Quantum Gravity with Purely Virtual Particles from Asymptotically Local Quantum Field Theory

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Abstract

We investigate the local limits of various classes of unitary, nonlocal quantum field theories. While it is easy to build nonlocal models with well-behaved asymptotics in Euclidean space, the Minkowskian correlation functions typically exhibit singular behaviors. We introduce "asymptotically local" quantum field theory (AL-QFT) as the class that encompasses unitary, nonlocal theories with well-defined local limits in Minkowski spacetime. The target models cannot propagate ghosts, but are allowed to contain purely virtual particles (PVPs). In the bubble diagram, the nonlocal deformation generates PVPs straightforwardly. In the triangle diagram, it does so possibly up to multi-threshold corrections, which may be adjusted by tuning the deformation itself. We build examples of AL-QFTs, including a deformation of quantum gravity with purely virtual particles. AL-QFT can serve various purposes, such as suggesting innovative approaches to off-shell physics, providing an alternative formulation for theories with PVPs, or smoothing out nonanalytic behaviors. We discuss its inherent arbitrariness and the implications for renormalizability.

1 Introduction

Locality is a guiding principle of quantum field theory (QFT). At the same time, quantum gravity (QG) challenges us to reconsider the principles that worked successfully so far for the standard model of particle physics. Adjusting the concept of locality is a possibility that must be taken into consideration by physicists aiming to classify the viable theoretical options.

The space $\mathcal{M}_{\rm NL}$ of nonlocal theories (NL-QFTs) is huge. The space $\mathcal{M}_{\rm L}$ of local theories (L-QFTs) may be understood as a subspace of its boundary $\partial\mathcal{M}_{\rm NL}$. It may be helpful to organize $\mathcal{M}_{\rm NL}$ into subspaces that have interesting properties, with special focus on the neighborhood $\mathcal{U}_{\rm NL/L}$ of $\mathcal{M}_{\rm L}$.

Various classes of nonlocal theories have been studied in the literature, starting from Efimov [1]. Krasnikov [2] had the idea to remove the ghosts of local theories from the spectrum by means of nonlocal deformations. Specifically, the ghost poles of the free propagators are canceled by "form factors", that is to say, entire functions with no zeros that appear instead of the unwanted poles. In so doing, unitarity is ensured [3, 4].

If the form factors are generic, they originate nonrenormalizable gauge and gravity interactions. Concentrating on gravity, Kuz'min [5] showed that renormalizability can be obtained by choosing entire functions that tend to polynomials at high energies. The parent and deformed theories are super-renormalizable. Precisely, the nonlocal propagators and vertices tend sufficiently fast to those of a higher-derivative (HD) theory in the ultraviolet limit [6]. Later Tomboulis [7] revived this idea and applied it to gauge theories. More recently, Modesto and others [8] elaborated it to a greater extent and built a variety of finite models. Properties and implications of more general form factors were studied in ref.s [9].

We take the class of super-renormalizable nonlocal theories just mentioned, which we call $\mathcal{M}_{\rm NL}^0$, as the starting point of our investigation. Assuming it makes sense to discuss a "local limit" in $\mathcal{M}_{\rm NL}^0$, we expect it to recover the parent HD-QFT. However, the relationship between nonlocal and local theories has not been investigated so far to the extent we aim to explore in this paper.

We assume that the manifolds $\mathcal{M}_{\rm NL}$ and $\mathcal{M}_{\rm L}$ are restricted to contain unitary theories only. The first step is to define what we mean by "local limit" in $\mathcal{M}_{\rm NL}$. We consider a family of theories in $\mathcal{M}_{\rm NL}$ parametrized by some extra variable λ. We arrange the λ dependence so that the limit $\lambda \to \infty$ returns a local theory in Euclidean space, where the problem is much simpler to handle. Then we inquire what happens when λ tends to infinity in Minkowski spacetime. In a class of treatable theories inspired by the ones of

$\mathcal{M}_{\rm NL}^0$ (but not quite those), we find severe singularities inside the correlation functions, such as integrals $\simeq \int \mathrm{d}^4p/|p^2 - m^2|$. We are unable, at this very moment, to say whether the nonlocal models of $\mathcal{M}_{\rm NL}^0$ admit local limits in Minkowski spacetime or not.

Still, the results we find suggest that the usual assumptions behind NL-QFT are too restrictive. We expect that if we adjust, or enlarge, the manifold of nonlocal models, we can include those that tend to local quantum field theories in Minkowski spacetime when λ tends to infinity.

A question that may help us identify the right extension is: which local models can be reached and which ones are out of reach? Since we want $\mathcal{M}_{\rm NL}$ to be unitary, the local models that contain ghosts are unreachable. Then, the natural candidates for the intersection $\partial\mathcal{M}_{\rm NL} \cap \mathcal{M}_{\rm L}$ are the local theories with purely virtual particles [10] (i.e., particles that are never on the mass shell), alongside physical particles.

We need to understand the relation between the local limit $\lambda \rightarrow \infty$, and the continuation E $\rightarrow$ M from Euclidean space (E) to Minkowski spacetime (M). Do these operations commute? If we take the local limit in E and then continue from E to M, do we find the local limit in M?

Unless we assume that it is analytic, the continuation E $\rightarrow$ M is not unique, even if we restrict it to $\mathcal{M}_{\rm L}$. However, the analytic continuation generically leads to models with ghosts (fields with kinetic terms multiplied by the wrong sign). This occurs in the cases of higher-derivative theories, for example. Since those models are unreachable from $\mathcal{M}_{\rm NL}$, we conclude that the continuation E $\rightarrow$ M, which is analytic for $\lambda < \infty$, cannot stay analytic after the local limit $\lambda \rightarrow \infty$. There remains the possibility that the target local limits are theories involving purely virtual particles (PVPs). Indeed, a nonanalytic continuation E $\rightarrow$ M is one way to formulate PVPs [11].

Two other ways to introduce purely virtual particles are available in the literature: one uses diagrammatic spectral optical identities [12] and the other one is based on special non-time ordered correlation functions [10]. A fourth way is the one emerging from this paper. Generically speaking, we can say that PVPs are defined by tweaking the usual diagrammatics so as to eradicate the ghosts, while remaining in the realm of local theories.

Taking advantage of the PVP concept, it is possible to build a local theory of quantum gravity [13] that is unitary and renormalizable at the same time. Its main prediction is a constrained window ($4/10000 \lesssim r \lesssim 3/1000$) for the value of the tensor-to-scalar ratio r [14] of primordial fluctuations, which essentially confirms the prediction of the Starobinsky $R + R^2$ model [15] ($r \simeq 3/1000$) within less than an order of magnitude. Other predictions can be derived as well (the running of scalar and tensor tilts, higher-order corrections, etc. [16]), but require much more effort to be tested, in the realm of current or planned observations [17]. In this paper we provide a nonlocal deformation of quantum gravity with PVPs, which returns it back in the local limit.

Since we know what to expect (i.e., local limits containing physical particles and PVPs), we can easily identify how to relax the defining assumptions of NL-QFT to enlarge the manifold $\mathcal{M}_{\rm NL}$ appropriately. It turns out that the form factors attached to the "propagators of non propagating fields", which eliminate the unwanted ghost poles, remain entire functions, but are allowed to have zeros. This generalization does not cause particular problems, since the associated fields have to be integrated out anyway, sooner or later. The Lagrangian is singular, strictly speaking, but the action is still regular.

We show that the so enlarged manifold $\mathcal{M}_{\rm NL}$ does include models that have regular local limits in Minkowski spacetime, and those limits are indeed theories that contain purely virtual particles in addition to physical particles (PVP-QFTs). We call the models of the subspace $\mathcal{U}_{\rm NL/L} \subset \mathcal{M}_{\rm NL}$ "asymptotically local" quantum field theories (AL-QFTs), and denote their space by $\mathcal{M}_{\rm AL}$ henceforth. AL-QFT can be used as an alternative formulation of theories with PVPs, or as an approximation that smooths out their typical nonanalytic behaviors.

Besides the tree level, where the local limit is straightforward, we study the local limits of the bubble and triangle diagrams. In the bubble diagram the nonlocal deformation "already knows", so to speak, how to generate PVPs in the limit, with no need of *ad hoc* adjustments. In the cases of triangle and more complicated diagrams, the local limit is "PVP ready" in the null and single-threshold sectors. The multi-threshold sectors, instead, are more difficult to handle. Possibly, the arbitrariness of the deformation must be invoked to fine-tune the limit appropriately.

The arbitrariness of AL-QFT is its weak point, which must be addressed. Nonlocal quantum field theory lacks a fundamental principle to select the nonpolynomial functions it is built upon, remove its inherent arbitrariness and identify the unique theory that describes nature. At the same time, this weakness is predicated on the very assumption that AL-QFT is ambitious to that point. This is not what we are claiming here: we are not proposing AL-QFT as a framework for fundamental theories of the universe. We just claim that AL-QFT is interesting *per se*, broadens our knowledge of QFT, and provides useful tools in support of local QFT.

To clarify this last point, let us recall another instance where a comparable degree of arbitrariness enters local QFT (without jeopardizing it as a framework for fundamental interactions): we are talking about off-the-mass shell physics [18]. In that case, the extra parameters are not rooted in the fundamental interactions, but describe the surrounding environment where the phenomenon is observed. We can say that they parametrize the quantum/classical interplay between the phenomenon, the observer, and the experimental apparatus. There is probably a map relating the arbitrariness of AL-QFT to the arbitrariness of off-shell physics. Yet, elucidating that map is beyond the scope of this paper. Here we just stress that the arbitrariness turned on by the deformation PVP-QFT $\rightarrow$ AL-QFT is less surprising when it is placed side by side with other forms of arbitrariness we are more accustomed to.

Summarizing, the lack of uniqueness of AL-QFT is not an issue, within our approach. AL-QFT is a tool to enlarge the set of quantum field theories we can treat, move beyond the common frameworks, possibly describe off-the-mass-shell physics in an alternative way, or propose new formulations of PVPs, and address peculiar aspects of PVPs themselves (such as the violation of microcausality [19] and the so called "peak uncertainty" [20]).

Another way to settle the non uniqueness problem of AL-QFT is as follows. We have said that nonlocal theories lack a selection criterion for the form factors, a sort of "minimum principle" in $\mathcal{M}_{\text{AL}}$. Yet, the requirement that they admit local limits provides a form of control, and possibly the missing principle itself. In this view, the "minima", i.e., the candidate theories to describe nature from AL-QFT, are nothing but the local limits themselves[1].

The paper is organized as follows. In section 2 we recall basic features of the nonlocal theories considered in the literature and outline how we generalize them. In section 3 we study the local limit in Euclidean space. In section 4 we describe the problem posed by the local limit in Minkowski spacetime. In section 5 we study the bubble and triangle diagrams. In section 6 we formulate the asymptotically local deformation of quantum gravity with purely virtual particles. Section 7 contains the conclusions. In the appendix we recall the calculation of the bubble diagram with PVPs.

When necessary, the dimensional regularization [21] is used for the explicit calculations, where $D = 4 - \varepsilon$ denotes the continued dimension around dimension 4.

2 Nonlocal theories

Nonlocal theories must be defined in Euclidean space and later analytically continued to Minkowski spacetime. We consider models with Euclidean Lagrangians of the form

$$\mathcal{L}_{\text{NL}} = \frac{\tau}{2}\phi(-p)Q(P(p))\phi(p) + \mathcal{O}(\phi^3), \tag{2.1}$$

[1]Strictly speaking, they are not contained in the manifold $\mathcal{M}_{\text{AL}}$ of asymptotically local theories, but in its border $\partial\mathcal{M}_{\text{AL}}$.

in momentum space, where ϕ denotes the fields, $Q(P)$ is a certain function of a polynomial P of the Euclidean momentum p, and $\tau = \pm 1$ (to describe nonlocal deformations of both physical particles and ghosts). Typically, we take

$$P(p) = p^2 + m^2 \tag{2.2}$$

where m is the "mass" in the local limit.

We require that $Q(P)$ tends to P in the local limit defined below and $1/Q(P)$ is entire in the complex P plane. The idea is that the field ϕ is itself purely virtual[2], since $Q(P)$ has no zeros. This makes it a natural candidate to become a PVP in the local limit.

The main difference with respect to the assumptions commonly adopted in the literature about nonlocal theories [5, 7, 8] is that the propagator $\tau/Q(P)$ is not required to never vanish in the complex P plane. Specifically, we allow $Q(P(p))$ to have singularities proportional to $(-(p_{\mathrm{M}} - k_{\mathrm{M}})^2 + M^2)^{-1}$, for Minkowskian momenta p_{M}, k_{M}, and real "masses" M. The Minkowskian action is then defined by means of the Cauchy principal value.

The Lagrangian (2.1) can describe the ϕ subsector of a more general theory, which may contain ordinary physical particles φ as well, as described by the extension

$$\mathcal{L}'_{\mathrm{NL}} = \frac{\tau}{2}\phi(-p)Q(P(p))\phi(p) + \frac{1}{2}\varphi(-p)(p^2 + m^2)\varphi(p) + \mathcal{L}'_{\mathrm{int}}, \tag{2.3}$$

where $\mathcal{L}'_{\mathrm{int}}$ collects the interactions. We focus our attention on ϕ here, since the propagators of the physical particles φ do not need to be nonlocally deformed.

The local limit is defined by rescaling P and Q by λ and λ^{-1}, respectively, where λ is a positive factor, and then letting λ tend to infinity. We assume that Q tends to P on the real axis:

$$\lim_{\lambda \to +\infty} \lambda^{-1} Q(\lambda P) = P, \qquad P \in \mathbb{R}. \tag{2.4}$$

The functions Q we consider in this paper are

$$Q(P) = h(P), \qquad Q(P) = \frac{h^2(P)}{P}, \tag{2.5}$$

where $h(P)$ is the entire function defined in formula (2.6) below, taken from the current literature on nonlocal theories [2, 5, 7, 8]. The second option allows us to build the asymptotically local theories of $\mathcal{M}_{\mathrm{AL}}$. With choices like (2.5), the property (2.4) extends to a double cone $\mathcal{C}$ around the real axis.

[2]We could call it "nonlocal purely virtual particle" (NL-PVP).

2.1 Approximating the absolute value

The absolute value of a complex number z can be approximated by the never vanishing entire function

$$h(z) \equiv \exp\left(\frac{1}{2}\int_0^{z^2}\frac{1-\mathrm{e}^{-w}}{w}\mathrm{d}w - \frac{\gamma_{\mathrm{E}}}{2}\right) = \exp\left(\frac{1}{2}\ln z^2 + \frac{1}{2}\Gamma(0,z^2)\right). \tag{2.6}$$

Precisely, in the double cone $\mathcal{C} = \{z : -\pi/4 < \arg[z] < \pi/4 \text{ or } 3\pi/4 < \arg[z] < 5\pi/4\}$, we have [22]

$$h(z) = \sqrt{z^2}\left[1 + \frac{\mathrm{e}^{-z^2}}{2z^2}\left(1+\mathcal{O}\left(\frac{\mathrm{e}^{-z^2}}{z^2}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\right], \tag{2.7}$$

so

$$\lim_{\lambda\rightarrow+\infty}\frac{h(\lambda z)}{\lambda} = \sqrt{z^2} \quad \text{for } z\in\mathcal{C}, \qquad \lim_{\lambda\rightarrow+\infty}\frac{h(\lambda x)}{\lambda} = |x| \quad \text{for } x\in\mathbb{R}. \tag{2.8}$$

It is important to stress that $h(z)$ has no zeros. Its reciprocal $1/h(z)$ is going to be useful to build the "propagators" that tend to purely virtual particles in the local limit.

The function $h(z)$ is even,

$$h(z) = h(-z), \tag{2.9}$$

and satisfies

$$h^*(z) = h(z^*). \tag{2.10}$$

Moreover, on the real axis it is positive and bounded from below by the absolute value, as illustrated in fig. 1:

$$h(x) \geqslant |x|, \qquad x\in\mathbb{R}. \tag{2.11}$$

Finally, (2.8) gives

$$\lim_{\lambda\rightarrow+\infty}\frac{h^2(\lambda z)}{\lambda^2 z} = z \quad \text{for } z\in\mathcal{C}. \tag{2.12}$$

2.2 Approximating the sign function

The sign function $\mathrm{sgn}(x)$ can be approximated on the complex plane by the entire function

$$\sigma(z) = \frac{z}{h(z)}, \tag{2.13}$$

which vanishes at the origin. In $\mathcal{C}$ the rescaled function $\sigma(\lambda z)$ tends to $z/\sqrt{z^2}$ for $\lambda\rightarrow\infty$, thus

$$\lim_{\lambda\rightarrow+\infty}\sigma(\lambda z) = \mathrm{sgn}(\mathrm{Re}[z]) \quad \text{for } z\in\mathcal{C}, \qquad \lim_{\lambda\rightarrow+\infty}\sigma(\lambda x) = \mathrm{sgn}(x) \quad \text{for } x\in\mathbb{R}. \tag{2.14}$$

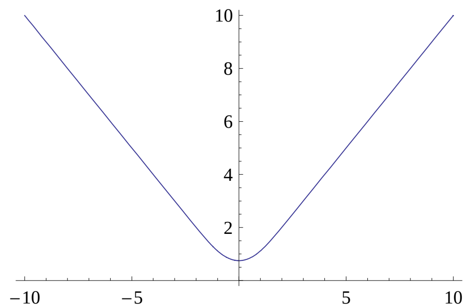

Figure 1: Plot of $h(x)$ for real x

Note that $\sigma(z)$ is odd in the complex plane:

$$\sigma(z) = -\sigma(-z). \tag{2.15}$$

Moreover, (2.11) gives

$$|\sigma(x)| \leqslant 1, \qquad x \in \mathbb{R}. \tag{2.16}$$

2.3 Approximating the principal value

We can use the functions σ and h to approximate the Cauchy principal value $\mathcal{P}$ on the real axis. Specifically, we prove that

$$\lim_{\lambda\to+\infty} \frac{\lambda\sigma(\lambda x)}{h(\lambda x)} = \mathcal{P}\frac{1}{x}, \qquad x \in \mathbb{R}, \tag{2.17}$$

in the sense of distributions.

First observe that

$$\lim_{\lambda\to+\infty} \frac{\lambda\sigma(\lambda x)}{h(\lambda x)} = \frac{1}{x} \tag{2.18}$$

for every real $x \neq 0$. This result follows from (2.8) and (2.14).

If $\varphi(x)$ denotes a real test function, consider the integral

$$\mathcal{I} \equiv \lim_{\lambda\to+\infty} \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\lambda\sigma(\lambda x)\varphi(x)}{h(\lambda x)}, \tag{2.19}$$

which can also be written as

$$\mathcal{I} = \lim_{\lambda\to+\infty} \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\lambda\sigma(\lambda x)(\varphi(x) - \varphi(-x))}{2h(\lambda x)},$$

thanks to (2.9) and (2.15). The modulus of the $\mathcal{I}$ integrand is bounded above by a λ-independent function that is integrable on $\mathbb{R}$. Indeed, the inequalities (2.11) and (2.16) give

$$\left|\frac{\lambda\sigma(\lambda x)(\varphi(x)-\varphi(-x))}{2h(\lambda x)}\right| \leqslant \frac{1}{2}\left|\frac{\varphi(x)-\varphi(-x)}{x}\right|, \tag{2.20}$$

for real x. Then the dominated convergence theorem allows us to exchange the limit with the integral. Given that (2.18) holds almost everywhere, we obtain

$$\mathcal{I} = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\varphi(x)-\varphi(-x)}{2x} = \mathcal{P}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x}\varphi(x), \tag{2.21}$$

as we wished to show.

2.4 Loop integrals in nonlocal theories

The propagator of the nonlocal Euclidean theory (2.1) is

$$G_{\mathrm{nl}}(p) = \frac{\tau}{Q(P(p))}, \tag{2.22}$$

where the polynomial $P(p)$ is given by (2.2). Associated with $G_{\mathrm{nl}}(p)$, we have a cone $\mathcal{C}$ where the expansion (2.7) applies.

Consider a Feynman diagram F. Let p, p_i denote the internal momenta and k, k_a the external ones. The loop integral associated with F is the integral of a product of propagators $G_{\mathrm{nl}}(p_i)$ times a product of functions $V(p_i, k_a)$ originated by the vertices. The latter are under control, as we explain in subsection 6.4, so we concentrate on the propagators.

For example, the bubble diagram with circulating ϕ fields has the form

$$\mathcal{B}(k) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} V_1(p,k)V_2(p,k)G_{\mathrm{nl}}(p)G_{\mathrm{nl}}(p-k). \tag{2.23}$$

The integrals must be defined in Euclidean space. This means that both p_i and k_a are Euclidean momenta. Only the external momenta k_a are later analytically continued to Minkowski spacetime. We denote their Minkowski versions by k_{M}.

It is easy to show that the integrals are convergent for Euclidean k_a, in the sense of the dimensional regularization. Consider, for example, the expression (2.23) for $\mathcal{B}(k)$. For Euclidean p and k, the arguments $P(p)$ and $P(p+k)$ are located inside the cones $\mathcal{C}$ and $\mathcal{C}'$ associated with the propagators $G_{\mathrm{nl}}(p)$ and $G_{\mathrm{nl}}(p+k)$. Formula (2.7) then ensures that (2.23) tends to the same integral with $G_{\mathrm{nl}}(p) \to P(p)$ and $G_{\mathrm{nl}}(p+k) \to P(p+k)$, which is indeed convergent in the sense of the dimensional regularization (assuming that the vertices do not invalidate the argument, see 6.4). What this means is that there exists an open set Ω of the complex D plane where the integral is convergent in the standard sense, or can be split into a finite sum of integrals that admit convergence domains Ω_i in the standard sense [23].

Now we prove that the loop integrals are convergent, in the sense of the dimensional regularization, for every complex external momenta k_a. Actually, they are entire functions of k_a, so it is possible to replace the external momenta k with their Minkowski versions k_{M} directly inside the integrals.

It is important to stress that the integrated momenta p_i remain Euclidean till the very end. The reason is that it is not convenient to make a Wick rotation on them. When we close integration paths by including arcs at infinity, we cross regions where the functions $G_{\mathrm{nl}}(p)$ and $G_{\mathrm{nl}}(p-k)$ behave in ways that are hard to control.

Consider (2.23) again. When k is deformed to complex values, the cone $\mathcal{C}'$, which is equal to $\mathcal{C}$ translated by k, is no longer centered along the Euclidean domain, but somewhere else, depending on the imaginary part of k:

$$P(p+k) = (p+k)^2 + m^2 = \rho \mathrm{e}^{i\theta}, \qquad \theta = \arctan \frac{\mathrm{Im}[k_4(k_4+2p_4)]}{\mathrm{Re}[(p+k)^2+m^2]}.$$

The phase θ of $P(p+k)$ tends to zero when the integrated momentum p tends to infinity, in any direction. This implies that the argument of $G_{\mathrm{nl}}(p+k)$ falls off fast enough inside $\mathcal{C}'$ for sufficiently large p, which makes the integral (2.24) convergent.

Together with the fact that the integrand is obviously regular everywhere, this argument proves that the function $\mathcal{B}(k)$ is entire. Then we can perform the analytic continuation E $\rightarrow$ M by replacing $k=(k_4,\mathbf{k})$ with its Minkowskian version $k_{\mathrm{M}} = (-ik^0, \mathbf{k})$ directly inside. The Minkowskian bubble diagram is thus

$$\mathcal{B}_{\mathrm{M}}(k_{\mathrm{M}}) = i\int \frac{\mathrm{d}^D p}{(2\pi)^D} V_1(p,k_{\mathrm{M}})V_2(p,k_{\mathrm{M}})G_{\mathrm{nl}}(p)G_{\mathrm{nl}}(p-k_{\mathrm{M}}), \qquad (2.24)$$

the factor i being due to the fact that p remains Euclidean.

The property just proved extends to more complex loop integrals, which also define entire functions of k_a. Note that it does not apply, on the contrary, to local quantum field theory. There, the entire functions G_{nl} are replaced by the reciprocals of polynomials. The integrands have poles, which can cross the integration path when k is deformed to complex values. Direct replacements $k \rightarrow k_{\mathrm{M}}$ jump *over* the integration paths, giving incorrect results.

When the theory contains physical particles, besides NL-PVPs, as in the example (2.3), one proceeds as usual in the physical sector, i.e., by Wick rotating the external momenta and keeping the internal ones Euclidean [3]. This procedure is also safe in loops that contain both physical particles and NL-PVPs.

To conclude this section, the analytic continuation E→M gives a well-defined map

$$\mathcal{M}_{\mathrm{NL}}^{\mathrm{E}} \quad \overset{\mathrm{E}\rightarrow\mathrm{M}}{\longrightarrow} \quad \mathcal{M}_{\mathrm{NL}}^{\mathrm{M}} \tag{2.25}$$

between the nonlocal Euclidean theories and their Minkowskian versions. It is sufficient to replace the external momenta k with their Minkowski versions k_{M} inside the loop integrals (or Wick rotate them), and adjust the overall factor.

3 Local limit in Euclidean space

In this section we study the local limit in Euclidean space, which does not pose significant challenges.

At the tree level, the assumption (2.4) ensures that the local limit of the Lagrangian (2.1) is the Lagrangian of a local theory:

$$\mathcal{L}_{\mathrm{NL}}(\lambda) \equiv \frac{\tau}{2}\phi(-p)\lambda^{-1}Q(\lambda P(p))\phi(p) + \mathcal{O}(\phi^3) \underset{\lambda\rightarrow+\infty}{\rightarrow} \mathcal{L}_{\mathrm{loc}} = \frac{\tau}{2}\phi(-p)P(p)\phi(p) + \mathcal{O}(\phi^3)\big|_{\mathrm{loc}}. \tag{3.1}$$

In what follows, we assume, again, that the vertices are local or their limits as $\lambda \rightarrow \infty$ are smooth enough to not affect our arguments. The theories we have in mind satisfy this assumption, as we demonstrate in subsection 6.4.

The propagator tends to the one of the local theory in both cases (2.5):

$$\lim_{\lambda\rightarrow+\infty} \frac{\tau\lambda}{Q(\lambda P(p))} = \frac{\tau}{P(p)}. \tag{3.2}$$

It is also straightforward to show that the correlation functions of the nonlocal theory tend to those of the local theory (3.1). Indeed, the loop integrals just involve the Euclidean domain. There, the values of the rescaled polynomial P are always real and larger than a positive number, $\lambda P(0) = \lambda m^2$, so the expansion (2.7) can be used to prove the statement.

Take for example (2.23), with the Green functions (2.22). Its local limit is

$$\lim_{\lambda\rightarrow+\infty} \lambda^2 \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V_1(p,k)V_2(p,k)}{Q(\lambda P(p))Q(\lambda P(p-k))}. \tag{3.3}$$

Since k is Euclidean, both $P(p)$ and $P(p-k)$ have Euclidean arguments.

Assume that the integral

$$\mathcal{B}_{\mathrm{L}}(k) \equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V_1(p,k)V_2(p,k)}{P(p)P(p-k)} \tag{3.4}$$

is convergent in the physical dimension $D=4$. The bounds (2.11) and (2.16) allow us to apply the dominated convergence theorem and conclude that the limit (3.3) gives $\mathcal{B}_{\mathrm{L}}(k)$.

If (3.4) is not convergent in $D=4$, we use the dimensional regularization technique. This means that we continue the spacetime dimension to complex values D, move to a domain Ω where $\mathcal{B}_{\mathrm{L}}(k)$ is convergent[3], take the limit $\lambda\rightarrow\infty$ there and then analytically continue the result to $D=4$.

This proves that the local limit is well defined in E. In other words, we have a map

$$\mathcal{M}_{\mathrm{NL}}^{\mathrm{E}}\overset{\mathrm{loc}}{\longrightarrow}\mathcal{M}_{\mathrm{L}}^{\mathrm{E}} \tag{3.5}$$

between nonlocal theories and local theories (3.1) in Euclidean space.

4 Local limit in Minkowski spacetime

The next task is to investigate the local limit of the Minkowskian correlation functions. We cannot just compose the map (3.5) with the continuation $\mathrm{E}\rightarrow\mathrm{M}$, because the continuation of a local theory admits a plurality of choices:

$$\begin{array}{ccc} & \nearrow & \mathcal{M}_{\mathrm{L}}^{\text{M-1}} \\ \mathcal{M}_{\mathrm{NL}}^{\mathrm{E}}\overset{\mathrm{loc}}{\longrightarrow}\mathcal{M}_{\mathrm{L}}^{\mathrm{E}}\overset{\mathrm{E}\rightarrow\mathrm{M}}{\longrightarrow} & \longrightarrow & \mathcal{M}_{\mathrm{L}}^{\text{M-2}} \\ & \searrow & \mathcal{M}_{\mathrm{L}}^{\text{M-3}}\cdots \end{array} \tag{4.1}$$

The simplest option is the analytic continuation, which is excluded by unitarity, as we show below in detail. Among the other options, a special place is reserved to the "average continuation", which defines purely virtual particles [11]. The worst possibility is that the local limit in Minkowski spacetime does not even exist. In particular, we cannot expect that the limit of (2.24) is just (3.4) with $k\rightarrow k_{\mathrm{M}}$, which is ill defined.

Ultimately, the task of identifying the right limit $\mathcal{M}_{\mathrm{L}}^{\mathrm{M}}$ of $\mathcal{M}_{\mathrm{NL}}^{\mathrm{M}}$ amounts to building the diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{NL}}^{\mathrm{E}} & \overset{\mathrm{loc}_{\mathrm{E}}}{\longrightarrow} & \mathcal{M}_{\mathrm{L}}^{\mathrm{E}} \\ \mathrm{E}\downarrow\mathrm{M} & & \mathrm{E}\downarrow\mathrm{M} \\ \mathcal{M}_{\mathrm{NL}}^{\mathrm{M}} & \overset{\mathrm{loc}_{\mathrm{M}}}{\longrightarrow} & \mathcal{M}_{\mathrm{L}}^{\mathrm{M}} \end{array} \tag{4.2}$$

[3]If $\mathcal{B}_{\mathrm{L}}(k)$ does not admit a convergence domain Ω, it can be split into a finite sum of terms $\mathcal{B}_{\mathrm{L}}^{(i)}(k)$ that separately admit convergence domains Ω_i. See [23]. Then the argument can be applied to each $\mathcal{B}_{\mathrm{L}}^{(i)}(k)$ separately.

by combining (2.25) and (3.5) with further maps that close the bottom right. The correct $\mathcal{M}_{\rm L}^{\rm M}$ must follow from the compatibility between the local limit and the continuation E $\rightarrow$ M.

Now we show that $\mathcal{M}_{\rm L}^{\rm M}$ cannot contain models with ghosts, while it can contain theories with purely virtual particles.

Assume for the moment that the vertices V_1 and V_2 are identically one. Making a reflection $p_4 \rightarrow -p_4$ in (2.24) we infer that $B_{\rm M}(k_{\rm M}) = B_{\rm M}(k_{\rm M}^*)$. Because of the property (2.10) and the overall factor i, the conjugate $B_{\rm M}^*(k_{\rm M})$ coincides with $-B_{\rm M}(k_{\rm M}^*)$. Hence, $B_{\rm M}(k_{\rm M}) = -[B_{\rm M}(k_{\rm M})]^*$. Given that the real part of $B_{\rm M}(k_{\rm M})$ vanishes identically, its local limit must vanish as well, if it exists.

Thus, the local limit of $B_{\rm M}(k_{\rm M})$ is purely imaginary. If it contained propagating particles or ghosts, its real part would be nontrivial. On the contrary, it is allowed to contain PVPs, because they give a purely imaginary bubble diagram (check the appendix).

If V_1 and V_2 are nontrivial, by unitarity they must be Hermitian in Minkowski spacetime. The argument just outlined extends straightforwardly when they are polynomial. More generally, the vertices may involve incremental ratios of the entire functions $1/Q$, as explained in subsection 6.4. Yet, the operations described above apply to those cases as well, and lead to the same conclusion.

A more general version of the argument is based on the optical theorem, which reads $-iT + iT^\dagger = TT^\dagger$, where $S = 1 + iT$ is the S matrix and iT collects the loop diagrams in matrix form. Consider the "empty" theory (2.1). We call it this way, because it does not propagate any degree of freedom, differently from (2.3). This means $TT^\dagger = 0$, hence the matrix T is Hermitian: all the loop diagrams are purely imaginary.

Since Re$[iT]$= 0 for arbitrary finite λ, the local limit $\lambda \rightarrow \infty$ can only return a local theory with Re$[iT]$= 0, that is to say, still an empty theory. No physical particles or ghosts can be generated by the limit. Purely virtual particles are allowed, precisely because they satisfy Re$[iT]$= 0.

Coming back to the bubble diagram, we separate the tentative (λ independent) local limit from the rest by writing

$$)\!\bigcirc\!(_{\rm nloc} =)\!\bigcirc\!(_{\rm loc} +)\!\bigcirc\!(_{\rm rest} . \tag{4.3}$$

The rest is supposed to tend to zero when λ tends to infinity.

We know that the real part of the left-hand side vanishes, so

$$0 = {\rm Re}\,[)\!\bigcirc\!(_{\rm nloc}] = {\rm Re}\,[)\!\bigcirc\!(_{\rm loc}] + {\rm Re}\,[)\!\bigcirc\!(_{\rm rest}] . \tag{4.4}$$

Assume, ad absurdum, that the local limit adds degrees of freedom. Then the optical theorem $-iT + iT^{\dagger} = TT^{\dagger} = -2\mathrm{Re}[iT]$, applied to the limit itself, tells us that $TT^{\dagger}$ is nonzero. For example, in the case of the ordinary bubble diagram with circulating physical particles, formula (A.1) gives

$$\mathrm{Re}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{loc}}] = -\int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\pi}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left[\delta(k^0 + \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{p}}) + \delta(k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}})\right].$$

Since the first term on the right hand side of (4.4) is nonvanishing and λ independent,

$$\mathrm{Re}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{loc}}] \neq 0, \qquad \frac{\partial}{\partial\lambda}\mathrm{Re}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{loc}}] = 0,$$

the rest is not negligible in the limit $\lambda \to +\infty$, which invalidates the assumption.

Assuming, instead, that the local limit gives a theory of PVPs,

$$)\!\!\bigcirc\!\!(_{\,\mathrm{nloc}} =)\!\!\bigcirc\!\!(_{\,\mathrm{PVP}} +)\!\!\bigcirc\!\!(_{\,\mathrm{rest}}, \tag{4.5}$$

and recalling that the bubble diagram (A.2) with circulating PVPs is purely imaginary, the real part of the rest vanishes,

$$0 = \mathrm{Re}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{nloc}}] = \mathrm{Re}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{rest}}],$$

which causes no problem when λ tends to ∞.

Taking the imaginary part of (4.5), we find

$$\mathrm{Im}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{nloc}}] = \mathrm{Im}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{PVPs}}] + \mathrm{Im}\,[\,)\!\!\bigcirc\!\!(_{\,\mathrm{rest}}].$$

The left-hand side is nonzero and λ dependent. The first term on the right-hand side is nonzero and λ independent. Hence the rest can be nonzero, λ dependent and tend to zero when $\lambda \to \infty$ with no contradiction.

We stress once again that the results of this section are predicated on the assumption that the local limit exists. It turns out that it does with the second choice of (2.5) for the function $Q(P)$, while it does not with the first choice.

5 Local limit of Minkowskian correlation functions

In this section we study the local limit in Minkowski spacetime with the options (2.5). We start from the tree level, then proceed to analyze the bubble and triangle diagrams in detail. We show that the local limit is singular with the left option and tends to a local theory with PVPs with the right option, albeit with some caveats.

The Euclidean and Minkowskian actions S_{E} and S_{M} are related by $S_{\mathrm{E}} = -iS_{\mathrm{M}}$ (since $\mathrm{e}^{-S_{\mathrm{E}}} \rightarrow \mathrm{e}^{iS_{\mathrm{M}}}$ inside the functional integral). We know that the integrated coordinates and momenta remain Euclidean. Thus, the propagator $G_{\mathrm{nl}}^{\mathrm{M}}(p_{\mathrm{M}})$ of the nonlocal Minkowski theory is (2.22) times $-i$:

$$G_{\mathrm{nl}}^{\mathrm{M}}(p_{\mathrm{M}}) = -\frac{i\tau}{Q(P(p_{\mathrm{M}}))}. \tag{5.1}$$

At the tree level, formulas (2.8) and (2.17) give

$$\lim_{\lambda\rightarrow+\infty} -\frac{i\tau\lambda}{Q(\lambda P(p_{\mathrm{M}}))} = \begin{cases} -\frac{i\tau}{|P(p_{\mathrm{M}})|} & \text{for } Q(P) = h(P), \\ -\mathcal{P}\frac{i\tau}{P(p_{\mathrm{M}})} & \text{for } Q(P) = \frac{h^2(P)}{P}. \end{cases} \tag{5.2}$$

The top result illustrates the problem with the left choice of (2.5): the $\lambda \rightarrow +\infty$ limit does not give an acceptable propagator for a local theory. The second choice for (2.5), on the other hand, gives the answer we expect for a PVP in the classical limit.

It is important to stress that the principal value is not the right answer for PVPs at the level of radiative corrections. If we adopt it as an alternative propagator in Feynman diagrams, we obtain a theory of "Wheelerons" [24], which propagates ghosts. For example, the bubble diagram ends up giving the integral

$$\mathcal{P}\int \frac{\mathrm{d}^D p_{\mathrm{M}}}{(2\pi)^D} \frac{1}{p_{\mathrm{M}}^2 - m^2} \frac{1}{(p_{\mathrm{M}} - k_{\mathrm{M}})^2 - m^2}, \tag{5.3}$$

which has a nonvanishing, unphysical absorptive part [25], absent in (A.2).

In a theory of PVPs, the radiative corrections are not given by the usual diagrams with a different propagator, but by new combinations of diagrams [10]. In view of this, before concluding that the local limit with the second choice of (2.5) gives a theory with PVPs, we must prove that the loop diagrams somehow "know what to do" by themselves. That is to say, for some mysterious reason they are already equipped with the instructions to implement the diagrammatic rules of [11, 12, 10], instead of those that lead to (5.3). Luckily, this turns out to be true[4].

5.1 Bubble diagram

Now we study the local limit of (2.24) in detail. We work with the second option of (2.5) for the function Q and later comment on the problems of the first option.

[4] Besides, we already know that they cannot give (5.3), because unitarity rules out local limits with ghosts.

It is convenient to split the calculation in two steps:

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = i \lim_{\substack{\lambda \rightarrow +\infty \\ \lambda' \rightarrow +\infty}} \lambda\lambda' \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{Q(\lambda' P(p)) Q(\lambda P(p - k_{\mathrm{M}}))},$$

where V denotes the vertices. As before, we focus on the propagators and assume that the vertices behave sufficiently well in the limit, so as not to invalidate the arguments. We recall that the polynomial P is the one of (2.2) and $k_{\mathrm{M}} = (-ik^0, \mathbf{k})$.

Since the loop momentum p is Euclidean, the limit $\lambda' \rightarrow +\infty$ can be evaluated immediately by means of (2.8), (2.17) or (3.2). We obtain

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = i \lim_{\lambda \rightarrow +\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{p^2 + m^2} \frac{\lambda}{Q(\lambda P(p - k_{\mathrm{M}}))}. \tag{5.4}$$

If $k^0 = 0$ the Euclidean and Minkowskian loop integrals coincide, so we can take the second limit right away, which gives the expected result.

If $k^0 \neq 0$, we can restrict to the case $k^0 > 0$, by symmetry. Then the second limit cannot be evaluated directly, since we do not have an easy control on the behavior of $Q(\lambda P(p - k_{\mathrm{M}}))$ away from the cone $\mathcal{C}$.

We briefly outline the strategy of the calculation. The function $Q(\lambda P(p - k_{\mathrm{M}}))$ is centered in a region translated by k_{M}. We would like to re-center it on the Euclidean domain by means of an opposite translation. The translation in question is complex, so it cannot be expressed as a mere change of variables in the integral.

We overcome the difficulty by adding and subtracting integration paths. Consider the p_4 integral. Its integration domain is the line C (see fig. 2 – left side). We can rewrite the integral on C as the sum of the integral on the closed curve γ, plus the integral on the line C' (fig. 2 – right side). The segments at infinity do not contribute, since they are located inside the cone $\mathcal{C}'$ where $\lambda^{-1} Q(\lambda P(p - k_{\mathrm{M}}))$ converges to $P(p - k_{\mathrm{M}})$, as follows from formula (2.12)[5].

The integrand of (5.4) has poles at $p^2 + m^2 = 0$ in the complex p_4 plane. One of them may fall inside the curve γ. We need to distinguish the case where this happens from the opposite case. The integral on C', instead, is centered on the Euclidean region, so we can take the limit $\lambda \rightarrow +\infty$ straightforwardly on it by means of (2.17).

Defining $p^0 = ip_4$, the poles of $p^2 + m^2 = 0$ are located at $p^0 = \pm\omega_{\mathbf{p}}$, where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, so $p^0 = \omega_{\mathbf{p}}$ is inside γ if $k^0 > \omega_{\mathbf{p}}$. No pole is inside γ if $k^0 < \omega_{\mathbf{p}}$. We can write

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = (a) + (b),$$

[5] Recall that we are using the dimensional regularization. This means that when an integral is ultraviolet divergent, we split it into a sum of integrals that admit convergence domains in the complex plane of the dimension D [23]. Then the arguments are applied to each of them separately.

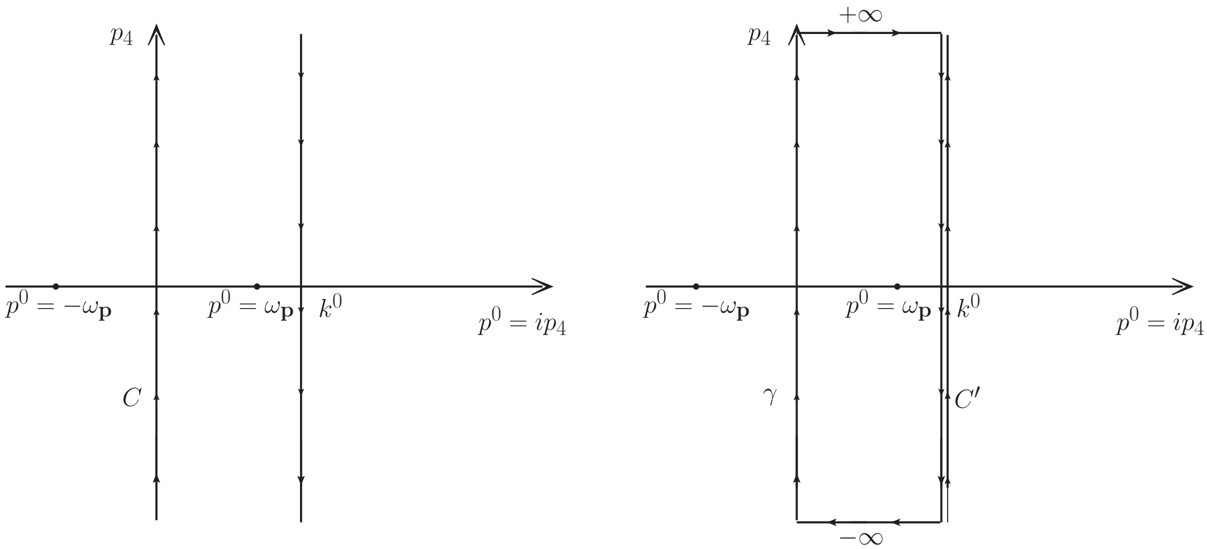

Figure 2: Dealing with the second limit (assuming $k^0 > 0$)

where

$$(a) = i \lim_{\lambda\to+\infty} \int_\gamma \frac{\mathrm{d}p_4}{2\pi} \int_{\omega_{\mathbf{p}}\leqslant k^0} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{p^2+m^2} \frac{\lambda}{Q(\lambda P(p-k_{\mathrm{M}}))}, \tag{5.5}$$

$$(b) = i \lim_{\lambda\to+\infty} \int_{C'} \frac{\mathrm{d}p_4}{2\pi} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{p^2+m^2} \frac{\lambda}{Q(\lambda P(p-k_{\mathrm{M}}))}. \tag{5.6}$$

Let us begin by evaluating (a) and (b) in the simplified case $\mathbf{k} = m = 0$. The residue theorem gives[6]

$$(a) = i \lim_{\lambda\to+\infty} \int_{|\mathbf{p}|\leqslant k^0} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{\lambda}{Q(\lambda k^0(2|\mathbf{p}|-k^0))}. \tag{5.7}$$

Since the argument of Q is real, we can take the limit $\lambda \to +\infty$ by means of formula (2.17). We obtain

$$(a) = -i\mathcal{P} \int_{|\mathbf{p}|\leqslant k^0} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0-2|\mathbf{p}|)}. \tag{5.8}$$

The integral (b) is centered on the Euclidean domain. It is convenient to make this fact apparent by relabeling the integration variable p_4 (which is not a change of variables) and writing

$$(b) = i \lim_{\lambda\to+\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p+k_{\mathrm{M}})^2} \frac{\lambda}{Q(\lambda P(p))}. \tag{5.9}$$

The limit $\lambda \to +\infty$ is now straightforward and gives

$$(b) = i \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p+k_{\mathrm{M}})^2} \frac{1}{p^2}. \tag{5.10}$$

[6]Note that the integral is in $p_4 = -ip^0$, not p^0, so there is an extra factor $-i$.

Note that two more poles are created by the limit. Moreover,(5.10) does not coincide with the loop integral of the usual bubble diagram, because when we apply the residue theorem to the p_4 integral, we end up including different poles.

Closing the path on the right side, we encircle two poles ($p^0 = |\mathbf{p}| - k^0$ and $p^0 = |\mathbf{p}|$) for $k^0 < |\mathbf{p}|$ and one ($p^0 = |\mathbf{p}|$) for $k^0 > |\mathbf{p}|$. The result is

$$(b) = -i \int_{k^0<|\mathbf{p}|} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0 - 2|\mathbf{p}|)} - i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0 + 2|\mathbf{p}|)}. \tag{5.11}$$

This expression does not need particular prescriptions, since the integrand is never singular.

Summing to (a), we find the total

$$\begin{aligned}\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) &= -i\mathcal{P} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0 - 2|\mathbf{p}|)} - i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0(k^0 + 2|\mathbf{p}|)} \\ &= i\mathcal{P} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{4|\mathbf{p}|^2} \left(\frac{1}{k^0 + 2|\mathbf{p}|} - \frac{1}{k^0 - 2|\mathbf{p}|} \right),\end{aligned} \tag{5.12}$$

which is precisely the result obtained in the case of PVPs (apart from the factor V), as given in formula (A.2).

We have obtained the result (5.12) by adopting the second option of (2.5). Let us see what happens, instead, when we adopt the first option. Everything proceeds as above in (b), because the arguments of Q are positive. The difference is visible in (a), where we obtain

$$(a) = i \int_{|\mathbf{p}|\leqslant k^0} \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V}{2|\mathbf{p}|} \frac{1}{k^0|k^0 - 2|\mathbf{p}||}, \tag{5.13}$$

instead of (5.8), which is clearly singular. Thus, the local limit is not well defined with the first option of (2.5) in Minkowski spacetime.

For completeness, let us treat the general case $\mathbf{k} \neq 0$, $m \neq 0$ with the second option (2.5) for Q. The residue theorem gives (5.5) again, but now on the pole $p^0 = \omega_{\mathbf{p}}$ we have

$$(p - k_{\mathrm{M}})^2 + m^2 = -(k^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{k}})(k^0 - \omega_{\mathbf{p}} + \omega_{\mathbf{p}-\mathbf{k}}). \tag{5.14}$$

What is important is that the argument of Q remains real, so we can still use formula (2.17) to evaluate the limit $\lambda \to +\infty$. The result is

$$(a) = -i\mathcal{P} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{V\theta(k^0 - \omega_{\mathbf{p}})}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left(\frac{1}{k^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{k}}} - \frac{1}{k^0 - \omega_{\mathbf{p}} + \omega_{\mathbf{p}-\mathbf{k}}} \right). \tag{5.15}$$

The singularity due to the first term inside the parentheses is regulated by the principal value inherited from (2.17). Instead, the second term is nonsingular for $\omega_{\mathbf{p}} \leqslant k^0$.

Centering the (b) integral as in (5.9), taking λ to infinity and translating $\mathbf{p}$ back to $\mathbf{p}-\mathbf{k}$, we obtain

$$(b) = i\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{V}{(p^0+k^0-\omega_{\mathbf{p}})(p^0+k^0+\omega_{\mathbf{p}})}\frac{1}{(p^0-\omega_{\mathbf{p}-\mathbf{k}})(p^0+\omega_{\mathbf{p}-\mathbf{k}})}. \tag{5.16}$$

Repeating the argument above, we close the path on the right side, thereby picking two poles ($p^0=\omega_{\mathbf{p}}-k^0$ and $p^0=\omega_{\mathbf{p}-\mathbf{k}}$) for $k^0<\omega_{\mathbf{p}}$ and one ($p^0=\omega_{\mathbf{p}-\mathbf{k}}$) for $k^0>\omega_{\mathbf{p}}$. The result is

$$(b) = -i\int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{V}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left(\frac{\theta(\omega_{\mathbf{p}}-k^0)}{k^0-\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{k}}}+\frac{\theta(k^0-\omega_{\mathbf{p}})}{k^0-\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}}-\frac{1}{k^0+\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}}\right).$$

Again, the integrand is never singular.

Summing the outcome to the (a) of (5.15), we finally get

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = -i\mathcal{P}\int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\frac{V}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}}\left(\frac{1}{k^0-\omega_{\mathbf{p}}-\omega_{\mathbf{p}-\mathbf{k}}}-\frac{1}{k^0+\omega_{\mathbf{p}}+\omega_{\mathbf{p}-\mathbf{k}}}\right),$$

which is the same as in the case of PVPs, formula (A.2).

If one particle is physical, like φ in (2.3), and the other one is ϕ with the second option of (2.5) for Q, the result does not change. The simplest parametrization of the circulating momenta gives (5.4) directly, whence the rest follows as before. If we start from the parametrization

$$\mathcal{B}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}) = i\lim_{\lambda\to+\infty}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{V}{(p+k_{\mathrm{M}})^2+m^2}\frac{\lambda}{Q(\lambda P(p))}, \tag{5.17}$$

we have to note that the integration on p_4 must be deformed to complex values in order to leave the right pole on one side and the left pole on the other side. Then the argument of Q is not everywhere real, so the limit $\lambda\to+\infty$ cannot be taken directly. On the other hand, one can easily see that (5.17) is equivalent to (5.4), because the difference is a closed path that encircles no pole.

5.2 Triangle diagram

Now we study the loop integral

$$\mathcal{T}_{\mathrm{M}}(k_{\mathrm{M}},q_{\mathrm{M}}) = i\int \frac{\mathrm{d}^D p}{(2\pi)^D}V(p,k_{\mathrm{M}},q_{\mathrm{M}})G_{\mathrm{nl}}(p)G_{\mathrm{nl}}(p-k_{\mathrm{M}})G_{\mathrm{nl}}(p-q_{\mathrm{M}}) \tag{5.18}$$

of the triangle diagram with the second option of (2.5) for Q. After possibly a translation of the internal momentum p, and assuming that the external momenta are generic, we can arrange the expression so that, say, $q^0>k^0>0$.

We introduce the parameter λ and split the local limit $\lambda \to +\infty$ in three steps. First, we take it inside $G_{\text{nl}}(p)$, then in $G_{\text{nl}}(p-k_{\text{M}})$ and finally in $G_{\text{nl}}(p-q_{\text{M}})$. The calculation will tell us to what extent it is legitimate to do so.

As before, the first limit is straightforward, since $G_{\text{nl}}(p)$ does not depend on k_{M} and q_{M}. We remain with

$$\mathcal{T}_{\text{M}}^{\text{loc}}(k_{\text{M}}, q_{\text{M}}) = i \lim_{\lambda' \to +\infty} \lim_{\lambda \to +\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{p^2+m^2} \frac{\lambda'}{Q(\lambda' P(p-k_{\text{M}}))} \frac{\lambda}{Q(\lambda P(p-q_{\text{M}}))}.$$

Now we use the residue theorem to integrate the loop energy $p^4 = -ip^0$ along the closed curve γ made of the lines $p^0 = 0$ and $p^0 = k^0$, plus segments at infinity. On the pole $p^0 = \omega_{\mathbf{p}}$, which contributes for $\omega_{\mathbf{p}} < k^0$, the arguments of the functions Q are real, so we can use formula (2.17). We then obtain the first contribution, which is

$$(a) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\theta(k^0 - \omega_{\mathbf{p}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}} \mathcal{Q}^{12}\mathcal{Q}^{13},$$

where

$$\mathcal{Q}^{ab} = \mathcal{P}^{ab} - \mathcal{P}\frac{1}{e_a - e_b - \omega_a + \omega_b}, \qquad \mathcal{P}^{ab} = \mathcal{P}\frac{1}{e_a - e_b - \omega_a - \omega_b},$$

$\mathcal{P}$ denotes the Cauchy principal value, and the subscripts $a, b, \ldots$ range over the values 1, 2 and 3, while $\{e_a\} = \{0, -k^0, -q^0\}$, $\{\omega_a\} = \{\omega_{\mathbf{p}}, \omega_{\mathbf{p}-\mathbf{k}}, \omega_{\mathbf{p}-\mathbf{q}}\}$.

We are left with the integral on the line $p^0 = k^0$, which we center on $p^0 = 0$ by means of a relabelling of the loop energy. After that, the limit $\lambda' \to +\infty$ acts on $\lambda'/Q(\lambda' P(p))$ and becomes straightforward. We remain with

$$(b) = i \lim_{\lambda \to +\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p^2+m^2)((p+k_{\text{M}})^2+m^2)} \frac{\lambda}{Q(\lambda P(p+k_{\text{M}}-q_{\text{M}}))}. \tag{5.19}$$

At this point, we write $(b) = (b_1) + (b_2)$, where (b_1) is the integral on the closed curve γ' made by the lines $p^0 = 0$ and $p^0 = q^0 - k^0$, plus segments at infinity, and (b_2) is the integral on the line $p^0 = q^0 - k^0$.

Using the residue theorem once more on (b_1), we obtain an integral on $\mathbf{p}$ that we do not report here. We just remark that, before taking the limit $\lambda \to \infty$, its integrand is regular. Specifically, the residues at the poles mutually compensate for every $\lambda < \infty$. This means that we can adopt the prescription we want for those poles. We choose the principal value. Then, we take the limit $\lambda \to \infty$ by means of (2.17), noting that the arguments of Q are real on the poles. We find that the contribution of γ' is

$$(b_1) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\theta(q^0 - \omega_{\mathbf{p}})\theta(\omega_{\mathbf{p}} - k^0)\mathcal{Q}^{12}\mathcal{Q}^{13} + \theta(q^0 - k^0 - \omega_{\mathbf{p}-\mathbf{k}})\mathcal{Q}^{23}\mathcal{Q}^{21}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}.$$

Centering the integral (b_2) on $p^0 = 0$ by means of a further relabelling of the loop energy, we obtain

$$(b_2) = i \lim_{\lambda \to +\infty} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{((p+q_\mathrm{M})^2 + m^2)((p - k_\mathrm{M} + q_\mathrm{M})^2 + m^2)} \frac{\lambda}{Q(\lambda P(p))}. \tag{5.20}$$

Now the limit $\lambda \to +\infty$ acts on $\lambda / Q(\lambda P(p))$. Nevertheless, we cannot evaluate it directly by means of (2.17), because if we do so, we find an unprescribed expression:

$$(b_2) = i \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{V}{(p^2 + m^2)((p - k_\mathrm{M} + q_\mathrm{M})^2 + m^2)((p+q_\mathrm{M})^2 + m^2)}. \tag{5.21}$$

Recall that p is Euclidean, so this formula does not correspond (for, e.g., $V = 1$) to the triangle diagram of local theories, due to the different sets of contributing poles. Using the residue theorem, the result of (5.21) is

$$(b_2) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\theta(\omega_\mathbf{p} - q^0)\hat{\mathcal{Q}}^{12}\hat{\mathcal{Q}}^{13} + \theta(k^0 - q^0 + \omega_{\mathbf{p}-\mathbf{k}})\hat{\mathcal{Q}}^{23}\hat{\mathcal{Q}}^{21} + \hat{\mathcal{Q}}^{32}\hat{\mathcal{Q}}^{31}}{8\omega_\mathbf{p}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}, \tag{5.22}$$

where $\hat{\mathcal{Q}}$ is the "unprescribed $\mathcal{Q}$", that is to say, the same as $\mathcal{Q}$ without the principal-value prescription.

Among the other things, the integrand of (5.22) involves an expression like

$$\frac{1}{xy} - \frac{1}{x(x+y)} - \frac{1}{y(x+y)}, \tag{5.23}$$

with $x = \omega_\mathbf{p} - \omega_{\mathbf{p}-\mathbf{q}} - q^0$ and $y = k^0 - \omega_\mathbf{p} + \omega_{\mathbf{p}-\mathbf{k}}$. We focus on the region around $x = y = 0$, where the θ functions appearing in some numerators leading to (5.23) are equal to one. Since the sum (5.23) is unprescribed, we do not know whether it is zero or not. For example, it is zero if we slightly move x and y to complex values, but it is nonzero if we take the principal value, due to the identity [12]

$$\mathcal{P}\left(\frac{1}{xy} - \frac{1}{x(x+y)} - \frac{1}{y(x+y)}\right) = -\pi^2 \delta(x)\delta(y). \tag{5.24}$$

Note that the double delta function on the right-hand side does not appear in Feynman diagrams (check [12] for details).

The problem just outlined indicates that we have jumped to (5.21) too quickly, since the integral of (5.20) is well defined before taking the limit. To avoid the trouble, we can evaluate (5.20) as follows. First, we move the external energies slightly away from the real axis: $k^0 \to k^0 + i\epsilon$, $q^0 \to q^0 + i\epsilon'$, with ϵ and ϵ' real (not necessarily positive) and small. Then we use (2.17) to evaluate the limit $\lambda \to +\infty$. When we apply the residue theorem, we find expressions like (5.23) with $x \to x - i\epsilon'$ and $y \to y + i\epsilon$, which do vanish.

Using the principal value and subtracting the right-hand side of (5.24), we can write

$$(b_2) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} V \frac{\theta(\omega_{\mathbf{p}} - q^0)\mathcal{Q}^{12}\mathcal{Q}^{13} + \theta(k^0 - q^0 + \omega_{\mathbf{p}-\mathbf{k}})\mathcal{Q}^{23}\mathcal{Q}^{21} + \mathcal{Q}^{32}\mathcal{Q}^{31}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}$$
$$+ i\pi^2 \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} V \frac{\delta(q^0 - \omega_{\mathbf{p}} + \omega_{\mathbf{p}-\mathbf{q}})\delta(k^0 - \omega_{\mathbf{p}} + \omega_{\mathbf{p}-\mathbf{k}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}.$$

The total gives

$$\mathcal{T}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}, q_{\mathrm{M}}) = (a) + (b) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} V \frac{\mathcal{Q}^{12}\mathcal{Q}^{13} + \mathcal{Q}^{23}\mathcal{Q}^{21} + \mathcal{Q}^{32}\mathcal{Q}^{31}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}$$
$$+ i\pi^2 \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} V \frac{\delta(q^0 - \omega_{\mathbf{p}} + \omega_{\mathbf{p}-\mathbf{q}})\delta(k^0 - \omega_{\mathbf{p}} + \omega_{\mathbf{p}-\mathbf{k}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}.$$

Rearranging it by means of (5.24), we obtain

$$\mathcal{T}_{\mathrm{M}}^{\mathrm{loc}}(k_{\mathrm{M}}, q_{\mathrm{M}}) = (a) + (b) = -i \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\mathcal{P}^{12}\mathcal{P}^{13} + \mathcal{P}^{21}\mathcal{P}^{31} + \mathrm{cycl}}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}$$
$$+ i\pi^2 \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\delta(q^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{q}})\delta(q^0 - k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}-\mathbf{q}})}{8\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}\omega_{\mathbf{p}-\mathbf{q}}}. \quad (5.25)$$

The expected result for a triangle with circulating PVPs is just the first line [12]. The second line is not correct.

The original integral (5.18) is symmetric (up to V) under exchanges of the three energies and the reflection $e \rightarrow -e$, but the second line of (5.25) is not. Indeed, the symmetric expression

$$\sum_{a \neq b \neq c \neq a} \delta(e_a - e_b - \omega_a - \omega_b)\delta(e_a - e_c - \omega_a - \omega_c) + (e \rightarrow -e),$$

specialized to our case, gives

$$\delta(q^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{q}})\delta(q^0 - k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}-\mathbf{q}}) + \delta(k^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{k}})\delta(q^0 - \omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{q}}), \quad (5.26)$$

but the second term is missing in (5.25). This means that the calculation fails in the multi-threshold sector, which is the one made by the double deltas. Splitting the limit $\lambda \rightarrow \infty$ into three distinct limits is only accurate up to those terms.

The first line of (5.25) is enough to show that there are no propagating degrees of freedom. Indeed, the outcome is purely imaginary. In particular, the single-delta contributions (those which contribute to the optical theorem and highlight the degrees of freedom propagating on-shell inside the diagram) are completely missing.

One may object that corrections proportional to (5.26) are also not acceptable, because they are on-shell. Currently, we lack computational tools that are powerful enough to determine whether they are actually present or not. At any rate, we can explain how we should proceed if they were. Basically, we would be forced to advocate the inherent arbitrariness of the nonlocal deformation to compensate for the extra contributions. The Lagrangian that gives the correct local limit, including the multi-threshold interaction sector, would have to be adjusted along the way by including suitable nonlocal, finite counter-vertices.

6 Asymptotically local quantum gravity

In this section we explain how to deform the (local) theory of quantum gravity with purely virtual particles (PVP-QG) [13] into a unitary, nonlocal theory that tends to it in the local limit. We call the latter “asymptotically local quantum gravity” (AL-QG). We work in Minkowski spacetime.

6.1 PVP-QG

The PVP-QG theory coupled to matter is described by the higher-derivative action [26]

$$S_{\mathrm{QG}}(g,\Phi)=-\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(2\Lambda+R-\frac{R^2}{6m_\phi^2}+\frac{\eta}{2m_\chi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)+S_m(g,\Phi), \quad (6.1)$$

where m_ϕ is the mass of the inflaton ϕ (introduced below), m_χ is the mass of the spin-2 massive mode $\chi_{\mu\nu}$ (due to the square of the Weyl tensor $C_{\mu\nu\rho\sigma}$), which must be treated as a PVP, Φ denotes the matter fields, with action $S_m(g,\Phi)$, and $\eta=m_\chi^2(3m_\phi^2+4\Lambda)/(m_\phi^2(3m_\chi^2-2\Lambda))$ is a parameter very close to one. The simplest option is to take S_m equal to the covariantized action of the standard model, equipped with the nonminimal couplings allowed by power counting. The resulting theory is renormalizable[7] and unitary.

The crucial point is the treatment of $\chi_{\mu\nu}$ as a PVP. To clarify how this works it is convenient to introduce ϕ and $\chi_{\mu\nu}$ explicitly as extra fields by eliminating the higher derivatives. The result is [26]

$$S_{\mathrm{QG}}(\tilde{g},\phi,\chi,\Phi)=\tilde{S}_{\mathrm{HE}}(\tilde{g})+S_\phi(\tilde{g}+\psi,\phi)+S_\chi(\tilde{g},\chi)+S_m(\tilde{g}\mathrm{e}^{\kappa_\phi\phi}+\psi\mathrm{e}^{\kappa_\phi\phi},\Phi), \quad (6.2)$$

[7]Renormalization works exactly as in the Stelle theory [27], where the spin-2 massive field is quantized in a conventional way and propagates a ghost.

The relation between the old metric $g_{\mu\nu}$ and the new metric $\tilde{g}_{\mu\nu}$ reads

$$g_{\mu\nu} = (\tilde{g}_{\mu\nu} + \psi_{\mu\nu})\mathrm{e}^{\kappa_\phi \phi}, \qquad \psi_{\mu\nu} \equiv 2\kappa_\chi \chi_{\mu\nu} + \kappa_\chi^2 \left(\chi_{\mu\nu}\chi_{\rho\sigma}\tilde{g}^{\rho\sigma} - 2\chi_{\mu\rho}\chi_{\nu\sigma}\tilde{g}^{\rho\sigma}\right), \tag{6.3}$$

and the constants are

$$\kappa_\phi = \frac{m_\phi\sqrt{16\pi G}}{\sqrt{4\Lambda + 3m_\phi^2}}, \qquad \kappa_\chi = \sqrt{8\pi\tilde{G}}, \qquad \tilde{G} = \frac{G}{\eta}.$$

Moreover,

$$\tilde{S}_{\mathrm{HE}}(g) = -\frac{1}{16\pi\tilde{G}}\int \mathrm{d}^4x\sqrt{-g}\,(2\Lambda + R),$$

is the Einstein-Hilbert action with the redefined Newton constant,

$$S_\phi(g,\phi) = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[D_\mu\phi D^\mu\phi - \frac{m_\phi^2}{\kappa_\phi^2}\left(1 - \mathrm{e}^{\kappa_\phi\phi}\right)^2\right], \tag{6.4}$$

is the inflaton action, and

$$S_\chi(\tilde{g},\chi) = \tilde{S}_{\mathrm{HE}}(\tilde{g}+\psi) - \tilde{S}_{\mathrm{HE}}(\tilde{g}) + \int \mathrm{d}^4x\left[\frac{m_\chi^2}{2}\sqrt{-g}(\chi_{\mu\nu}\chi^{\mu\nu} - \chi^2) - 2\kappa_\chi\chi_{\mu\nu}\frac{\delta\tilde{S}_{\mathrm{HE}}(g)}{\delta g_{\mu\nu}}\right]_{g\rightarrow\tilde{g}+\psi} \tag{6.5}$$

is the $\chi_{\mu\nu}$ action. Specifically, one finds

$$S_\chi(g,\chi) = -S_{\mathrm{PF}}(g,\chi,m_\chi^2) + S_\chi^{(>2)}(g,\chi), \tag{6.6}$$

where

$$\begin{aligned} S_{\mathrm{PF}}(g,\chi,m_\chi^2) = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,[&D_\rho\chi_{\mu\nu}D^\rho\chi^{\mu\nu} - D_\rho\chi D^\rho\chi + 2D_\mu\chi^{\mu\nu}D_\nu\chi - 2D_\mu\chi^{\rho\nu}D_\rho\chi_\nu^\mu \\ &-m_\chi^2(\chi_{\mu\nu}\chi^{\mu\nu} - \chi^2) + R^{\mu\nu}(\chi\chi_{\mu\nu} - 2\chi_{\mu\rho}\chi_\nu^\rho)] \end{aligned} \tag{6.7}$$

is the covariantized Pauli-Fierz action with a nonminimal term, and $S_\chi^{(>2)}(g,\chi)$ are corrections at least cubic in χ.

The theory (6.2) is renormalizable, but not manifestly. This means that, when we calculate its Feynman diagrams, "miraculous" cancellations make it possible to subtract the divergences by means of field redefinitions and renormalizations of the parameters already contained in (6.2).

The crucial problem is the minus sign in front of S_{PF}. If $\chi_{\mu\nu}$ is treated conventionally, it propagates a ghost, and unitarity is violated. For analogous reasons, the theory (6.1)-(6.2) is not acceptable as a classical theory.

The situation changes radically when $\chi_{\mu\nu}$ is understood as a PVP. The field $\chi_{\mu\nu}$ is projected away by integrating it out according to the diagrammatic rules of PVPs, briefly recalled in the appendix. No $\chi_{\mu\nu}$ external legs are considered, and the modified diagrams guarantee that $\chi_{\mu\nu}$ does not give on-shell contributions to the radiative corrections.

Expanding around the flat-space metric $\eta_{\mu\nu}$ =diag$(1,-1,-1,-1)$, the $\chi_{\mu\nu}$ propagator of (6.2) is

$$\langle\chi_{\mu\nu}(p)\,\chi_{\rho\sigma}(-p)\rangle_0=-\left.\frac{i}{p^2-m_\chi^2}\right|_{\text{PVP}}\left(\frac{\pi_{\mu\rho}\pi_{\nu\sigma}+\pi_{\mu\sigma}\pi_{\nu\rho}}{2}-\frac{1}{3}\pi_{\mu\nu}\pi_{\rho\sigma}\right),\qquad \pi_{\mu\nu}\equiv\eta_{\mu\nu}-\frac{p_\mu p_\nu}{m_\chi^2}, \tag{6.8}$$

where and the subscript "PVP" is there to remind us that $\chi_{\mu\nu}$ must be treated as a PVP inside diagrams.

The projection applies at the classical level as well. The true, classical theory is neither (6.1) nor (6.2). It is obtained by collecting the tree diagrams with no $\chi_{\mu\nu}$ external legs [19]. The result is

$$S_{\text{cl}}(g,\Phi)=-\frac{1}{16\pi G}\int\mathrm{d}^4x\sqrt{-g}\left(2\Lambda+R-\frac{R^2}{6m_\phi^2}\right)+S_m(g,\Phi)+\Delta S_{\text{nl}}(g,\Phi),$$

where ΔS_{nl} collects nonlocal vertices that are negligible at energies lower than m_χ.

The prices to pay to have renormalizability and unitarity at the same time in quantum gravity are the impossibility to distinguish past, present and future at distances smaller, or intervals shorter, than $1/m_\chi$, as well as a certain "peak uncertainty": in the processes where the PVP is supposed to be "detected", significant complications arise due to the inherent impossibility of its detection [20].

6.2 Kinetic Lagrangians of Proca and Pauli-Fierz AL-QFTs

The residue of the propagator (6.8) at $p^2=m_\chi^2$ has the wrong sign. If treated conventionally, it gives a ghost. We know that the solution, in the realm of local quantum field theory, is to treat it as a PVP. An alternative option is to alter the propagator (6.8) by means of entire functions, so as to eliminate the zero in the denominator.

The simplest deformation amounts to turning (6.8) into

$$\langle\chi_{\mu\nu}(p)\,\chi_{\rho\sigma}(-p)\rangle_0=\frac{i}{Q(-p^2+m_\chi^2)}\left(\frac{\tilde\pi_{\mu\rho}\tilde\pi_{\nu\sigma}+\tilde\pi_{\mu\sigma}\tilde\pi_{\nu\rho}}{2}-\frac{1}{3}\tilde\pi_{\mu\nu}\tilde\pi_{\rho\sigma}\right), \tag{6.9}$$

where Q is the second option of formula (2.5) and

$$\tilde\pi_{\mu\nu}\equiv\eta_{\mu\nu}-\frac{p_\mu p_\nu}{m_\chi^2}\sigma^2(-p^2+m_\chi^2).$$

As anticipated, the new "propagator" (6.9) has no pole, hence it does not actually propagate degrees of freedom. We can view it as the propagator of a NL-PVP. In the local limit, the function σ^2 tends to one, so (6.9) tends to (6.8) (at the tree level), by formula (2.17).

The choice (6.9) corresponds to the nonlocal kinetic Lagrangian

$$S_{\mathrm{PF}}' = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,[\chi_{\mu\nu}Q(D_\rho D^\rho + m_\chi^2)\chi^{\mu\nu} - \chi Q(D_\rho D^\rho + m_\chi^2)\chi \\ +2\chi^{\mu\nu}D_\mu\tilde{Q}(D_\rho D^\rho + m_\chi^2)D_\nu\chi - 2\chi^{\rho\nu}D_\mu\tilde{Q}(D_\rho D^\rho + m_\chi^2)D_\rho\chi_\nu^\mu \\ - R^{\mu\nu}(\chi\chi_{\mu\nu} - 2\chi_{\mu\rho}\chi_\nu^\rho)]\,, \tag{6.10}$$

where

$$\tilde{Q}(x) = \frac{x}{m_\chi^2 + (x - m_\chi^2)\sigma^2(x)}.$$

It is not necessary, at this stage, to modify the nonminimal coupling (last line).

The lagrangian of S_{PF}' is singular for $x = 0$ and $(m_\chi^2 - x)\sigma^2(x) = m_\chi^2$, where $x = m_\chi^2 - p^2$. The singularities are simple poles in x, and can be prescribed by means of the Cauchy principal value. This keeps S_{PF}' convergent and real.

For reasons similar to those explained in the case of PVPs, the action S_{PF}' is not the true classical action, but a sort of "interim" action. The field $\chi_{\mu\nu}$ must be integrated out, so the singularity of the Lagrangian is harmless. What is important is that the propagator (6.9) is regular. Below we show that the vertices are regular as well.

If we use $\pi_{\mu\nu}$ in (6.9), instead of $\tilde{\pi}_{\mu\nu}$, we have (6.10) with $\tilde{Q} \to \sigma^{-2}$. Then the Lagrangian has singularities $\sim 1/x^2$, which are more severe, to the extent that the action S_{PF}' becomes also singular. Again, what is important is that the propagator and the vertices are regular.

For reference, let us consider the action

$$S_{\mathrm{Proca}} = \frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}\,[g^{\mu\rho}g^{\nu\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) - 2m^2 g^{\mu\nu}A_\mu A_\nu]$$

of a PVP Proca vector A_μ. Using the same notation as above with $m_\chi \to m$, the propagator

$$\langle A_\mu(p)\,A_\nu(-p)\rangle_0 = \left.\frac{i\pi_{\mu\nu}}{p^2 - m^2}\right|_{\mathrm{PVP}}$$

can be deformed into

$$\langle A_\mu(p)\,A_\nu(-p)\rangle_0 = -\frac{i}{Q(-p^2 + m^2)}\tilde{\pi}_{\mu\nu},$$

which is derived from the modified action

$$S_{\mathrm{Proca}}' = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\nu}A_\mu Q(D_\rho D^\rho + m^2)A_\nu + A_\mu D^\nu\tilde{Q}(D_\rho D^\rho + m^2)D^\mu A_\nu)\right].$$

6.3 AL-QG

Summarizing, asymptotically local quantum gravity is the theory described by the action

$$S_{\rm QG}(\tilde{g},\phi,\chi,\Phi)=\tilde{S}_{\rm HE}(\tilde{g})+S_{\chi}^{\prime}(\tilde{g},\chi)+S_{\phi}(\tilde{g}+\psi,\phi)+S_{m}(\tilde{g}\mathrm{e}^{\kappa_{\phi}\phi}+\psi\mathrm{e}^{\kappa_{\phi}\phi},\Phi), \tag{6.11}$$

where

$$S_{\chi}(g,\chi)=-S_{\rm PF}^{\prime}(g,\chi,m_{\chi}^{2})+S_{\chi}^{(>2)}(g,\chi). \tag{6.12}$$

It is obtained from (6.2) by replacing the Pauli-Fierz $\chi_{\mu\nu}$ action with (6.10).

In fact, (6.11) is just the starting action, because, as we have shown in the previous section, the multi-threshold sectors of involved diagrams may need to be adjusted along the way, in order to reach the correct local limit. Moreover, the deformed theory is not guaranteed to be renormalizable, so further adjustments may be required. We discuss this issue below.

The field $\chi_{\mu\nu}$ does not need a special prescription in AL-QG, since the nonlocal deformation (6.9) of the propagator is self-sufficient. Modulo the adjustments just mentioned, the local limit of (6.12) is the theory of quantum gravity with purely virtual particles [13], that is to say, (6.1) or (6.2), with $\chi_{\mu\nu}$ treated as a PVP.

6.4 Vertices

Now we show that the vertices obtained by expanding (6.12) around flat space are well defined (see [28] and [6]). Write

$$D_{\mu}D^{\mu}+m^{2}=\Box+V_{\mu}\partial^{\mu}+W,$$

where $\Box$ is the flat-space D'Alembertian. Let $f(z)=\sum_{n=0}^{\infty}a_{n}z^{n}$ denote a generic entire function. We assume that $f(D_{\mu}D^{\mu}+m^{2})$ belongs to an expression where the derivatives can be integrated by parts. Expanding, the first order in V and W is

$$\begin{aligned}\sum_{n=0}^{\infty}a_{n}\sum_{k=0}^{n-1}\Box^{k}(V_{\mu}\partial^{\mu}+W)\Box^{n-k-1}&=(V_{\mu}\partial^{\mu}+W)\sum_{n=0}^{\infty}a_{n}\sum_{k=0}^{n-1}\overleftarrow{\Box}^{k}\Box^{n-k-1}\\ &=(V_{\mu}\partial^{\mu}+W)\sum_{n=0}^{\infty}a_{n}\frac{\overleftarrow{\Box}^{n}-\Box^{n}}{\overleftarrow{\Box}-\Box}=(V_{\mu}\partial^{\mu}+W)f(\overleftarrow{\Box},\Box),\end{aligned} \tag{6.13}$$

where

$$f(x,y)=\frac{f(x)-f(y)}{x-y}$$

denotes the incremental ratio of the function f, and the arrow on $\overleftarrow{\Box}$ means that the box acts to the very left, beyond $V_\mu \partial^\mu + W$. In momentum space, the ratio is calculated with respect to the right and left momenta.

Since the incremental ratio of an entire function is an entire function (of two variables), no singularity appears. Moreover, with $f = Q$ the ratio converges to

$$\frac{Q(-p^2+m_\chi^2) - Q(-q^2+m_\chi^2)}{-p^2+q^2} \rightarrow \mathcal{P}\frac{\frac{1}{p^2-m_\chi^2} - \frac{1}{q^2-m_\chi^2}}{p^2-q^2} = -\mathcal{P}\frac{1}{p^2-m_\chi^2}\frac{1}{q^2-m_\chi^2} \tag{6.14}$$

in the cone $\mathcal{C}$ fast enough to validate the arguments of the previous sections. Note that (6.14) is the correct result for PVPs at the tree level[8].

To the second order in some operator δ, we find, with $r = \Box$,

$$f(r+\delta) = \sum_{n=0}^{\infty} a_n (r+\delta)^n \rightarrow \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n-2} \sum_{l=0}^{n-k-2} r^k \delta r^l \delta r^{n-k-l-2} = \delta_{12}\delta_{23} \frac{f(r_1, r_3) - f(r_2, r_3)}{r_1 - r_2}, \tag{6.15}$$

where the subscripts have been introduced to keep track of the ordering of the various ingredients, which is $r_1 \delta_{12} r_2 \delta_{23} r_3$. For example, in momentum space the ordering gives

$$r(p+k+q)\delta(k)r(p+q)\delta(q)r(p),$$

where k and q are the incoming momenta of the insertions δ_{12} and δ_{23}, respectively.

Formula (6.15) shows that the second-order vertex is the incremental ratio of the incremental ratio, keeping one variable fixed (which one being immaterial). In particular, the vertex is well defined and tends to its local limit fast enough. One can proceed similarly for the rest of the expansion.

To derive the loop integrals of the theory (6.12), we proceed as follows. First, we integrate out $\chi_{\mu\nu}$. This generates "functional diagrams" built by means of the covariantized versions of the propagators (6.9), where Q and σ are functions of $D_\mu D^\mu + m^2$. Later, we expand the metric tensor $g_{\mu\nu}$ around flat space. The expansion also acts inside the covariantized $\chi_{\mu\nu}$ propagators, and is treated by means of identities like (6.13). Once this is done, we arrive at the loop integrals, which are well defined.

6.5 Arbitrariness and renormalization

The AL-QG theory (6.12) is unitary, and so is its local limit, which is, by construction, the PVP-QG theory of quantum gravity described by the action (6.2), with $\chi_{\mu\nu}$ treated

[8] It is not correct inside loop diagrams, but we know from section 5 that the loop integrals make the right PVP expressions appear there as well, possibly up to multi-thresholds.

as a PVP. Given that (6.2) is renormalizable, although not manifestly, it is mandatory to inquire whether AL-QG is also renormalizable or not.

The nonlocal theories of the literature [5, 7, 8] are super-renormalizable. Unfortunately, their renormalization properties do not extend to (6.1), which is strictly renormalizable [6]. In view of this, AL-QG is not expected to be renormalizable either. For the reasons that we explain below, we do not think that this is a serious liability.

Although PVP-QG is unique [13], the nonlocal extension PVP-QG $\rightarrow$ AL-QG is not. Any entire function like those considered in the literature [5, 7, 8] can be used for h inside the second option for the function Q of (2.5). This leads to an infinite arbitrariness. The arbitrariness likely turned on by renormalization, together with the one associated with the adjustments of the multi-threshold contributions mentioned in the previous section, is not worse than that.

Moreover, the problem of arbitrariness is predicated on the assumption that the nonlocal theories are fundamental ones, but this is not what we claim here. We merely view AL-QFTs as tools to move beyond common frameworks in quantum field theory.

The main weakness of nonlocal quantum field theory is the lack of a fundamental principle for selecting the form factors that modify the propagators. In our approach this problem is addressed by the very existence of the local limit in Minkowski spacetime. If we view *that* as the missing guiding principle, the candidate theories of the universe are the local limits themselves.

The arbitrariness of AL-QFT is reminiscent of the one of off-the-mass shell physics [18]. In the latter, the extra parameters are not properties of the fundamental interactions, but describe the environment where the phenomenon is observed, such as the experimental apparatus and the observer itself. We could say that they describe the quantum/classical interplay between the phenomenon and the rest of the universe. It would be interesting to uncover the map relating the arbitrariness of AL-QFT to the one of the off-shell approaches to QFT of ref.s [18]. We postpone this task, because it is beyond the scope of this paper.

We stress that the non uniqueness of the nonlocal deformation does not necessarily imply a lack of predictivity. The number of parameters impacting the phenomenon we want to observe is hopefully finite. Once they are identified, they can be fixed by sacrificing an equal number of initial measurements, after which every other measurement is predicted efficiently.

7 Conclusions

We have studied the local limits of nonlocal quantum field theories. Moving along the manifold $\mathcal{M}_{\rm NL}$ of NL-QFTs by varying a certain parameter λ, we have inquired whether a target local model is reached when λ tends to infinity. When it is so, the nonlocal models can be seen as nonlocal deformations of their target local limits.

The nonlocal deformations are encoded into form factors that multiply the propagators of local theories and remove poles that normally propagate ghosts. We have shown that the form factors inspired by the models mostly studied in the current literature give theories that have well-defined limits only in Euclidean space. Singular behaviors appear in the Minkowskian correlation functions when $\lambda \rightarrow \infty$.

To overcome this difficulty, we have relaxed certain requirements and defined a new class $\mathcal{M}_{\rm AL}$ of unitary, asymptotically local theories, so-called because they have well-defined local limits in Minkowski spacetime. Unitarity forbids target models with ghosts and privileges models with purely virtual particles.

Inside the bubble diagram, the nonlocal deformation generates PVPs directly, in the local limit. In the triangle diagram, it does so possibly up to multi-threshold corrections, which can be adjusted by tuning the deformation itself.

The asymptotically local deformation of a local theory is not unique, and not renormalizable. In our approach, this is not a liability, because we are not proposing AL-QFTs as candidate fundamental theories of nature, but merely as tools to provide alternative formulations of theories with PVPs, approximations to study the violations of microcausality and the peak uncertainties, or alternative approaches to off-shell physics, more commonly treated by restricting QFT to a finite interval of time and a compact space manifold. In that case, the non uniqueness is associated with the possibility of introducing parameters that describe the apparatus, the classical environment surrounding the experiment, and the observer itself.

When this type of arbitrariness is present, the first thing to do is identify the parameters that are relevant to the phenomenon under observation. If they are finitely many, as is reasonable to expect in normal circumstances (or cleverly arranged setups), one sacrifices an equal number of initial measurements to fix them. At that point, one can verify whether the subsequent measurements are correctly predicted or not.

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Appendices

A Bubble diagram with PVPs

We review the calculation of the bubble diagram with one or two circulating PVPs, in the formulation of ref. [12]. We work in Minkowski spacetime, so here p and k denote Minkowskian momenta.

We start from the integral

$$\mathcal{B}_{\text{ph}} \equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon'}$$

of the standard bubble with circulating physical particles. First, we decompose the propagators

$$\frac{1}{q^2 - m^2 + i\epsilon} \to \frac{1}{2\omega_{\mathbf{q}}} \left(\frac{1}{q^0 - \omega_{\mathbf{q}} + i\epsilon} - \frac{1}{q^0 + \omega_{\mathbf{q}} - i\epsilon} \right)$$

by isolating the particle and antiparticle poles, where $\omega_{\mathbf{q}} = \sqrt{\mathbf{q}^2 + m^2}$ is the frequency. Then we expand the integrand and integrate on p^0 by means of the residue theorem. The result is

$$\mathcal{B}_{\text{ph}} = \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{i}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left(\frac{1}{k^0 + \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{p}} - i(\epsilon + \epsilon')} - \frac{1}{k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}} + i(\epsilon + \epsilon')} \right).$$

At this point, we use

$$\frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$$

for each term inside the parentheses, where $\mathcal{P}$ is the Cauchy principal value. We obtain

$$\begin{aligned} \mathcal{B}_{\text{ph}} = {} & \mathcal{P} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{i}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left(\frac{1}{k^0 + \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{p}}} - \frac{1}{k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}}} \right) \\ & - \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{\pi}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left[\delta(k^0 + \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{p}}) + \delta(k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}}) \right]. \end{aligned} \tag{A.1}$$

The bubble diagram $\mathcal{B}_{\text{PVP}}$ with one or two circulating PVPs is obtained from $\mathcal{B}_{\text{ph}}$ by dropping the delta terms:

$$\mathcal{B}_{\text{PVP}} = \mathcal{P} \int \frac{\mathrm{d}^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{i}{4\omega_{\mathbf{p}}\omega_{\mathbf{p}-\mathbf{k}}} \left(\frac{1}{k^0 + \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{p}}} - \frac{1}{k^0 - \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{p}}} \right). \tag{A.2}$$

For the generalization to the other diagrams, see [12].

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