SISSA/ISAS 90/94/EP
hepth@xxx/9407023
July, 1994

MORE ON THE SUBTRACTION ALGORITHM [1]

Damiano Anselmi

International School for Advanced Studies (ISAS), via Beirut 2-4, I-34100 Trieste, Italia
and Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Trieste, Trieste, Italia

Abstract

We go on in the program of investigating the removal of divergences of a generical quantum gauge field theory, in the context of the Batalin-Vilkovisky formalism. We extend to open gauge-algebræ a recently formulated algorithm, based on redefinitions $\delta\lambda$ of the parameters λ of the classical Lagrangian and canonical transformations. The key point is to generalize a well-known conjecture on the form of the divergent terms to the case of open gauge-algebræ. We also show that it is possible to reach a complete control on the effects of the subtraction algorithm on the space $\mathcal{M}_{gf}$ of the gauge-fixing parameters. We develop a differential calculus on $\mathcal{M}_{gf}$ providing an intuitive geometrical description of the fact the on shell physical amplitudes cannot depend on $\mathcal{M}_{gf}$. A principal fiber bundle $\mathcal{E} \to \mathcal{M}_{gf}$ with a connection ω_1 is defined, such that the canonical transformations are gauge transformations for ω_1. A geometrical description of the effect of the subtraction algorithm on the space $\mathcal{M}_{ph}$ of the physical parameters λ is also proposed. At the end, the full subtraction algorithm can be described as a series of diffeomorphisms on $\mathcal{M}_{ph}$, orthogonal to $\mathcal{M}_{gf}$ (under which the action transforms as a scalar), and gauge transformations on $\mathcal{E}$. In this geometrical context, a suitable concept of predictivity is formulated. Finally, we give some examples of (unphysical) toy models that satisfy this requirement, though being neither power counting renormalizable, nor finite.

[1]Partially supported by EEC, Science Project SC1*-CT92-0789.

1 Introduction

The Batalin-Vilkovisky formalism [1] provides a useful set-up for the quantization of Lagrangian field theories. It is now attracting more and more attention, since the proofs of known results can be considerably simplified and generalizations are straightforward (see for example [2, 3]).

In a recent paper [4], we formulated, in the framework of the Batalin-Vilkovisky formalism, an algorithm for removing the divergences of a quantum gauge field theory. It preserves a suitable generalization of gauge-invariance and BRS-invariance, keeps the on-shell physical amplitudes independent of the gauge-fixing and permits to get a compete control on the involved arbitrariness. The algorithm is based on a series of step-by-step redefinitions of the parameters of the classical Lagrangian and canonical transformations of fields and BRS sources.

In the present paper, we generalize the results of ref. [4] and give a geometrical description of the subtraction algorithm. Such a description is expected to be a source of insight for the classification of predictive quantum field theories.

In ref. [4] the starting theory was supposed to generate a closed gauge-algebra, i.e. an off-shell nilpotent BRS operator s. Moreover, the initial BRS action Σ_0 was assumed to be of the form

$$\Sigma_0(\Phi, K) = \mathcal{L}_{class}(\lambda, \phi) + s\Psi(\Phi) + K_A s\Phi^A. \tag{1.1}$$

In particular, Σ_0 is $\hbar$-independent and satisfies both $(\Sigma_0, \Sigma_0) = 0$ and $\Delta\Sigma_0 = 0$ [and, consequently, also the master equation $(\Sigma_0, \Sigma_0) = 2i\hbar\Delta\Sigma_0$].

One of the purposes of this paper is to show that the results of [4] hold even without assumption (1.1).

We shall use the same notation that was convenient in ref. [4]. For any further detail, see directly ref. [4]. In (1.1) λ denote the parameters that multiply the gauge-invariant terms $\mathcal{G}_i$ of the classical Lagrangian: $\mathcal{L}_{class} = \sum_i \lambda_i \mathcal{G}_i$. $(\,.\,,\,.\,)$ denote the antibrackets and Δ is the Batalin-Vilkovisky delta-operator. ϕ are the classical fields, while Φ is the full set of fields, containing classical fields, ghosts, antighosts, Lagrange multipliers and so on. $\Psi(\Phi)$ is the gauge-fermion and K are the BRS sources. The starting action Σ_0 only depends on K and Ψ *via* the antifield-combination

$$\Phi^*_A = K_A + \frac{\partial \Psi}{\partial \Phi^A}. \tag{1.2}$$

Σ denotes the action, while $Z[J, K]$ is the partition function, $W[J, K]$ is the logarithm of Z and $\Gamma(\Phi, K)$ is the Legendre transform of W with respect to the field sources J. A subscript k marking the functionals (Σ_k, Z_k, W_k, Γ_k and so on) refers to the theory in which the divergences have been removed up to the k^{th}-loop order included: $\Gamma_k = \text{finite} + \mathcal{O}(\hbar^{k+1})$. Moreover, $\Gamma^{(k+1)}_{div}$ denotes the $(k+1)^{th}$-loop divergences of the effective action Γ_k: $\Gamma_k = \text{finite} + \Gamma^{(k+1)}_{div} + \mathcal{O}(\hbar^{k+2})$. $\Gamma^{(k+1)}_{div}$ is a local functional. $< \ldots >_J$ denotes the average of a functional at nonzero sources J.

The operation that removes the order $\hbar^n$ divergences when all lower order divergences are assumed to have already been removed is denoted by $\mathcal{L}_n$. The operation $\mathcal{R}_n = \mathcal{L}_n \circ \mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_1$ then removes all the divergences up to order $\hbar^n$ included.

The property of Σ_0 of depending on K and Ψ only *via* the antifield-combination (1.2) is not guaranteed to hold for Σ_k, $k \geq 1$. Were it so, the gauge-fixing independence of the physical amplitudes would be a direct consequence of the master equation [1]. Instead, the master equation only assures the gauge-fixing independence of the *divergent* amplitudes and the problem is to show that the subtraction algorithm is compatible with this, so that the finite physical amplitudes are also gauge-fixing independent.

In [4] it was also useful to define the nilpotent operator $\sigma = \mathrm{ad}\,\Sigma_0 = (\,.\,, \Sigma_0)$, since $\Gamma^{(n)}_{div}$ was constrained to satisfy

$$\sigma \Gamma^{(n)}_{div} = 0. \tag{1.3}$$

The general form of $\Gamma^{(n)}_{div}$, solution to the above equation, was assumed to be

$$\Gamma^{(n)}_{div} = \mathcal{G}^{(n)}(\phi) + \sigma R^{(n)}(\Phi, K), \tag{1.4}$$

where $\mathcal{G}^{(n)}(\phi) = \sum_i \delta_n \lambda_i \mathcal{G}_i$ is a gauge-invariant functional of the classical fields ϕ. Such a conjecture was first proved for Yang-Mills theories in ref. [6]. There, the usual Lorentz gauge was used. However, the statement is independent on the gauge-choice, since the cohomology of σ is invariant under the canonical transformations that leave the fields Φ unchanged, i.e. such that the generating functional $F(\Phi, K')$ has the form

$$F(\Phi, K') = K'_A \Phi^A + f(\Phi). \tag{1.5}$$

Indeed, due to the fact that the starting action Σ_0 depends on K and Ψ only *via* the antifield combination (1.2), the variation of the gauge-fermion $\Psi(\Phi) \to \Psi'(\Phi)$ corresponds to a peculiar canonical transformation of the form (1.5) with $f = \Psi - \Psi'$. The set of canonical transformations of the form (1.5) will be called the *little group* of canonical transformations.

Property (1.4) was assumed by Stelle in [7] for higher derivative quantum gravity. It has been recently proved for matter coupled Yang-Mills theory with a semi-simple gauge-group in ref. [2], where some peculiar features of abelian gauge groups are also pointed out.

Notice that conjecture (1.4) is not independent of the choice of variables, since it is not invariant under the most general canonical transformation. When quantizing a classical theory, a “boundary condition” [1] is imposed on the action Σ when solving the master equation

$$(\Sigma, \Sigma) = 2i\hbar\Delta\Sigma, \tag{1.6}$$

namely the condition that the order zero part S of the action Σ (let us write $\Sigma = S + \sum_{n=1}^{\infty} \hbar^n M_n$) reduces to the classical Lagrangian $\mathcal{L}_{class}(\phi, \lambda)$, when the antifields Φ^* are set to zero:

$$S|_{\Phi^*=0} = \mathcal{L}_{class}(\phi, \lambda). \tag{1.7}$$

This condition is invariant under the little group of canonical transformations, but not under the full group, precisely as (1.4). Thus, condition (1.4) is a statement about the compatibility of the variables that appear in $\mathcal{G}^{(n)}$ and in the boundary condition (1.7).

In section 5 we shall give a geometrical interpretation of (1.4) according to which $\mathcal{G}^{(n)}(\phi)$ is the Lie derivative of S along a vector field tangent to the space of the parameters λ and orthogonal to the space of gauge-fixing parameters. Compatibility of (1.4) and (1.7) then means that the content of the σ-cohomology should be fully encoded in the classical Lagrangian $\mathcal{L}_{class}$.

The meaning of (1.4) is that the divergent terms can be of two types: the "Lagrangian type" terms $\mathcal{G}^{(n)}(\phi)$, that are removed by redefining the parameters λ of the classical Lagrangian and the "gauge-fermion type" terms $\sigma R^{(n)}(\Phi, K)$, that are removed by a canonical transformation on fields Φ and BRS sources K, with a generating functional

$$F^{(n)}(\Phi, K') = K'_A \Phi^A + R^{(n)}(\Phi, K'). \tag{1.8}$$

In ref. [4] we proved a "no-mixing" theorem, that states that the redefinitions of λ are independent of the gauge-fermion Ψ. In other words, the gauge-fermion does not mix with the classical Lagrangian, although the classical Lagrangian mixes with the gauge-fermion (i.e. $R^{(n)}$ depends on λ, although $\mathcal{G}^{(n)}$ does not depend on Ψ). This fact assures that the on-shell physical amplitudes are independent of the gauge-fixing parameters.

We identified the following sequence of generalizations of gauge-transformations

$$\delta_{gauge} \rightarrow s \rightarrow \sigma \rightarrow \Omega \rightarrow \mathrm{ad}\,\Gamma. \tag{1.9}$$

Every operator, except for the gauge-transformation operator δ_{gauge}, is nilpotent:

$$s^2 = 0, \quad \sigma^2 = 0, \quad \Omega^2 = 0, \quad (\mathrm{ad}\,\Gamma)^2 = 0\,. \tag{1.10}$$

The BRS operator s only acts on the fields Φ and not on the BRS sources K. For closed gauge-algebræ it coincides with δ_{gauge} on the classical fields ϕ. σ is a generalization of s to the space of fields and BRS sources. It coincides with s on the fields Φ. As we have anticipated, the cohomological content of the operator σ is invariant under the little group of canonical transformations (1.5). $\Omega = \mathrm{ad}\,\Sigma - i\hbar\Delta$ is a further generalization of σ, in the sense that the Ω-cohomology is invariant under the most general canonical transformation. All the previous operators act on "integrand functionals". The operator $\mathrm{ad}\,\Gamma$ represents a generalization of Ω acting on "integrated functionals", i.e. the average values of the "integrand functionals".

The nice feature of the algorithm is that what happens is clearly *visible*, even when some nonrenormalizable vertices[2] are present (of whatever origin: exotic gauge-fixing, highly dimensioned composite operators, genuine nonrenormalizability of the theory). We showed that the useful formulæ holding for the initial action Σ_0 can be naturally

[2]To avoid misunderstandings, our concept of nonrenormalizability refers to power-counting nonrenormalizability [5].

"propagated" (*via* the subtraction algorithm) to the "renormalized" action Σ_∞ and to the Γ-functional.

We noticed that the algorithm can be viewed as a way of implementing the *principle of correspondence*: one starts from some convenient *classical* variables and looks for the *correct* quantum variables and quantum parameters. The assumptions made in ref. [4] assure that the divergences can be reabsorbed by simply redefining those quantities (fields, parameters and sources) that are already present at the classical level [3]: the idea is that nothing can be added, but anything can be adjusted.

The requirement of a precise *correspondence* between classical and quantum worlds is quite natural: indeed, since we are only able to explore the quantum world by means of classical instruments, what we observe is only the part of the quantum world that has a correspondence with the classical one. Since we are not allowed to say that there exists more than what we see, we conclude that the whole quantum world is in correspondence with the classical one.

The first purpose of this paper is to show that some of the assumptions that were made in ref. [4] can be relaxed, i.e. that the subtraction algorithm can be applied in more general cases than those considered in ref. [4]. For example, the results also apply to open gauge-algebræ, i.e. gauge field theories such that the BRS operator s is not off-shell nilpotent, rather $s^2 = 0$ only on shell. In this case, the starting action Σ_0 is not linear in K, so that it is not possible to restrict to theories in which Σ_0 has the form (1.1). Moreover, Σ_0 is not required to satisfy both $(\Sigma_0, \Sigma_0) = 0$ and $\Delta\Sigma_0 = 0$ separately, but only the full master equation $(\Sigma_0, \Sigma_0) = 2i\hbar\Sigma_0$. The order $\hbar^0$ part of this equation gives

$$(S, S) = 0, \tag{1.11}$$

i.e. the order $\hbar^0$ part S of the starting action Σ_0 satisfies the so-called "classical master equation". The correct definition of the operator σ is then

$$\sigma = \mathrm{ad}\, S, \tag{1.12}$$

instead of $\mathrm{ad}\,\Sigma_0$. Indeed $\sigma^2 = 0$ is assured by (1.11) and not by the master equation (which assures the nilpotency of Ω). Moreover, the BRS operator s corresponds to

$$s\Phi^A = \left.\frac{\partial_l S}{\partial K_A}\right|_{K=0}. \tag{1.13}$$

We see that σ no more coincides with the BRS operator s on the fields Φ, rather, $s\Phi^A = \sigma\Phi^A|_{K=0}$. Consequently, the classical Lagrangian $\mathcal{L}_{class} = \sum_i \lambda_i \mathcal{G}_i$ is in general not σ-closed. This fact implies that a suitable generalization of conjecture (1.4) should be found. All this is extensively discussed in section 2, where we show explicitly that any argument of ref. [4] can be adapted to the case of open algebræ [or to the case of closed algebræ such that Σ_0 has not the form (1.1)]. Provided the above remarks are taken into account, sequence (1.9) and its properties are unaltered.

[3] The whole initial BRS action Σ_0 can be considered *classical*, since it is $\hbar$-independent.

The second purpose of this paper is to search for a geometrical description of the subtraction algorithm, since it could be a source of insight in the program of classifying those nonrenormalizable quantum field theories that are predictive. We develop arguments stressing the fact that our subtraction algorithm extends very easily suitable identities that hold for the initial action Σ_0 to any step of the subtraction procedure and so to the convergent effective action Γ_∞. In other words, the algorithm is *tractable* even when (like in the case of nonrenormalizable theories) the structure of the counterterms is very complicated: it is possible to isolate the meaningful properties of the subtraction procedure from the contingent complications.

We develop a complete differential calculus on the manifold $\mathcal{M}_{gf}$ of the gauge-fixing parameters (section 3), permitting to reach a complete control of the effects of the subtraction algorithm on the gauge-fixing sector. This is achieved by introducing a certain set of differential forms ω_k of degrees $k = 0, \ldots m$ on $\mathcal{M}_{gf}$, the zero-form ω_0 being the effective action Γ. ω_k satisfy a certain set of differential identities (*cascade equations*) that start with the Ward identity $(\Gamma, \Gamma) = 0$ and that are preserved by the subtraction algorithm. In section 4 we show that the cascade equations imply descent equations for the divergent parts $\omega_{k\,div}$ of ω_k and this property permits to choose the canonical transformation (1.8) so as to make any ω_k convergent, while preserving the gauge-fixing independence of the redefinitions $\lambda_i - \delta_n \lambda_i$ of the parameters λ_i ("no mixing theorem").

In ref. [4] this was shown focusing on *one* gauge-fixing parameter κ only, instead of a generic manifold $\mathcal{M}_{gf}$. Precisely, a functional $S_n =< \chi_n >_J$ such that

$$\frac{\partial \Gamma_n}{\partial \kappa} = (S_n, \Gamma_n), \qquad \frac{\partial \Sigma_n}{\partial \kappa} = \Omega_n \chi_n, \tag{1.14}$$

was introduced. Initially, the gauge-fixing parameter κ only enters in the gauge-fermion: $\Psi = \Psi(\Phi, \kappa)$. χ_n is a local functional and its order zero part coincides with $\frac{\partial \Psi}{\partial \kappa}$. We showed that it is possible to choose the canonical transformations (1.8) so as to make S_n convergent to order $\hbar^n$ (provided S_{n-1} is inductively assumed to be convergent up to $\hbar^{n-1}$). Due to this, equations (1.14) can be extended to any step of the subtraction procedure and permit to show that the on-shell physical amplitudes are κ-independent.

We gave two different proofs of the above property, stressing different features of the subtraction algorithm. The first method has the property that it can be easily generalized to a generic manifold $\mathcal{M}_{gf}$, but its disadvantage is that it is only applyable in the context of a regularization technique, like the dimensional one, where it is possible to set $\Delta = 0$ on local functionals. The second method, instead, does not suffer from this restriction, but its generalization to a manifold $\mathcal{M}_{gf}$ is less straightforward. This fact can be turned into a positive feature, since it leads to the development of the interesting differential calculus on $\mathcal{M}_{gf}$ that we anticipated.

In section 5, we give a geometrical description of the subtraction algorithm and of the independence from the gauge-fixing parameters. We define a principal fiber bundle $\mathcal{P}$, whose fiber is isomorphic to the group of canonical transformations and whose base manifold is $\mathcal{M}_{gf}$. The form ω_1 is a connection on $\mathcal{P}$ and its field strength vanishes on shell. The canonical transformations are the gauge transformations for ω_1.

We also give the building blocks for the geometrical description of the effects of the subtraction algorithm on the space $\mathcal{M}_{ph}$ of the physical parameters λ. The set of redefinitions $\delta\lambda$ is viewed as a diffeomorphism on $\mathcal{M}_{ph}$, orthogonal to $\mathcal{M}_{gf}$. We translate into this language the requirement that finitely many parameters are sufficient for removing all the divergences, by formulating a suitable concept of predictivity. Finally, in section 6 we investigate how this could happen, by means of suitable toy Lagrangians that are not power counting renormalizable.

Let us write $S_n^{(\kappa)}$, $\chi_n^{(\kappa)}$ and so on, when it is convenient to denote the gauge-fixing parameter explicitly. We introduce a useful concept of "covariant derivative" D_κ with respect to the gauge-fixing parameter κ,

$$D_\kappa = \frac{\partial}{\partial\kappa} - \mathrm{ad}_l \, \frac{\partial\Psi}{\partial\kappa}, \tag{1.15}$$

where ad_l denotes the left-adjoint operation, i.e. $\mathrm{ad}_l \, X \, Y = (X, Y)$. The nice property of D_κ is that it sends σ-closed and σ-exact functionals into σ-closed and σ-exact functionals, i.e. it commutes with σ:

$$[\sigma, D_\kappa] = 0. \tag{1.16}$$

This fact is a simple consequence of the covariant constancy of the order $\hbar^0$-part $S(\Phi, K)$ of the starting action $\Sigma_0(\Phi, K)$:

$$D_\kappa S = 0. \tag{1.17}$$

It is also simple to prove that the manifold $\mathcal{M}_{gf}$ of the gauge-fixing parameters is flat with respect to the above covariant derivative, namely

$$[D_\kappa, D_\alpha] = 0, \tag{1.18}$$

for any couple of gauge-fixing parameters κ and α.

Let $\{\kappa_1, \ldots, \kappa_m\}$ denote the set of gauge-fixing parameters which the gauge-fermion Ψ depends on ($m = \dim \mathcal{M}_{gf}$). We define an exterior derivative on $\mathcal{M}_{gf}$:

$$d = d\kappa_i \, \frac{\partial}{\partial\kappa_i}\bigg|_{\Phi,K}. \tag{1.19}$$

Then, the covariant derivative (1.15) and properties (1.16), (1.17) and (1.18) read

$$D = d - \mathrm{ad}_l \, d\Psi, \qquad [D, \sigma] = 0, \qquad D^2 = 0, \qquad DS = 0. \tag{1.20}$$

We can also define the connection

$$\omega_n = S_n^{(\kappa_i)} dk_i \tag{1.21}$$

and the curly covariant derivative

$$\mathcal{D}_n = d - \mathrm{ad}_l \, \omega_n. \tag{1.22}$$

From (1.14) we have

$$\mathcal{D}_n \Gamma_n = 0, \tag{1.23}$$

that generalizes $DS = 0$. We see that Γ is covariantly constant with respect to the curly covariant derivative.

2 Extension to open gauge-algebræ

In this section we show that the results of ref. [4] can be extended to theories corresponding to open gauge-algebræ as well as to closed gauge-algebræ such that Σ_0 is not of the form (1.1). It is useful to stress the assumptions that are relaxed and those that are maintained. The gauge-fermion Ψ is still supposed to be a functional of the fields Φ only and not of the BRS sources K. It may depend on some gauge-fixing parameters κ_i, but we assume that it is independent of the parameters λ of the classical Lagrangian. Any such dependence would represent an undesired identification between gauge-fixing parameters and Lagrangian parameters: eventually, it is better to make this identification at the very end, i.e. in the final convergent theory. In this way, one has a clearer perception of what happens in the subtraction procedure. Moreover, $\Psi(\Phi, \kappa_i)$ is assumed to be local, convergent, independent of $\hbar$ and such as to produce standard propagators (i.e. gauge-conditions, like the axial-gauge, that produce non-local divergent counterterms, should be treated apart). Ψ is not constrained by power-counting requirements.

The starting action $\Sigma_0(\Phi, K)$ is assumed to satisfy the master equation

$$(\Sigma_0, \Sigma_0) = 2i\hbar\Delta\Sigma_0. \tag{2.1}$$

Moreover, Σ_0 depends on the BRS sources K and the gauge-fermion Ψ only *via* the antifield-combination (1.2), so that the derivative of Σ_0 with respect to the gauge-fixing parameters is

$$d\Sigma_0 = d\left(\frac{\partial\Psi}{\partial\Phi^A}\right)\frac{\partial_l\Sigma_0}{\partial K_A} = (d\Psi, \Sigma_0) = \Omega_0 d\Psi. \tag{2.2}$$

One can also write $D\Sigma_0 = 0$, with D being defined by (1.20). In general, let $\Sigma_0 = S + \sum_{n=1}^{\infty}\hbar^n M_n$ be the expansion of Σ_0 as a power series in $\hbar$. S satisfies the classical master equation (1.11) and the boundary condition (1.7). The classical Lagrangian $\mathcal{L}_{class} = S|_{\Phi^*=0}$ is written as

$$\mathcal{L}_{class} = \sum_i \lambda_i \mathcal{G}_i, \tag{2.3}$$

where $\mathcal{G}_i$ is a basis of gauge-invariant functionals of the classical fields: $\delta_{gauge}\mathcal{G}_i = 0$. The BRS operator s is defined by (1.13) and σ by (1.12). This means that it is no longer true that $\sigma\mathcal{G}_i = 0$: $\mathcal{G}_i$ cannot represent a basis of the σ cohomology and conjecture (1.4) has to be suitably generalized. In a moment we shall see how it is reasonable to extend it.

Since Ψ is assumed to be $\hbar$-independent, S also depends on K and Ψ *via* the antifield-combination (1.2). It satisfies the classical master equation (1.11) and the property $DS = 0$ (1.17), that can be derived taking the order $\hbar^0$ part of (2.2). As anticipated, we no longer introduce assumptions on the form of Σ_0 or S or on the BRS invariance of the functional measure.

In the remainder of the section we generalize the arguments of ref. [4]. However, we do not repeat the complete derivations, for brevity. First notice that, under the inductive assumption that Γ_{n-1} is finite up to order $\hbar^{n-1}$ included, the Ward identity

$$(\Gamma_{n-1}, \Gamma_{n-1}) = 0, \tag{2.4}$$

assures that

$$(\Gamma^{(n)}_{div}, S) = \sigma\Gamma^{(n)}_{div} = 0. \tag{2.5}$$

As anticipated, conjecture (1.4) is no more correct, since $\mathcal{G}_i$ are not σ-closed, but only s-closed. Differentiating (1.11) with respect to λ_i, we obtain

$$(\mathcal{S}_i, S) = \sigma\mathcal{S}_i = 0, \tag{2.6}$$

where $\mathcal{S}_i \equiv \frac{\partial S}{\partial\lambda_i}$. Consequently, it is natural to conjecture that $\mathcal{S}_i$ is a basis of the σ-cohomology, i.e. that the most general solution to equation (2.5) is

$$\Gamma^{(n)}_{div} = \sum_i \delta_n\lambda_i\mathcal{S}_i + \sigma R^{(n)}. \tag{2.7}$$

We shall still denote the sum $\sum_i \delta_n\lambda_i\mathcal{S}_i$ by $\mathcal{G}^{(n)}$. Notice that

$$\mathcal{S}_i|_{\Phi^*=0} = \mathcal{G}_i \tag{2.8}$$

and that $\mathcal{S}_i$ do not necessarily depend on the classical fields only, but can also depend on the other fields and the BRS sources. Clearly, they depend on the BRS sources K and the gauge-fermion Ψ only *via* the combination (1.2). Moreover, although $d\mathcal{G}_i = 0$, since no gauge-fixing is involved in the determination of the basis of gauge-invariant functionals $\mathcal{G}_i$, the same is no longer true for $\mathcal{S}_i$, rather, as in formula (2.2),

$$D\mathcal{S}_i = d\mathcal{S}_i - (d\Psi, \mathcal{S}_i) = 0, \tag{2.9}$$

a result that can be also obtained by differentiating $DS = 0$ with respect to λ_i and noticing that $\frac{\partial d\Psi}{\partial\lambda_i} = 0$. So, $\mathcal{S}_i$ are covariantly constant with respect to D.

If (2.7) holds, then it is possible to proceed as in the case of closed gauge-algebræ: $\mathcal{G}^{(n)}$ is removed by a redefinition $\lambda_i \to \lambda_i - \delta_n\lambda_i = \lambda_i + \mathcal{O}(\hbar^n)$ of the parameters λ_i of the classical Lagrangian. Indeed,

$$\begin{aligned}\Sigma_{n-1}(\Phi, K, \lambda_i - \delta_n\lambda_i) &= \Sigma_{n-1}(\Phi, K, \lambda_i) - \sum_i \delta_n\lambda_i\frac{\partial\Sigma_{n-1}}{\partial\lambda_i} + \mathcal{O}(\hbar^{n+1}) \\ &= \Sigma_{n-1}(\Phi, K, \lambda) - \sum_i \delta_n\lambda_i\mathcal{S}_i + \mathcal{O}(\hbar^{n+1}).\end{aligned} \tag{2.10}$$

Then, $\sigma R^{(n)}$ is removed by a canonical transformation generated by (1.8) with no difference with respect to the case considered in ref. [4].

Let us now present the modifications to the proof of independence from the gauge-fermion, i.e.

$$d\delta_m\lambda_i = 0 \quad \forall m \quad \forall i. \tag{2.11}$$

This was done, in ref. [4], for a single gauge-fixing parameter κ, not for a generic manifold $\mathcal{M}_{gf}$. In this section, we generalize the proof of ref. [4] for a single parameter to the case of open-algebræ. In the next sections, we generalize the proof to the full manifold $\mathcal{M}_{gf}$.

Assuming inductively that (2.11) holds up to $m = n-1$, we can extend equation (2.2) to

$$\frac{\partial \Sigma_{n-1}}{\partial \kappa} = \Omega_{n-1} \chi^{(\kappa)}_{n-1}, \tag{2.12}$$

for some local functional $\chi^{(\kappa)}_{n-1}$, whose zeroth order part is $\frac{\partial \Psi}{\partial \kappa}$. Consequently, we also have [4]

$$\frac{\partial \Gamma_{n-1}}{\partial \kappa} = (S^{(\kappa)}_{n-1}, \Gamma_{n-1}), \tag{2.13}$$

where $S^{(\kappa)}_{n-1} =< \chi^{(\kappa)}_{n-1} >_J$. $S^{(\kappa)}_{n-1}$ is inductively assumed to be finite up to order $\hbar^{n-1}$: $S^{(\kappa)}_{n-1} = \text{finite} + S^{(n)}_{\kappa\, div} + \mathcal{O}(\hbar^{n+1})$, $S^{(n)}_{\kappa\, div}$ denoting the order $\hbar^n$-divergent part, which is local.

The order $\hbar^n$ divergent part of equation (2.13) gives, as in ref. [4],

$$\frac{\partial \Gamma^{(n)}_{div}}{\partial \kappa} = \left(\frac{\partial \Psi}{\partial \kappa}, \Gamma^{(n)}_{div} \right) + \sigma S^{(n)}_{\kappa\, div}, \tag{2.14}$$

that can be also written as

$$D_\kappa \Gamma^{(n)}_{div} = \sigma S^{(n)}_{\kappa\, div}. \tag{2.15}$$

Using (2.7) and (1.16) we have

$$D_\kappa \mathcal{G}^{(n)} = \sigma S^{(n)}_{\kappa\, div} - D_\kappa \sigma R^{(n)} = \sigma (S^{(n)}_{\kappa\, div} - D_\kappa R^{(n)}). \tag{2.16}$$

At this point, equation (2.9) permits to write

$$D_\kappa \mathcal{G}^{(n)} = D_\kappa \left(\sum_i \delta_n \lambda_i \mathcal{S}_i \right) = \sum_i \frac{\partial \delta_n \lambda_i}{\partial \kappa} \mathcal{S}_i = \sigma (S^{(n)}_{\kappa\, div} - D_\kappa R^{(n)}). \tag{2.17}$$

Since by assumption $\mathcal{S}_i$ is a basis of the σ cohomology, we conclude that both sides of equation (2.17) vanish, so that

$$\frac{\partial \delta_n \lambda_i}{\partial \kappa} = 0 \quad \forall i. \tag{2.18}$$

This assures that (2.12) and (2.13) can be extended to order $\hbar^n$, giving (1.14).

Finally, in order to fully reproduce the inductive assumptions, one also has to prove that $S^{(\kappa)}_n$ can be chosen finite up to order $\hbar^n$ included. This can be done exactly as in ref. [4] by means of a suitable choice of the functional $R^{(n)}$ in the canonical transformation $F^{(n)}$ (1.8).

We recall that (2.7) determines $R^{(n)}$ only up to additions of σ-closed functional $T^{(n)}$:

$$R^{(n)} \to R^{(n)} + T^{(n)}, \qquad \sigma T^{(n)} = 0. \tag{2.19}$$

A corresponding freedom characterizes the canonical transformation (1.8). If $T^{(n)} = \mathcal{O}(\hbar^n)$, such additions do not change the action Σ_n up to order $\hbar^n$ included, so that the order $\hbar^n$-change of the average of a functional can only be due to the change of the functional and not to the change of the average. Clearly, Γ_n is convergent up to order $\hbar^n$ for any $T^{(n)} = \mathcal{O}(\hbar^n)$.

We noticed in ref. [4] that under the canonical transformation generated by $F^{(n)}$, $\chi_{n-1}^{(\kappa)}$ changes into

$$\tilde{\chi}_{n-1}^{(\kappa)} - \frac{\partial F^{(n)}}{\partial \kappa}. \tag{2.20}$$

We recall [4] that the tilde means that the old variables $\{\Phi, K\}$ have to be replaced with the new ones $\{\Phi', K'\}$, considered as functions of the old ones: $\{\Phi'(\Phi, K), K'(\Phi, K)\}$. $\chi_n^{(\kappa)}$ is thus obtained from $\chi_{n-1}^{(\kappa)}$ by letting $\lambda_i \to \lambda_i - \delta_n \lambda_i$ and performing the canonical transformation (1.8) according to (2.20).

Let us assume that $\Gamma_{div}^{(n)}$ has been removed and $T^{(n)}$ is of order $\hbar^n$ and divergent. Under a variation (2.19) of the generating functional (1.8), $\chi_n^{(\kappa)}$ varies of $-D_\kappa T^{(n)} + \mathcal{O}(\hbar^{n+1})$, so that the order $\hbar^n$ divergent part $\mathcal{S}_{\kappa\, div}^{(n)}$ of $S_n^{(\kappa)} =< \chi_n^{(k)} >_J$ varies as

$$\mathcal{S}_{\kappa\, div}^{(n)} \to \mathcal{S}_{\kappa\, div}^{(n)} - D_\kappa T^{(n)}. \tag{2.21}$$

We also recall that $\mathcal{S}_{\kappa\, div}^{(n)}$ is σ-closed

$$\sigma \mathcal{S}_{\kappa\, div}^{(n)} = 0, \tag{2.22}$$

due to the first equation of (1.14), i.e. $\frac{\partial \Gamma_n}{\partial \kappa} = (S_n^{(\kappa)}, \Gamma)$. Given $\mathcal{S}_{\kappa\, div}^{(n)}$, the condition

$$\mathcal{S}_{\kappa\, div}^{(n)} = D_\kappa T^{(n)} \tag{2.23}$$

for $T^{(n)}$ is solved (perturbatively in κ) by

$$T^{(n)} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \kappa^j}{j!} D_\kappa^{j-1} \mathcal{S}_{\kappa\, div}^{(n)}. \tag{2.24}$$

Due to (1.16) and (2.22), $T^{(n)}$ is σ-closed, as desired. Moreover, it is order $\hbar^n$ and divergent. The above choice of $T^{(n)}$ produces a functional $S_n^{(\kappa)}$ that is convergent up to order $\hbar^n$ included.

We conclude noticing that if the starting action $\Sigma_0(\Phi, \Phi^*)$ is power counting renormalizable and the most general power counting renormalizable classical Lagrangian $\mathcal{L}_{class}(\lambda, \Phi)$ depends on a finite number of parameters λ, then the above result combined with power counting assures that the quantum theory is predictive, namely that all the divergences are removed by canonical transformations and redefinitions of the parameters λ and that the on-shell physical amplitudes depend on a finite number of parameters. In other words, the fact that the algebra is open has no dramatic consequence, provided the generalization (2.7) of the usual conjecture (1.4) holds.

3 Covariant treatment of the gauge-fixing parameters

In this section, we develop a differential calculus on $\mathcal{M}_{gf}$. We define differential forms ω_k of degree $k = 1, \ldots m = \dim \mathcal{M}_{gf}$, the zero-form ω_0 being the effective action Γ and the one form ω_1 being (1.21). ω_k satisfy certain improved descent equations, called by us *cascade equations* due to their æsthetic aspect, that are preserved by the subtraction algorithm and possess some *cascade invariances* that will be fundamental in section 4, where we shall apply these properties to show that $F^{(n)}$ (1.8) can be chosen in order to make any ω_k finite up to order $\hbar^n$ and $d\delta_n\lambda_i = 0\ \forall i$. In section 5 we shall define a fiber bundle $\mathcal{P}$ on $\mathcal{M}_{gf}$ that provides an intuitive geometrical description of the fact that the subtraction algorithm is not able to pick up any information from $\mathcal{M}_{gf}$. ω_1 is a connection on $\mathcal{P}$.

Our purpose is to generalize the construction of the previous section to the m-dimensional manifold $\mathcal{M}_{gf}$. In the remainder, apart from the situations of possible misunderstanding, we omit the suffix n in Γ_n, Σ_n, χ_n, S_n, and so on.

It is easy to prove, from equation (2.20), the following formula

$$d\chi = \frac{1}{2}(\chi, \chi). \tag{3.1}$$

Indeed, this equation holds for $n = 0$ and is canonically preserved (this fact will be a straightforward consequence of a computation that will be made later on). Moreover, due to (2.18), the redefinitions of the parameters λ do not affect it, so that (3.1) is also preserved by $\mathcal{L}_n$ and by the operation $\mathcal{R}_n$ that removes the divergences up to order n included. This proves (3.1).

Let us write $\omega_{-1} = 0$, $\omega_0 = \Gamma$, $\omega_1 =< \chi >_J$ (now ω_1 denotes what we previously called ω_n). We have, from the first of (1.14),

$$d\omega_0 = (\omega_1, \omega_0). \tag{3.2}$$

We want to prove that

$$d\omega_1 = \frac{1}{2}(\omega_1, \omega_1) - (\omega_2, \omega_0), \tag{3.3}$$

where

$$\omega_2 = \frac{i}{2\hbar} < \chi\chi >_J - \frac{i}{2\hbar}\omega_1\omega_1 - \frac{1}{2}\{\omega_1, \omega_1\}. \tag{3.4}$$

ω_2 is a two-form on $\mathcal{M}_{gf}$. The symbol of wedge product among forms on $\mathcal{M}_{gf}$ is understood. The curly brackets do not denote an anticommutator but a different notion of brackets, that is convenient to analyse explicitly. Precisely,

$$\{X, Y\} = \left.\frac{\partial_r X}{\partial \Phi^A}\right|_K \left.\frac{\partial_l Y}{\partial J_A}\right|_K. \tag{3.5}$$

They will be called *mixed brackets*, due to the fact that they mix derivatives with respect to Φ and J, while K is kept constant. They possess a nice diagrammatical meaning, that will be illustrated in the sequel. It is easy to prove that the mixed brackets satisfy the following properties

$$\begin{aligned} \{X,Y\} &= (-1)^{\varepsilon(X)\varepsilon(Y)+d_X d_Y}\{Y,X\}, \\ \{X,YZ\} &= \{X,Y\}Z+(-1)^{\varepsilon(Y)\varepsilon(Z)+d_Y d_Z}\{X,Z\}Y. \end{aligned} \tag{3.6}$$

Here d_X denotes the form degree of X. The factors like $(-1)^{d_X d_Y}$ are due to the fact that X and Y have been interchanged. One must also keep into account that when applying identities for the antibrackets to differential forms on $\mathcal{M}_{gf}$, similar corrections involving the form degrees are necessary whenever the order of the forms is changed.

Another property of the mixed brackets (3.5) is

$$(X,Y)+\{X,(\omega_0,Y)\}+\{(X,\omega_0),Y\}=(-1)^{\varepsilon(X)}(\omega_0,\{X,Y\}). \tag{3.7}$$

The proof of this identity is more involved and will be given explicitly.

Let us write the antibrackets in the form

$$\begin{aligned} (X,Y) &= \left.\frac{\partial_r X}{\partial J_B}\right|_K \left.\frac{\partial_r J_B}{\partial \Phi^A}\right|_K \left.\frac{\partial_l Y}{\partial K_A}\right|_\Phi - \left.\frac{\partial_r X}{\partial K_A}\right|_\Phi \left.\frac{\partial_l Y}{\partial \Phi^A}\right|_K \\ &= \left.\frac{\partial_r X}{\partial J_B}\right|_K (J_B,Y) - \left.\frac{\partial_r X}{\partial K_A}\right|_J \left.\frac{\partial_l J_A}{\partial \Phi^B}\right|_K \left.\frac{\partial_l Y}{\partial J_B}\right|_K (-1)^{\varepsilon_A+\varepsilon_B+\varepsilon_A\varepsilon_B}. \end{aligned} \tag{3.8}$$

Now, notice that the first term of the above expression can be written as

$$\begin{aligned} \left.\frac{\partial_r X}{\partial J_B}\right|_K (J_B,Y) &= -(-1)^{\varepsilon_B}\left.\frac{\partial_r X}{\partial J_B}\right|_K \left.\frac{\partial_l (\Gamma,Y)}{\partial \Phi^B}\right|_K + \left.\frac{\partial_r X}{\partial J_B}\right|_K \left(\Gamma, \left.\frac{\partial_l Y}{\partial \Phi^B}\right|_K\right) \\ &= -\{X,(\Gamma,Y)\} + \left.\frac{\partial_r X}{\partial J_B}\right|_K \left(\Gamma, \left.\frac{\partial_l Y}{\partial \Phi^B}\right|_K\right). \end{aligned} \tag{3.9}$$

On the other hand, the second term is

$$\begin{aligned} &-\left.\frac{\partial_r X}{\partial K_A}\right|_J \left.\frac{\partial_l J_A}{\partial \Phi^B}\right|_K \left.\frac{\partial_l Y}{\partial J_B}\right|_K (-1)^{\varepsilon_A+\varepsilon_B+\varepsilon_A\varepsilon_B} \\ &= -(-1)^{\varepsilon_A+\varepsilon_B\varepsilon(X)}\left\{\left.\frac{\partial_l}{\partial\Phi^B}\left(\left.\frac{\partial_r X}{\partial K_A}\right|_J J_A\right)\right|_K - \left.\frac{\partial_l}{\partial\Phi^B}\left(\left.\frac{\partial_r X}{\partial K_A}\right|_J\right)\right|_K J_A\right\}\left.\frac{\partial_l Y}{\partial J_B}\right|_K. \end{aligned} \tag{3.10}$$

Remembering that [4]

$$(-1)^{\varepsilon_A}\left.\frac{\partial_r Z}{\partial K_A}\right|_J J_A = (Z,\Gamma) \qquad \forall Z, \tag{3.11}$$

we get

$$\begin{aligned} -\left.\frac{\partial_r X}{\partial K_A}\right|_J \left.\frac{\partial_l J_A}{\partial \Phi^B}\right|_K \left.\frac{\partial_l Y}{\partial J_B}\right|_K (-1)^{\varepsilon_A+\varepsilon_B+\varepsilon_A\varepsilon_B} &= -(-1)^{\varepsilon_B(\varepsilon(X)+\varepsilon(Y)+1)}\left.\frac{\partial_l (X,\Gamma)}{\partial \Phi^B}\right|_K \left.\frac{\partial_r Y}{\partial J_B}\right|_K \\ +(-1)^{\varepsilon(X)\varepsilon_B}\left.\frac{\partial_l J_C}{\partial \Phi^B}\right|_K \left(\left.\frac{\partial_l X}{\partial J_C}\right|_K, \Gamma\right)\left.\frac{\partial_l Y}{\partial J_B}\right|_K. \end{aligned} \tag{3.12}$$

At this point, collecting (3.9) and (3.12), it is simple to arrive at the desired result, i.e. formula (3.7).

It is, instead, immediate to prove that

$$dX = d_J X - \{d\omega_0, X\} = d_J X - \{(\omega_1, \omega_0), X\}, \tag{3.13}$$

where d_J differs from d for the fact that $\{J, K\}$ are kept constant instead of $\{\Phi, K\}$.

We are now ready to prove (3.3). Using (3.13) with $X = \omega_1$, we get

$$d\omega_1 = d_J\omega_1 - \{(\omega_1, \omega_0), \omega_1\}. \tag{3.14}$$

On the other hand, (3.7) with $X = Y = \omega_1$ gives

$$\{(\omega_1, \omega_0), \omega_1\} = -\frac{1}{2}(\omega_1, \omega_1) - \frac{1}{2}(\omega_0, \{\omega_1, \omega_1\}). \tag{3.15}$$

Moreover, a straightforward differentiation permits to write

$$d_J\omega_1 = d_J < \chi >_J = < d\chi >_J - \frac{i}{\hbar} < \chi \, d\Sigma >_J + \frac{i}{\hbar} < \chi >_J < d\Sigma >_J . \tag{3.16}$$

Using (3.1) and $d\Sigma = \Omega\chi$ [which is the second of (1.14)], we obtain

$$d_J\omega_1 = -\frac{i}{2\hbar} < \Omega(\chi\chi) >_J + \frac{i}{\hbar}\omega_1(\omega_1, \omega_0) = -\frac{i}{2\hbar}(< \chi\chi >_J - \omega_1\omega_1, \omega_0). \tag{3.17}$$

Collecting (3.14), (3.15) and (3.17) we arrive directly at (3.3).

Now, let us come to the diagrammatical meaning of the expression ω_2 given in formula (3.4). The term

$$\frac{i}{2\hbar} < \chi\chi >_J \tag{3.18}$$

collects a set of Feynmann diagrams that contain two insertions of the composite operator χ. These diagrams are connected and irreducible as for the action vertices, but are neither connected nor irreducible as for the vertices that represent the χ-insertions. Thus, it is natural to conjecture that ω_2 represents the set of connected irreducible graphs with two χ insertions, i.e. that the remaining terms of expression (3.4) remove the disconnected and reducible contributions that are contained in (3.18). This is indeed true. The term

$$-\frac{i}{2\hbar}\omega_1\omega_1 \tag{3.19}$$

subtracts the disconnected contributions, i.e. all the graphs that are a product of two separate diagrams, each one containing a single χ-insertion. On the other hand, the term

$$-\frac{1}{2}\{\omega_1, \omega_1\} = -\frac{1}{2} \left.\frac{\partial\omega_1}{\partial\Phi^A}\right|_K \left.\frac{\partial_l^2 W[J,K]}{\partial J_A \partial J_B}\right|_K \left.\frac{\partial\omega_1}{\partial\Phi^B}\right|_K \tag{3.20}$$

represents the set of connected reducible diagrams, i.e. those diagrams in which a single leg connects the two χ-insertions. Indeed, the derivatives with respect to Φ represent Φ-legs, while $\left.\frac{\partial_l^2 W[J,K]}{\partial J_A \partial J_B}\right|_K$ is the propagator connecting them. As promised, this also provides a nice diagrammatical interpretation of the mixed brackets.

The diagrammatical meaning of ω_2 is crucial, in the sense that it guarantees that the overall divergences of ω_2 are local, when all subdivergences have been removed. This fact will be important is the sequel.

Identity (3.3) can be rewritten as

$$\mathcal{F} = d\omega_1 - \frac{1}{2}(\omega_1, \omega_1) = -\mathrm{ad}\,\Gamma\ \omega_2. \tag{3.21}$$

Its meaning is that the field strength $\mathcal{F}$ of the connection ω_1 does not vanish, however it is $\mathrm{ad}\,\Gamma$-exact (and so, it vanishes on shell). Notice that, due to (1.15), the field strength of the connection $d\Psi$ is instead zero. Similarly, due to (3.1), the field strength of χ is also zero. However, as we shall discuss later on, the latter fact is related to the special form of χ that we have chosen. Indeed, if we let χ go into $\chi' = \chi + \Omega\delta$, δ being some local functional, [this is allowed, because it does not affect the second equation of (1.14) that defines χ], then the field strength of χ' is no more zero, but it is Ω-exact.

We can collect equations (3.3), (3.2) and the Ward identity $(\Gamma, \Gamma) = 0$ into the formula

$$d\omega_i = \frac{1}{2}(-1)^{j+1}(\omega_j, \omega_{i-j+1}), \tag{3.22}$$

where $i, j = -1, 0, 1$ and the sum over j is understood. We therefore are lead to conjecture that there exist k-forms ω_k on $\mathcal{M}_{gf}$, for $k = 3, \ldots m$, such that (3.22) holds for $i, j = -1, \ldots m$. It is also natural to conjecture that the k-form ω_k, $k = 3, \ldots m$ represents the connected irreducible Feynman diagrams with k χ-insertions. Equations (3.22) will be called *cascade equations* due to their aspect and are a generalization of the descent equations that would read, in this case,

$$d\omega_i = \mathrm{ad}\,\Gamma\ \omega_{i+1}. \tag{3.23}$$

Such a formula is correct only for $i = -1, 0$. All the other cases are improved by the fact that the exterior derivative d and the nilpotent operator $\mathrm{ad}\,\Gamma$ do not commute nor anticommute, rather

$$[d, \mathrm{ad}\,\Gamma]X = (-1)^{d_X}(X, d\Gamma) \tag{3.24}$$

(the square brackets still denote a commutator). On the other hand, the operator $\mathrm{ad}\,\Gamma$ and the covariant derivative $\mathcal{D} = d - \mathrm{ad}_l\,\omega_1$ have the following properties

$$(\mathrm{ad}\,\Gamma)^2 = 0, \qquad [\mathcal{D}, \mathrm{ad}\,\Gamma] = 0, \qquad \mathcal{D}^2 = (\mathrm{ad}\,\Gamma \text{ something}, \,.\,), \tag{3.25}$$

so that $\mathcal{D}^2 = 0$ on shell. This is the difference with respect to a double-complex and is responsible of the generation of cascade equations instead of descent equations. A structure like (3.25) will be called a *quasi* double-complex.

Let us discuss the properties of the cascade equations (3.22), beginning from self-consistency. Interchanging the forms in the right-hand side we get

$$d\omega_i = \frac{1}{2}(-1)^{i-j}(\omega_{i-j+1}, \omega_i), \tag{3.26}$$

which is consistent with the replacement $j \to i - j + 1$. Moreover, taking the exterior derivative and using eq.s (3.22) themselves, we get

$$(\omega_{i-j+1}, (\omega_k, \omega_{j-k+1}))(-1)^k = 0, \tag{3.27}$$

which is indeed true (it is sufficient to use the Jacobi identity for the antibrackets and help oneself with replacements of indices).

The cascade equations possess some "invariance", similar to the invariance under $\omega_i \to \omega_i + \sigma\Delta_i + d\Delta_{i-1}$ of the usual descent equations, however much more complicated. Let us consider the replacement $\omega_1 \to \omega_1' = \omega_1 + (\Delta_1, \omega_0)$. One checks that there is invariance under

$$\begin{aligned}
\omega_0' &= \omega_0, \\
\omega_1' &= \omega_1 + (\Delta_1, \omega_0), \\
\omega_2' &= \omega_2 - d\Delta_1 + (\Delta_1, \omega_1) + \frac{1}{2!}(\Delta_1, (\Delta_1, \omega_0)), \\
\omega_3' &= \omega_3 - \frac{1}{2!}(\Delta_1, d\Delta_1) + (\Delta_1, \omega_2) + \frac{1}{2!}(\Delta_1, (\Delta_1, \omega_1)) + \frac{1}{3!}(\Delta_1, (\Delta_1, (\Delta_1, \omega_0))), \\
&\ldots
\end{aligned} \tag{3.28}$$

Similar formulæ hold starting from every ω_i with $\omega_i \to \omega_i + (\Delta_i, \omega_0)$, $i = -1, \ldots m$. For example, there is invariance under

$$\begin{aligned}
\omega_0' &= \omega_0, \\
\omega_1' &= \omega_1, \\
\omega_2' &= \omega_2 + (\Delta_2, \omega_0), \\
\omega_3' &= \omega_3 + d\Delta_2 + (\Delta_2, \omega_1), \\
\omega_4' &= \omega_4 + (\Delta_2, \omega_2) + \frac{1}{2!}(\Delta_2, (\Delta_2, \omega_0)), \\
\omega_5' &= \omega_5 + \frac{1}{2!}(\Delta_2, d\Delta_2) + (\Delta_2, \omega_3) + \frac{1}{2!}(\Delta_2, (\Delta_2, \omega_1)), \\
&\ldots
\end{aligned} \tag{3.29}$$

The proof of invariance under (3.28) and (3.29) is straightforward. Again, due to their aspect, invariances (3.28), (3.29) and similar ones will be called *cascade invariances* or *cascade transformations*. The cascade transformation that starts from $\omega_i \to \omega_i + (\Delta_i, \omega_0)$ will be called *cascade transformation of degree* i.

Let us discuss a particular case of the above transformations. Let us consider a cascade transformation of degree one (3.28) with

$$\Delta_1 =< \delta_1 >_J, \tag{3.30}$$

δ_1 being a local functional of fields and BRS sources. Since $\omega_1 =< \chi >_J$, (3.28) gives

$$\omega_1' =< \chi' >_J \equiv < \chi + \Omega\delta_1 >_J \ . \tag{3.31}$$

Thus, it is natural to expect that $\chi \to \chi'$ generates the same transformation as (3.28). We now show that it is not precisely so, however $\chi \to \chi'$ generates (3.28) up to a cascade transformation of degree two, eq.s (3.29), Δ_2 being a suitable set of connected irreducible diagrams with two insertions of certain local composite operators.

Let us first notice that equation (3.1) is modified into

$$d\chi' = \frac{1}{2}(\chi', \chi') + \Omega\left\{d\delta_1 - (\delta_1, \chi) - \frac{1}{2}(\delta_1, \Omega\delta_1)\right\}. \tag{3.32}$$

The transformed field strength is no more zero, however it is Ω-exact [we anticipated this fact soon after formula (3.21)]. Repeating the argument of (3.14), (3.15) and (3.17) with the primed functionals and using the appropriate expression for $d\chi'$ given in (3.32), one finds

$$\begin{aligned}\omega_2' = & \frac{i}{2\hbar} < \chi'\chi' >_J - \frac{i}{2\hbar}\omega_1'\omega_1' - \frac{1}{2}\{\omega_1', \omega_1'\} \\ & - < d\delta_1 >_J + < (\delta_1, \chi) >_J + \frac{1}{2!} < (\delta_1, \Omega\delta_1) >_J \ .\end{aligned} \tag{3.33}$$

This form of ω_2' differs from the one given in (3.28) and it is clearly the sum of connected irreducible Feynman graphs. The difference between the expressions for ω_2' given in (3.28) and in (3.33) is easily shown to be equal to

$$(\mathcal{U}_1 + \mathcal{U}_2, \omega_0), \tag{3.34}$$

where

$$\begin{aligned}\mathcal{U}_1 = & \frac{i}{\hbar}(< \delta_1\chi >_J - \Delta_1\omega_1) - \{\Delta_1, \omega_1\}, \\ \mathcal{U}_2 = & -\frac{i}{2\hbar}\,(< \Omega\delta_1\,\delta_1 >_J - (\Delta_1, \omega_0)\Delta_1) + \frac{1}{2}\{(\Delta_1, \omega_0), \Delta_1\}.\end{aligned} \tag{3.35}$$

Now that we know how to express the set of connected irreducible graphs with two insertions of local composite operators [see (3.4)], it is evident that both $\mathcal{U}_1$ and $\mathcal{U}_2$ represent such situations. We conclude that the two expressions for ω_2' differ by a transformation of the kind (3.29) with

$$\Delta_2 = \mathcal{U}_1 + \mathcal{U}_2. \tag{3.36}$$

Δ_2, expression (3.34) and both the expressions of ω'_2 of eq. (3.28) and eq. (3.33), have the property that their order $\hbar^n$ divergences are local, when all subdivergences have been removed.

This discussion illustrates that the cascade transformations of the kind $\Delta_i =< \delta_i >_J$ do not spoil the property that the $\mathcal{O}(\hbar^n)$ of ω_k are local when all the subdivergences have been removed. In particular, notice that if X and Y are sums of connected irreducible diagrams, then (X, Y) has the same property. Indeed, antibrackets connect a K_A-leg of X with a Φ^A-leg of Y, and viceversa, *without* any propagator on the Φ^A- and K_A-legs. This defines new vertices obtained by shrinking the Φ^A-K_A leg to a point and the graphs of (X, Y) constructed with such vertices are irreducible.

The cascade equations are also invariant under canonical transformations. Let $F(\Phi, K')$ denote the generating functional. Then one has

$$\omega'_0 = \tilde{\omega}_0, \qquad \omega'_1 = \tilde{\omega}_1 - d'F, \qquad \omega'_2 = \tilde{\omega}_2, \qquad \omega'_3 = \tilde{\omega}_3, \qquad \ldots \tag{3.37}$$

where d' is a derivative at constant $\{\Phi, K'\}$. As we see, only ω_1 has a strange transformation rule, which is nothing but the analogue of (2.20) and is very similar to a gauge transformation. It will be further investigated in section 4. (3.37) can be proved starting from the properties

$$\begin{aligned} d\tilde{X} &= \widetilde{dX} - (d'F, \tilde{X}), \\ dd'F &= -\frac{1}{2}(d'F, d'F). \end{aligned} \tag{3.38}$$

The first equation is analogous to the formula that was proved in the appendix of ref. [4] and can be derived following similar steps. The second equation is derived as follows. We write

$$\begin{aligned} dd'F &= d\kappa_i \left.\frac{\partial d'F}{\partial \kappa_i}\right|_{\Phi,K} \\ &= d\kappa_i \left.\frac{\partial d'F}{\partial \kappa_i}\right|_{\Phi,K'} + d\kappa_i \left.\frac{\partial K'_A}{\partial \kappa_i}\right|_{\Phi,K} \left.\frac{\partial d'F}{\partial K'_A}\right|_{\Phi} \\ &= d'd'F - \left.\frac{\partial d'F}{\partial \Phi'^A}\right|_{K'} \left.\frac{\partial d'F}{\partial K'_A}\right|_{\Phi} \\ &= - \left.\frac{\partial d'F}{\partial \Phi'^A}\right|_{K'} \left.\frac{\partial d'F}{\partial K'_A}\right|_{\Phi'} - \left.\frac{\partial d'F}{\partial \Phi'^A}\right|_{K'} F^{BA} \left.\frac{\partial d'F}{\partial \Phi'^B}\right|_{K'}. \end{aligned} \tag{3.39}$$

The first term in the last expression is equal to $-\frac{1}{2}(d'F, d'F)$, while the second term (where $F^{BA} = \frac{\partial_l^2 F}{\partial K'_A \partial K'_B}$) vanishes. This can be easily proved by interchanging the various factors and showing that the expression is equal to the opposite of itself.

The same formulæ prove that the field strength $\mathcal{F}$ of (3.21) is sent into $\tilde{\mathcal{F}}$, a fact that has a nice interpretation: the canonical transformations are the gauge-transformations for the connection ω_1. A completely analogous argument proves that (3.1) is canonically preserved, a property that we left without proof before. Indeed, the transformation rule for χ (2.20) is formally analogous to that of ω_1.

Equations (3.22) where rigorously derived for $i = -1, 0, 1$ and then conjectured for the other values of i. Although they seem very natural (and their properties together with the applications that will be examined in the next section give a stronger support to this), we want to conclude this section with the explicit proof of the case $i = 2$. This provides an expression for ω_3 that permits to interpret it as the set of connected irreducible Feynman diagrams with three χ-insertions. Moreover, the explicit calculation of $d\omega_2$ is sufficient to illustrate how to proceed in the proofs of any case $i > 2$ of (3.22). Let us start by applying equations (3.13) and (3.7), obtaining

$$d\omega_2 = (\omega_1, \omega_2) - (\{\omega_1, \omega_2\}, \omega_0) + d_J\omega_2 + \{\omega_1, (\omega_2, \omega_0)\}. \tag{3.40}$$

A direct computation and a series of manipulations that by now should have become standard give

$$\begin{aligned} d_J\omega_2 = & -\frac{1}{3!\hbar^2}(<\chi\chi\chi>, \omega_0) + \frac{1}{2\hbar^2} <\chi\chi> (\omega_1, \omega_0) + \{\omega_1, d\omega_1\} - \frac{i}{\hbar}\{d\Gamma, \omega_1\}\omega_1 \\ & -\frac{i}{2\hbar}(\omega_1, \omega_1)\omega_1 + \frac{i}{\hbar}(\omega_2, \omega_0)\omega_1 - \frac{1}{2}\left\{d\Gamma, \left.\frac{\partial_r\omega_1}{\partial\Phi^A}\right|_K\right\} \left.\frac{\partial_l\omega_1}{\partial J_A}\right|_K \\ & +\frac{1}{2}\{\omega_1, \{d\Gamma, \omega_1\}\}. \end{aligned} \tag{3.41}$$

Collecting the above two formulæ and using (3.4), we arrive at

$$\begin{aligned} d\omega_2 - (\omega_1, \omega_2) + (\Delta\omega_3, \omega_0) = & \frac{1}{2}\{\omega_1, (\omega_1, \omega_1)\} + \frac{1}{2}\{\omega_1, \{d\Gamma, \omega_1\}\} \\ & -\frac{1}{2}\left\{d\Gamma, \left.\frac{\partial_r\omega_1}{\partial\Phi^A}\right|_K\right\} \left.\frac{\partial_l\omega_1}{\partial J_A}\right|_K, \end{aligned} \tag{3.42}$$

where $\Delta\omega_3$ is the first set of contributions to ω_3, precisely

$$\Delta\omega_3 = \frac{1}{3!\hbar^2} <\chi\chi\chi> - \frac{1}{3!\hbar^2}\omega_1\omega_1\omega_1 + \{\omega_1, \omega_2\} + \frac{i}{\hbar}\omega_2\omega_1 + \frac{i}{2\hbar}\{\omega_1, \omega_1\}\omega_1. \tag{3.43}$$

It is not difficult to prove that

$$-\frac{1}{2}\left\{d\Gamma, \left.\frac{\partial_r\omega_1}{\partial\Phi^A}\right|_K\right\} \left.\frac{\partial_l\omega_1}{\partial J_A}\right|_K = -\frac{1}{4}\{d\Gamma, \{\omega_1, \omega_1\}\} + \frac{1}{4}\{d\Gamma, \omega_1, \omega_1\}, \tag{3.44}$$

where we have introduced the following notion of *triple brackets*

$$\{X, Y, Z\} = \left.\frac{\partial_r X}{\partial\Phi^A}\right|_K \left.\frac{\partial_l^3 W}{\partial J_A \partial J_B \partial J_C}\right|_K \left.\frac{\partial_l Y}{\partial\Phi^C}\right|_K \left.\frac{\partial_r Z}{\partial\Phi^B}\right|_K (-1)^{\varepsilon_B(\varepsilon(Y)+\varepsilon(Z)+1)}. \tag{3.45}$$

The triple brackets have the following properties

$$\begin{aligned}\{Y,X,Z\}&=(-1)^{\varepsilon(X)\varepsilon(Y)+d_X d_Y}\{X,Y,Z\}\\ \{X,Z,Y\}&=(-1)^{\varepsilon(Y)\varepsilon(Z)+d_Y d_Z}\{X,Y,Z\}.\end{aligned}\tag{3.46}$$

The diagrammatical meaning of the triple brackets is that, when X, Y and Z are average values of local composite operators, $\{X,Y,Z\}$ is the set of connected reducible diagrams where the insertions of the three composite operators are only connected *via* the three-vertex

$$\left.\frac{\partial_l^3 W}{\partial J_A \partial J_B \partial J_C}\right|_K .\tag{3.47}$$

This will be illustrated in more detail later on. An identity generalizing (3.7) for $(\{X,Y,Z\},\omega_0)$ can be surely proved for the triple brackets, however we only need the case $X=Y=Z=\omega_1$, in which it happens that

$$\frac{1}{3}(\{\omega_1,\omega_1,\omega_1\},\omega_0)=\{\omega_1,(\omega_1,\omega_1)\}+(\omega_1,\{\omega_1,\omega_1\})+\{d\Gamma,\omega_1,\omega_1\}.\tag{3.48}$$

Indeed, one has

$$(\omega_1,\{\omega_1,\omega_1\})+\{\omega_1,(\omega_1,\omega_1)\}=(-1)^{\varepsilon_A(1+\varepsilon_B)}\left(\omega_1,\left.\frac{\partial_r^2\Gamma}{\partial\Phi^A\partial\Phi^B}\right|_K\right)\left.\frac{\partial\omega_1}{\partial J_B}\right|_K\left.\frac{\partial\omega_1}{\partial J_A}\right|_K .\tag{3.49}$$

Moreover,

$$\begin{aligned}(\{\omega_1,\omega_1,\omega_1\},\Gamma)=J_A\left.\frac{\partial_r\{\omega_1,\omega_1,\omega_1\}}{\partial K_A}\right|_J&=3\{d\Gamma,\omega_1,\omega_1\}\\ &-3(-1)^{(\varepsilon_A+\varepsilon_C)(1+\varepsilon_B)+\varepsilon_A\varepsilon_C}\left.\frac{\partial\omega_1}{\partial K_C}\right|_J\left.\frac{\partial_r^3\Gamma}{\partial\Phi^C\partial\Phi^A\partial\Phi^B}\right|_K\left.\frac{\partial\omega_1}{\partial J_B}\right|_K\left.\frac{\partial\omega_1}{\partial J_A}\right|_K\\ &+(-1)^{\varepsilon_B}\left.\frac{\partial\omega_1}{\partial J_A}\right|_K\left.\frac{\partial\omega_1}{\partial J_B}\right|_K\left.\frac{\partial\omega_1}{\partial J_C}\right|_K\left(\left.\frac{\partial_r^3\Gamma}{\partial\Phi^A\partial\Phi^B\partial\Phi^C}\right|_K,\Gamma\right)\end{aligned}\tag{3.50}$$

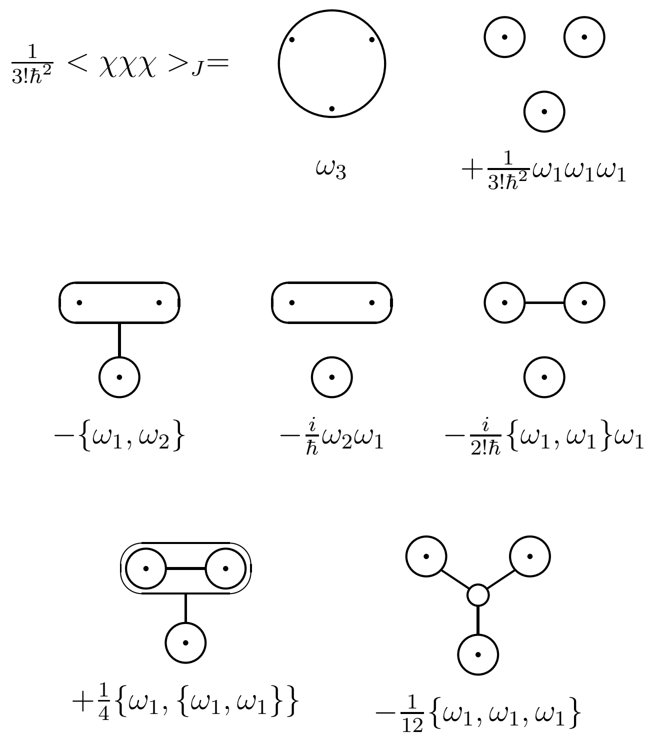
Differentiating $(\Gamma,\Gamma)=0$ three times with respect to Φ, one finds

$$\begin{aligned}(-1)^{\varepsilon_B}\left(\left.\frac{\partial_r^3\Gamma}{\partial\Phi^A\partial\Phi^B\partial\Phi^C}\right|_K,\Gamma\right)&=-(-1)^{\varepsilon_A+\varepsilon_B}\left(\left.\frac{\partial_r^2\Gamma}{\partial\Phi^B\partial\Phi^C}\right|_K,\left.\frac{\partial_r\Gamma}{\partial\Phi^A}\right|_K\right)\\ -(-1)^{\varepsilon_A}\left(\left.\frac{\partial_r\Gamma}{\partial\Phi^C}\right|_K,\left.\frac{\partial_r^2\Gamma}{\partial\Phi^A\partial\Phi^B}\right|_K\right)&-(-1)^{\varepsilon_A\varepsilon_B}\left(\left.\frac{\partial_r^2\Gamma}{\partial\Phi^A\partial\Phi^C}\right|_K,\left.\frac{\partial_r\Gamma}{\partial\Phi^B}\right|_K\right).\end{aligned}\tag{3.51}$$

Using this equation, one finally arrives at (3.48). Collecting (3.42), (3.44) and (3.48), one finally gets the desired equation, namely

$$d\omega_2=(\omega_1,\omega_2)+(\omega_3,\omega_0),\tag{3.52}$$

Figure 1: ω_3 is a sum of connected irreducible Feynmann diagrams.



where

$$\omega_3 = \frac{1}{3!\hbar^2}(< \chi\chi\chi > -\omega_1\omega_1\omega_1) + \{\omega_1, \omega_2\} + \frac{i}{\hbar}\omega_2\omega_1 + \frac{i}{2\hbar}\{\omega_1, \omega_1\}\omega_1 \\ -\frac{1}{4}\{\omega_1, \{\omega_1, \omega_1\}\} + \frac{1}{12}\{\omega_1, \omega_1, \omega_1\}. \tag{3.53}$$

The diagrammatical meaning of each term is illustrated in Fig. 1. The blobs denote sets of connected irreducible diagrams, the dots denote the χ-insertions. It is a useful exercise to check that each coefficient and each power of $\hbar$ is in agreement with the diagrammatical meaning of ω_3.

4 Application of the covariant formalism

It is now time to apply the formalism developed in the previous section in order to find a functional $R^{(n)}$ that makes every ω_k finite up to order $\hbar^n$, under the inductive hypothesis that ω_k are finite up to order $\hbar^{n-1}$. This inductive assumption (that suitably extends the one that was needed for a single gauge-fixing parameter κ) is clearly satisfied for $n = 0$: indeed, for $n = 0$ (i.e. for the theory described by the action Σ_0), the order $\hbar^0$ of ω_0 is

S, which is clearly finite, the order $\hbar^0$ of ω_1 is $d\Psi$, which is also finite, while the order $\hbar^0$ parts of ω_k for $k>1$ are zero. The reason of this last fact is that for $k>1$ the number of χ-insertions is greater that one and of course there is no connected irreducible tree diagram with more that one χ-insertion.

Let us recall what is the situation. We possess cascade equations for the "$n-1$-theory", namely the theory described by Σ_{n-1}, that is convergent up to order $\hbar^{n-1}$ included. The forms $\omega_k^{(n-1)}$ are finite up to order $\hbar^{n-1}$, by induction. We want to make them finite up to order $\hbar^n$. The effective action $\Gamma_{n-1}=\omega_0^{(n-1)}$ is promptly made finite up to order $\hbar^n$ by suitably redefining the parameters λ_i and by performing a canonical transformation generated by (1.8), as explained in section 2. This only changes $\omega_k^{(n-1)}$ up to order $\hbar^n$, so that the inductive assumption is preserved: $\omega_k^{(n-1)}$ is finite up to order $\hbar^{n-1}$ $\forall k$ and moreover $\omega_0^{(n-1)}$ is turned into $\Gamma_n=\omega_0^{(n)}$, which is finite up to order $\hbar^n$. This is sufficient, as in section 2, to show that

$$d\delta_n\lambda_i=0\quad\forall i, \tag{4.1}$$

since the argument of section 2 is trivially extended to an arbitrary number of gauge-fixing parameters κ. (4.1) permits to derive cascade equations of the "n-theory", i.e. for the forms $\omega_k^{(n)}$ [from now on, we suppress the superscript (n)]. Clearly, we can choose ω_k such that $\omega_k=\omega_k^{(n-1)}+\mathcal{O}(\hbar^n)=\text{finite}+\mathcal{O}(\hbar^n)$. Thus, we remain with the task of showing how to make ω_k finite up to order $\hbar^n$ for $k\geq 1$ in order to fully reproduce the inductive assumption to order $\hbar^n$. The argument given at the end of section 2 for a single gauge-fixing parameter is not immediately generalizable to the case of many gauge-fixing parameters, as anticipated. We need to combine the properties of the cascade equations together with the arbitrariness (2.19).

Before doing this, we need to discuss the implications of the cascade equations for ω_k and the properties satisfied by their order $\hbar^n$ divergent parts $\omega_{k\,div}$. First of all, we notice that the cascade equations for ω_k imply descent equations for $\omega_{k\,div}$. The quasi double-complex generates a double complex. It is sufficient to take the order $\hbar^n$ divergent parts of eq.s (3.22). We get

$$D\omega_{k\,div}=(-1)^k\sigma\omega_{k+1\,div}. \tag{4.2}$$

These equations are indeed descent equations, since $D^2=0$, $D\sigma=\sigma D$ and $\sigma^2=0$.

Choosing the functionals $\Delta_k=\mathcal{O}(\hbar^n)$, $k=1,\ldots m$, to be averages of local functionals, the cascade invariances reduce to the usual invariances of the descent equations, namely

$$\omega'_{k\,div}=\omega_{k\,div}-(-1)^k D\Delta_{k-1\,div}+\sigma\Delta_{k\,div}, \tag{4.3}$$

where $\Delta_{k\,div}$ denotes the (local) order $\hbar^n$ divergent part of Δ_k and we have set $\Delta_{-1}=\Delta_0=0$.

The fact that the covariant derivative D is flat, $D^2=0$, permit to "integrate" a D-closed form ω, $D\omega=0$, i.e. to find (perturbatively in the gauge-fixing parameters

$\{\kappa_1, \ldots, \kappa_m\}$) a form δ such that $\omega = D\delta$. An example of this kind was given at the end of section 2, where (2.23) solved (2.24). Analogous formulæ can be found for a generic ω, using the "integrability condition" $D\omega = 0$. The general solution is obtained as follows. Let us introduce the vector field

$$v = \kappa_i \frac{\partial}{\partial \kappa_i} \tag{4.4}$$

and let i_v denote the operator of contraction with v. Furthermore, we need the following operator

$$\Theta = \kappa_i D_i. \tag{4.5}$$

It is easy to prove the following properties

$$[D, i_v]_+ \omega_p = (p + \Theta)\omega_p, \qquad [D, \Theta]\omega_p = D\omega_p, \qquad [i_v, \Theta]\omega_p = -i_v\omega_p, \tag{4.6}$$

for any p-form ω_p, while σ commutes with D, Θ and i_v. Let us define, for $p \geq 1$,

$$\Omega_p = (p-1)! \sum_{n=0}^{\infty} \sum_{i_1 \cdots i_n} (-1)^n \frac{\kappa_{i_1} \cdots \kappa_{i_n}}{(n+p)!} D_{i_1} \cdots D_{i_n} \omega_p. \tag{4.7}$$

Using $D\omega_p = 0$, it is immediate to prove that Ω_p is also D-closed: $D\Omega_p = 0$. Moreover, notice that, if $\sigma\omega_p = 0$, then $\sigma\Omega_p = 0$. The solution to our problem is ($p \geq 1$)

$$\delta_p = i_v \Omega_p. \tag{4.8}$$

Indeed, using the first of (4.6), one finds

$$D\delta_p = Di_v\Omega_p = -i_v D\Omega_p + (p + \Theta)\Omega_p = (p + \Theta)\Omega_p = \omega_p. \tag{4.9}$$

The integrability property that we have shown permits to find k-forms δ_k, $k = 0, \ldots m$, such that

$$\omega_{k\,div} = (-1)^k D\delta_{k-1} - \sigma\delta_k. \tag{4.10}$$

One starts from ω_m, which satisfies $D\omega_m = 0$. Let us set $\delta_m = 0$. There exists a δ_{m-1} such that $\omega_m = (-1)^m D\delta_{m-1}$. Then, (4.2) gives, for ω_{m-1}, $D(\omega_{m-1} + \sigma\delta_{m-1}) = 0$, so that there exists a δ_{m-2} such that $\omega_{m-1} = -\sigma\delta_{m-1} + (-1)^m D\delta_{m-2}$. Repeating this argument, one proves (4.10). At the last step one has $D(\omega_{1\,div} + \sigma\delta_1) = 0$, that is solved by $\omega_{1\,div} = -\sigma\delta_1 - D\delta_0$. Moreover, since $\sigma\omega_{1\,div} = D\omega_{0\,div} = 0$ (due to $\omega_{0\,div} = 0$) and $\sigma(\omega_{1\,div} + \sigma\delta_1) = 0$, the explicit form of the solution δ_0 [formulæ (4.7) and (4.8) with $\omega = \omega_{1\,div} + \sigma\delta_1$] and the fact that σ commutes with i_v show that $\sigma\delta_0 = 0$, precisely as in (2.24). Clearly, δ_k are of order $\hbar^n$ and divergent.

Let us now define $\Delta_k =< \delta_k >_J$. We see that it is possible to use the cascade invariances of degrees $k = 1, \ldots m-1$ to make $\omega_{k\,div} = 0$ for $k = 2, \ldots m$. This is easily seen from (4.3), since the order $\hbar^n$ divergent part $\Delta_{k\,div}$ of Δ_k coincides with δ_k. At the end, we have a $\omega_{1\,div}$ that is σ-closed and has the form

$$\omega_{1\,div} = -D\delta_0, \quad \sigma\delta_0 = 0. \tag{4.11}$$

So, we have solved our problem for any ω_k except for ω_1. The final step is to use the freedom (2.19) with $T^{(n)} = -\delta_0$, which permits to cancel $\omega_{1\,div}$ (and has no effect on the order $\hbar^n$ divergent parts of the other ω_k, as can be easily verified).

We conclude that the full set of cascade equations survives the subtraction algorithm, allowing a complete control on the gauge-fixing sector and proving the gauge-fixing independence of the on-shell physical amplitudes [4].

5 A geometrical description

Let us define a principal fiber bundle with $\mathcal{M}_{gf}$ as the base manifold. The fibers are isomorphic to the group of canonical transformations. The "Lie" algebra is the algebra of antibrackets. To be more precise, let $\mathcal{I}$ denote the space of fields Φ and BRS sources K. Let us give the name *scalar functionals* to those functionals $X(\Phi, K)$ of fields and BRS sources that transform as $X' = \tilde{X}$ under canonical transformations. Let $\mathcal{Z}$ denote the set of scalar functionals. Let g be the operator that represents the action of a canonical transformation on a scalar functional X: $gX = X' = \tilde{X}$ and let $\mathcal{G}$ be the set of such g's. $\mathcal{G}$ is obviously a group and the map

$$\begin{aligned} R &: \mathcal{G} \times \mathcal{Z} \to \mathcal{Z} \\ R(g, X) &= gX \end{aligned} \tag{5.1}$$

is a representation of $\mathcal{G}$.

$\mathcal{Z}$ is obviously a vector space. Equation (3.37) shows that $\omega_i \in \mathcal{Z}\ \forall i \neq 1$. We define a product $\cdot$ in $\mathcal{Z}$, represented by antibrackets

$$\begin{aligned} \cdot \quad &: \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z} \\ X \cdot Y &= (X, Y) \end{aligned} \tag{5.2}$$

In this way, $\mathcal{Z}$ becomes an algebra. To make connections with usual notions, we notice that antibrackets replace Lie brackets and the above concept of algebra replaces the concept of Lie algebra. In analogy with this, $\mathcal{Z}$ will be called *antialgebra*. Similarly, $\mathcal{G}$ will be called *antigroup* and replaces the Lie group. Clearly, $\mathcal{Z}$ and $\mathcal{G}$ are infinite-dimensional. The space $\mathcal{I}$ corresponds to the set of "Lie algebra"-indices a, so that $X(\Phi, K)$ corresponds to the adjoint representation ϕ^a of the Lie algebra.

Let us define the principle fiber bundle

$$\mathcal{P} = (\mathcal{E}, \mathcal{M}_{gf}, \pi, \mathcal{G}). \tag{5.3}$$

As anticipated, $\mathcal{M}_{gf}$ is the base manifold. $\mathcal{E}$ is such that the sections of $\mathcal{P}$ are obtained by letting the canonical transformations $g \in \mathcal{G}$ depend on the points κ of the base manifold $\mathcal{M}_{gf}$: $g = g(\kappa)$ and $g(\kappa) \in \mathcal{G}\ \forall \kappa$. $\pi : \mathcal{E} \to \mathcal{M}_{gf}$ is the projection onto the base manifold: $\pi(g(\kappa)) = \kappa$. $\pi^{-1}(\kappa)$ is isomorphic to $\mathcal{G}$.

When a functional $X \in \mathcal{Z}$ is $\mathcal{M}_{gf}$-dependent, $X = X(\Phi, K, \kappa)$, then it can be described as a section of the fiber bundle $\mathcal{B}$ with typical fiber $\mathcal{Z}$ associated with $\mathcal{P}$ by the representation R and corresponds to the familiar Lie algebra-valued scalar field $\phi^a(x)$.

On the bundle $\mathcal{P}$, a connection can be introduced: it is ω_1. Under a canonical transformation represented by g on $\mathcal{Z}$, associated to the generating functional F_g (roughly speaking, $g \sim \mathrm{e}^{F_g}$), we have from (3.37) $\omega_1' = \tilde{\omega}_1 - d'F$. For an infinitesimal transformation $g = 1 + \varepsilon h$, $F_g = \Phi K' + \varepsilon f_h$, we have

$$\begin{aligned} \omega_1' &= \omega_1 - \varepsilon(df_h - (\omega_1, f_h)) + \cdots = \omega_1 - \varepsilon \mathcal{D} f_h + \cdots \\ X' &= gX = X - \varepsilon(f_h, X) + \cdots \end{aligned} \tag{5.4}$$

where $X \in \mathcal{Z}$. The above formulæ are very similar to the common ones for a gauge-theory and justify the name "scalar functionals" for the elements of $\mathcal{Z}$.

The effective action Γ is a scalar functional and, moreover, it is covariantly constant. The off-shell covariant constancy of Γ implies the on-shell constancy, since $\mathcal{D}\Gamma - d\Gamma = -\mathrm{ad}\,\Gamma\ \omega_1$, which is zero on-shell.

The above geometrical description permits to get an intuitive perception of the fact that physical amplitudes remain independent of the gauge-fixing parameters. Indeed, the subtraction algorithm is made of two basic ingredients: the redefinitions of the parameters λ, that are "orthogonal" to $\mathcal{M}_{gf}$ in the sense that they do not depend on the point on $\mathcal{M}_{gf}$, and a canonical transformation. The canonical transformation, on the other hand, is a gauge-transformation in the principal fiber bundle $\mathcal{P}$, i.e. it is "vertical" with respect to the base manifold $\mathcal{M}_{gf}$. For this reason, it cannot pick up any information from the base manifold itself. Thus, we can say that the full subtraction algorithm is orthogonal to the manifold $\mathcal{M}_{gf}$ of gauge-fixing parameters.

With this, we think that we have reached a satisfactory control on what happens to the gauge-fixing parameters. Of course, more important is to have control on what happens to the physical parameters λ. An analogous covariant treatment should be introduced as a starting point for solving the problem of classification of predictive nonrenormalizable quantum field theories.

Let us call $\mathcal{M}_{ph}$ the manifold of the physical parameters $\{\lambda_i\}$. In general, $\mathcal{M}_{ph}$ is infinite dimensional (for nonrenormalizable theories), while the dimension of $\mathcal{M}_{gf}$ depends on the gauge-fixing choice. One (at least) of the parameters λ of the classical Lagrangian (let us call it λ_0) is peculiar, since it multiplies the gauge-invariant functional $\mathcal{G}_0$ that defines the propagator. So, there is no perturbative expansion in λ_0 and the hyperplane $\lambda_0 = 0$ does not belong to $\mathcal{M}_{ph}$.

From now on, the differentiation (1.19) on $\mathcal{M}_{gf}$ will be denoted by d_{gf}, while we introduce a differentiation

$$d_{ph} = d\lambda_i \left. \frac{\partial}{\partial \lambda_i} \right|_{\Phi, K} \tag{5.5}$$

on the manifold $\mathcal{M}_{ph}$. Moreover, we write

$$d = d_{ph} + d_{gf}, \tag{5.6}$$

which is the differential operator on the manifold $\mathcal{M} = \mathcal{M}_{ph} \otimes \mathcal{M}_{gf}$. Notice that, since the gauge coupling constants are not redefined by the subtraction algorithm [4], they do not belong to $\mathcal{M}$.

The redefinitions $\lambda_i \to \lambda_i - \delta_n\lambda_i$ of the parameters λ_i can be described as a diffeomorphism in $\mathcal{M}_{ph}$. Indeed, $v_n = \delta_n\lambda_i \frac{\partial}{\partial\lambda_i}$ is a vector field on $\mathcal{M}_{ph}$, $v_n \in \mathcal{T}\mathcal{M}_{ph}$ ($\mathcal{T}$ denoting the tangent bundle) and we can write $\mathcal{G}^{(n)} = \sum_i \delta_n\lambda_i\mathcal{S}_i = v_n\mathcal{S}$ or equivalently

$$\mathcal{G}^{(n)} = l_n\mathcal{S}, \tag{5.7}$$

where $l_n = di_{v_n} + i_{v_n}d$ is the Lie derivative along the vector field v_n. Thus, the subtraction algorithm can be described as a composition of diffeomorphisms l_n on $\mathcal{M}_{ph}$, independent of $\mathcal{M}_{gf}$, and canonical transformations. (5.7) means that the action $\mathcal{S}$ transforms as a scalar under l_n.

We can now formulate the requirement that finitely many parameters are sufficient to remove the divergences in this geometrical framework. A theory is predictive if there exists a finite dimensional submanifold $\mathcal{V} \subset \mathcal{M}_{ph}$, such that $v_n|_{\mathcal{V}}$ are vector fields of $\mathcal{V}$ $\forall n$:

$$v_n|_{\mathcal{V}} \in \mathcal{T}\mathcal{V}\ \forall n, \qquad \dim\mathcal{V} < \infty. \tag{5.8}$$

Applying the subtraction algorithm to a quantum field theory, one gets a set of vector fields v_n on $\mathcal{M}_{ph}$. In particular, to each point λ of $\mathcal{M}_{ph}$ a set of vectors $\{v_1(\lambda), \ldots, v_n(\lambda), \ldots\}$ is associated. Let us call $V(\lambda)$ the vector space spanned by these vectors. Let us call $\mathcal{V}_k$ the subset of $\mathcal{M}_{ph}$ (it is not guaranteed that it is a manifold, but let us suppose that it is) where $\dim V(\lambda) = k$,

$$\mathcal{V}_k = \{\lambda \in \mathcal{M}_{ph} : \dim V(\lambda) = k\}. \tag{5.9}$$

A necessary condition for predictivity is that $\exists k < \infty$ such that $\mathcal{V}_k \neq \emptyset$. Then, if $v_n(\lambda) \in \mathcal{T}\mathcal{V}_k$ $\forall n$ and $\forall \lambda \in \mathcal{V}_k$, we can take $\mathcal{V} = \mathcal{V}_k$. In general, however, this does not hold. So, one can define a sequence of subspaces

$$\mathcal{V}_k^{(i)} = \{\lambda \in \mathcal{V}_k^{(i-1)} : v_n(\lambda) \in \mathcal{T}\mathcal{V}_k^{(i-1)}\}, \tag{5.10}$$

where $\mathcal{V}_k^{(0)} = \mathcal{V}_k$. The search stops at an i such that $\mathcal{V}_k^{(i+1)} = \mathcal{V}_k^{(i)}$.

Supposing that a suitable submanifold $\mathcal{V}$ has been found, one can apply the same construction to $\mathcal{V}$ itself, in order to see if it is possible to *reduce* the (now finite) number of independent parameters that are necessary for predictivity. One has to check if there exists no submanifold of $\mathcal{V}$ to which one can consistently restrict. A quantum field theory with a $\mathcal{V}$ such that $\dim\mathcal{V} =$ minimum can reasonably be called *irreducible*. In the cases where no gauge symmetry is present, the irreducible theory is free (reduction to the single parameter λ_0 plus the eventual mass), obtained by removing all interactions. When there is some gauge-invariance, on the other hand, the free theory is obtained by letting the gauge coupling constant g going to zero, which, however, does not correspond to a reduction in the above sense, since the gauge coupling constant does not belong to the set $\mathcal{M}_{ph}$. So, the above definition of irreducibility is nontrivial for gauge-theories. The irreducible Yang-Mills theories are the ordinary renormalizable versions. The irreducible theory of gravity is unknown.

In the next section, we discuss some predictive (but unphysical) toy models, that, for simplicity, have no gauge-symmetry.

6 Toy models

The removal of $\mathcal{G}^{(n)}$ with redefinitions of λ requires, in general, the presence of infinitely many λ's, so that predictivity is lost. The algorithm defined in [4] and generalized in the present paper is also applyable to non-renormalizable gauge-field theories and it is not so naïve to pretend that in general the removal of divergences can be performed with a finite number of parameters (in various algorithms appeared in the literature, instead, a preferred choice for infinitely many parameters is hidden in some peculiar regularization technique or in the renormalization prescription, loosing control on the involved arbitrariness). Our algorithm keeps complete control on the arbitrariness introduced in the subtraction procedure. However, one cannot *a priori* discard the possibility that, in the context of the algorithm that we have formulated, the divergences of a nonrenormalizable theory can be removed with only a *finite* number of parameters λ, while keeping a complete control on the involved arbitrariness. The distinction between two subsets of divergences ($\mathcal{G}^{(n)}$ and $\sigma R^{(n)}$) that have different roles and properties clarifies that the problem of predictivity only concerns $\mathcal{G}^{(n)}$. As we shall see, one can reformulate the concept of predictivity of the previous section by means of a condition on $\mathcal{G}^{(n)}$ such that it is sufficient to redefine a finite number of parameters in order to remove $\mathcal{G}^{(n)}$ itself. The remaining divergent terms (i.e. $\sigma R^{(n)}$) are not constrained to have any particular form: indeed, infinitely many new counterterms can appear through $R^{(n)}$, but, whatever $R^{(n)}$ is, it can always be removed by a canonical transformation, without effects on the physical amplitudes. This section is devoted to the search for examples of toy models of power counting nonrenormalizable theories in which a finite number of parameters is sufficient to remove the divergences. Since the piece $\sigma R^{(n)}$ has no influence on predictivity, we focus for now on non-gauge field theories, where $\sigma R^{(n)}$ is absent: only redefinitions of the parameters λ are involved and no canonical transformation at all. In a first example we show that it is possible to construct a power counting nonrenormalizable theory that is polynomial and such that the counterterms are in a finite number of types and have the same form as the terms of the classical Lagrangian. The theory is protected by a diagrammatics that is simplified by the fact that the propagator is off-diagonal. Due to this, however, it is nonphysical, since the kinetic action is not positive definite. Nevertheless, we think that its properties with respect to the subtraction algorithm deserve attention. We then develop a method for producing certain predictive nonrenormalizable theories by means of a "change of variables" that has to be performed on suitable renormalizable theories in which some composite operator is introduced, coupled to an external source K.

Before entering into details, let us briefly point out some differences between our subtraction algorithm and other algorithms that one can find in the literature.

For example, in ref. [8] one simply redefines the action Σ by subtracting, order by order, the divergent part $\Gamma^{(n)}_{div}$, i.e.

$$\Sigma_n = \Sigma_{n-1} - \Gamma^{(n)}_{div}. \tag{6.1}$$

The dimensional regularization technique is used. The algorithm (6.1) breaks the master equation, however one can prove that this breaking, at least in the dimensional regularization framework, is under control. At the end, the "renormalized" action Σ_∞ satisfies the classical master equation $(\Sigma_\infty, \Sigma_\infty) = 0$. The algorithm (6.1), apart from the restriction on the regularization technique, is completely general. The "philosophy of the method" is the following: whenever a divergent term is found, it has to be subtracted away. One does not wonder whether the divergent term is of a new type or not. The limit of the algorithm, however, is that one does not have a direct control on the arbitrariness of the procedure.

On the other hand, if one is not satisfied with the simple subtraction of the divergent terms, but one requires this subtraction to be implemented by a redefinition of parameters and fields, then the classical Lagrangian is, in general, demanded to possess an infinite number of parameters λ. In this case, predictivity is lost.

Now, we go on by reformulating the predictivity requirement of the end of section 5 in more concrete terms. Suppose that the λ_i are certain functions of a finite number of new parameters α_j, i.e.

$$\mathcal{L}_{class} = \mathcal{L}_{class}(\phi, \alpha) = \sum_i \lambda_i(\alpha)\, \mathcal{G}_i. \tag{6.2}$$

Then, $\delta_n \lambda_i$ are also functions of α:

$$\mathcal{G}^{(n)} = \sum_i \delta_n \lambda_i(\alpha)\, \mathcal{S}_i(\lambda(\alpha)). \tag{6.3}$$

If there exist suitable $\Delta_n \alpha_j$, of order $\hbar^n$, such that

$$\mathcal{G}^{(n)} = \sum_j \Delta_n \alpha_j \frac{\partial S}{\partial \alpha_j} = \sum_{j,i} \Delta_n \alpha_j \frac{\partial \lambda_i(\alpha)}{\partial \alpha_j} \mathcal{S}_i, \tag{6.4}$$

then it is possible to cancel $\mathcal{G}^{(n)}$ by simply redefining the α_j. Indeed,

$$\begin{aligned}\Sigma_{n-1}(\Phi, K, \alpha_j - \Delta_n \alpha_j) &= \Sigma_{n-1}(\Phi, K, \alpha) - \sum_j \Delta_n \alpha_j \frac{\partial \Sigma_{n-1}}{\partial \alpha_j} + \mathcal{O}(\hbar^{n+1}) \\ &= \Sigma_{n-1}(\Phi, K, \alpha) - \sum_j \Delta_n \alpha_j \frac{\partial S}{\partial \alpha_j} + \mathcal{O}(\hbar^{n+1}) \\ &= \Sigma_{n-1} - \mathcal{G}^{(n)} + \mathcal{O}(\hbar^{n+1}).\end{aligned} \tag{6.5}$$

The term $\sigma R^{(n)}(\Phi, K)$ is then cancelled in the known way by a canonical transformation.

Condition (6.4) is equivalent to

$$\delta_n \lambda_i(\alpha) = \sum_j \Delta_n \alpha_j \frac{\partial \lambda_i(\alpha)}{\partial \alpha_j}. \tag{6.6}$$

These are nontrivial conditions (they should be satisfied for any n) on the functions $\lambda_i(\alpha)$. It is clear that the power-counting renormalizable theories trivially satisfy condition (6.6): the sets $\{\lambda_i\}$ and $\{\alpha_j\}$ coincide.

The contact with the geometrical formulation of the predictivity requirement given at the end of the previous section is that α_j are coordinates on the finite dimensional manifold $\mathcal{V}$ and

$$\lambda_i = \lambda_i(\alpha_j) \tag{6.7}$$

are the equations embedding $\mathcal{V}$ in $\mathcal{M}_{ph}$, while (6.6) corresponds to the condition $v_n(\lambda) \in \mathcal{TV}\ \forall \lambda \in \mathcal{V}$, equivalent to $v_n(\lambda(\alpha)) \in \mathcal{TV}\ \forall \alpha$.

We emphasize again that a possible source of insight is to investigate this kind of "stability" with respect to the subtraction algorithm, or a stability "in the sense of the correspondence principle". Instead, we never refer to stability with respect to the renormalization group.

The next task is to elaborate some toy models of nonrenormalizable theories that satisfy (6.6) in order to show that we are dealing with something nonempty and nontrivial.

Let us start with a polynomial theory. It consists of two scalar fields ϕ_1 and ϕ_2 with (nonpositive definite) kinetic action $-\phi_1 \Box \phi_2 + m^2 \phi_1 \phi_2$. Consider the Lagrangian

$$\mathcal{L}_{class}(\phi_1, \phi_2) = -\phi_1 \Box \phi_2 + m^2 \phi_1 \phi_2 + \frac{\lambda}{4} \phi_1^2 \phi_2^2. \tag{6.8}$$

It is an exotic ϕ^4-type theory. We work with the dimensional regularization technique. It is clear that the theory is ultraviolet renormalizable (but not finite): there exist suitable constants Z, δm^2 and Z_λ such that the renormalized Lagrangian

$$\mathcal{L}_{ren}(\phi_1, \phi_2) = -Z \phi_1 \Box \phi_2 + Z(m^2 + \delta m^2) \phi_1 \phi_2 + \frac{\lambda Z_\lambda}{4} Z^2 \phi_1^2 \phi_2^2, \tag{6.9}$$

gives a convergent generating functional W of the connected Green functions. Let us now introduce a ϕ^5-type term, precisely,

$$\mathcal{L}_{class}(\phi_1, \phi_2) = -\phi_1 \Box \phi_2 + m^2 \phi_1 \phi_2 + \frac{\lambda}{4} \phi_1^2 \phi_2^2 + \frac{\alpha}{4!} \phi_2 \phi_1^4. \tag{6.10}$$

Despite the appearance, only a finite number of types of divergent graphs are generated, because the diagrammatics is very simple. One can easily check that there are two-loop divergent diagrams with three external ϕ_1-legs. The dimensions are such that the power in the external momenta is two. The required counterterms have the form $\phi_1^2 \Box \phi_1$ and ϕ_1^3. Moreover, one loop divergent diagrams with six external ϕ_1-legs can easily been constructed, so that a counterterm ϕ_1^6 is also required. For a reason that will be clear in a moment, let us also introduce a ϕ_2^3 vertex. One is thus lead to consider the Lagrangian

$$\mathcal{L}_{class}(\phi_1,\phi_2) = -\phi_1\Box\phi_2 + m^2\phi_1\phi_2 + \frac{\eta}{3!}\phi_1^3 + \frac{\lambda}{4}\phi_1^2\phi_2^2 + \frac{\alpha}{4!}\phi_2\phi_1^4 + \frac{\beta}{3!}\phi_1^2\Box\phi_1 + \frac{\gamma}{6!}\phi_1^6 + \frac{\zeta}{3!}\phi_2^3, \tag{6.11}$$

where η, λ, α, β, γ and ζ are independent (and "small") coupling constants. The diagrammatics is so simple that it is easy to check that no other counterterms are generated. Indeed, the kinetic action $-\phi_1\Box\phi_2 + m^2\phi_1\phi_2$ has been chosen precisely to simplify the diagrammatics. Let us give the explicit proof. Let G denote a graph with E_1 external ϕ_1-legs, E_2 external ϕ_2-legs, I internal legs and L loops. Let n_1, n_2, n_3, n_4, n_5 and n_6 denote the number of vertices of the forms ϕ_1^3, $\phi_1^2\phi_2^2$, $\phi_1^4\phi_2$, $\phi_1^2\Box\phi_1$, ϕ_1^6 and ϕ_2^3, respectively. Finally, let $\omega(G)$ denote the superficial degree of divergence of the graph G. We have

$$\begin{aligned} I + E_1 &= 3n_1 + 2n_2 + 4n_3 + 3n_4 + 6n_5, \\ I + E_2 &= 2n_2 + n_3 + 3n_6, \\ L &= I - n_1 - n_2 - n_3 - n_4 - n_5 - n_6 + 1. \end{aligned} \tag{6.12}$$

The first two formulæ give the total numbers of ϕ_1- and ϕ_2-legs. Each propagator connects a ϕ_1-leg with a ϕ_2-leg. The superficial degree of divergence turns out to be

$$\begin{aligned} \omega(G) &= 4L - 2I + 2n_4 = 2I - 4n_1 - 4n_2 - 4n_3 - 2n_4 - 4n_5 - 4n_6 + 4 \\ &= 4 + 2I - \frac{2}{3}(I + E_1 + 2I + 2E_2) - 2n_1 = 4 - \frac{2}{3}(E_1 + 2E_2) - 2n_1. \end{aligned} \tag{6.13}$$

As we see, $\omega(G)$ is bounded. We have to show that no new counterterm is required.

For $\omega(G) = 0$ there are two possibilities:

i) n_1=0. In this case, we have $E_1 = 6$, $E_2 = 0$, or $E_1 = 4$, $E_2 = 1$, or $E_1 = 2$, $E_2 = 2$, or $E_1 = 0$, $E_2 = 3$;

b) $n_1 = 1$. Now, it can only be $E_1 = 3$, $E_2 = 0$, or $E_1 = 1$, $E_2 = 1$.

Instead, $\omega(G) = 2$ is only consistent with $E_1 = 3$, $E_2 = 0$, or with $E_1 = 1$, $E_2 = 1$ at $n_1 = 0$. All these divergences have the form of the quadratic part of the Lagrangian or of the vertices. Equations (6.6) are trivially satisfied [in the present case, the α-parameters of (6.2) coincide with the λ-parameters]. Thus the theory is predictive, nonrenormalizable, nonfinite, polynomial and has a nonpositive definite kinetic action.

It we set $\zeta = 0$, the theory remains predictive with five parameters (η, λ, α, β and γ). Indeed, when ϕ_2^3 is absent ($n_6 = 0$), eq. (6.12) gives $E_1 \geq E_2$, so that the solution $E_1 = 0$, $E_2 = 3$ has to be discarded: the ϕ_2^3 vertex is not radiatively generated, if it is initially absent.

We can do even more, namely we can fix some parameters as suitable functions of the others, while preserving predictivity. In particular, for a reason that we shall discuss in a moment, the theory with Lagrangian

$$\mathcal{L}_{class}(\phi_1,\phi_2,\lambda,\alpha,m^2) = -\phi_1\Box\phi_2 + m^2\phi_1\phi_2 - \alpha m^2\phi_1^3 + \frac{\lambda}{4}\phi_1^2\phi_2^2$$

$$-\frac{\lambda\alpha}{2}\phi_2\phi_1^4+\alpha\phi_1^2\Box\phi_1+\frac{\lambda\alpha^2}{4}\phi_1^6 \tag{6.14}$$

is predictive. Now eqs. (6.6) are still verified, but in a nontrivial way. The independent parameters have been reduced to three: m, λ and α. To check that the theory (6.14) is indeed predictive, let us introduce the sources explicitly:

$$\begin{aligned}\mathcal{L}_{class}(\phi_1,\phi_2)=&-\phi_1\Box\phi_2+m^2\phi_1\phi_2-\alpha m^2\phi_1^3+\frac{\lambda}{4}\phi_1^2\phi_2^2-\frac{\lambda\alpha}{2}\phi_2\phi_1^4\\&+\alpha\phi_1^2\Box\phi_1+\frac{\lambda\alpha^2}{4}\phi_1^6+J_1\phi_1+J_2\phi_2.\end{aligned} \tag{6.15}$$

We now perform the following change of variables in the functional integral (the Jacobian determinant being one)

$$\begin{aligned}\varphi_1&=\phi_1,\\ \varphi_2&=\phi_2-\alpha\phi_1^2,\end{aligned} \tag{6.16}$$

so that (6.15) can be rewritten as

$$\mathcal{L}'_{class}=-\varphi_1\Box\varphi_2+m^2\varphi_1\varphi_2+\frac{\lambda}{4}\varphi_1^2\varphi_2^2+J_1\varphi_1+J_2(\varphi_2+\alpha\varphi_1^2). \tag{6.17}$$

Now the Lagrangian has a quite honest aspect, however, the sources do not appear in the conventional way $J_1\varphi_1+J_2\varphi_2$. Nevertheless, let us call $W[J_1,J_2,\lambda,\alpha,m^2]$ the generating functional of the connected Green functions for the theory (6.15) or (6.17). We can write

$$W[J_1,J_2,\lambda,\alpha,m^2]=\tilde{W}[J_1,J_2,K,\lambda,m^2]|_{K=\alpha J_2}, \tag{6.18}$$

where $\tilde{W}[J_1,J_2,K,\lambda,m^2]$ is the generating functional of the theory with Lagrangian

$$\tilde{\mathcal{L}}_{class}=-\varphi_1\Box\varphi_2+m^2\varphi_1\varphi_2+\frac{\lambda}{4}\varphi_1^2\varphi_2^2+J_1\varphi_1+J_2\varphi_2+K\varphi_1^2. \tag{6.19}$$

This is simply the renormalizable theory (6.8) with the introduction of the composite operator φ_1^2, with source K. If G is a graph with E_1 external φ_1-legs, E_2 external φ_2-legs, I internal legs, L loops and E_K external K-legs, the superficial degree of divergence $\omega(G)$ turns out to be $\omega(G)=4-2E_1$. Moreover, it is easy to show that $E_1=E_2+2E_K$. Thus, the only divergent graphs [$\omega(G)=0$ and $\omega(G)=2$] have the form of the terms of $\tilde{\mathcal{L}}_{class}$ (6.19). Consequently, there exist constants Z, δm^2, Z_λ and Z_α such that the renormalized Lagrangian

$$\tilde{\mathcal{L}}_{ren}=-Z\varphi_1\Box\varphi_2+Z(m^2+\delta m^2)\varphi_1\varphi_2+\frac{\lambda Z_\lambda}{4}Z^2\varphi_1^2\varphi_2^2+J_1\varphi_1+J_2\varphi_2+Z_\alpha Z^{1/2}K\varphi_1^2, \tag{6.20}$$

gives a finite generating functional $\tilde{W}[J_1,J_2,K,\lambda,m^2]$. Setting $K=\alpha J_2$ and going back with the renormalized inverse change of variables, namely

$$\begin{aligned}\phi_1&=\varphi_1,\\ \phi_2&=\varphi_2+\alpha Z_\alpha Z^{1/2}\varphi_1^2,\end{aligned} \tag{6.21}$$

the theory

$$\begin{aligned}\mathcal{L}_{ren}(\phi_1,\phi_2,\lambda,\alpha,m^2) = & -Z\phi_1\Box\phi_2 + Z(m^2+\delta m^2)\phi_1\phi_2 - \alpha Z_\alpha Z^{3/2}(m^2+\delta m^2)\phi_1^3 \\ & + \frac{\lambda Z_\lambda}{4}Z^2\phi_1^2\phi_2^2 - \frac{\lambda\alpha Z_\lambda Z_\alpha}{2}Z^{5/2}\phi_2\phi_1^4 + \alpha Z_\alpha Z^{3/2}\phi_1^2\Box\phi_1 \\ & + \frac{\lambda\alpha^2 Z_\lambda Z_\alpha^2}{4}Z^3\phi_1^6 + J_1\phi_1 + J_2\phi_2, \end{aligned} \tag{6.22}$$

corresponds to a finite generating functional $W_{ren}[J_1, J_2, \lambda, \alpha, m^2]$. Notice that (6.15) and (6.17) are the same theory, while (6.17) and (6.19) are different theories. We see that $\mathcal{L}_{ren}(\phi_1,\phi_2,\lambda,\alpha,m^2) = \mathcal{L}_{class}(Z^{1/2}\phi_1, Z^{1/2}\phi_2, \lambda Z_\lambda, \alpha Z_\alpha, m^2+\delta m^2)$.

Let us rewrite $\mathcal{L}_{class}$ and $\mathcal{L}_{ren}$ in a form that is more similar to (2.3). Let us introduce a parameter ζ in front of the kinetic Lagrangian. ζ is not "small", i.e. we do not make a perturbative expansion in ζ. We write

$$\begin{aligned}\mathcal{L}_{class}(\zeta,\lambda,\alpha) = & -\zeta\phi_1\Box\phi_2 + \zeta m^2\phi_1\phi_2 - \zeta\alpha m^2\phi_1^3 + \frac{\lambda}{4}\phi_1^2\phi_2^2 \\ & - \frac{\lambda\alpha}{2}\phi_2\phi_1^4 + \zeta\alpha\phi_1^2\Box\phi_1 + \frac{\lambda\alpha^2}{4}\phi_1^6. \end{aligned} \tag{6.23}$$

Then, there exist factors $\tilde{Z}_\zeta$, $\tilde{\delta}m^2$, $\tilde{Z}_\lambda$ and $\tilde{Z}_\alpha$ such that the renormalized Lagrangian is

$$\mathcal{L}_{ren}(\zeta,\lambda,\alpha,m^2) = \mathcal{L}_{class}(\zeta\tilde{Z}_\zeta, \lambda\tilde{Z}_\lambda, \alpha\tilde{Z}_\alpha, m^2+\tilde{\delta}m^2). \tag{6.24}$$

In other words, $\mathcal{L}_{ren}$ is obtained from $\mathcal{L}_{class}$ precisely with suitable redefinitions of the four parameters ζ, m^2, λ and α. Restoring $\zeta=1$, one has $\tilde{Z}_\zeta|_{\zeta=1} = Z$, $\tilde{\delta}m^2|_{\zeta=1} = \delta m^2$, $\tilde{Z}_\lambda|_{\zeta=1} = Z_\lambda Z^2$ and $\tilde{Z}_\alpha|_{\zeta=1} = Z_\alpha Z^{1/2}$.

What we have elaborated is a method for constructing certain nonrenormalizable predictive theories from renormalizable ones. It is worth stopping for a moment and giving a clear description of this method. One starts from a renormalizable theory of certain fields Φ. Let the corresponding sources be denoted by J: in the functional integral, J only appear in the linear term $J\Phi$ that is added to the action. Then, one introduces some suitable composite operators $\mathcal{O}(\Phi)$, coupled to external sources K. So, J and K appear in the form $J\Phi + K\mathcal{O}(\Phi)$. For simplicity, let us adopt the convention that the sum of the Lagrangian plus $J\Phi + K\mathcal{O}(\Phi)$ is still called the "Lagrangian". Things have to be arranged in such a way that only counterterms that are linear in K are generated, i.e. such that the renormalized Lagrangian is still linear in K. Afterwards, one identifies K with J: $K = \alpha J$, α being a parameter that, in general, is negatively dimensioned. The sources J appear now in the form $J[\Phi + \alpha\mathcal{O}(\Phi)]$, that can be turned to the standard form $J\tilde{\Phi}$ by a change of variables $\tilde{\Phi} = \Phi + \alpha\mathcal{O}(\Phi)$. A simple diagrammatic analysis [11] shows that if the functional integral is convergent in the initial variables, then it is also convergent in the new variables. In general, the new Lagrangian $\tilde{\mathcal{L}}(\tilde{\Phi})$ contains nonrenormalizable vertices, due to the negative dimension of α. $\tilde{\mathcal{L}}(\tilde{\Phi})$ describes the physical content of the new theory. In particular, the new field $\tilde{\Phi}$ is the elementary field of the theory $\tilde{\mathcal{L}}(\tilde{\Phi})$ and a composite field of the theory $\mathcal{L}(\Phi)$. Viceversa for Φ. Since the generating functional $W[J,K]$ of the initial theory was convergent [even in presence of the composite operators $\mathcal{O}(\Phi)$], an identification between J and K produces a new convergent generating functional $\tilde{W}[J]=W[J,\alpha J]$, which is, in fact, the generating functional of the new theory $\tilde{\mathcal{L}}(\tilde{\Phi})$. The new theory is predictive, as the initial one. Moreover, if the initial theory is not finite, then the new theory is also not finite.

Notice that it is more convenient to deal with W than with the effective action Γ (i.e. the Legendre transform of W with respect to J), since the Legendre transform changes nontrivially when K is identified with J (and, in fact, the physical meaning of the effective action changes correspondingly). The new effective action $\tilde{\Gamma}$ is convergent, since it is the Legendre transform of a convergent functional $\tilde{W}[J]$. $\tilde{\Gamma}$ is the generating functional of the irreducible graphs of the theory described by $\tilde{\mathcal{L}}(\tilde{\Phi})$.

Now, the reason why we chose a kinetic action of the form $-\phi_1\Box\phi_2+m^2\phi_1\phi_2$ is clear: it was to avoid counterterms quadratic in K, that would in general be required when introducing a ϕ^2-type composite operator coupled to the source K. Only if linearity in K is preserved, we can safely apply the above procedure by identifying K with J. Instead, if there are counterterms that are nonlinear in K, let us say quadratic, then the identification $K=\alpha J$ produces quadratic terms in J. Then, it is easy to check that the convergence of $W[J,\alpha J]$ only means that the connected diagrams converge, while the connected *irreducible* ones do not converge, in general. Indeed, due to the nonlinearity of $\mathcal{L}$ in J, the Legendre transform Γ of W is *not* the set of connected irreducible graphs.

To further illustrate the method, let us see what happens when setting $K=\alpha J_1$ in $\tilde{W}[J_1,J_2,K,\lambda,m^2]$. This means that we are considering a theory described by the Lagrangian

$$\mathcal{L}'_{class}(\varphi_1,\varphi_2)=-\varphi_1\Box\varphi_2+m^2\varphi_1\varphi_2+\frac{\lambda}{4}\varphi_1^2\varphi_2^2+J_1(\varphi_1+\alpha\varphi_1^2)+J_2\varphi_2. \tag{6.25}$$

Then, we perform the change of variables

$$\begin{aligned}\phi_1&=\varphi_1+\alpha\varphi_1^2,\\ \phi_2&=\varphi_2.\end{aligned} \tag{6.26}$$

We assume to use the dimensional regularization technique, so that the Jacobian determinant is still trivial. Let $\varphi_1(\phi_1,\alpha)$ be the inverse of $\phi_1=\varphi_1+\alpha\varphi_1^2$ (to be intended as a power series in α). We get a theory described by

$$\mathcal{L}_{class}(\phi_1,\phi_2,\lambda,\alpha)=-\varphi_1(\phi_1,\alpha)\Box\phi_2+m^2\varphi_1(\phi_1,\alpha)\phi_2+\frac{\lambda}{4}\varphi_1^2(\phi_1,\alpha)\phi_2^2+J_1\phi_1+J_2\phi_2. \tag{6.27}$$

In this example the Lagrangian is nonpolynomial. The first nonrenormalizable vertices are

$$-5m^2\alpha^3\phi_1^4\phi_2+\alpha\phi_1^2\Box\phi_2-\frac{1}{2}\lambda\alpha\phi_1^3\phi_2^2. \tag{6.28}$$

Nevertheless, the coefficients of the infinitely many nonrenormalizable counterterms are related in such a way that eqs. (6.6) are satisfied. Indeed, reasoning in a similar way as before, setting $\phi_1 = \varphi_1 + \alpha Z_\alpha Z^{1/2}\varphi_1^2$ and $\phi_2 = \varphi_2$, we find that the "renormalized" Lagrangian is

$$\begin{aligned}\mathcal{L}_{ren}(\phi_1,\phi_2) &= -Z\varphi_1(\phi_1,\alpha Z_\alpha Z^{1/2})\Box\phi_2 + Z(m^2+\delta m^2)\varphi_1(\phi_1,\alpha Z_\alpha Z^{1/2})\phi_2 \\ &+ \frac{\lambda Z_\lambda}{4}Z^2\varphi_1^2(\phi,\alpha Z_\alpha Z^{1/2})\phi_2^2 \\ &= \mathcal{L}'_{class}(Z^{1/2}\phi_1, Z^{1/2}\phi_2, \lambda Z_\lambda, \alpha Z_\alpha, m^2+\delta m^2). \end{aligned} \tag{6.29}$$

Up to know, we do not know whether the nonpositive definiteness of the kinetic action is an essential requirement for the above mechanism to work. Surely, this aspect deserves attention.

Let us conclude with some brief remarks and comments about the concept of predictivity that is formulated in this and in the previous section of the paper.

One can wonder if there is some hidden symmetry that protects the above theories and makes it possible to have predictivity in presence of power-counting nonrenormalizable interactions. Renormalizable theories are also protected by a "symmetry", that is power counting. On the other hand, the "symmetry" that protects some of the predictive nonrenormalizable theories that we exhibited is purely "diagrammatical", i.e. the impossibility of constructing many divergent graphs. This was the criterion with which we constructed the theory (6.11): in particular, the quadratic part $-\phi_1\Box\phi_2 + m^2\phi_1\phi_2$ was responsible of the limited number of divergent graphs. As one can see, there is no need of a sophisticated symmetry (like a local symmetry or a supersymmetry) to have predictivity. This fact suggests that the set of predictive theories is not so small. In the general case, we do not possess, up to now, any description of the "symmetry principle" contained in equations (6.6), simpler than eq.s (6.6) themselves.

What about the finite counterterms that one can attach to the divergent ones? Eq. (6.2) imposes a relation among the coefficients λ_i of the terms of the classical Lagrangian $\mathcal{L}_{class}(\phi)$ and eqs. (6.6) express the consistency between λ_i and the coefficients $\delta_n\lambda_i$ of $\mathcal{G}^{(n)}(\phi) = \sum_i \delta_n\lambda_i\mathcal{G}_i(\phi)$. Now, it is not permitted to add finite terms $\sum_i f_i\mathcal{G}_i(\phi)$, with arbitrary finite coefficients f_i, since such coefficients are in general infinitely many and thus predictivity is lost. Stated differently, such an addition of $\sum_i f_i\mathcal{G}_i(\phi)$ is equivalent to redefine $\delta_n\lambda_i$ as $\delta_n\lambda_i - f_i$ and consequently relations (6.6) are in general not preserved: the f_i cannot be reabsorbed as a redefinition of α_j and at the subsequent orders the divergent terms are out of control. Thus, the finite terms that we are allowed to add are not completely arbitrary, rather they are restricted to be of the form

$$\sum_{j,i} f_j \frac{\partial\lambda_i(\alpha)}{\partial\alpha_j}\mathcal{G}_i(\phi), \tag{6.30}$$

which corresponds to the shifts $\Delta_n\alpha_j \to \Delta_n\alpha_j - f_j$. Moreover, the f_i should be independent of the gauge-fixing parameters, otherwise an accidental dependence on these parameters would be introduced.

In order to compare this situation with a more familiar one, let us go back to quantum gravity. Goroff and Sagnotti proved [10] that Einstein gravity is not two-loop finite, because of a divergent term equal to

$$\frac{1}{\varepsilon}\frac{209}{2880(4\pi)^4}\sqrt{g}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}, \tag{6.31}$$

where $\varepsilon = 4 - d$. This divergence is not reabsorbable with a redefinition of the metric tensor and so, one is forced to introduce a higher derivative term in the Lagrangian. However, let us suppose, for a moment, that the divergence (6.31) was miraculously absent. In other words, let us consider a theory in which some divergent term $\mathcal{D}$ is *a priori* allowed, but effectively absent. Moreover, let us suppose that the absence of $\mathcal{D}$ is fundamental for finiteness. Although $\mathcal{D}$ is not *a priori* discarded, one should avoid to introduce it as a finite gauge-invariant counterterm, since finiteness would be destroyed. This means that one has to provide an *ad hoc* restriction on the finite counterterms in order to avoid terms like $\mathcal{D}$. Indeed, the addition of such finite terms corresponds to an unwanted modification of the initial Lagrangian, in particular to the introduction of new coupling constants and the idea of finiteness would be meaningless.

More or less the same situation characterizes the predictive theories that we have defined. A restriction on the finite terms, like (6.30), is necessary, otherwise predictivity is lost. Strictly speaking, an analogous restriction is also present in the case of renormalizable theories: one restricts the admissible finite counterterms with the criterion of power-counting. In other words, the finite counterterms have to respect the "symmetry" that protects the theory, otherwise an infinite degree of arbitrariness is introduced. Since the "symmetry" that protects our predictive theories is represented by equations (6.6), it is quite natural to restrict the finite counterterms correspondingly.

Sometimes, when dealing with predictivity, one pays attention to the "number of measurements" that is necessary to fix the theory. Instead, we have never mentioned this, so far. Rather, our criterion for predictivity is the correspondence principle. Indeed, the number of measurements that fix uniquely the theory can be a misleading concept. Since the physical amplitudes of our predictive theories are supposed to depend on a finite number of parameters λ plus the gauge-couplings [4], it is clear that a finite number of measurements is sufficient to fix them and consequently determine the theory uniquely. However, the classical Lagrangian can be nonpolynomial, i.e. it can contain an infinite number of terms, so that one could say that it is necessary to check experimentally the consistency of an infinite number of interactions with the coefficients $\lambda_i(\alpha)$ of the corresponding vertices in the Lagrangian. Thus, at least when the theory is nonpolynomial, an infinite number of measurements would be necessary. However, again strictly speaking, one should conclude that an infinite number of measurements is also necessary when the theory is renormalizable. Indeed, in that case, there are infinitely many possible Lagrangian terms (the power-counting nonrenormalizable ones) that are multiplied "by the coefficient 0" and testing these values 0 experimentally would be a check of the power counting criterion, but it would require infinitely many measurements. Instead, the usual power-counting criterion is an *assumption* and our concept of predictivity can be viewed as a generalization of it. If one accepts hat the good criterion is the correspondence principle, then one has a *conceptual* restriction on the set of physical theories; *after* that, the experimental measurements further project this set onto the set of realized theories.

Some of the examples that we have constructed are suitable "change of variables" of renormalizable theories, some others, like (6.11), are not. What is the general way of proceeding, at least in principle, to investigate whether or not a given classical nonrenormalizable theory can be made predictive? First, we have to specify what we mean by "classical theory" in this context. Indeed, the functions $\lambda_i(\alpha)$ are not known *a priori*, so that the classical Lagrangian is itself not known. Thus the classical theory is identified by the field content and the gauge symmetry. The classical Lagrangian should be constructed *together* with the quantum theory. It is identified as the classical Lagrangian that permits the implementation of the correspondence principle (with a finite number of free parameters λ). At the present stage, we cannot say more about the set of solutions: it can be empty or there can be a single solution or eventually more solutions. The problem of classifying the quantum field theories that are predictive according to (6.6) is surely deserving of interest. One should start from the most general classical Lagrangian $\mathcal{L}_{class} = \sum_i \lambda_i \mathcal{G}_i(\phi)$. Then one should compute the one loop divergences $\mathcal{G}^{(1)}(\phi) = \sum_i \delta_1 \lambda_i \mathcal{G}_i(\phi)$. Making a suitable ansatz about the number of parameters α_j, one should then solve eqs. (6.6) for $\lambda_i(\alpha)$. If the solution is a deep property of the theory, eqs. (6.6) should be solved for any n.

7 Conclusions

In this paper we have extended a previously formulated subtraction algorithm and we have reached a satisfactory control on the effects of the subtraction procedure on the "easy part" of the problem, namely the gauge-fixing sector. This investigation can be useful, to our opinion, for establishing a convenient framework for the study of the "difficult part" of the problem, to get a satisfactory knowledge on the effects of the subtraction algorithm on the physical parameters. We should classify the theories that can be quantized with a *finite* number of parameters, a problem that surprisingly has not been considered with sufficient attention, so far. We have noticed that, in order to do this, one has to determine the classical Lagrangian and the full quantum theory *contemporarily*. The classical Lagrangian is a sum of (possibly) infinitely many terms, whose coefficients are suitable functions of a finite number of parameters. Lagrangians of this type could be furnished as effective Lagrangians of some more fundamental theory (like string theory). The tree level part of the effective Lagrangian could be considered as the classical Lagrangian of a quantum field theory. Then, one should check whether conditions (6.6) are satisfied or not.

References

[1] I.A. Batalin and G. Vilkovisky, Phys. Rev. D 28 (1983) 2567; J. Math. Phys. 26 (1985) 172; Nucl. Phys. B234 (1984) 106; Phys. Lett. 69B (1977) 309; 102B (1981) 27.

[2] G. Barnich and M. Henneaux, preprint ULB-PMIF-93/11 and hepth/9312206.

[3] W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, Nucl. Phys. B333 (1990) 727.

[4] D. Anselmi, preprint SISSA/ISAS 147/93/EP hepth/9309085, September 1993, to appear in Class. and Quantum Grav.

[5] C. Itzykson and J.B. Zuber, "Quantum Field Theory", McGraw-Hill Book Company, New York, 1980.

[6] S.D. Joglekar and B.W. Lee, Ann. Phys. 97 (1976) 160.

[7] K.S. Stelle, Phys. Rev. D 16 (1977) 953.

[8] P.M. Lavrov, I.V. Tyutin and B.L. Voronov, Sov. J. Nucl. Phys. 36 (1982) 292.

[9] G. 't Hooft and M.J. Veltman, Ann. Inst. H. Poincaré, 20 (1974) 69.

[10] M.H. Goroff and A. Sagnotti, Nucl. Phys. B266 (1986) 709.

[11] G. 't Hooft and M.J. Veltman in "Particle Interactions at Very High Energies", Edited by D. Speiser, F. Halzen and J. Weyers (Plenum, New York, 1974).