the mixing between spin 3 and spin 1 currents, we observe that (2.5) and $\partial_{\mu} J_{\mu}=0$ allow for the correlator

$$
\begin{equation*}
<J_{\alpha}(x) \mathcal{J}_{\mu \nu \rho}^{(3)}(0)>=\left(\pi_{\mu \nu} \pi_{\rho \alpha}+\pi_{\mu \rho} \pi_{\nu \alpha}+\pi_{\nu \rho} \pi_{\mu \alpha}\right)\left(\frac{f_{31}(t)}{|x|^{4}}\right) \tag{3.36}
\end{equation*}
$$

to be nonzero, in violation of the orthogonality formula (2.15).
We are now ready to write the two-point function off-criticality. One finds three independent invariants satisfying (2.5), namely

$$
\begin{align*}
<\mathcal{J}_{\mu \nu \rho}^{(3)}(x) \mathcal{J}_{\alpha \beta \gamma}^{(3)}(0)>=\quad & \prod_{\mu \nu \rho, \alpha \beta \gamma}^{(3)}\left(\frac{c_{3}(t)}{x^{4}}\right)+\sum_{\text {symm }} \pi_{\mu \nu} \pi_{\rho \gamma} \pi_{\alpha \beta}\left(\frac{c_{3,1}(t)}{x^{4}}\right) \\
& +P_{\mu \nu \rho}^{(3)} P_{\alpha \beta \gamma}^{(3)}\left(\frac{c_{3,0}(t)}{x^{4}}\right) \tag{3.37}
\end{align*}
$$

For the moment, we do not make the RG dependences of the functions $c$ explicit (in particular, do they depend only on the running coupling constant or not?). This issue will be one of our concerns later on.

The first term of (3.37) is the pure spin- 3 term, the middle term gives the spin- 3 -spin- 1 mixing, coming from (3.35), or, equivalently, (3.36). The third term is the spin3-spin-0 mixing. It is the only factorizable invariant. Indeed the differential operator

$$
P_{\mu \nu \rho}^{(3)}=\partial_{\mu} \partial_{\nu} \partial_{\rho}-\frac{1}{2} \square\left(\delta_{\mu \nu} \partial_{\rho}+\delta_{\mu \rho} \partial_{\nu}+\delta_{\nu \rho} \partial_{\mu}\right)
$$

satisfies (2.5) identically. We have explicitly checked that the three invariants in (3.37) exhaust all the possibilities.
(3.37) should be compared to the stress tensor two-point function, formula (3.32), which exhibits two invariants. One is the pure spin-2 invariant $\prod_{\mu \nu, \rho \sigma}^{(2)}$ and the other is the factorized invariant $\pi_{\mu \nu} \pi_{\rho \sigma}$ describing the spin-2-spin-0 mixing. The improvement term $\pi_{\mu \nu}\left(\varphi^{i} \varphi^{i}\right)$ is the analogue of (3.33).

Despite the renormalization mixing between $\mathcal{J}_{\mu \nu \rho}^{(3)}$ and its own improvement terms, one can isolate the function $c_{3}(t)$ by formally extracting the traceless part via a nonlocal projection, namely

$$
\begin{align*}
\hat{\mathcal{J}}_{\mu \nu \rho}^{(3)} \equiv & \mathcal{J}_{\mu \nu \rho}^{(3)}+\frac{1}{5 \square^{2}}\left[\square\left(\pi_{\mu \nu} \mathcal{J}_{\alpha \alpha \rho}^{(3)}+\pi_{\rho \nu} \mathcal{J}_{\alpha \alpha \mu}^{(3)}+\pi_{\mu \rho} \mathcal{J}_{\alpha \alpha \nu}^{(3)}\right)\right. \\
& \left.-\frac{1}{4} \square\left(\delta_{\mu \nu} \partial_{\rho}+\delta_{\mu \rho} \partial_{\nu}+\delta_{\nu \rho} \partial_{\mu}\right) \partial_{\beta} \mathcal{J}_{\alpha \alpha \beta}^{(3)}-\frac{1}{2} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\beta} \mathcal{J}_{\alpha \alpha \beta}^{(3)}\right] . \tag{3.38}
\end{align*}
$$

Indeed, $\hat{\mathcal{J}}_{\mu \nu \rho}^{(3)}$ satisfies both (2.5) and (2.7). It should be compared with the nonlocal traceless projection of the stress-energy tensor (see for example formula (1.8) of [4]). Therefore,

$$
<\hat{\mathcal{J}}_{\mu \nu \rho}^{(3)}(x) \hat{\mathcal{J}}_{\alpha \beta \gamma}^{(3)}(0)>=\prod_{\mu \nu \rho, \alpha \beta \gamma}^{(3)}\left(\frac{c_{3}(t)}{x^{4}}\right) .
$$

For the hypothetical theories with spin-3 flavor symmetry, this construction is sufficient to isolate the desired central function $c_{3}(t)$, that will depend only on the running coupling constant.

Before proceeding, it is useful to repeat the above analysis for spin- 4 currents.
In terms of a symmetric tensor $T_{\mu \nu}$ (not necessarily traceless and not necessarily conserved) we can always construct a spin-4 operator that satisfies (2.4) and (2.5) identically, but not (2.7):

$$
\begin{align*}
\Delta_{\mu \nu \rho \sigma}^{(4,2)}= & \pi_{\mu \nu} T_{\rho \sigma}+\pi_{\mu \rho} T_{\nu \sigma}+\pi_{\mu \sigma} T_{\nu \rho}+\pi_{\rho \sigma} T_{\mu \nu}+\pi_{\nu \sigma} T_{\mu \rho}+\pi_{\nu \rho} T_{\mu \sigma}  \tag{3.39}\\
& -\frac{1}{2}\left(\delta_{\mu \nu} \partial_{\rho} \partial_{\alpha} T_{\alpha \sigma}+\operatorname{perms}_{12}\right)+\frac{1}{12}\left(\delta_{\mu \nu} \delta_{\rho \sigma}+\delta_{\mu \rho} \delta_{\nu \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}\right)\left(5 \square T+4 \partial_{\alpha} \partial_{\beta} T_{\alpha \beta}\right),
\end{align*}
$$

where $T=T_{\mu \nu}$. For clarity we have specified via a subscript the total number of terms that one obtains after permutation (we shall use the same convention in other cases). Note that several improvement terms found in section 2 are of this type.

Secondly, one can construct an object with the same properties by means of the spin-1 current $J_{\mu}$, namely

$$
\begin{align*}
\Delta_{\mu \nu \rho \sigma}^{(4,1)}= & P_{\mu \nu \rho}^{(3)} J_{\sigma}+P_{\nu \rho \sigma}^{(3)} J_{\mu}+P_{\rho \sigma \mu}^{(3)} J_{\nu}+P_{\sigma \mu \nu}^{(3)} J_{\rho}-\frac{1}{3}\left(\delta_{\mu \nu} \partial_{\rho} \partial_{\sigma}+\mathrm{perms}_{6}\right) \partial_{\alpha} J_{\alpha}  \tag{3.40}\\
& +\frac{1}{2}\left(\delta_{\mu \nu} \delta_{\rho \sigma}+\delta_{\mu \rho} \delta_{\nu \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}\right) \square \partial_{\alpha} J_{\alpha} .
\end{align*}
$$

Finally, the only factorizable invariant is generated by the forth-order differential operator

$$
\begin{align*}
P_{\mu \nu \rho \sigma}^{(4)}= & \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma}-\frac{1}{3} \square\left(\partial_{\mu} \partial_{\nu} \delta_{\rho \sigma}+\partial_{\mu} \partial_{\rho} \delta_{\nu \sigma}+\partial_{\mu} \partial_{\sigma} \delta_{\nu \rho}+\partial_{\nu} \partial_{\rho} \delta_{\mu \sigma}+\partial_{\nu} \partial_{\sigma} \delta_{\mu \rho}+\partial_{\rho} \partial_{\sigma} \delta_{\mu \nu}\right) \\
& +\frac{1}{8} \square^{2}\left(\delta_{\mu \nu} \delta_{\rho \sigma}+\delta_{\mu \rho} \delta_{\nu \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}\right), \tag{3.41}
\end{align*}
$$

the unique one that satisfies identically (2.4) and (2.5) and represents the mixing between spin 4 and spin zero. For example, $\Delta_{\mu \nu \rho \sigma}^{(4,0)}=P_{\mu \nu \rho \sigma}^{(4)}\left[\varphi^{2}\right]$ is a spin-0 local operator mixing with $\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}$, $\varphi$ being a scalar field.

Off-criticality, the two-point function $\left\langle\mathcal{J}^{(4)} \mathcal{J}^{(4)}\right\rangle$ contains four invariants.
i) One invariant is the pure spin-4 invariant $\prod^{(4)}$ of (2.27).
ii) A spin-2 invariant comes from $<\Delta^{(4,2)} \Delta^{(4,2)}>$ and has the form $\pi \prod^{(2)} \pi$;
iii) $<\Delta^{(4,1)} \Delta^{(4,1)}>$ produces a spin-1 invariant the form $P^{(3)} \pi P^{(3)}$.
iv) Finally, there is the spin-0 invariant $\left(P^{(4)}\right)^{2}$.

Correlators of the type $\left\langle\Delta^{(4,2)} \Delta^{(4,1)}\right\rangle,\left\langle\mathcal{J}^{(4)} \Delta^{(4,1)}\right\rangle$ and so on, do not produce anything new. For example, $<T_{\mu \nu} T_{\rho \sigma}>$ contains also a spin-0 invariant of the form $\pi \pi$, as we know. Therefore, $<\Delta^{(4,2)} \Delta^{(4,2)}>$ contributes itself a spin-0 invariant of the form $\pi \pi \pi \pi+\pi \pi \square^{2} \delta \delta$. One can verify that this spin-0 invariant is the same as $\left(P^{(4)}\right)^{2}$.

The claimed four invariants are clearly independent. We have carefully checked that they exhaust all possibilities. In summary, the most general spin-4 two-point function in a spin-4 flavor symmetric theory is

$$
\begin{align*}
<\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}(x) \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)}(0)> & =\prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\left(\frac{c_{4}(t)}{x^{4}}\right)+\sum_{\text {symm }} \pi_{\mu \nu} \prod_{\rho \sigma, \alpha \beta}^{(2)} \pi_{\gamma \delta}\left(\frac{c_{4,2}(t)}{x^{4}}\right) \\
& +\sum_{\text {symm }} P_{\mu \nu \rho}^{(3)} \pi_{\sigma \alpha} P_{\beta \gamma \delta}^{(3)}\left(\frac{c_{4,1}(t)}{x^{4}}\right)+P_{\mu \nu \rho \sigma}^{(4)} P_{\alpha \beta \gamma \delta}^{(4)}\left(\frac{c_{4,0}(t)}{x^{4}}\right) . \tag{3.42}
\end{align*}
$$

These and other formulas in our paper might appear rather implicit, but after some experience one easily recognizes the structure of the terms. For example, the fourth term of (3.42) is completely factorized. The third and second term are partially factorized (which means that each term in the sum is factorized, but the sum itself is not factorized) - in particular, the third term in (3.42) is, so to speak, $3 / 4$-factorized, while the second term is $1 / 2$-factorized. Finally, the first term exhibits no factorization. These structures reflect the spin content of each invariant.

It is important to show that there exists a unique nonlocal projection isolating the pure spin-4 central function $c_{4}(t)$ from the mixing functions $c_{4, i}(t), i \leq s-2$, like in (3.38). The result is

$$
\begin{align*}
\hat{\mathcal{J}}_{\mu \nu \rho \sigma}^{(4)}= & \mathcal{J}_{\mu \nu \rho \sigma}^{(4)}+\frac{1}{7} \frac{1}{\square}\left[\pi_{\mu \nu} \mathcal{J}_{\alpha \alpha \rho \sigma}^{(4)}+\mathrm{perms}_{6}\right] \\
& -\frac{2}{21} \frac{1}{\square^{2}}\left[\partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\alpha} \mathcal{J}_{\sigma \alpha \beta \beta}^{(4)}+\operatorname{perms}_{4}+\frac{1}{4} \square \delta_{\mu \nu} \partial_{\rho} \partial_{\alpha} \mathcal{J}_{\sigma \alpha \beta \beta}^{(4)}+\mathrm{perms}_{12}\right] \\
& +\frac{2}{105} \frac{1}{\square^{2}}\left[\delta_{\mu \nu} \partial_{\rho} \partial_{\sigma}+\operatorname{perms}_{6}+\frac{1}{4} \square\left(\delta_{\mu \nu} \delta_{\rho \sigma}+\delta_{\mu \rho} \delta_{\nu \sigma}+\delta_{\mu \sigma} \delta_{\nu \rho}\right)\right] \partial_{\alpha} \partial_{\beta} \mathcal{J}_{\alpha \beta \gamma \gamma}^{(4)} \\
& +\frac{4}{105} \frac{1}{\square^{3}} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \partial_{\alpha} \partial_{\beta} \mathcal{J}_{\alpha \beta \gamma \gamma}^{(4)} . \tag{3.43}
\end{align*}
$$

Finally, we can throw away all sorts of complicacies by writing the projected two-point function

$$
<\hat{\mathcal{J}}_{\mu \nu \rho \sigma}^{(4)}(x) \hat{\mathcal{J}}_{\alpha \beta \gamma \delta}^{(4)}(0)>=\prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\left(\frac{c_{4}(t)}{x^{4}}\right),
$$

which is what we wanted to arrive at.
The analysis can be easily generalized to spin $s$. We present here the result. The most general two-point function has the form

$$
\begin{equation*}
<\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}(x) \mathcal{J}_{\nu_{1} \cdots \nu_{s}}^{(s)}(0)>=\prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}\left(\frac{c_{s}(t)}{x^{4}}\right)+\sum_{s^{\prime}=0}^{s-2} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{\left(s, s^{\prime}\right)}\left(\frac{c_{s, s^{\prime}}(t)}{x^{4}}\right), \tag{3.44}
\end{equation*}
$$

where the sum runs over the spin mixings. There is no $s-(s-1)$ spin mixing and a unique $s-s^{\prime}$ spin mixing for any $s^{\prime}=s-2, \ldots, 0$. There exists a unique nonlocal projection $\hat{\mathcal{J}}^{(s)}$ isolating the pure spin- $s$ part, such that

$$
\begin{equation*}
<\hat{\mathcal{J}}_{\mu_{1} \cdots \mu_{s}}^{(s)}(x) \hat{\mathcal{J}}_{\nu_{1} \cdots \nu_{s}}^{(s)}(0)>=\prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}\left(\frac{c_{s}(t)}{x^{4}}\right) \tag{3.45}
\end{equation*}
$$

At criticality $\hat{\mathcal{J}}_{\mu_{1} \cdots \mu_{s}}^{(s)}=\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}$ and $c_{s}(t)=c_{s *}$ is a constant. (3.45) reduces then to (2.12).

### 3.1 Renormalization group analysis.

The relationship between bare and renormalized operators mixes the higher spin current with all its improvement terms and is therefore of the form

$$
\begin{equation*}
\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s) R}=\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s) B}+\sum_{s^{\prime}=0}^{s-2} A_{s, s^{\prime}} \Delta_{\mu_{1} \cdots \mu_{s}}^{\left(s, s s^{\prime}\right) B} \tag{3.46}
\end{equation*}
$$

We recall that we are still assuming that the spin-s tensor current $\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}$ satisfies (2.4) and (2.5) on shell and the above formula follows from this assumption.

In particular, the coefficient in front of $\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s) B}$ is one. Indeed, conditions (2.4) and (2.5) are assumed to be violated by terms proportional to the field equations: terms of this type are finite operators, this being a well-known classical result (see for example the book by Collins [17]). Therefore, the renormalization constants of the operators

$$
\mathcal{J}^{(s)}{ }^{\mu}{ }^{\mu} \nu \quad \mu_{5} \cdots \mu_{s}
$$

and

$$
\partial^{\mu} \mathcal{J}_{\mu \mu_{2} \cdots \mu_{s}}^{(s)}-\frac{1}{2(s-1)(s-2)} \sum_{i<j=2}^{s} \delta_{\mu_{i} \mu_{j}} \partial^{\mu} \mathcal{J}^{(s)}{ }_{\mu \cdots \mu_{i-1} \alpha \mu_{i+1} \cdots \mu_{j-1} \alpha \mu_{j+1} \cdots \mu_{s}}
$$

equal one, as we wanted to prove. This does not imply that $\mathcal{J}_{\mu_{1} \ldots \mu_{s}}^{(s)}$ is itself finite, since (2.4) and (2.5) are identically satisfied (i.e. satisfied off-shell) by certain operators that we call $\Delta_{\mu_{1} \ldots \mu_{s}}^{\left(s, s s^{\prime}\right.}$. The operators $\Delta_{\mu_{1} \cdots \mu_{s}}^{\left(s, s^{\prime}\right.}$ are constructed iteratively via the lower spin tensor currents $\mathcal{J}_{\mu_{1} \cdots \mu s^{\prime}}^{\left(s^{\prime}\right)}$, $s^{\prime}=s-2, \ldots, 0$, (or in general non-conserved lower spin operators) by acting with derivatives in various ways, like in the formulas that we have worked out explicitly for $s=3$ and $s=4$, (3.33), (3.40), (3.41) and so on. Moreover, as we have proved, $\Delta_{\mu_{1} \cdots \mu_{s}}^{\left(s, s^{\prime}\right)}$ are the only operators with this property and therefore they renormalize among themselves. $A_{s, s^{\prime}}$ are the appropriate
renormalization mixing constants. It follows from our assumptions that the set of operators formed by $\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s) B}$ and $\Delta_{\mu_{1} \cdots \mu_{s}}^{\left(s, s^{\prime}\right) B}, s^{\prime}=s-2, \ldots, 0$, is closed under renormalization mixing.

Equations (3.46) assure that the $s \times s$ renormalization matrix $Z_{i j}, i, j=s, s-2, \ldots, 0$, of the set of operators $\left(\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s) B}, \Delta_{\mu_{1} \cdots \mu_{s}}^{\left(s, s^{\prime}\right) B}\right)$ is triangular and $Z_{11}=1$. Then the analysis of section 1 of ref. [4] can be repeated straightforwardly to prove that the functions $c_{s}$ depend just on the running coupling constant, $c_{s}(t)=c_{s}[g(t)]$, and therefore are good central functions under the assumption of higher spin flavor symmetry.

### 3.2 Trace anomaly and higher spin anomalies.

Taking the scale derivative $\mu \partial / \partial \mu$ of (2.12) one gets the expression for the integrated trace anomaly. The integrated trace anomaly reads at criticality

$$
\begin{equation*}
\int \Theta=\sum_{s=1}^{\infty} c_{s *} \int h_{s}^{\mu_{1} \cdots \mu_{s}} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)} h_{s}^{\nu_{1} \cdots \nu_{s}}+\mathcal{O}\left(h^{3}\right) \tag{3.47}
\end{equation*}
$$

Note that the $c_{s}$-term is the higher spin analogue of the square of the Weyl tensor for gravity. It is the conformal term of the higher derivative action for higher spin fields. At intermediate energies one has instead

$$
\begin{equation*}
\int \Theta=\sum_{s=1}^{\infty} \int h_{s}^{\mu_{1} \cdots \mu_{s}}\left[\tilde{c}_{s}(g) \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}+\sum_{s^{\prime}=0}^{s-2} \tilde{c}_{s, s^{\prime}}(g) \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{\left(s, s^{\prime}\right)}\right] h_{s}^{\nu_{1} \cdots \nu_{s}}+\mathcal{O}\left(h^{3}\right) \tag{3.48}
\end{equation*}
$$

The relationship between the tilded functions of this formula to the untilded ones of the previous subsections can be found in section 1 of ref. [4]. The complete higher derivative action for higher spin fields is also encoded in (3.44) and reads:

$$
\begin{equation*}
\mathcal{S}_{s}=\int h_{s}^{\mu_{1} \cdots \mu_{s}}\left[\alpha_{s} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}+\sum_{s^{\prime}=0}^{s-2} \alpha_{s, s^{\prime}} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{\left(s, s^{\prime}\right)}\right] h_{s}^{\nu_{1} \cdots \nu_{s}}+\mathcal{O}\left(h^{3}\right), \tag{3.49}
\end{equation*}
$$

$\alpha_{s}, \alpha_{s, s^{\prime}}$ being the coupling constants. The higher orders in $h$ should be determined by the gauge invariance (2.6) (itself suitably modified by higher orders in $h$ ) and are encoded in the divergent parts of the $n$-point correlators

$$
\begin{equation*}
<\prod_{i=1}^{n} \mathcal{J}_{\mu_{1}^{i} \cdots \mu_{s}^{i}}^{(s)}\left(x_{i}\right)> \tag{3.50}
\end{equation*}
$$

The higher derivative higher spin action is expected to be power-counting renormalizable and therefore generalize the familiar higher derivative action for quantum gravity,

$$
\int \sqrt{g}\left(\alpha R^{2}+\beta R_{\mu \nu} R^{\mu \nu}\right)
$$

to which the lower derivative action $\left(\int \sqrt{g} R\right.$ for gravity, the action of ref.s [12, 13] for higher spin fields) can be added without spoiling renormalizability.

The unintegrated trace anomaly can contain topological terms, at least for even $s$. In particular at criticality we have

$$
\begin{align*}
\Theta= & \sum_{s=1}^{\infty} c_{s *} h_{s}^{\mu_{1} \cdots \mu_{s}} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)} h_{s}^{\nu_{1} \cdots \nu_{s}}  \tag{3.51}\\
& +\sum_{s=\text { even }}^{\infty} a_{s *} \partial_{\mu_{1} \cdots \mu_{s}}\left[h_{s}^{\nu_{1} \cdots \nu_{s}} \partial_{\rho_{1} \cdots \rho_{s}} h_{s}^{\sigma_{1} \cdots \sigma_{s}}\right] \prod_{i=1}^{s}\left(\varepsilon_{\mu_{i} \nu_{i} \rho_{i} \sigma_{i}}\right)+\mathcal{O}\left(h^{3}\right)
\end{align*}
$$

where $\partial_{\mu_{1} \cdots \mu_{s}}=\partial_{\mu_{1}} \cdots \partial_{\mu_{s}}$. Off-criticality the topological invariant is multiplied by a function $a_{s}(g)$. Therefore the complete unintegrated trace anomaly contains $s+1$ terms in total, those of (3.49) plus the topological invariant appearing in (3.51). For $s$ odd the topological invariant that we have written in (3.51) violates parity and appears in the axial anomalies.

The ones that we have described are all the allowed higher derivative invariants, as we now prove. Summarizing, they are $s+1$ in total, one being topological, one being conformal, the other $s-1$ invariants disappearing at criticality.

To complete the proof that we have listed all possible invariants for the trace anomaly (the analysis of the previous subsection suffices to prove this fact up to total derivatives), we need to show that there exists no other independent topological term. We can proceed as follows. Let us recall that there exists a "Riemann" tensor for higher spin fields. To our knowledge, the first expression of this tensor appeared in [13]. Nevertheless, here we write the form of ref. [10], more explicit for our purpose, namely

$$
R_{\mu_{1} \nu_{1} \cdots \mu_{s} \nu_{s}}=\partial_{\alpha_{1} \cdots \alpha_{s}} h_{s}^{\beta_{1} \cdots \beta_{s}} \prod_{i=1}^{s}\left(\delta_{\mu_{i} \alpha_{i}} \delta_{\nu_{i} \beta_{i}-} \delta_{\mu_{i} \beta_{i}} \delta_{\nu_{i} \alpha_{i}}\right)
$$

We can introduce the "Weyl" tensor by extracting all traces from $R_{\mu_{1} \nu_{1} \cdots \mu_{s} \nu_{s}}$ and we can define $s$ "Ricci" tensors by tracing $R_{\mu_{1} \nu_{1} \cdots \mu_{s} \nu_{s}}$ in all possible ways. In total, one counts $s+1$ independent tensors. Squaring these tensors one reproduces exactly $s+1$ higher derivative terms. These are the $s$ terms of the higher derivative action (3.49) and the topological term contained in (3.51). Indeed, expanding the topological term of (3.51) by using the properties of the $\varepsilon$-tensor, one can re-express it as a linear combination of the squares of the "Weyl" tensor and the $s$ "Ricci" tensors (this is the "Gauss-Bonnet" theorem for higher spin fields).

We do not re-derive our classification of invariants following this procedure, since we have already obtained it in a different manner. We leave this completion as an exercise for the reader. It is enough to note that the counting proves that all $s+1$ possibilities have been accounted for, as claimed. Topological terms linear in the curvature (like $\square R$ for spin-2) are not listed.

We conclude that there exist precisely one central charge of type $c_{s}$ and one central charge of type $a_{s}$ for each spin $s$. The quantities $a_{s}$ are expected to satisfy irreversibility properties (" $a$-theorems") analogous to the property satisfied by the coefficient of the Euler density in the expression of the trace anomaly in external gravity (the quantity $a_{2}[2]$ ). These $a$-theorems are expressed by inequalities $a_{S U V}-a_{s I R} \geq 0$.

The properties of the quantities $a_{s}$ will be studied in detail in a forthcoming publication [16], where other higher spin anomalies will also be treated quantitatively. We observe, in this
context, that the simple trace $\mathcal{J}_{\mu_{1} \ldots \mu_{s-2} \alpha \alpha}^{(s)}$ of a spin- $s$ current $\mathcal{J}^{(s)}$ (the role played by $\Theta$ for the stress-tensor) is a dimension $2+s$ simply traceless operator, not conserved, with $s-2$ indices and therefore $(s-1)^{2}$ components in total. It can be decomposed into a sum

$$
\begin{equation*}
\mathcal{J}_{\mu_{1} \cdots \mu_{s-2} \alpha \alpha}^{(s)}=\sum_{s^{\prime}=0}^{s-2} \Theta_{\mu_{1} \cdots \mu_{s-2}}^{\left(s, s^{\prime}\right)}, \tag{3.52}
\end{equation*}
$$

each term being simply traceless and having a definite spin- $s^{\prime}$ content. With this we mean that $\Theta_{\mu_{1} \cdots \mu_{s-2}}^{\left(s, s s^{\prime}\right)}$ is an operator with the same quantum numbers as the traces $\Delta_{\mu_{1} \cdots \mu_{s-2 \alpha \alpha}}^{\left(s, s{ }^{\prime}\right)}$ of the improvement operators. For example, for $s=2$, we have $s^{\prime}=0$ and $\Theta$ has indeed the same quantum numbers as the trace of $\pi_{\mu \nu} \varphi^{2}$, the unique improvement term for the stresstensor. Similarly, the two-point functions $<\Theta_{\mu_{1} \cdots \mu_{s-2}}^{\left(s, s^{\prime}\right)}(x) \Theta_{\nu_{1} \cdots \nu_{s-2}}^{\left(s, s^{\prime}\right)}(0)>$ are of the same form as $<\Delta_{\mu_{1} \cdots \mu_{s-2 \alpha \alpha}}^{\left(s, s^{\prime}\right)}(x) \Delta_{\mu_{1} \cdots \mu_{s-2 \beta \beta}}^{\left(s, s^{\prime}\right.}(0)>$.

Decomposition (3.52) can be proved by observing that taking the simple trace of the twopoint function (3.44) one remains only with the claimed mixing terms. Each of them describes the mixing with a unique lower spin operator, with spin $s^{\prime}$ between 0 and $s-2$. The counting indeed matches: $2 s^{\prime}+1$ components for each $s^{\prime}$ make in total $\sum_{s^{\prime}=0}^{s-2}\left(2 s^{\prime}+1\right)=(s-1)^{2}$ components.

This concludes the discussion of higher spin currents in theories with higher spin flavor symmetries. Higher spin currents, however, are useful tools to study certain aspects of ordinary renormalizable quantum field theories, where no flavor higher spin symmetry is present, and indeed the major purpose of the paper is to improve our knowledge of such aspects. The results of the present section will have to be combined with other results in order to achieve this goal.

### 3.3 Treatment of the $h$-violation.

The $h$-violation is responsible for one additional term in formula (3.44). We perform a simple counting to explain this fact. We know that a symmetric $k$-indexed tensor $S_{\mu_{1}, \ldots \mu_{k}}$ has a number of components equal to the number of decompositions of the integer $k$ into the sum of four integer numbers, $k=n_{1}+n_{2}+n_{3}+n_{4}$. This is the number of ways to assign the values $1,2,3,4$ to the indices $\mu_{1}, \ldots \mu_{k}$ : there will be $n_{1} 1$ 's, $n_{2} 2$ 's, and so on. It follows from a simple calculation that a symmetric traceless $k$-indexed tensor has $(k+1)^{2}$ components. A doubly traceless $s$-indexed tensor $\mathcal{J}^{(s)}$ is the sum of an $s$-indexed traceless tensor and an $(s-2)$-indexed traceless tensor and therefore has $(s+1)^{2}+(s-1)^{2}=2\left(s^{2}+1\right)$ components, which we can write as

$$
2\left(s^{2}+1\right)=(s+1)^{2}+(s-1)^{2}=\sum_{s^{\prime}=0}^{s}\left(2 s^{\prime}+1\right)+\sum_{s^{\prime}=0}^{s-2}\left(2 s^{\prime}+1\right)
$$

in order to exhibit the spin decomposition. The divergence operator (2.5) is an $(s-1)$-indexed traceless tensor, therefore it has $s^{2}$ components. The difference is

$$
\begin{equation*}
s^{2}+2=(2 s+1)+\sum_{s^{\prime}=0}^{s-2}\left(2 s^{\prime}+1\right) \tag{3.53}
\end{equation*}
$$

This is precisely the spin decomposition of formula (3.44): a spin-s term, with projector $\prod^{(s)}$, plus mixing terms of the type $\left(s, s^{\prime}\right)$ for $s^{\prime}=s-2, \ldots 0$, with projectors $\prod^{\left(s, s^{\prime}\right)}$.

So far we have re-described the results of the previous subsections. Now we proceed differently. We do not impose the conservation condition, which indeed does not hold in general interacting theories, but we nevertheless enforce the simple trace condition (2.7). We can always do this since this condition is purely algebraic. We remain with

$$
\begin{equation*}
(s+1)^{2}=\sum_{s^{\prime}=0}^{s}\left(2 s^{\prime}+1\right)=(2 s+1)+(2 s-1)+\sum_{s^{\prime}=0}^{s-2}\left(2 s^{\prime}+1\right) \tag{3.54}
\end{equation*}
$$

components. We have one component more than in (3.53), precisely the spin $s-1$ term, for which we could not find a projector $\Pi^{(s, s-1)}$. Now we understand the deep reason of this fact: if such a projector existed then we would have automatically conservation.
(3.44) will contain precisely one additional term. We observe that two different types of violations of the conservation condition appear in (3.54): the terms of the sum $\sum_{s^{\prime}=0}^{s-2}\left(2 s^{\prime}+\right.$ 1) are the violations that we call anomalous, since they can be moved to the simple trace condition (via finite local counterterms) and be described by the projectors $\prod^{\left(s, s^{\prime}\right)}$; the term $(2 s-1)$ is the explicit violation, responsible for the anomalous dimension $h$. This term is not anomalous, because it cannot be moved to the simple trace condition, since there exists no projector $\prod^{(s, s-1)}$.

Anomalous terms are moved from the conservation condition to the trace condition or vice versa by the simple redefinition

$$
\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)} \rightarrow \mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}+\sum_{\text {symm }} \delta_{\mu_{s-1} \mu_{s}} \Theta_{\mu_{1} \cdots \mu_{s-2}}
$$

where $\Theta_{\mu_{1} \cdots \mu_{s-2}}$ is traceless (compare this with formula (3.52)). It is clear that these additions cannot have spin higher than $s-2$ and accordingly there is no possibility to cancel the $h$ violation.

So, let us move the anomalous violations to the simple trace condition. The terms $(2 s+1)$ and $(2 s-1)$ can be described by a sum of the right hand sides of (2.12) and (2.16). We relabel the functions and conclude that the most general two-point correlator reads

$$
\begin{equation*}
<\mathcal{J}_{\mu_{1} \ldots \mu_{s}}^{(s)}(x) \mathcal{J}_{\nu_{1} \cdots \nu_{s}}^{(s)}(0)>=\frac{c_{s}(t)}{|x|^{4+2 s}} \mathcal{I}_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{(s)}(x)+\sum_{s^{\prime}=0}^{s-1} \prod_{\mu_{1} \cdots \mu_{s}, \nu_{1} \cdots \nu_{s}}^{\left(s, s^{\prime}\right)}\left(\frac{c_{s, s^{\prime}}(t)}{x^{4}}\right) . \tag{3.55}
\end{equation*}
$$

Here $\prod^{(s, s-1)}$ is defined to be $\Pi^{(s)}$, with a certain abuse of notation. The first term (3.55) is the one that survives at criticality:

$$
c_{S}(t) \sim \frac{c_{S *}}{|x|^{2 h_{s}}} .
$$

The other functions vanish at criticality. Precisely, this means

$$
\begin{equation*}
\frac{c_{s, s^{\prime}}(t)}{c_{s}(t)} \rightarrow 0, \quad s^{\prime}=s-1, \ldots, 0 \tag{3.56}
\end{equation*}
$$

If $h_{s}=0$ (free fixed point) then $\prod^{(s)}\left(1 /|x|^{4}\right)$ is the same as $\mathcal{I}^{(s)}(x) /|x|^{4}$. In this case the function $c_{s, s-1}$ is defined to satisfy property (3.56), in order to avoid overcounting. This is the sense in which $c_{s, s-1}$ is "declassed" to describe the spin- $s$-spin- $(s-1)$ mixing.

In order to familiarize with (3.55), let us recall that in the simplest case, $s=1$, formula (3.55) becomes

$$
<\mathcal{J}_{\mu}(x) \mathcal{J}_{\nu}(0)>=c_{1}(t) \frac{\mathcal{I}_{\mu \nu}(x)}{|x|^{6}}+\pi_{\mu \nu}\left(\frac{c_{1,0}(t)}{x^{4}}\right)
$$

which can be understood as a decomposition into the sum of the "conformal" part and the "conserved" part of the correlator. This formula can be used, for example, for the axial-current two-point function.

One of the by-products of the investigation of the present paper is that higher spin tensor currents and the axial current fall in the same class. Indeed, axial currents and higher spin currents have various properties in common: the violations of their conservation conditions vanish only in the free fixed point; nonvanishing anomalous dimensions survive in the interacting fixed points (see [5]); they appear in the same context, in particular in the TT operator product expansion (see next section).

We remark that the usual anomalous violation of the axial-current conservation, $\partial \mu j_{5} \mu=$ $-\frac{e^{2}}{16 \pi^{2}} F \tilde{F}$, is called in our language "explicit violation", since it does not vanish at a generic fixed point, but only at a free fixed point. Instead, the usual trace anomaly of the stress-tensor, $\Theta=-\frac{\beta}{4 \alpha} F^{2}$, is of the type that we call "anomalous", since it vanishes at both fixed points.

We have worked out free-field formulas and classified improvement terms, projectors, twopoint functions off-criticality, as well as anomalous and explicit violations of the trace and conservation conditions. But we have not given the concrete expressions of the currents offcriticality. There is a certain amount of arbitrariness (for example, how to weight the scalar, vector and spinor contributions?). We will see in the next sections that the relevant higher spin currents are defined by the TT OPE, but before proceeding further, let us make some final observation.

One would like to define an interpolating ("primary") current such that its UV and IR limits are conformal, which amounts to separate the good current from its own improvement terms. Indeed, these are not conformal operators, but descendants of conformal operators. For example, a redefinition like

$$
\begin{equation*}
\mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)} \rightarrow \mathcal{J}_{\mu_{1} \cdots \mu_{s}}^{(s)}+\sum_{s^{\prime}=0}^{s-2} h_{s, s^{\prime}}(g) \Delta_{\mu_{1} \cdots \mu_{s}}^{\left(s, s^{\prime}\right)}, \tag{3.57}
\end{equation*}
$$

where $h_{s, s^{\prime}}(g)$ are arbitrary functions of the coupling constants, finite and vanishing in the free-field limit $\left(h_{s, s^{\prime}}(0)=0\right)$, is always allowed. It does not change the free (UV) current, uniquely determined in section 2 . However, it does change the IR limit of $\mathcal{J}^{(s)}$ if $h_{s, s^{\prime}}\left(g_{I R}\right) \neq 0$. The correlators of $\Delta_{\mu_{1} \cdots \mu_{s}}^{\left(s, s^{\prime}\right)}$ are not conformal (see for example (3.35)) and so one can spoil conformality of the two-point function $<\mathcal{J}^{(s)} \mathcal{J}^{(s)}>$ in the IR, in particular property (3.56). Only the precise knowledge of the dynamics of the theory can fix (3.57) so that it interpolates appropriately between two conformal currents. For spin-1, on the other hand, where there is no
improvement term, the current (vector or axial) is uniquely fixed. For higher spin the ambiguity survives even in presence of a higher spin symmetry. The stress tensor, for example, can be redefined as $T_{\mu \nu} \rightarrow T_{\mu \nu}+h(g) \pi_{\mu \nu}\left[\varphi^{2}\right]$ and the trace as $\Theta \rightarrow \Theta-3 h(g) \square \varphi^{2}$. This corresponds to a redefinition of the action in external gravity, $\mathcal{L} \rightarrow \mathcal{L}+h(g) R \varphi^{2}$, via a finite counterterm. The additional vertex generates divergent counterterms of the type $R^{2}$. As a consequence, the trace anomaly in external gravity has a new coefficient in front of $R^{2}$. We recall that this coefficient is the function that we called $\tilde{c}_{2,0}(g)$ (or $c_{2,0}(g)$ at the level of the two-point function $<T T>$ ). In conclusion, redefinition (3.57) affects the functions $c_{s, s^{\prime}}(g), s^{\prime}<s$, spoiling eventually (3.56), but leaving the relevant central function $c_{s}(g)$ unchanged. This is a "mild" effect and can for most purposes be neglected. One interpolates between UV and IR with any preferred form (3.57) of the current. If the IR limit of $\left\langle\mathcal{J}^{(s)} \mathcal{J}^{(s)}\right\rangle$, formula (3.55), is not conformal, one extracts the conformal part either via the nonlocal projection $\mathcal{J}^{(s)} \rightarrow \hat{\mathcal{J}}^{(s)}$ or via the spin decomposition (3.55).

## 4 The $T T$ operator product expansion.

Higher spin currents appear in the operator product expansion of the stress energy tensor. This is true in any quantum field theory in more that two dimensions. Only in two dimensions the $T T$ OPE closes with the identity operator and $T$ itself, several other operators appearing in general dimension [18]. In paper [5], the first of our series, it was stressed that this mixing can have interesting applications, in interacting conformal field theories as well as in quantum field theory out of the critical points. It is therefore mandatory to deepen our knowledge on this issue. In the present section we work out the $T T$ OPE for free fields in complete detail and use these results to write down the most general form of the $T T$ OPE in interacting conformal field theories. In the next section we shall combine the result of the present and previous sections to achieve our goal and define higher spin central charges and central functions. We focus on symmetric tensors in our analysis of the OPE.

Our results are consistent with those found in earlier works, in particular in the context of deep inelastic scattering [7], where the operator product light-cone expansion of two electromagnetic currents was extensively studied. Comparison with the results of ref.s [7, 19] makes it apparent that the ligh-cone and Euclidean expansions are very different, in the sense that the terms are organized in a different way. For example, infinitely many operators, including their descendants, have the same light-cone singularity in the free-field limit and different singularities in the Euclidean framework. The space-time structure of the singular terms is correspondingly rearranged. The TT OPE was not previously considered in detail.

| Singularity | $1 / x^{8}$ | $1 / x^{7}$ | $1 / x^{6}$ | $1 / x^{5}$ | $1 / x^{4}$ | $1 / x^{3}$ | $1 / x^{2}$ | $1 / x$ | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Spin | - | - | 0 | 1 | 2 | 3 | 4 | 5 | $\ldots$ |
| Operator | 1 | - | $\Sigma$ | $\mathcal{A}_{\mu}^{(1)}$ | $T_{\mu \nu}$ | $\mathcal{A}_{\mu \nu \rho}^{(3)}$ | $\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}$ | $\mathcal{A}_{\mu \nu \rho \sigma \alpha}^{(5)}$ | regular |
|  |  |  |  |  |  |  | $\partial \mathcal{A}^{(3)}$ | $\partial \mathcal{J}^{(4)}$ |  |
| Descendants | - | - | - | $\partial \Sigma$ | $\partial \mathcal{A}^{(1)}$ | $\partial T$ | $\partial^{2} \mathcal{A}^{(3)}$ |  |  |
|  |  |  |  |  | $\partial^{2} \Sigma$ | $\partial^{(1)}$ | $\partial^{2} T$ | $\partial^{3} \mathcal{A}^{(1)}$ | $\partial^{3} T$ |
| $\ldots$ | $\partial^{4} \mathcal{A}^{(1)}$ |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  | $\partial^{4} \Sigma$ | $\partial^{5} \Sigma$ |  |

A quick look at the singularity structure of the $T T$ OPE reveals that it contains the operators shown in the table. In particular, there are operators of spin from 0 to 5 before the regular terms. However, this table does not say how many independent operators of each spin there are. To answer this question we need to classify all possible invariants for free scalars, spinors and vectors.

### 4.1 Scalar field.

We have the propagator (using $\square\left(1 /|x|^{2}\right)=-4 \pi^{2} \delta(x)$ )

$$
<\varphi(x) \varphi(y)>=\frac{1}{4 \pi^{2}} \frac{1}{|x-y|^{2}},
$$

while the stress tensor is given in formula (1.2). The TT OPE contains currents with even spin: $0,2,4$. Precisely,

$$
\begin{align*}
T_{\mu \nu}(x) T_{\rho \sigma}(y)= & \frac{1}{60}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x-y|^{4}}\right) \\
& +\frac{1}{6} \frac{1}{4 \pi^{2}} \Sigma \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x-y|^{2}}\right) \\
& +\frac{1}{4 \pi^{2}} T_{\alpha \beta} \prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right)  \tag{4.59}\\
& +\frac{1}{4 \pi^{2}} \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)} \prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma \delta}^{(2,24)}\left(|x-y|^{6} \ln |x-y|^{2} M^{2}\right) \\
& + \text { descendants + regular terms. }
\end{align*}
$$

The operators that appear on the right hand side of the OPE are located in the point $(x+y)$ / 2 (this will not be written explicitly in order to simplify the formulas). The general term (for $s<6$, in order to be singular) is therefore of the form

$$
\begin{equation*}
\mathcal{J}_{\alpha_{1} \cdots \alpha_{s}}^{(s)} \prod_{\mu \nu, \rho \sigma ; \alpha_{1} \cdots \alpha_{s}}^{(2,2 ; s)}\left[|x-y|^{2 s-2}\left(\ln |x-y|^{2} M^{2}\right)^{\sigma(2 s-2)}\right] . \tag{4.60}
\end{equation*}
$$

The logarithm is present only if $s \geq 1: \sigma(2 s-2)=1$ for $s \geq 1$ and $\sigma(2 s-2)=0$ for $s=0$. The scale $M$ disappears after taking the derivatives and has no physical meaning. The differential
operator $\prod_{\mu \nu, \rho \sigma ; \alpha_{1} \cdots \alpha_{s}}^{(2,2 ; s)}$ is fixed according to the following rules: it is of order $s+4$ in derivatives, symmetric in $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu \nu \leftrightarrow \rho \sigma$ and in $\alpha_{1} \cdots \alpha_{s}$ (in total, $2^{3} s$ ! exchanges); it is completely traceless in $\alpha_{1} \cdots \alpha_{s}$, as well as in $\mu \nu$ and $\rho \sigma$; finally, it is conserved in $\mu \nu \rho \sigma$ (but not necessarily in $\alpha_{1} \cdots \alpha_{s}$ ). A term like (4.60) will be called "primary".

In addition to the primary terms, we have their descendants and regular terms. Most of this section is devoted to the primary terms and before starting their classification, let us illustrate the descendants.

The descendant terms are uniquely fixed by the primary terms and the symmetry constraints on the operator product expansion. For example, let us consider the first descendant of the spin-0 operator $\Sigma=\varphi^{2}$. The primary term

$$
\begin{equation*}
\frac{1}{6} \frac{1}{4 \pi^{2}} \Sigma\left(\frac{x+y}{2}\right) \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x-y|^{2}}\right) \tag{4.61}
\end{equation*}
$$

is conserved only up to $\partial \Sigma$-terms. The first descendant is fixed by requiring conservation up to $\partial^{2} \Sigma$-terms. Let us write the first descendant of (4.61) as

$$
\frac{1}{6} \frac{1}{4 \pi^{2}} \partial_{\alpha} \Sigma\left(\frac{x+y}{2}\right) \prod_{\mu \nu, \rho \sigma ; \alpha}^{(\mathrm{des})}\left(\ln |x-y|^{2} M^{2}\right)
$$

The differential operator $\prod_{\mu \nu, \rho \sigma ; \alpha}^{(\mathrm{des})}$ is allowed to contain five derivatives, it is traceless in $\mu \nu$ and $\rho \sigma$, but not conserved, rather it satisfies

$$
\begin{equation*}
\partial_{\alpha} \prod_{\alpha \nu, \rho \sigma ; \mu}^{(\mathrm{des})}\left(\ln |x-y|^{2} \mu^{2}\right)+\frac{1}{2} \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x-y|^{2}}\right)=0 \tag{4.62}
\end{equation*}
$$

Finally, symmetry under the exchange $(\mu \nu, x) \leftrightarrow(\rho \sigma, y)$ imposes

$$
\begin{equation*}
\prod_{\mu \nu, \rho \sigma ; \alpha}^{(\mathrm{des})}+\prod_{\rho \sigma, \mu \nu ; \alpha}^{(\mathrm{des})}=0 \tag{4.63}
\end{equation*}
$$

(4.62) and (4.63) produce the unique answer

$$
\begin{equation*}
\prod_{\mu \nu, \rho \sigma ; \alpha}^{(\mathrm{des})}=\frac{1}{8}\left(\prod_{\mu \nu, \rho \alpha}^{(2)} \partial_{\sigma}+\prod_{\mu \nu, \alpha \sigma}^{(2)} \partial_{\rho}-\prod_{\mu \alpha, \rho \sigma}^{(2)} \partial_{\nu}-\prod_{\alpha \nu, \rho \sigma}^{(2)} \partial_{\mu}\right) \tag{4.64}
\end{equation*}
$$

This formula gives the projector of the first descendant in terms of the projector of the primary. Similar formulas hold in all other cases, but one (a special term SP that we shall discuss at length). We call the objects $\prod^{(2,2 ; s)}$ "projector-primaries" and $\prod^{(\text {des })}$ "projector-descendants". We conclude that the projector-descendants are generated algorithmically from the projectorprimaries. Therefore, we do not need to illustrate the descendant terms any further, our main concern being now to classify the primary terms.

A useful property to work out our classification in the following. We observe that the set of derivatives $\partial_{\left\{\mu_{1} \cdots \mu_{m}\right.} \mathcal{J}_{\left.\nu_{1} \cdots \nu_{n}\right\}}^{(n)}$, completely symmetrized in the indices and with $m$ and $n$ even, form a basis for the quadratic monomials of the form $\partial_{\left\{\mu_{1} \cdots \mu_{m}\right.} \varphi \partial_{\left.\nu_{1} \cdots \nu_{n}\right\}} \varphi, m+n=$ even. Indeed, it is easy to show that the correspondence between the two sets of operators is one-to-one. The
condition of simple tracelessness (2.7) is important to avoid overcounting: for example, a traceful stress tensor $T_{\mu \nu}$ can only be proportional to $\partial_{\alpha} \varphi \partial_{\alpha} \varphi$, but this is equal to $\frac{1}{2} \square \Sigma$. Actually, all the terms with contracted derivatives, like $\partial_{\alpha} \varphi \partial_{\alpha} \varphi=\frac{1}{2} \square \Sigma$, either vanish using the field equations or can be expressed trivially by acting with "ם" on simpler operators. Similarly, a spin-4 current satisfying the double trace condition but not the simple trace condition would have a simple trace equal to a linear combination of $\partial_{\{\mu} \partial_{\nu} T_{\rho \sigma\}}, \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \Sigma$ and various contractions of these descendants. This fact follows by counting the possible terms. In the table we show the correspondence in the first few cases.

We conclude that each monomial $\partial_{\left\{\mu_{1} \cdots \mu_{m}\right.} \varphi \partial_{\left.\nu_{1} \cdots \nu_{n}\right\}} \varphi$ appearing on the right hand side of the OPE can be reexpressed as the sum of a primary term of spin $m+n$ plus descendants of operators with lower spins. In this way, its primary content is easily identified. However, this argument does not say anything, yet, about the number of independent operators with the same spin appearing in the OPE, for which we need to classify the projectors $\prod^{(2,2 ; s)}$.

| $\varphi^{2}$ | $\Sigma$ |
| :---: | :---: |
|  | $\partial_{\mu} \varphi \partial_{\nu} \varphi$ |
| $\varphi \partial_{\mu} \partial_{\nu} \varphi$ | $T_{\mu \nu}+\frac{1}{6} \partial_{\mu} \partial_{\nu} \Sigma+\frac{1}{12} \delta_{\mu \nu} \square \Sigma$ |
| $\sum_{\mathrm{s}} \partial_{\mu} \partial_{\nu} \varphi \partial_{\rho} \partial_{\sigma} \varphi$ | $\frac{1}{16} \mathcal{J}_{\mu \nu \rho \sigma}+K_{\mu \nu \rho \sigma}+\frac{2}{7} H_{\mu \nu \rho \sigma}+\frac{1}{30} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \Sigma$ |
| $\sum_{\mathrm{s}} \partial_{\mu} \varphi \partial_{\nu} \partial_{\rho} \partial_{\sigma} \varphi$ | $-\frac{1}{16} \mathcal{J}_{\mu \mu \rho \sigma}-K_{\mu \nu \rho \sigma}+\frac{3}{14} H_{\mu \nu \rho \sigma}+\frac{1}{20} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \Sigma$ |
| $\varphi \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \varphi$ | $\frac{1}{16} \mathcal{J}_{\mu \nu \rho \sigma}+K_{\mu \nu \rho \sigma}-\frac{12}{7} H_{\mu \nu \rho \sigma}+\frac{1}{5} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \Sigma$ |
| notation: $H_{\mu \nu \rho \sigma}$ | $\sum_{\mathbf{s}}\left(\partial_{\mu} \partial_{\nu} T_{\rho \sigma}+\frac{1}{12} \delta_{\mu \nu} \partial_{\rho} \partial_{\sigma} \square \Sigma\right)$ |
| notation: $K_{\mu \nu \rho \sigma}{ }^{(4)}$ | $\frac{3}{70} \square \sum_{\mathbf{s}} \delta_{\mu \nu}\left(5 T_{\rho \sigma}+\frac{1}{6} \partial_{\rho} \partial_{\sigma} \Sigma+\frac{7}{24} \delta_{\rho \sigma} \square \Sigma\right)$ |

Let us start with $\prod^{(2,2 ; 2)}$. With six uncontracted derivatives two obvious candidates for $\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}$ are

$$
\begin{equation*}
\prod_{\mu \nu \alpha, \beta \rho \sigma}^{(3)}+\prod_{\mu \nu \beta, \alpha \rho \sigma}^{(3)}+\frac{7}{10} \delta_{\alpha \beta} \square \prod_{\mu \nu, \rho \sigma}^{(2)}, \quad \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\partial_{\alpha} \partial_{\beta}-\frac{1}{4} \square \delta_{\alpha \beta}\right) \tag{4.66}
\end{equation*}
$$

In the first candidate we have used (2.19) to extract the trace. Note that terms like the first one exhibit a spin- 3 content in the three-point function of the stress- tensor. The above projectors can be generated in a way that we now illustrate and that generalizes straightforwardly to $\prod_{\mu \nu, \rho \sigma ; \alpha_{1} \cdots \alpha_{s}}^{(2,2 ; s)}$.

We have an OPE term containing a projector $\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}$ with 6 indices. An OPE term is the two-point limit of a three-point function, so let us regard it as a peculiar two-point function, a two-point function whose spin content has still to be determined and where one operator is actually the product of two operators. Certainly in $\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}$ there can be a spin- 3 term, whose two-point function is described by our familiar projector $\prod^{(3)}$, completely symmetric in its indices. The indices $\mu \nu, \rho \sigma, \alpha \beta$ can be distributed apparently in different manners. Actually, one can check that, after imposing the correct symmetries and taking traces out, one remains with a single independent choice. We have written this term in the form $\prod_{\mu \nu \alpha, \beta \rho \sigma}^{(3)}$ shown in (4.66). This term satisfies more properties than required (for example, it is conserved in $\alpha \beta$ )
and therefore it cannot be the unique solution. Six indices can also describe the two-point function of a product operator of spin (2-1), and this is the second term in (4.66). There is no other possibility, since there is no other decomposition of three indices.

To show that this argument generalizes further, let us now discuss the case $s=4$. The first candidate is the spin- 4 projector $\Pi^{(4)}$. As before, there is a unique independent way of distributing the indices. The other decompositions of four indices are the products (3-1) and (2-2). In total, three independent tensors, a fact that we have carefully checked. The invariants are therefore

$$
\begin{equation*}
\prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}, \quad \sum_{\substack{\operatorname{symm} \\(\alpha \beta \gamma \delta)}} \prod_{\mu \nu \alpha, \rho \sigma \beta}^{(3)}\left(\partial_{\gamma} \partial_{\delta}-\frac{1}{4} \square \delta_{\gamma \delta}\right), \quad \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta}-\operatorname{tr}_{(\alpha \beta \gamma \delta)}\right) \tag{4.67}
\end{equation*}
$$

In general, one takes the total number of indices of the projector $\left(s+4\right.$ for $\prod^{(2,2 ; s)}$ - we assume $s=$ even for simplicity), divides it by two $s^{\prime}=s / 2$ (since a projector for a two-point function has twice as many indices as the current) and counts the number of ways of splitting $s^{\prime}$ into two (since a term in the OPE is a two-point limit of a three-point function): $s^{\prime}=m+n$, $m \leq n$.

The three invariants of (4.67) do reproduce the appropriate term in the OPE (4.59) (see below). However, this is not true for the two terms (4.66). The reason is that there is one projector more in the case for $\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}$, that we have neglected by applying the argument outlined above.

So far we have classified the projectors by imposing the conservation condition and the condition of tracelessness in a strict sense. However, there is one case in which these properties can be satisfied up to local terms, i.e. up to a delta-function. This can only happen for the singularity $1 /|x|^{4}$ : lower singularities cannot produce any $\delta^{(4)}(x)$, while higher singularities, which would eventually produce $\square \delta^{(4)}(x)$ or $\partial \delta^{(4)}(x)$, have already been discussed in detail. The special additional invariant for $\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2,2)}$ is unique, quadratic in derivatives, acts on $1 /|x|^{2}$ and factorizes a $\square$ when tracing or appropriately differentiating,

$$
\begin{aligned}
\mathrm{SP}_{\mu \nu, \rho \sigma ; \alpha \beta}= & \frac{1}{2}\left[\left(\delta_{\mu \nu} \delta_{\rho \sigma}-\delta_{\mu \rho} \delta_{\nu \sigma}-\delta_{\mu \sigma} \delta_{\nu \rho}\right) \partial_{\alpha} \partial_{\beta}-2 \delta_{\alpha \mu} \delta_{\beta \nu} \partial_{\rho} \partial_{\sigma}-2 \delta_{\alpha \rho} \delta_{\beta \sigma} \partial_{\mu} \partial_{\nu}\right. \\
& +\left(\delta_{\beta \rho} \delta_{\nu \sigma}+\delta_{\beta \sigma} \delta_{\nu \rho}-\delta_{\beta \nu} \delta_{\rho \sigma}\right) \partial_{\alpha} \partial_{\mu}+\left(\delta_{\beta \rho} \delta_{\mu \sigma}+\delta_{\beta \sigma} \delta_{\mu \rho}-\delta_{\beta \mu} \delta_{\rho \sigma}\right) \partial_{\alpha} \partial_{\nu} \\
& +\left(\delta_{\beta \mu} \delta_{\nu \sigma}+\delta_{\beta \nu} \delta_{\mu \sigma}-\delta_{\beta \sigma} \delta_{\mu \nu}\right) \partial_{\alpha} \partial_{\rho} \\
& \left.+\left(\delta_{\beta \mu} \delta_{\nu \rho}+\delta_{\beta \nu} \delta_{\mu \rho}-\delta_{\beta \rho} \delta_{\mu \nu}\right) \partial_{\alpha} \partial_{\sigma}\right]+(\alpha \leftrightarrow \beta)-\operatorname{tr}_{\alpha \beta} .
\end{aligned}
$$

This term is the one which closes the Poincaré algebra (we will check this explicitly later on).
By explicit computation, the projector $\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}$ is in the case of the scalar field the following linear combination:

$$
\begin{align*}
& \prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right)=\mathrm{SP}_{\mu \nu, \rho \sigma ; \alpha \beta}\left(\frac{1}{|x-y|^{2}}\right) \\
& +\frac{1}{6} \prod_{\mu \nu, \rho \sigma}^{(2)} \partial_{\alpha} \partial_{\beta}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right) \\
& -\frac{5}{16} \prod_{\mu \nu \alpha, \beta \rho \sigma}^{(3)}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right) . \tag{4.68}
\end{align*}
$$

For $s=4$, instead, the projector is

$$
\begin{aligned}
\prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma \delta}^{(2,2 ; 4)}= & -\frac{1}{2^{11} \cdot 3^{3} \cdot 5}\left[\prod_{\mu \nu, \rho \sigma}^{(2)}\left(\partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta}-\operatorname{tr}_{(\alpha \beta \gamma \delta)}\right)\right. \\
& \left.-\frac{75}{28} \sum_{\substack{\operatorname{symm} \\
(\alpha \beta \gamma \delta)}} \prod_{\mu \nu \alpha, \rho \sigma \beta}^{(3)}\left(\partial_{\gamma} \partial_{\delta}-\frac{1}{4} \square \delta_{\gamma \delta}\right)-\frac{15}{8} \prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\right]
\end{aligned}
$$

The spin- 4 two-point functions is

$$
<\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}(x) \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)}(0)>=\frac{2^{4}}{3^{2} \cdot 5 \cdot 7}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\left(\frac{1}{|x|^{4}}\right)
$$

while the spin- 0 two point function is simply

$$
<\Sigma(x) \Sigma(0)>=\frac{1}{4 \pi^{2}} \frac{2}{|x|^{4}} .
$$

Summarizing, there are three invariants in both cases $s=2$ and $s=4$. The explicit coefficients with which they appear in the operator product expansion depend on the nature of the field (scalar, spinor, vector). Therefore there are three spin-2 central charges, three spin-4 central charges, and so on. This conclusion should be compared with the conclusion of the previous section, which stated that at the level of the trace anomaly (and under the assumption of higher spin flavor symmetry) there are only two central charges for any spin. This is not a contradiction, since it corresponds to two different descriptions, although not unrelated.

Before concluding this section, let us make one further remark. In the spin-2 projector, $\Pi^{(2,2 ; 2)}$, the existence of the three invariants that we have listed reflects the known fact [20] that the three-point function of the stress tensor contains three independent structures, corresponding precisely to scalar, spinor and vector fields. This can be explained as follows. Primary and descendant terms are sufficient to reconstruct the entire three-point correlator. We know that the descendants are uniquely fixed by the primary terms. Therefore the three-point function is fully encoded in the primary term and the number of structures of the three-point correlator has to be the same as the number of structures appearing in the OPE. This idea is completely general and can be used as an algorithm to classify all the structures of the three-point functions, for any spin, as we have done explicitly up to $s=4$.

### 4.2 Vector field.

Let us now repeat the analysis in the case of the vector field. The situation is somewhat simpler, because the $T T$ OPE closes up to $1 /|x|^{4}$-terms. However, odd spin currents appear for lower singularities. We have

$$
T_{\mu \nu}=F_{\mu \alpha} F_{\nu \alpha}-\frac{1}{4} \delta_{\mu \nu} F^{2}=2 F_{\mu \alpha}^{+} F_{\nu \alpha}^{-}, \quad<A_{\mu}(x) A_{\nu}(0)>=\frac{1}{4 \pi^{2}} \frac{\delta_{\mu \nu}}{|x|^{2}} .
$$

The OPE reads

$$
\begin{aligned}
& T_{\mu \nu}(x) T_{\rho \sigma}(y)=\frac{1}{5}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x-y|^{4}}\right) \\
& \quad+\frac{1}{4 \pi^{2}} T_{\alpha \beta} \operatorname{SP}_{\mu \nu, \rho \sigma ; \alpha \beta}\left(\frac{1}{|x-y|^{2}}\right) \\
& \quad-\frac{1}{4 \pi^{2}} \mathcal{A}_{\alpha \beta \gamma}^{(3)} \frac{1}{2^{3}} \sum_{\text {symm }} \varepsilon_{\mu \rho \alpha \delta}\left(\pi_{\nu \beta} \pi_{\sigma \gamma}-\pi_{\nu \sigma} \partial_{\beta} \partial_{\gamma}\right) \partial_{\delta}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right) \\
& \quad+\frac{1}{4 \pi^{2}} \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)} \prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma \delta}^{(2,2 ; 4)}\left(|x-y|^{6} \ln |x-y|^{2} M^{2}\right) \\
& \quad-\frac{1}{4 \pi^{2}} \mathcal{A}_{\alpha \beta \gamma \delta \varepsilon}^{(5)} \frac{1}{2^{12} \cdot 3^{2}} \sum_{\text {symm }} \varepsilon_{\mu \rho \alpha \zeta}\left(\pi_{\nu \beta} \pi_{\sigma \gamma}-\pi_{\nu \sigma} \partial_{\beta} \partial_{\gamma}\right) \partial_{\delta} \partial_{\varepsilon} \partial_{\zeta}\left(|x-y|^{6} \ln |x-y|^{2} M^{2}\right) \\
& \quad+\text { descendants }+ \text { regular terms. }
\end{aligned}
$$

The odd spin terms have been written down explicitly, since there is a unique projector for each of them (see below). We see that there is only the special invariant SP for $s=2$ and that it multiplies the stress tensor with the same coefficient as in the scalar case. The conversion table now reads

| $F_{\mu \alpha}^{+} F_{\nu \alpha}^{-}$ | $\frac{1}{2} T_{\mu \nu}$ |
| :---: | :---: |
| $F_{\mu \nu}^{+} F_{\rho \sigma}^{-}+F_{\rho \sigma}^{+} F_{\mu \nu}^{-}$ | $\frac{1}{2}\left[\delta_{\mu \rho} T_{\nu \sigma}-\delta_{\mu \sigma} T_{\nu \rho}-\delta_{\nu \rho} T_{\mu \sigma}+\delta_{\nu \sigma} T_{\mu \rho}\right]$ |
| $\sum_{\mathrm{s}}\left(F_{\rho \alpha}^{+} \partial_{\mu} \partial_{\nu} F_{\sigma \alpha}^{-}+\partial_{\mu} \partial_{\nu} F_{\rho \alpha}^{+} F_{\sigma \alpha}^{-}\right)$ | $-\frac{1}{2} \mathcal{J}_{\mu \nu \rho \sigma}^{(4)}+\frac{1}{28} \sum_{\mathrm{s}} \pi_{\rho \sigma} T_{\mu \nu}+\frac{1}{4} \sum_{\mathrm{s}} \partial_{\rho} \partial_{\sigma} T_{\mu \nu}$ |
| $\sum_{\mathrm{s}}\left(\partial_{\mu} F_{\rho \alpha}^{+} \partial_{\nu} F_{\sigma \alpha}^{-}+\partial_{\nu} F_{\rho \alpha}^{+} \partial_{\mu} F_{\sigma \alpha}^{-}\right)$ | $\frac{1}{2} \mathcal{J}_{\mu \nu \rho \sigma}^{(4)}-\frac{1}{28} \sum_{\mathrm{s}} \pi_{\rho \sigma} T_{\mu \nu}+\frac{1}{4} \sum_{\mathrm{s}} \partial_{\rho} \partial_{\sigma} T_{\mu \nu}$ |

Again, the correspondence is one-to-one.
In the analysis of the terms in the OPE some simplification comes from the parity symmetry under exchange of $F^{+}$and $F^{-}$(and $\varepsilon_{\mu \nu \rho \sigma} \rightarrow-\varepsilon_{\mu \nu \rho \sigma}$ ). Moreover, one has to use systematically the special identity appearing in the second line of the table and similar identities obtained by differentiating it. This relationship follows by the properties of the $\varepsilon$-tensor (in particular that the product of two $\varepsilon$-tensor is a quartic polynomial is $\delta_{\mu \nu}$ ). Moreover, the existence of this relationship is guaranteed by the following argument. $F^{+}$and $F^{-}$contain 3 independent components each. Therefore the bilinear $F_{\mu \nu}^{+} F_{\rho \sigma}^{-}+F_{\rho \sigma}^{+} F_{\mu \nu}^{-}$contains nine independent components. Then it is necessarily re-expressible algebraically via the stress-tensor, since the stress tensor contains precisely nine independent components and it is bilinear in $F^{+}-F^{-}$.

To read the odd-spin content one has to use replacements of the type

$$
F_{\mu \nu}^{+} \overleftrightarrow{\partial_{\alpha}} F_{\rho \sigma}^{-}-F_{\rho \sigma}^{+} \overleftrightarrow{\partial_{\alpha}} F_{\mu \nu}^{-} \rightarrow-\frac{1}{2}\left(\varepsilon_{\mu \nu \rho \beta} \mathcal{A}_{\alpha \beta \sigma}^{(3)}-\varepsilon_{\mu \nu \sigma \beta} \mathcal{A}_{\alpha \beta \rho}^{(3)}-\varepsilon_{\rho \sigma \mu \beta} \mathcal{A}_{\alpha \beta \nu}^{(3)}+\varepsilon_{\rho \sigma \nu \beta} \mathcal{A}_{\alpha \beta \mu}^{(3)}\right)
$$

The projector $\prod^{(2,2 ; 4)}$ is the unique linear combination of the three invariants (4.67) that is a polynomial of degree at most four in uncontracted derivatives. This is easily seen by counting the free indices and by the structure of the vector propagator. Therefore we have for a vector

$$
\prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma \delta}^{(2,2 ; 4)}=\frac{1}{2^{9} \cdot 3 \cdot 5}\left[\prod_{\mu \nu, \rho \sigma}^{(2)}\left(\partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta}-\operatorname{tr}_{(\alpha \beta \gamma \delta)}\right)\right.
$$

$$
\left.-\frac{25}{14} \sum_{\substack{\text { symm } \\(\alpha \beta \gamma \delta)}} \prod_{\mu \nu \alpha, \rho \sigma \beta}^{(3)}\left(\partial_{\gamma} \partial_{\delta}-\frac{1}{4} \square \delta_{\gamma \delta}\right)+\frac{5}{24} \prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\right]
$$

Let us now discuss the new terms, i.e. those with odd spin. The unique spin-1 invariant is axial and has the form

$$
\begin{equation*}
\prod_{\mu \nu, \rho \sigma ; \alpha}^{(2,2 ; 1)}=\sum_{\text {symm }} \varepsilon_{\mu \rho \alpha \delta} \pi_{\nu \sigma} \partial_{\delta} \tag{4.70}
\end{equation*}
$$

The symmetrization acts also on $\mu, \nu$ and $\rho, \sigma$. However, there is no spin- 1 axial current for the vector field. For the spin-3 invariant $\prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma}^{(2,2 ; 3)}$ there are two candidates:

$$
\sum_{\mathrm{symm}} \varepsilon_{\mu \rho \alpha \delta} \pi_{\nu \sigma} \partial_{\beta} \partial_{\gamma} \partial_{\delta}, \quad \sum_{\mathrm{symm}} \varepsilon_{\mu \rho \alpha \delta} \pi_{\nu \beta} \pi_{\sigma \gamma} \partial_{\delta}
$$

The combination appearing in (4.72) is the one with the least numer of uncontracted derivatives. Similarly, the spin- 5 invariant $\prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma \varepsilon \delta}^{(2,2 ; 5)}$ admits two candidates, obtained by acting with two derivatives on the spin- 3 ones:

$$
\sum_{\operatorname{symm}} \varepsilon_{\mu \rho \alpha \zeta} \pi_{\nu \sigma} \partial_{\beta} \partial_{\gamma} \partial_{\delta} \partial_{\varepsilon} \partial_{\zeta}, \quad \sum_{\text {symm }} \varepsilon_{\mu \rho \alpha \zeta} \pi_{\nu \beta} \pi_{\sigma \gamma} \partial_{\delta} \partial_{\varepsilon} \partial_{\zeta}
$$

Again, the combination in (4.72) has the minimum number of derivatives. Three odd-spin invariants (spin-1, spin-3 and spin-5) are not used for the vector field: they will appear in the $T T$ OPE for the spinor. Indeed, the general rule is that all independent invariants that one can construct correspond to a nontrivial OPE term.

The spin-3, 4 and 5 two-point functions are

$$
\begin{align*}
<\mathcal{A}_{\mu \nu \rho}^{(3)}(x) \mathcal{A}_{\alpha \beta \gamma}^{(3)}(0)> & =\frac{1}{2^{2} \cdot 3 \cdot 7}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu \rho, \alpha \beta \gamma}^{(3)}\left(\frac{1}{|x|^{4}}\right) \\
<\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}(x) \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)}(0)> & =\frac{1}{2^{2} \cdot 3^{2} \cdot 7}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\left(\frac{1}{|x|^{4}}\right)  \tag{4.71}\\
<\mathcal{A}_{\mu \nu \rho \sigma \tau}^{(5)}(x) \mathcal{A}_{\alpha \beta \gamma \delta \varepsilon}^{(5)}(0)> & =\frac{1}{2^{2} \cdot 3 \cdot 5 \cdot 11}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu \rho \sigma \tau, \alpha \beta \gamma \delta \varepsilon}^{(5)}\left(\frac{1}{|x|^{4}}\right)
\end{align*}
$$

### 4.3 Spinor.

The case of the spinor is the most involved. The stress tensor

$$
T_{\mu \nu}=\frac{1}{4}\left(\bar{\psi} \gamma_{\mu} \overleftrightarrow{\partial}_{\nu} \psi+\bar{\psi} \gamma_{\nu} \overleftrightarrow{\partial}_{\mu} \psi\right), \quad<\psi(x) \bar{\psi}(0)>=\frac{1}{2 \pi^{2}} \frac{\not x}{|x|^{4}}
$$

generates the OPE

$$
\begin{align*}
T_{\mu \nu}(x) T_{\rho \sigma}(y)= & \frac{1}{10}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x-y|^{4}}\right) \\
& -\frac{1}{4 \pi^{2}} \mathcal{A}_{\alpha}^{(1)} \frac{1}{2} \sum_{\text {symm }} \varepsilon_{\mu \rho \alpha \beta} \pi_{\nu \sigma} \partial_{\beta}\left(\frac{1}{|x-y|^{2}}\right) \\
& +\frac{1}{4 \pi^{2}} T_{\alpha \beta} \prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right) \\
& -\frac{1}{4 \pi^{2}} \mathcal{A}_{\alpha \beta \gamma}^{(3)} \frac{1}{2^{6}} \sum_{\text {symm }} \varepsilon_{\mu \rho \alpha \delta} \pi_{\nu \beta} \pi_{\sigma \gamma} \partial_{\delta}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right)  \tag{4.72}\\
& +\frac{1}{4 \pi^{2}} \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)} \prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma \delta}^{(2,2 ; 4)}\left(|x-y|^{6} \ln |x-y|^{2} M^{2}\right) \\
& -\frac{1}{4 \pi^{2}} \mathcal{A}_{\alpha \beta \gamma \delta \varepsilon}^{(5)} \frac{1}{2^{13} \cdot 3^{2}} \sum_{\text {symm }} \varepsilon_{\mu \rho \alpha \zeta} \pi_{\nu \beta} \pi_{\sigma \gamma} \partial_{\delta} \partial_{\varepsilon} \partial_{\zeta}\left(|x-y|^{6} \ln |x-y|^{2} M^{2}\right) \\
& + \text { descendants }+ \text { regular terms. }
\end{align*}
$$

Let us describe the even-spin terms first.
The term containing the stress tensor is a polynomial of degree at most four in uncontracted derivatives. This fixes the relative coefficient of the second and third invariant in the sum

$$
\begin{align*}
& \prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right)=\mathrm{SP}_{\mu \nu, \rho \sigma ; \alpha \beta}\left(\frac{1}{|x-y|^{2}}\right) \\
& +\frac{9}{64} \prod_{\mu \nu, \rho \sigma}^{(2)} \partial_{\alpha} \partial_{\beta}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right) \\
& -\frac{15}{64} \prod_{\mu \nu \alpha, \beta \rho \sigma}^{(3)}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right) . \tag{4.73}
\end{align*}
$$

Another check is the universality of the coefficient of SP.
The spin- 4 term is explicitly computed to give

$$
\begin{aligned}
\prod_{\mu \nu, \rho \sigma ; \alpha \beta \gamma \delta}^{(2,2 ; 4)}= & -\frac{1}{2^{14} \cdot 3 \cdot 5}\left[\prod_{\mu \nu, \rho \sigma}^{(2)}\left(\partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta}-\operatorname{tr}_{(\alpha \beta \gamma \delta)}\right)\right. \\
& \left.-\frac{25}{7} \sum_{\substack{\text { symm } \\
(\alpha \beta \gamma \delta)}} \prod_{\mu \nu \alpha, \rho \sigma \beta}^{(3)}\left(\partial_{\gamma} \partial_{\delta}-\frac{1}{4} \square \delta_{\gamma \delta}\right)+\frac{10}{3} \prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\right] .
\end{aligned}
$$

The odd-spin terms are identified by the projector-invariants classified in the previous subsection. Here there is no restriction on the number of uncontracted derivatives and the spin-1 current is also present. The results confirm the prediction that there is some OPE term for all existing projector-invariants with the given properties.

The derivation of the OPE for the fermion, although more complicated than the previous ones, is done following the same strategy. The higher spin operators appearing on the right hand side form a basis for the monomials $\partial^{p} \bar{\psi} \gamma_{\mu} \partial^{q} \psi, p+q=$ odd, and $\partial^{p} \bar{\psi} \gamma_{5} \gamma_{\mu} \partial^{q} \psi, p+q=$ even.

Here proper symmetrizations in the indices are understood and all derivatives are meant uncontracted, both among themselves and with the index $\mu$ of the Dirac matrix. The reason for the restrictions on the parity of $p+q$ is that the operators that do not satisfy it, like for example the vector current $\bar{\psi} \gamma_{\mu} \psi(p+q=0=$ even $)$ do not appear in the operator product expansion. This is clearly seen by decomposing the Dirac fermion $\psi=\psi_{1}+i \psi_{2}$ into its Majorana components $\psi_{1,2}$ and observing that the stress tensor and the propagator do not mix the two. In conclusion, a one-to-one correspondence table like (4.65) and (4.69) can be worked out. We do not write the complete table. It is sufficient to report the primary spin- $(p+q+1)$ contents of each monomial $\partial^{p}\left(\gamma_{5}\right) \bar{\psi} \gamma_{\mu} \partial^{q} \psi$. We observe that $\partial^{p} \bar{\psi}\left(\gamma_{5}\right) \gamma_{\mu} \partial^{q} \psi$ is equivalent to $-\partial^{p-1} \bar{\psi}\left(\gamma_{5}\right) \gamma_{\mu} \partial^{q+1} \psi$ for our purposes, since the difference of the two is a descendant of a spin- $(p+q)$ operator. Up to descendants and permutations of the indices, we have, very simply,

| $\bar{\psi} \gamma_{5} \gamma_{\mu} \psi$ | $\bar{\psi} \gamma_{\mu} \partial_{\nu} \psi$ | $\bar{\psi} \gamma_{5} \gamma_{\mu} \partial_{\nu} \partial_{\rho} \psi$ | $\bar{\psi} \gamma_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \psi$ | $\bar{\psi} \gamma_{5} \gamma_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \partial_{\alpha} \psi$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathcal{A}_{\mu}^{(1)}$ | $T_{\mu \nu}$ | $\frac{1}{4} \mathcal{A}_{\mu \nu \rho}(3)$ | $\frac{1}{8} \mathcal{A}_{\mu \nu \rho \sigma}^{(4)}$ | $\frac{1}{16} \mathcal{A}_{\mu \nu \rho \sigma \alpha}^{(5)}$ |

We have computed the two-point functions. The results are

$$
\begin{aligned}
<\mathcal{A}_{\mu}^{(1)}(x) \mathcal{A}_{\alpha}^{(1)}(0)> & =\frac{2^{2}}{3}\left(\frac{1}{4 \pi^{2}}\right)^{2} \pi_{\mu \alpha}\left(\frac{1}{|x|^{4}}\right) \\
<\mathcal{A}_{\mu \nu \rho}^{(3)}(x) \mathcal{A}_{\alpha \beta \gamma}^{(3)}(0)> & =\frac{2^{4}}{3 \cdot 5 \cdot 7}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu \rho, \alpha \beta \gamma}^{(3)}\left(\frac{1}{|x|^{4}}\right), \\
<\mathcal{J}_{\mu \nu \rho \sigma}^{(4)}(x) \mathcal{J}_{\alpha \beta \gamma \delta}^{(4)}(0)> & =\frac{2^{2}}{3^{2} \cdot 7}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu \rho \sigma, \alpha \beta \gamma \delta}^{(4)}\left(\frac{1}{|x|^{4}}\right), \\
<\mathcal{A}_{\mu \nu \rho \sigma \tau}^{(5)}(x) \mathcal{A}_{\alpha \beta \gamma \delta \varepsilon}^{(5)}(0)> & =\frac{2^{5}}{3 \cdot 5 \cdot 7 \cdot 11}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu \rho \sigma \tau, \alpha \beta \gamma \delta \varepsilon}^{(5)}\left(\frac{1}{|x|^{4}}\right) .
\end{aligned}
$$

The number theoretical relationship between the spin of a current and the prime factors is natural: no prime number grater than $2 s+1$ appears for spin $s$ (the one-loop values of Feynman diagrams are computable algebraically). Finally, observe that our two-point functions are reflection positive.

### 4.4 Classical and quantum conformal algebras.

All the invariants appearing in the OPE, but SP, are annihilated algebraically by $\partial_{\mu}$ or by the trace contraction $\delta_{\mu \nu}$. Instead, SP is not annihilated algebraically and a contact term survives. Explicitly,

$$
\partial_{\mu} \mathrm{SP}_{\mu \nu, \rho \sigma ; \alpha \beta}=\frac{1}{2}\left[-2 \delta_{\alpha \rho} \delta_{\beta \sigma} \partial_{\nu}+\partial_{\alpha}\left(\delta_{\beta \rho} \delta_{\nu \sigma}+\delta_{\beta \sigma} \delta_{\nu \rho}-\delta_{\beta \nu} \delta_{\rho \sigma}\right)\right] \square+(\alpha \leftrightarrow \beta)-\operatorname{tr}_{\alpha \beta} .
$$

The surviving local terms describe the symmetries of the classical conformal algebra. Let us show this explicitly.

Keeping the contact terms originated by $\square\left(1 /|x|^{2}\right)$ we obtain

$$
\begin{equation*}
T(x) T_{\rho \sigma}(y)=2 \delta(x-y) T_{\rho \sigma}, \tag{4.74}
\end{equation*}
$$

and

$$
\begin{aligned}
\partial_{\mu} T_{\mu \nu}(x) T_{\rho \sigma}(y)= & -\partial_{\alpha} \delta(x-y)\left[\delta_{\nu \sigma} T_{\alpha \rho}+\delta_{\nu \rho} T_{\alpha \sigma}-\delta_{\rho \sigma} T_{\alpha \nu}\right] \\
& +2 \partial_{\nu} \delta(x-y) T_{\rho \sigma}+1^{s t} \text { descendant. }
\end{aligned}
$$

We have to study the first descendant, since $\partial T$ can multiply $\delta(x-y)$. Further descendants can be neglected, because they cannot multiply local terms (accordingly, the transformation rule does cannot contain derivatives of $T$ beyond the first). One can check that the first descendant can be set to zero if $T$ is located in $(x+y) / 2$.

We have $T_{\mu \nu}=\frac{2}{\sqrt{g}} \frac{\delta \mathcal{S}}{\delta g^{\mu \nu}}$, where $\mathcal{S}$ denotes the action. In the linear approximation, $\mathcal{S}=$ $\mathcal{S}_{0}-\frac{1}{2} \phi_{\mu \nu} T_{\mu \nu}$, where $g_{\mu \nu}=\delta_{\mu \nu}+\phi_{\mu \nu}$. Dilatations correspond to the variation $\delta_{\Lambda} \phi_{\mu \nu}=2 \Lambda \delta_{\mu \nu}$. Consequently, acting on the exponential $\mathrm{e}^{-\mathcal{S}}$ in the functional integral, the operator $\delta_{\Lambda}$ inserts $\int d^{4} x \Lambda T$ in the correlators. Therefore, $\delta_{\Lambda} T_{\mu \nu}=2 \Lambda T_{\mu \nu}$, from (4.74).

Diffeomorphisms are expressed by $\delta_{\xi} \phi_{\mu \nu}=\partial_{\mu} \xi_{\nu}+\partial_{\nu} \xi_{\mu}$ : the operator $\delta_{\xi}$ inserts $\int d^{4} x\left(\partial_{\mu} \xi_{\nu} T_{\mu \nu}\right)$ in the correlators. Therefore,

$$
\begin{align*}
\delta_{\xi} T_{\rho \sigma}(y) & =\int d^{4} x \partial_{\mu} \xi_{\nu}(x) T_{\mu \nu}(x) T_{\rho \sigma}(y)=-\int d^{4} x \xi_{\nu}(x) \partial_{\mu} T_{\mu \nu}(x) T_{\rho \sigma}(y) \\
& =\xi^{\alpha} \partial_{\alpha} T_{\rho \sigma}-\partial_{\alpha} \xi_{\rho} T_{\alpha \sigma}-\partial_{\alpha} \xi_{\sigma} T_{\rho \alpha}+2 \partial \cdot \xi T_{\rho \sigma}+\delta_{\rho \sigma} \partial_{\alpha} \xi_{\beta} T_{\alpha \beta} \tag{4.75}
\end{align*}
$$

after integrating by parts. The transformation rule is not the expected one. The reason is that there can be local terms in the OPE other than those obtained by the above procedure. i.e. SP is defined up to the local terms:

$$
\begin{aligned}
T_{\mu \nu}(x) T_{\rho \sigma}(y) \rightarrow & T_{\mu \nu}(x) T_{\rho \sigma}(y)+A \delta(x-y)\left(\delta_{\mu \nu} T_{\rho \sigma}+\delta_{\rho \sigma} T_{\mu \nu}\right) \\
& +B \delta(x-y)\left(\delta_{\mu \rho} T_{\nu \sigma}+\delta_{\mu \sigma} T_{\rho \nu}+\delta_{\nu \rho} T_{\mu \sigma}+\delta_{\nu \sigma} T_{\mu \rho}\right) .
\end{aligned}
$$

(4.74) was correct and in order to preserve it one has to impose $A+B=0$. Choosing $B=1$ in order to cancel the last term of (4.75), we obtain

$$
\delta_{\xi} T_{\rho \sigma}(y)=\xi^{\alpha} \partial_{\alpha} T_{\rho \sigma}+\partial_{\rho} \xi^{\alpha} T_{\alpha \sigma}+\partial_{\sigma} \xi^{\alpha} T_{\rho \alpha}+\partial \cdot \xi T_{\rho \sigma}
$$

The quantities that appear in the OPE (central charge, operators of various spin, as well as descendants and regular terms) constitute the quantum conformal algebra [5]. In two dimensions the difference between quantum and classical conformal algebras is just the identity operator (and therefore there is a unique, primary, central charge). In general dimension the difference is a tower of operators, classified by their spin (and another label $I$ ), defining infinitely many (secondary) central charges. In the next subsection we write down the general quantum conformal algebra for interacting conformal field theories and discuss its properties.

### 4.5 Interacting critical theories.

The free-field OPE can be easily generalized to give the structure of the TT OPE in the most general conformal field theory. We recall that the projectors $\prod_{\mu \nu, \rho \sigma ; \alpha_{1} \cdots \alpha_{s}}^{(2,2 ; s)}$ do not satisfy
conservation in the indices $\alpha_{1} \cdots \alpha_{s}$ and this means that the operators appearing on the right hand side of the OPE can indeed be non-conserved. For the reasons that we have explained in sect. 2 , the only violation of conservation allowed at criticality is the explicit $h$-violation due to the anomalous dimension $h$ of the operator in question. Indeed, the structure of the OPE remains unchanged in presence of an anomalous dimension $h$, since the projectors that we have classified can act on any power of $|x-y|$. Actually, this is true for all of them but the special one, which has necessarily to act on $1 /|x|^{2}$ in order to produce the desired $\delta$-functions. This statement is precisely the finiteness of the stress tensor, implied by the Poincaré algebra. Therefore the special term (and its descendants) is fixed, while all other terms depend on the theory. There are as many independent operators as projectors $\prod^{(2,2 ; s)}$, at least.

The operators associated with the projectors $\prod^{(2,2 ; s)}$ are primary and correlators of primary operators are conformal. Traces, which are indeed primary operators of lower spin (and the same dimension), are appropriately extracted from $\prod^{(2,2 ; s)}$. Improvement terms á la (3.33) are descendants of lower spin currents and therefore cannot appear multiplied by $\Pi^{(2,2 ; s)}$ (this would violate conformal symmetry, see (3.35)). They can appear only multiplied by projector descendants à la (4.64). Projector-primaries identify primary-operators and vice versa. This solves the problem of identifying the appropriate currents at criticality, see the end of section 3.

We have seen that for a given $s$ the projector $\prod^{(2,2 ; s)}$ is not unique, actually in general there is a triple degeneracy in the free-field limit. Actually, this triple degeneracy can be enhanced in presence of internal symmetries, when there are matter fields in various different irreducible representations of the gauge group, etc. Therefore we have to introduce a second label $I$ and sum over it. Correspondingly, there will be central charges $c_{s}^{I}$ for each channel and each spin (see next section). The resulting structure is

$$
\begin{align*}
& T_{\mu \nu}(x) T_{\rho \sigma}(y)=\frac{1}{60}\left(\frac{1}{4 \pi^{2}}\right)^{2} \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{c_{2}}{|x-y|^{4}}\right)+\frac{1}{4 \pi^{2}} T_{\alpha \beta} \tilde{\operatorname{SP}}_{\mu \nu, \rho \sigma ; \alpha \beta}(x-y) \\
&+\frac{1}{4 \pi^{2}} \sum_{\substack{s, I: \\
s-6+h_{s, I}<0}} \mathcal{J}_{\alpha_{1} \cdots \alpha_{s}}^{(s, I)} \prod_{\mu \nu, \rho \sigma ; \alpha_{1} \cdots \alpha_{s}}^{(2,2 ; s, I}\left[|x-y|^{2 s-2+h_{s, I}} \mu^{h_{s, I}}\left(\ln |x-y|^{2} M^{2}\right)^{\sigma\left(2 s-2+h_{s, I}\right)}\right] \\
& \quad+\text { descendants }+ \text { regular terms. } \tag{4.76}
\end{align*}
$$

This section is devoted to the analysis of this formula.
The factor $\mu^{h_{s, I}}$ is needed to match the dimensions and assures the compatibility of the renormalization group equations $\mathrm{d}(T T) / \mathrm{d} \ln \mu=0$ and $\mathrm{d} \mathcal{J}^{(s, I)} / \mathrm{d} \ln \mu=-h_{s, I} \mathcal{J}^{(s, I)}$.

The label $I$ runs over the projectors of formula (4.66) for $s=2$ and over those of formula (4.67) for $s=4$, as well as over eventual internal indices. Kinematically, projectors (4.66) and (4.67) are sufficient to emphasize the presence of the $I$-degeneracy, but only dynamics can determine it precisely. Let us explain this point in detail.

Operators with different spin are orthogonal, according to property (2.15). However, the situation is more complicated for operators with the same spin. There is no unique definition
of the projectors $\prod_{\mu \nu, \rho \sigma ; \alpha_{1} \cdots \alpha_{s}}^{(2,2 ; s, I)}$ : any linear combination of the projectors of (4.66) and (4.67) is allowed. Correspondingly, the true operators $\mathcal{J}_{\alpha_{1} \ldots \alpha_{s}}^{(s, I)}$ are linear combinations of the $\mathcal{J}_{\alpha_{1} \cdots \alpha_{s}-}^{(s)}$ operators of scalar, spinor and vector fields. Only dynamics can determine the precise nature of the labels $I$, and so remove this degeneracy of the free-field limit.

In general, orthogonal combinations $\mathcal{J}_{\alpha_{1} \ldots \alpha_{s}}^{(s, I)}$ with the same $s$ will be distinguished by their different anomalous dimensions $h_{s, I}$. Operators with different dimensions are clearly orthogonal and define different channels $I$, but in interacting $\mathrm{CFT}_{4}$ 's there might be operators with the same spin and the same dimension. In that case it is more convenient to triangulate the matrix of two-point functions $M_{s}^{I I^{\prime}}=<\mathcal{J}^{(s, I)} \mathcal{J}^{\left(s, I^{\prime}\right)}>$ rather than diagonalize it. Finally, there is the case of operators with the same spin, the same dimension and the same channel (i.e. the same $I)$. There is now way to remove this residual degeneracy, which is the mixing studied in sect. 2 of [4]: the set of mixing operators $\mathcal{O}_{i}$ with these properties define a unique central charge. Issues like this will be debated at length in the next section.

The tilde on the special projector-invariant SP denotes that SP itself is identified up to linear combinations of the other two projector-invariants $\prod^{(2,2 ; 2)}$. The combination depends on the theory.

From the general form of the OPE we observe that the higher spin symmetry is precisely the infinite symmetry characterizing free field theory, since only when all $h$ 's are zero the higher spin currents are conserved. The set of operators appearing in the OPE is different for scalar, spinor and vector fields, so we can say that there are three known types of such infinite symmetries. There could be more, in particular in the domain of higher derivative theories [21], where it might be possible to preserve some flavor higher spin symmetry in presence of certain interactions. This research is under consideration.

In conclusion, we identify the primary currents of spin $s$ via the $T T$ OPE. This we can do, for the moment, in the UV and IR limits, i.e. we know how to study the initial and final "positions" $\mathrm{CFT}_{U V}$ and $\mathrm{CFT}_{I R}$ of the RG trajectory. In the next section we will discuss the off-critical extension of this analysis. The reader should keep in mind the general viewpoint described in the introduction, since we are proceeding precisely as represented in the second part of Fig. 1.

We now make several comments on interacting conformal field theories. Concrete examples in which one can study these issues using perturbation theory are provided by supersymmetric theories. The most singular term in the TT OPE after the central charge contains the lowest component $\bar{\varphi} \varphi=\Sigma$ of the Konishi superfield [5], which is indeed anomalous dimensioned,

$$
T T \sim \frac{c}{|x|^{8}}+\frac{\bar{\varphi} \varphi}{|x|^{6-h}}+\cdots
$$

This is true in any $\mathrm{N}=1$ supersymmetric theory (so, also off-criticality; see [6] for the geeral expression of $h$ to the lowest order). Superfield formulas are presented in the appendix and they relate the free field currents of scalar, spinor and vector fields. Other components of the Konishi $\mathrm{N}=1$ superfield are the Konishi current (which is basically $\mathcal{A}^{(1)}$ in the notation of the present paper) and the Kinetic Lagrangian for matter multiplets. Therefore these operators are
not conserved and have the same anomalous dimension $h$.
Now, let us consider $\mathrm{N}=4$ supersymmetric Yang-Mills theory. The extended supersymmetry relates $\bar{\varphi} \varphi$ to various other operators (the components of a hypothetical "N=4 superfield") not contained in the $\mathrm{N}=1$ Konishi superfield [23]. Among these operators there is one of the three spin- 4 currents (already reduced to two by supersymmetry) that appear in the OPE, as well as a spin-3 current and other components.

Therefore, the "secondary central charge" $c^{\prime}$ studied in [6] at the second loop order in perturbation theory is also one of the spin- 4 central charges in the language of the present paper. Moreover, the anomalous dimension $h$ computed in [5] and [6] for the Konishi operator is also the anomalous dimension of the spin-4 current [23]. Since $h$ is nonvanishing already at the one loop level, precisely $h=3 N_{c} \alpha / \pi$ [5], this clearly shows that the spin- 4 current is not conserved even if the theory is conformal. The Konishi $\mathrm{N}=4$ supermultiplet does not contain a spin-5 current [23], so there is necessarily a further superfield appearing in the singular terms of the OPE. Actually, one can show that there are two [16].


Fig. 2: Dynamical phenomena of higher spin currents: exemplification.

In the strongly coupled large $N_{c}$ limit, according to a recent conjecture [8], there is a certain correspondence between $\mathrm{N}=4$ supersymmetric Yang-Mills theory with gauge group $S U\left(N_{c}\right)$ and supergravity on Anti de Sitter space. Calculations based on this correspondence [24] show that $h$ should grow indefinitely in this limit and therefore that the mixing terms should disappear from the OPE [23]. Assuming that this is correct, we see that the OPE (4.76) with $h_{s, I}=\infty$ will contain only the identity operator and the stress tensor itself (the special term SP), together with its descendants. We conclude that conformal field theories in the strongly coupled large $N_{c}$ limit are the best analogues of two dimensional conformal field theories. Correspondingly, they should be much more symmetric than ordinary conformal field theories in four dimensions. It is compulsory to look for an eventual "infinite symmetry" characterizing these special theories [25], presumably rather different from the higher spin symmetries of free field theories. Higher spin flavour symmetry could be the key-concept for a complete classification of conformal field theories in four dimensions.

We have stressed that an interesting phenomenon occurring in our investigation is the splitting of the $I$-degeneracy, depicted in Fig. 2. We do not have examples in families of conformal field theories. Presumably it is possible to construct such examples in finite $\mathrm{N}=2$ theories. We do have examples for RG flows interpolating between pairs of $\mathrm{CFT}_{4}$ 's. In particular, there is one case in which this phenomenon is evident [5, 26], also at the nonperturbative level. We briefly describe it.

Consider $\mathrm{N}=1$ supersymmetric QCD with $N_{c}$ colors and $N_{f}$ flavors. In the conformal window $3 / 2 N_{c}<N_{f}<3 N_{c}$ this theory (called "electric") has an IR fixed point, expected to be equivalent to the IR fixed point of a very different theory, the "magnetic" dual [1]. The magnetic theory contains gluons, quarks and mesons, and a superpotential. In the neighborhood $N_{f} \lesssim 3 N_{c}$, the IR fixed point is weakly coupled from the electric point of view and strongly coupled from the magnetic point of view. From the electric point of view, the TT OPE outlines a single operator $\Sigma$, with a nonvanishing small anomalous dimension $h$. One can indeed use perturbation theory and the unicity of $\Sigma$ follows from power counting. See [5] for the value of $h$. Instead, the magnetic theory exhibits a case of $I$-degeneracy. In the UV limit the $T T$ OPE contains a single $\Sigma$ operator, which is however the sum of two Konishi currents, of the quarks and of the mesons. These two Konishi currents have no reason to be parts of a single operator. Perturbative calculations around the magnetic UV fixed point show that indeed the two components of $\Sigma$ are splitted by acquiring different (and both nonvanishing) anomalous dimensions.

The RG flow of the magnetic theory to the IR limit is beyond perturbation theory. It can be studied by assuming electric-magnetic duality, which implies that one (and only one) of the anomalous dimensions remains small, while the other one, say $h_{m}$, grows enough to move the mesonic Konishi operator $\Sigma_{m}$ very far away, actually in the regular terms, as we now prove. This does not mean that $\Sigma_{m}$ has no electric companion, rather that the electric partner is a higher dimensioned composite operator (according to the identification $M_{j}^{i}=Q^{i} \tilde{Q}_{j}$ of the meson field $M_{j}^{i}$ as a product of electric quark fields). In the limit $N_{f}=3 N_{c}$, where the electric theory is free, $\Sigma_{m}$ disappears from the electric OPE's (a free OPE does not contain it) and
therefore $h_{m}$ tends to infinity in this limit. This phenomenon is similar to the one occurring in the large $N_{c}$ limit of $N=4$ supersymmetric Yang-Mills theory [24].

We have depicted this behavior in Fig. 2 (the two lines exiting from the dot labelled "spin 0 "). Regular terms can become singular and other similar situations can occur. The other situations appearing in Fig. 2 are purely illustrative.

The operators appearing in the TT OPE are natural deformations of the theory [5]. This is particularly relevant in the supersymmetric case where the anomalous dimension of the Konishi operator $\Sigma$ equals the derivative of the beta-function at criticality. In the non-supersymmetric case $\Sigma=\varphi^{2}$ is just the mass deformation. The other operators represent nontrivial axialcurrent deformations, spin- 2 deformations, higher spin deformations. We finally remark that the equivalence of two conformal field theories necessarily implies equivalence of the quantum field theories obtained by these deformations (this is the idea of e-m universality, see ref. [4]: the observation that a critical theory carries nontrivial information about the quantum field theories of which it can be the limit).

## 5 Higher spin central charges and central functions.

The definition of central functions $c_{s, I}(g)$ interpolating between the critical values of the higher spin central charges proceeds as in sect. 2 of ref. [4]. One identifies the higher spin current $\mathcal{J}^{(s, I)}$ as a channel of the stress-tensor four-point function $\left.<T_{1} T_{2} T_{3} T_{4}\right\rangle$ in the limit in which $T_{1}$ is close to $T_{2}$ and $T_{3}$ is close to $T_{4}$, see Fig. 3. The general form of the two-point function $\left\langle\mathcal{J}^{(s, I)} \mathcal{J}^{(s, I)}\right\rangle$, formula (3.55), has to be combined with the $T T$ OPE studied in the previous section. We do not repeat here the cut-and-paste construction of $c_{s, I}(g)$, but we assume and use it. We just recall that this construction assures that $c_{s, I}(g)$ depends only on the running coupling constant $g$ notwithstanding the $h$-violation of the conservation condition, so that the problem of sect. 3 is circumvented. We recall that the discussion of most of that section cannot be applied to ordinary quantum field theory as it stands, since it assumes the existence of a higher spin flavor symmetry.

Identifying $\mathcal{J}^{(s, I)}$ as a channel of correlators of conserved currents eliminates a certain amount of arbitrariness in the definitions of $\mathcal{J}^{(s, I)}$ itself. For example, the overall constants of the quantities $c_{s, I}(g)$ are fixed, since the external legs of our correlators are always stresstensors. Secondly, we had no way to fix the relative factors in front of the scalar, spinor and vector contributions to the currents. Now we know that this problem is related to the $I$-degeneracy and that it is removed by stating that the appropriate combinations of projectors and operators have to be read via the $T T$ OPE. As we said, the dynamics of the theory is deeply involved, so one needs to go at least to two loops to appreciate this effect. Thirdly, other nuisances, like scheme dependence, are automatically bypassed by this construction and the functions $c_{s, I}(g)$ are physical.

In practice, to make the comparison between our construction and the notions commonly considered in the theory of deep inelastic scattering [7] we are replacing the electron and parton with two or more gravitons, in order to turn those techniques into useful tools for a theoretical


Figure 3: Construction of $c_{4}$.
investigation of conformal windows and conformal field theories. The central functions of [4] have therefore this "physical" interpretation.

We consider a theory with $N_{0}$ real scalars, $N_{1 / 2}$ Dirac spinors and $N_{1}$ vector fields. The $\Sigma$-channel was studied extensively in [6], where the central charge $c_{0}$ was called $c^{\prime}$ :

$$
c_{0}=N_{0} .
$$

Precisely, le structure of the channel is

$$
\frac{1}{18}\left(\frac{1}{4 \pi^{2}}\right)^{4} \prod_{\mu \nu, \rho \sigma}^{(2)}\left(\frac{1}{|x-y|^{4}}\right) \frac{N_{0}}{|(x+y-z-w) / 2|^{4}} \prod_{\alpha \beta, \gamma \delta}^{(2)}\left(\frac{1}{|z-w|^{4}}\right) .
$$

In the spin- 2 channel we observe the first case if $I$-degeneracy. There are three operators, the stress tensors $T_{0}, T_{1 / 2}$ and $T_{1}$ of scalar, spinor and vector fields. One combination is precisely the stress-tensor of the theory, $T=T_{0}+T_{1 / 2}+T_{1}$, identified by the projector SP , with canonical dimension. $T$ does not mix with the other two for obvious reasons and so we know a priori how this kind of $I$-degeneracy is removed.

The central function $c_{(2)}^{T}$ associated with the $T$-channel of the four-point function of the stress-tensor is just the function $c(g)$ of (3.32). The one-loop value is

$$
c_{2}^{T}=N_{0}+6 N_{1 / 2}+12 N_{1}
$$

and the structure of the channel can be read by combining the appropriate OPE term with the two-point function itself,

$$
\frac{1}{60}\left(\frac{1}{4 \pi^{2}}\right)^{4} \mathrm{SP}_{\mu \nu, \rho \sigma ; \varepsilon \zeta}(x-y) \prod_{\varepsilon \zeta, \iota \kappa}^{(2)}\left(\frac{c_{2}^{T}}{|(x+y-z-w) / 2|^{4}}\right) \mathrm{SP}_{\alpha \beta, \gamma \delta ; \kappa \kappa}(z-w)
$$

As we said, there are other two spin-2 central charges, corresponding to combinations of the two invariants of (4.66). We cannot say which precise combinations split the degeneracy since our analysis is one-loop, but we can make some observations. We choose to present the one-loop formulas for the operators $\mathcal{O}_{2}=T_{0}+T_{1 / 2}$ and $\mathcal{O}_{3}=T_{0}$ (we put $\mathcal{O}_{1}=T$ ) and we call the respective charges $c_{2}^{(0+1 / 2)}$ and $c_{2}^{(0)}$. The corresponding projectors, $\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)(s c+f)}$ and
$\prod_{\mu \nu, \rho \sigma ; \alpha \beta}^{(2,2 ; 2)(s c)}$ are, respectively, the sum of the second and third term of formula (4.73) and the difference between (4.68) and (4.73).

The purpose of this choice is to triangulate the matrix of two-point functions, actually to impose an even further constraint $<\mathcal{O}_{i} \mathcal{O}_{j}>=<\mathcal{O}_{i} \mathcal{O}_{i}>$ for $i<j$. This constraint reduces the set of independent entries to the diagonal ones. Instead, orthogonality $<\mathcal{O}_{i} \mathcal{O}_{j}>=0$ for $i \neq j$ would require awkward denominators like $1 / N_{0}$ and appears therefore meaningless (the simplest orthogonal operators are just $T_{0}, T_{1 / 2}$ and $T_{1}$ but dynamics makes it compulsory to isolate $T$, due to its special nature; this shows that the $I$-degeneracy is a nontrivial issue). We have simply

$$
c_{2}^{(s c+f)}=N_{0}+6 N_{1 / 2}, \quad c_{2}^{(s c)}=N_{0} .
$$

The structure of the channel reads

$$
\begin{align*}
\frac{1}{60}\left(\frac{1}{4 \pi^{2}}\right)^{4} & \prod_{\mu \nu, \rho \sigma ; \varepsilon \zeta}^{(2,2 ; 2, I)}\left(|x-y|^{2} \ln |x-y|^{2} M^{2}\right) \prod_{\varepsilon \zeta, \iota \kappa}^{(2)}\left(\frac{c_{2}^{(I)}}{|(x+y-z-w) / 2|^{4}}\right) \\
\times & \prod_{\alpha \beta, \gamma \delta ; \iota \kappa}^{(2,2 ; 2 I)}\left(|z-w|^{2} \ln |z-w|^{2} M^{2}\right), \tag{5.77}
\end{align*}
$$

with $I=s c+f$ or $I=s c$. We stress again that the issue of $I$-degeneracy demands for a two-loop analysis.

The discussion can be repeated in the other cases. For odd spin there is no $I$-degeneracy at the level of our investigation, for spin-4 there is a triple degeneracy. The formulas for the channels and charges can be written straightforwardy following the general recipe.

Off-criticality the OPE contains several quantities and operators that vanish or disappear at criticality. First of all, many more projectors appear, since $T_{\mu \nu}$ is not traceless. These are new channels of the many-point functions of the stress tensor. We can nevertheless focus on the terms carried by our traceless projectors and treat the other ones apart (i.e. in the OPE's $T_{\mu \nu}(x) T(y)$ and $\left.T(x) T(y)\right)$. More importantly, there are new functions in the same channel, say $\mathcal{J}^{(s)}$. These are the $c_{s, s^{\prime}}$ of section 3, formula (3.55). They change the internal structure of the channel (the middle factor of (5.77)), but not the external structure. Combining the discussions of sections 3 and 4 , the full set of off-critical functions is labelled by the spin $s$, the mixing spin $s^{\prime}$ and the index $I$. Precisely, $c_{s, s^{\prime}}^{I}, s^{\prime}=s, \ldots 0, c_{s, s}^{I}=c_{s}^{I}$. Each function will have its own operator in the OPE, the associated operator component of the $\mathcal{J}^{(s, I)}$-divergence (see also (3.52)). For example, $c_{2,0}^{T}$ is the function called $f$ in formula (3.32). Its operator is the trace $\Theta$ of the stress-tensor (which indeed represents a spin-2-0 mixing). $c_{2,0}^{T}$ is the central function associated with the $\Theta$-channel of the four-point function of the stress tensor.

Only the quantities $c_{s}^{I}$ interpolate between nonvanishing critical values. The other quantities $c_{s, s^{\prime}}^{I}, s^{\prime} \leq s-2$, typically vanish like $\beta^{2}$ at criticality. Examples are the function $c_{2,0}^{T}$ or, in two dimensions, the derivative of Zamolodchikov's $c$-function [27]. These functions might be relevant to prove the $a$-theorems of sect. 3 and [2].

By considering many-point functions of the stress tensor it is possible to generate channels of arbitrary spin. Taking the limits which pairs of $T$ 's become close, one generates, for example,


Figure 4: Higher spin central charges from many-point functions of the stress tensor.
spin-4 operators $\mathcal{J}^{(4)}$. Then taking the limit in which two $\mathcal{J}^{(4)}$ 's become close one can generate spin-8 operators $\mathcal{J}^{(8)}$ and so on. Two $\mathcal{J}^{(4)}$ 's also generate lower spin operators, for example some $\mathcal{J}^{(2)}$ 's (recall that two stress tensors produce also a spin- 0 operator $\Sigma$ ), which are clearly linear combinations of the known ones. In conclusion, we have in general three central charges for each even spin greater than zero (vector, spinor and scalar), two central charges for each odd spin grater than one (vector and spinor), one central charge for spin 0 (scalar) and one central charge for spin 1 (spinor).

## 6 Outlook.

In this paper we have worked out a general set-up for the application of techniques similar to those familiar in the context of deep inelastic scattering to conformal field theories in four dimensions and quantum field theories interpolating between pairs of them.

The various issues that we have considered constitute in some sense the ground level of our project, which is the investigation of quantum field theory as a radiative interpolation between pairs of conformal field theories. We have revealed the nature of the "secondary central charges" introduced in [5, 6, 4], but several questions remain unanswered and a large part of the work remains to be done. In particular: two-loop calculations like the one of 6] would be desirable; several parts of the analysis presented here in four dimensions should be performed also in the $\varepsilon \rightarrow 0$ limit starting from $4-\varepsilon$ dimensions; higher spin current OPE's and many- $T$-point functions should be investigated; higher spin anomalies should be understood better, in particular the quantities $a_{s}$; theories with higher spin flavor symmetries (admitting that they exist) should be classified; etc. Non-perturbative arguments like e-m duality or the AdS/CFT correspondence might prove useful to reach a better understanding of the phenomena that we have discussed, in particular the splitting of the $I$-degeneracy, and maybe to compute exact IR values of the secondary central charges. It would be great to achieve for the higher spin central charges the same success achieved in [2] for the spin-1 and spin-2 ones. Numerical
computations are under consideration. Last but not least, we mention again the main objective, which is to provide a suitable framework for the formulation of an action principle for the RG trajectory connecting the UV and the IR conformal limits. We will report soon on some of these developments.

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## 7 Appendix: higher spin tensor currents and supersymmetry.

We use the conventions of the book Superspace [28]. Spinor derivatives are defined by

$$
D_{\alpha}=\frac{\partial}{\partial \theta^{\alpha}}+\frac{i}{2} \theta^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}}, \quad D_{\dot{\alpha}}=\frac{\partial}{\partial \theta^{\dot{\alpha}}}+\frac{i}{2} \theta^{\alpha} \partial_{\alpha \dot{\alpha}}
$$

Chiral fields $\Phi$ satisfy $D_{\dot{\alpha}} \Phi=D_{\alpha} \bar{\Phi}=0$. The field equations read $\bar{D}^{2} \bar{\Phi}=D^{2} \Phi=0$. For vector chiral superfields we have $D_{\dot{\alpha}} W_{\alpha}=D_{\alpha} W_{\dot{\alpha}}=0$. The reality condition, combined with the field equations, imposes $D_{\dot{\alpha}} W^{\dot{\alpha}}=D_{\alpha} W^{\alpha}=0$. We recall that $\left\{D_{\alpha}, D_{\beta}\right\}=\left\{D_{\dot{\alpha}}, D_{\dot{\beta}}\right\}=0$ and $\left\{D_{\alpha}, D_{\dot{\alpha}}\right\}=i \partial_{\alpha \dot{\alpha}}$.

The stress tensors superfields for vector and matter multiplets can be written as

$$
J_{\alpha \dot{\alpha}}=W_{\alpha} W_{\dot{\alpha}}, \quad J_{\alpha \dot{\alpha}}=\frac{1}{3} \bar{\Phi} \overleftrightarrow{D}_{\dot{\alpha}} \overleftrightarrow{D}_{\alpha} \Phi
$$

The latter is a condensed notation that we use in this appendix. It has to be meant as follows. One writes down all possible terms, starting from $\bar{\Phi} D_{\dot{\alpha}} D_{\alpha} D_{\dot{\beta}} D_{\beta} \cdots \Phi$ and moving derivatives according only to their statistics (i.e. neglecting (anti-)commutators) plus an additional minus sign any time a derivative is moved from $\Phi$ to $\bar{\Phi}$. In this process, terms like $\bar{\Phi} D_{\dot{\alpha}} D_{\alpha} D_{\beta} D_{\dot{\beta}} \Phi$ are not meant to vanish, rather derivatives have to be anticommuted till they give the only nonvanishing combination. Concretely, $\bar{\Phi} D_{\dot{\alpha}} D_{\alpha} D_{\beta} D_{\dot{\beta}} \Phi \rightarrow-\bar{\Phi} D_{\dot{\alpha}} D_{\alpha} D_{\dot{\beta}} D_{\beta} \Phi$.

The spin- $3 \mathrm{~N}=1$ superfields are, for matter and vector multiplets respectively,

$$
\begin{aligned}
J_{\alpha \beta \dot{\alpha} \dot{\beta}}= & \bar{\Phi} \overleftrightarrow{D}_{\dot{\alpha}} \overleftrightarrow{D}_{\alpha} \overleftrightarrow{D}_{\dot{\beta}} \overleftrightarrow{D}_{\beta} \Phi=\bar{\Phi} D_{\dot{\alpha}} D_{\alpha} D_{\dot{\beta}} D_{\beta} \Phi-2 D_{\dot{\alpha}} \bar{\Phi} D_{\alpha} D_{\dot{\beta}} D_{\beta} \Phi \\
& -4 D_{\alpha} D_{\dot{\alpha}} \bar{\Phi} D_{\dot{\beta}} D_{\beta} \Phi+2 D_{\dot{\alpha}} D_{\alpha} D_{\dot{\beta}} \bar{\Phi} D_{\beta} \Phi+D_{\alpha} D_{\dot{\alpha}} D_{\beta} D_{\dot{\beta}} \bar{\Phi} \Phi \\
J_{\alpha \dot{\alpha} \dot{\beta}}= & D_{\dot{\beta}} D_{\beta} W_{\alpha} W_{\dot{\alpha}}-\frac{1}{2} D_{\beta} W_{\alpha} D_{\dot{\beta}} W_{\dot{\alpha}}-W_{\alpha} D_{\beta} D_{\dot{\beta}} W_{\dot{\alpha}}
\end{aligned}
$$

The spin-4 superfield currents are

$$
\begin{aligned}
J_{\alpha \beta \gamma \dot{\alpha} \dot{\beta} \dot{\gamma}}= & \bar{\Phi} \overleftrightarrow{D}_{\dot{\alpha}} \overleftrightarrow{D}_{\alpha} \overleftrightarrow{D}_{\dot{\beta}} \overleftrightarrow{D}_{\beta} \overleftrightarrow{D}_{\dot{\gamma}} \overleftrightarrow{D}_{\gamma} \Phi \\
J_{\alpha \beta \gamma \dot{\alpha} \dot{\beta} \dot{\gamma}}= & D_{\dot{\beta}} D_{\beta} D_{\dot{\gamma}} D_{\gamma} W_{\alpha} W_{\dot{\alpha}}-D_{\beta} D_{\dot{\gamma}} D_{\gamma} W_{\alpha} D_{\dot{\beta}} W_{\dot{\alpha}}-3 D_{\dot{\gamma}} D_{\gamma} W_{\alpha} D_{\beta} D_{\dot{\beta}} W_{\dot{\alpha}} \\
& +D_{\gamma} W_{\alpha} D_{\dot{\gamma}} D_{\beta} D_{\dot{\beta}} W_{\dot{\alpha}}+W_{\alpha} D_{\beta} D_{\dot{\beta}} D_{\gamma} D_{\dot{\gamma}} W_{\dot{\alpha}}
\end{aligned}
$$

Complete symmetrization in dotted and undotted indices are understood. Decomposing these superfields one obtains automatically the improved versions of the currents. Similar formulas can be written down for generic spin $s$. Conservation and simple tracelessness are imposed by symmetry in the indices and by the superfield condition $D^{\alpha_{i}} J_{\alpha_{1} \alpha_{1} \cdots \alpha_{s} \alpha_{s}}=0 \forall i$.

## References

[1] N. Seiberg, Electric-Magnetic Duality in Supersymmetric Non-Abelian Gauge Theories, Nucl. Phys. B435 (1995) 129.
[2] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, Nonperturbative formulas for central functions in supersymmetric theories, Nucl. Phys. B and hepth/9708042.
[3] D. Anselmi, J. Erlich, D.Z. Freedman and A.A. Johansen, Positivity constraints on anomalies in supersymmetric gauge theories, Phys. Rev. D57 (1998) 7570 and hepth/9711035.
[4] D. Anselmi, Central functions and their physical implications, JHEP 05 (1998) 005 and hep-th/9702056.
[5] D. Anselmi, M.T. Grisaru and A.A. Johansen, A critical behaviour of anomalous currents, electric-magnetic universality and $\mathrm{CFT}_{4}$, Nucl. Phys. B 491 (1997) 221 and hepth/9601023.
[6] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A Johansen, Universality of the operator product expansions in $\mathrm{SCFT}_{4}$, Phys. Lett. B324 (1997) 329.
[7] See for example T. Muta, Foundations of quantum chromodynamics an introduction to perturbative methods in gauge theories, World Scientific, Singapore, 1987.
[8] J.M. Maldacena, The large N limit of superconformal field theories and supergravity, hepth/9705104.
[9] R. Jackiw in Current algebra and anomalies, S.B. Treiman et al., World Scientific, Singapore, 1985.
[10] F.A. Berends, G.J.H. Burgers and H. Van Dam, Explicit construction of conserved currents for massless fields of arbitrary spin, Nucl. Phys. B 271 (1986) 429;
On the theoretical problem in constructing interactions involving higher-spin massless particles, Nucl. Phys. B 260 (1985) 295.
[11] M.A. Vasiliev, Higher spin gauge theories in four, three and two dimensions, Int. J. Mod. Phys. D5 (1996) 763 and hep-th/9611024.
[12] L.P.S. Singh and C.R. Hagen, Lagrangian formulation for arbitrary spin. I. The boson case, Phys. Rev. D9 (1974) 898;
C. Fronsdal, Massless fields with integer spin, Phys. Rev. D18 (1978) 3624;
C. Fronsdal, Singleton and massless, integral spin fields on de Sitter space (elementary particles in a curved space VII), Phys. Rev. D20 (1979) 848;
T. Curtright, Massless spin supermultiplets with arbitrary spin, Phys. Lett. B85 (1979) 219.
M.A. Vasiliev, Sov. J. Nucl. Phys. 32 (1980) 855 (p. 439 in English trasl.);
C. Aragone and S. Deser, Higher spin vierbein gauge fermions and hypergravities, Nucl. Phys. B170 [FS1] (1980) 329.
[13] B. de Wit and D.Z. Freedman, Systematics of higher spin gauge fields, Phys. Rev. D21 (1980) 358.
[14] S. Ferrara, R. Gatto and A.F. Grillo, Conformal algebra in space-time, Springer-Verlag, Berlin, 1967;
S. Ferrara, A.F. Grillo and G. Parisi, The shadow operator formalism for conformal algebra. Vacuum expectation values of operator products, Lett. Nuovo Cimento 4 (1972) 115;
S. Ferrara, R. Gatto, A.F. Grillo and G. Parisi,Canonical scaling and conformal invariance, Phys. Lett. B38 (1972) 333.
[15] See for example, E.J. Schreier, Conformal symmetry and three-point functions, Phys. Rev. D3 (1971) 980;
M. Baker and K. Johnson, Applications of conformal symmetry in quantum electrodynamics, Physica 96A (1979) 120.
[16] D. Anselmi, Higher spin anomalies, to appear.
[17] J.C. Collins, Renormalization, Cambridge University Press, 1985.
[18] See [20, 5] and A.C. Petkou, Conserved currents, consistency relations and operator product expansions in the conformally invariant O(N) vector model, Ann. Phys. 249 (1996) 180.
[19] S. Ferrara, R. Gatto and A.F. Grillo, Conformal invariance on the light cone and canonical dimensions, nucl. Phys. B 34 (1971) 349.
[20] Y. Stanev, Stress energy tensor and U(1) current operator product expansions in conformal QFT, Bulgarian Journal of Physics 15 (1988) 93.
[21] I. Antoniadis, P.O. Mazur, E. Mottola, Physical states of the quantum conformal factor, Phys. Rev. D55 (1997) 4770.
[22] K. Konishi, Anomalous supersymmetry transformations of some composite operators in SQCD, Phys. Lett. B135 (1984) 439.
[23] S. Ferrara and A. Zaffaroni, Bulk gauge fields in AdS supergravity and supersingletons, hepth/9807090.
[24] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Gauge theory correlators from non-critical string theory, hepth/9802109.
[25] I am grateful to L. Palacios about this remark.
[26] This observation is taken from an old unpuplished correspondence with A.A. Johansen.
[27] A.B. Zamolodchikov, "Irreversibility" of the Flux of the Renormalization Group in a 2D Field Theory, JETP Lett. 43 (1986) 730.
[28] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, Superspace, Addison Wesley, Reading, MA (1983).


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[^1]:    ${ }^{2}$ See for example [6] for a three-point correlator constructed with this technique.

