

Propagators and Widths of Physical and Purely Virtual Particles in a Finite Interval of Time

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Abstract

We study the free and dressed propagators of physical and purely virtual particles in a finite interval of time τ and on a compact space manifold Ω , using coherent states. In the free-field limit, the propagators are described by the entire function $(e^z - 1 - z)/z^2$, whose shape on the real axis is similar to the one of a Breit-Wigner function, with an effective width around $1/\tau$. The real part is positive, in agreement with unitarity, and remains so after including the radiative corrections, which shift the function into the physical half plane. We investigate the effects of the restriction to finite τ on the problem of unstable particles vs resonances, and show that the muon observation emerges from the right physical process, differently from what happens at $\tau = \infty$. We also study the case of purely virtual particles, and show that, if τ is small enough, there exists a situation where the geometric series of the self-energies is always convergent. The plots of the dressed propagators show testable differences: while physical particles are characterized by the usual, single peak, purely virtual particles are characterized by twin peaks.

1 Introduction

Widths are key quantities in quantum field theory, and a link between perturbative and nonperturbative quantum field theory. A perturbatively stable particle may decay after the resummation of its self-energies into the so-called dressed propagator. Yet, the resummation, which is normally considered a straightforward operation, has unexpected features, when it comes to explain the observation of long-lived unstable particles, like the muon [1].

The S matrix amplitudes allow us to study scattering processes between asymptotic states, which are separated by an infinite amount of time. In this scenario, a long-lived unstable particle always has enough time to decay, before being actually observed. Although it is possible to make room for the muon observation in a rough and ready way within the usual frameworks, too many important details are missed along the way by doing so. It is much better to study the problem where it belongs, which is quantum field theory in a finite interval of time.

It is possible to formulate quantum field theory in a finite time interval τ , and on a compact space manifold Ω , by moving most details about such restrictions away from the internal sectors of the diagrams into external sources [2]. Then the diagrams are the same as usual, apart from the discretization of the loop momenta, and the presence of sources attached to the vertices. Most known properties of the usual S matrix amplitudes generalize straightforwardly, and allow us to study the systematics of renormalization and unitarity [2]. The formulation is well-suited to be generalized so as to include purely virtual particles, i.e., particles that do not exist on the mass shell at any order of the perturbative expansion. At $\tau = \infty$, $\Omega = \mathbb{R}^3$, they are introduced by removing the on-shell contributions of a physical particle (or a ghost, which is a particle with the wrong sign in front of its kinetic term) from the internal parts of the diagrams [3]. At finite τ and on a compact Ω , they are introduced by removing the same on-shell parts from the core diagrams, and choosing trivial initial and final conditions for them [2]. The evolution operator of the resulting theory is unitary.

In this paper, we study the propagators of physical and purely virtual particles in a finite interval of time τ , and on a compact space manifold Ω . In the free-field limit, the typical pole $1/z$ of the usual propagator at $\tau = \infty$ is replaced by an entire function, which is $f(z) = (e^z - 1 - z)/z^2$. Although $f(z)$ is very different from $1/z$ (and from a Breit-Wigner function) in most of the complex plane, its shape on the real axis $z = ix$, $x \in \mathbb{R}$, does remind the one of a Breit-Wigner function, with an effective width equal to

$16/(3\tau)$. When we include the radiative corrections, the function $f(z)$ is shifted into the physical half plane, where the real part of the propagator remains positive, consistently with unitarity. The width is enlarged by an amount equal to Γ (the usual width at $\tau = \infty$).

The muon observation emerges rather naturally from the right physical process: there is no need to confuse the observation of an unstable particle with the observation of its decay products, as one normally does to adjust the matter at $\tau = \infty$.

In the case of purely virtual particles, we show that, for τ small enough, there is an arrangement where the geometric series of the self-energies is always convergent. In that situation, we can resum the radiative corrections rigorously to the very end, and obtain the dressed propagator. Comparing the plot of its real part with the one of physical particles, testable differences emerge: while the physical particles are characterized by the common, single peak, purely virtual particles are characterized by two twin peaks.

The results confirm the ones of ref. [1], where they were derived by arguing, on general grounds, what the main effects of the restriction to finite τ were going to be.

Both physical particles and ghosts can be rendered purely virtual. At the same time, purely virtual particles are not Lee-Wick ghosts [4]¹, as shown in [6]. In particular, they do not need to have nonvanishing widths, and decay. And even if they have a nonvanishing width Γ_f , its meaning is not the reciprocal of a lifetime, nor the actual width of a peak. In the case studied here, where the resummation of the dressed propagator can be done rigorously to the very end, Γ_f is a measure of the height of the twin peaks, while their distance is universally fixed to 2π (in suitable units). In every other case, the “peak region” of a purely virtual particle is nonperturbative. Certain arguments suggest that Γ_f may measure a “peak uncertainty” $\Delta E > \Gamma_f/2$, telling us that, when we approach the peak region too close in energy, identical experiments may give different results [1]. At the phenomenological level, purely virtual particles may have interesting applications, because they evade many constraints that are typical of normal particles (see [7] and references therein).

The paper is organized as follows. In section 2 we study the free propagator at finite τ . In section 3 we resum the self-energies into the dressed propagator. In section 4 we study the free and dressed propagators of purely virtual particles. In section 5 we investigate the problem of unstable particles. Section 6 contains the conclusions. We work on bosonic fields, since the generalization to fermions and gauge fields does not present problems.

¹For Lee-Wick ghosts in quantum gravity, see [5]

2 Free propagator in a finite interval of time

In this section, we study the free propagator in a finite interval of time τ . For most purposes of this paper, we can Fourier transform the space coordinates, understand the integrals on the loop momenta, and concentrate on time and energy. This means that we can basically work with quantum mechanics, where the coordinates $q(t)$ stand for fields $\phi(t, \mathbf{x})$. We assume that the Lagrangian has the form

$$L_\lambda(q, \dot{q}) = \frac{1}{2} (\dot{q}^2 - \omega^2 q^2) - V_\lambda(q, \dot{q}), \quad (2.1)$$

where $V_\lambda(q)$ is proportional to some coupling λ . If the space manifold Ω is compact, the frequencies ω are restricted to a discrete set $\omega_{\mathbf{n}}$, for some label \mathbf{n} . This affects the propagator only in a minor way. Effects like these will be understood, from now on, so the formulas we write look practically the same as on $\Omega = \mathbb{R}^3$.

We use coherent “states” [8] (so we call them, although we work in the functional-integral approach)

$$z = \frac{1}{2} \left(q + i \frac{p}{\omega} \right), \quad \bar{z} = \frac{1}{2} \left(q - i \frac{p}{\omega} \right), \quad (2.2)$$

where $p = \partial L_\lambda / \partial \dot{q}$ is the momentum². So doing, we double the number of coordinates, or fields, lower the number of time derivatives from two to one, and treat the poles of the propagator

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{2\omega} \left(\frac{1}{e - \omega + i\epsilon} - \frac{1}{e + \omega - i\epsilon} \right) \quad (2.3)$$

separately³, where $k^\mu = (e, \mathbf{k})$ is the four-momentum and $\omega = \sqrt{\mathbf{k}^2 + m^2}$ denotes the frequency.

The first pole gives the propagator

$$G^+(t, t') = \langle z(t) \bar{z}(t') \rangle_0 = \theta(t - t') \frac{e^{-i\omega(t-t')}}{2\omega}, \quad (2.4)$$

while the other pole gives $G^-(t, t') = \langle \bar{z}(t) z(t') \rangle_0 = G^+(t', t)$. Moreover, $\langle z(t) z(t') \rangle_0 = \langle \bar{z}(t) \bar{z}(t') \rangle_0 = 0$. The sum

$$G(t, t') = G^+(t, t') + G^-(t, t') = \frac{e^{-i\omega|t-t'|}}{2\omega} \quad (2.5)$$

is indeed the Fourier transform of the Feynman propagator (2.3).

²We use the notation of [2], where details about the switch to coherent states can be found.

³A redefinition on ϵ is understood between the left- and right-hand sides of (2.2).

When $\tau = \infty$, the propagators are (2.3) and (2.4) for all real values of t and t' . When τ is finite, the propagators are unaffected, in the coherent-state approach, apart from the restrictions of t and t' to the interval (t_i, t_f) . To make this restriction explicit, we multiply both sides of $G^\pm(t, t')$ and $G(t, t')$ by projectors $\Pi_\tau(t) \equiv \theta(t_f - t)\theta(t - t_i)$ and $\Pi_\tau(t')$. The projected propagators are then

$$G_\tau(t, t') = \Pi_\tau(t)G(t, t')\Pi_\tau(t'), \quad G_\tau^\pm(t, t') = \Pi_\tau(t)G^\pm(t, t')\Pi_\tau(t'). \quad (2.6)$$

For simplicity, we take $t_f = \tau/2$, $t_i = -\tau/2$.

It is interesting to study the Fourier transforms of (2.6), which can be calculated by assuming that ω has a small, negative imaginary part. We start from

$$\tilde{G}_\tau^+(e, e') = \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt G_\tau^+(t, t') e^{i(et+e't')}. \quad (2.7)$$

Due to the lack of invariance under time translations, the result does not factorize the usual energy-conservation delta function $(2\pi)\delta(e + e')$. Instead, we can factorize a

$$\frac{2 \sin\left(\frac{e+e'}{2}\tau\right)}{e + e'}, \quad (2.8)$$

which is the Fourier transform of $\Pi_\tau(t)$ with energy $e + e'$. Furthermore, we assume that τ is large enough, so that we can restrict the coefficient of (2.8) in $\tilde{G}_\tau^+(e, e')$ to $e + e' = 0$. Factorizing a $2\omega/\tau$ for convenience, we approximate $\tilde{G}_\tau^+(e, e')$ to

$$\tilde{G}_\tau^+(e, e') \simeq \frac{2 \sin\left(\frac{e+e'}{2}\tau\right)}{e + e'} \frac{\tau}{2\omega} f(z), \quad (2.9)$$

where $z = i(e - \omega)\tau$. We find

$$f(z) = 2 \lim_{e' \rightarrow -e} \frac{\omega(e + e') \tilde{G}_\tau^+(e, e')}{\tau \sin\left(\frac{e+e'}{2}\tau\right)} = \frac{2\omega}{\tau^2} \lim_{e' \rightarrow -e} \tilde{G}_\tau^+(e, e') = \frac{e^z - 1 - z}{z^2}.$$

Interestingly enough, $f(z)$ is an entire function: the propagator at finite τ has no pole, and no other type of singularity.

Writing $z = ix$, it is useful to single out the real and imaginary parts:

$$f(z) = \frac{1 - \cos(x)}{x^2} + i \frac{x - \sin(x)}{x^2}.$$

To verify that the limit $\tau \rightarrow \infty$ gives the usual result, we first rescale τ by a factor λ and then let λ tend to infinity by means of the identities

$$\lim_{\lambda \rightarrow \infty} \frac{\sin(\lambda x)}{x} = \pi \delta(x), \quad \lim_{\lambda \rightarrow \infty} \frac{1 - \cos(\lambda x)}{\lambda x^2} = \pi \delta(x), \quad \lim_{\lambda \rightarrow \infty} \frac{\lambda x - \sin(\lambda x)}{\lambda x^2} = \mathcal{P} \frac{1}{x} \quad (2.10)$$

(which can be easily proved by studying them on test functions), where \mathcal{P} denotes the Cauchy principal value. Thus,

$$\lim_{\tau \rightarrow \infty} \tilde{G}_\tau^+(e, e') = (2\pi)\delta(e + e') \frac{1}{2\omega} \frac{i}{e - \omega + i\epsilon} = \tilde{G}^+(e, e').$$

Summing it to $\tilde{G}_\tau^-(e, e') = \tilde{G}_\tau^+(e', e)$, we go back to the Feynman propagator at $\tau = \infty$:

$$\tilde{G}(e, e') = \tilde{G}^+(e, e') + \tilde{G}^-(e, e') = (2\pi)\delta(e + e') \frac{i}{k^2 - m^2 + i\epsilon}. \quad (2.11)$$

Finally, the Fourier transform of the total propagator $G_\tau(t, t')$ at finite τ is

$$\tilde{G}_\tau(e, e') \simeq \frac{2\tau^2 \sin\left(\frac{e+e'}{2}\tau\right)}{e + e'} h(x_+, x_-), \quad (2.12)$$

where

$$h(x_+, x_-) \equiv \frac{f(ix_+) + f(-ix_-)}{x_- - x_+}, \quad x_\pm \equiv (e \mp \omega)\tau.$$

We see that the propagator at finite τ is encoded into the key function $f(z)$. It is convenient to compare it to a ‘‘twin’’ Breit-Wigner (BW) function $f_{\text{BW}}(z)$, determined so that $f(z)$ and $f_{\text{BW}}(z)$ have the same values at $z = 0$ and the same $L^2(\mathbb{R})$ norms on the real axis (by which we mean for $z = ix$, $x \in \mathbb{R}$). We find

$$f_{\text{BW}}(z) = \frac{4}{8 - 3z}, \quad \lim_{z \rightarrow 0} f_{\text{BW}}(z) = \lim_{z \rightarrow 0} f(z), \quad \int_{-\infty}^{+\infty} |f_{\text{BW}}(ix)|^2 dx = \int_{-\infty}^{+\infty} |f(ix)|^2 dx. \quad (2.13)$$

The width Γ_{eff} of the twin function $f_{\text{BW}}(z)$ is a good measure of the effective width of the function $f(z)$ on the real axis. We find

$$\Gamma_{\text{eff}} = \frac{16}{3\tau}, \quad f_{\text{BW}}(i(e - \omega)\tau) = \frac{4i}{3\tau(e - \omega + i\frac{\Gamma_{\text{eff}}}{2})}. \quad (2.14)$$

In fig. 1 we compare the square moduli, the real parts and the imaginary parts of $f(z)$ and $f_{\text{BW}}(z)$. We see that their slices on the real axis are similar, although the functions differ a lot in the rest of the complex plane.

It is also possible to approximate the total propagator (2.12) by replacing the function f with the twin BW function f_{BW} . The approximation is good enough when the distance $x_- - x_+$ between the two peaks (the one of the particle and the one of the antiparticle) is large. When $x_- - x_+$ decreases, effects due to the superposition between the two peaks start to become important, although the approximation remains good quantitatively.

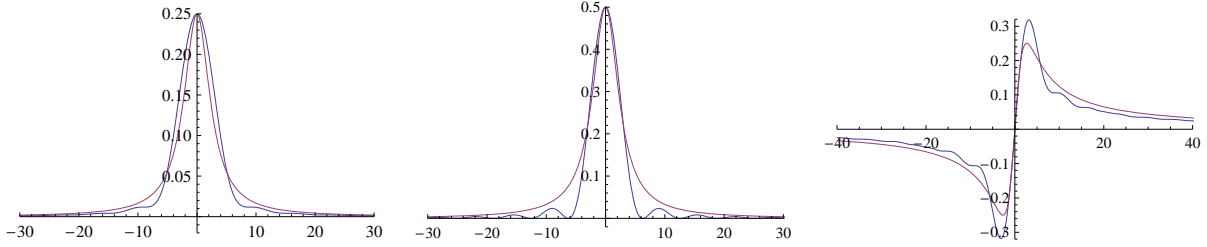


Figure 1: Comparison between $f(z)$ (in blue) and $f_{\text{BW}}(z)$ (in red): square modulus (left), real part (middle) and imaginary part (right)

Now we describe $f(z)$ for generic complex z . We shift $z = ix$ by a real constant a , with x also real, and compare parallel slices $f_a(ix) \equiv f(ix + a)$. The typical behaviors of the real and imaginary parts of $f_a(ix)$ are shown in figure 2, for positive and negative a . We see that the real part is always positive for $a < 0$, but can have both signs for $a > 0$.

For $a < 0$, the function $\text{Re}[f_a(ix)]$ still looks like the real part of a BW function, but with a larger width. The physical meaning of this behavior is explained by the radiative corrections. Specifically, we show that a negative a originates from the resummation of the self-energy diagrams into the dressed propagator, and is ultimately proportional to $-\Gamma$, where Γ is the usual particle width.

3 Dressed propagator

In this section we study the dressed propagator, by resumming the corrections due to the self-energy diagrams.

Let $\Sigma(t, t')$ denote the usual self-energy (at $\tau = \infty$) and $\Sigma_\tau(t, t')$ the one at finite τ . For what we are going to say, it is sufficient to focus on the one-loop corrections in the simplest case, where $-i\Sigma(t, t')$ is the bubble diagram in coordinate space, that is to say, the product of two propagators between the same (non coinciding) points. Then, $\Sigma_\tau(t, t') = \Pi_\tau(t)\Sigma(t, t')\Pi_\tau(t')$.

The dressed propagator $G_{\tau\text{d}}(t, t')$, obtained from the mentioned resummation, reads

$$\begin{aligned}
 G_{\tau\text{d}}(t, t') &= \Pi_\tau(t)\hat{G}_{\tau\text{d}}(t, t')\Pi_\tau(t'), \\
 \hat{G}_{\tau\text{d}}(t, t') &\equiv G(t, t') + \int dt_1 dt_2 G(t, t_1)(-i)\Sigma_\tau(t_1, t_2)G(t_2, t') \\
 &\quad + \int dt_1 dt_2 dt_3 dt_4 G(t, t_1)(-i)\Sigma_\tau(t_1, t_2)G(t_2, t_3)(-i)\Sigma_\tau(t_3, t_4)G(t_4, t') + \dots, \quad (3.1)
 \end{aligned}$$

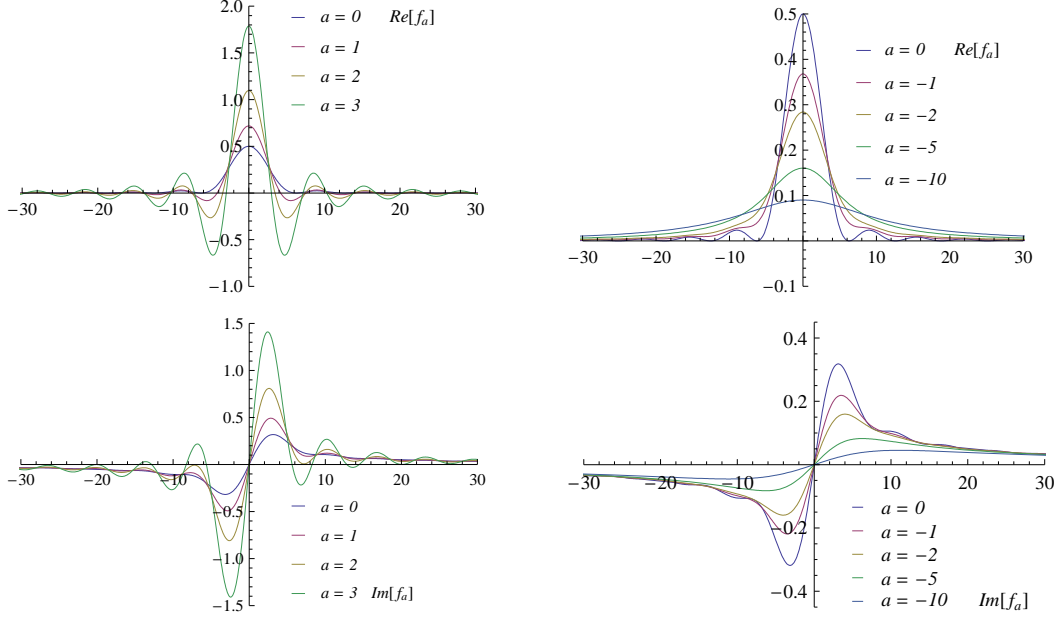


Figure 2: Real and imaginary parts of the function $f_a(x) = f(ix + a)$ for various values of a

where $\hat{G}_{\tau d}(t, t')$ is a sort of unprojected dressed propagator.

We can work out the resummation in two ways, which are equivalent within the approximations we are making here.

In the first method we first show that Σ_τ can be replaced by Σ inside $\hat{G}_{\tau d}(t, t')$. This makes $\hat{G}_{\tau d}(t, t')$ coincide with the usual dressed propagator $G_d(t, t')$ at $\tau = \infty$. Then, $G_{\tau d}(t, t')$ is the projected version of $G_d(t, t')$, which can be worked out as we did in the previous section.

In Fourier transforms, the usual bubble diagram can be approximated by a constant around the peak, which encodes the mass renormalization Δm^2 and the (nonnegative) width Γ :

$$\Sigma(t, t') \simeq \delta(t - t')(\Delta m^2 - im_{\text{ph}}\Gamma), \quad \tilde{\Sigma}(e, e') \simeq (2\pi)\delta(e + e')(\Delta m^2 - im_{\text{ph}}\Gamma), \quad (3.2)$$

where $m_{\text{ph}}^2 \equiv m^2 + \Delta m^2$. We ignore the radiative corrections to the normalization factor Z of the propagator, since we can reinstate Z at a later time. Using the approximation (3.2) as the whole self-energy, the Fourier transform of $\Sigma_\tau(t, t')$ is

$$\tilde{\Sigma}_\tau(e, e') \simeq \frac{2 \sin\left(\frac{e+e'}{2}\tau\right)}{e + e'}(\Delta m^2 - im_{\text{ph}}\Gamma). \quad (3.3)$$

As before, we neglect the energy nonconservation at the vertices, by assuming that τ is large enough so that we can replace the factor in front by $(2\pi)\delta(e + e')$. We obtain $\tilde{\Sigma}_\tau(e, e') \simeq \tilde{\Sigma}(e, e')$, which means that the restriction to finite τ has negligible effects on $\hat{G}_{\tau d}(t, t')$, and we can replace it with $G_d(t, t')$. Then (3.1) gives

$$G_{\tau d}(t, t') = \Pi_\tau(t)G_d(t, t')\Pi_\tau(t'). \quad (3.4)$$

Resumming the geometric series, the Fourier transform of $G_d(t, t')$, which is

$$\tilde{G}_d(e, e') = (2\pi)\delta(e + e')\frac{i}{k^2 - m_{\text{ph}}^2 + im_{\text{ph}}\Gamma}, \quad (3.5)$$

is the same as $\tilde{G}(e, e')$, formula (2.11), with the replacement $m^2 \rightarrow m_{\text{ph}}^2 - im_{\text{ph}}\Gamma$. Then, by comparing the first formula of (2.6) with (3.4), and using (2.12), we conclude that the Fourier transform of $G_{\tau d}(t, t')$ is

$$\tilde{G}_{\tau d}(e, e') \simeq \tilde{G}_\tau(e, e')\Big|_{m^2 \rightarrow m_{\text{ph}}^2 - im_{\text{ph}}\Gamma} = \frac{2\tau^2 \sin\left(\frac{e+e'}{2}\tau\right)}{e + e'} h(x_{+\text{ph}}, x_{-\text{ph}}), \quad (3.6)$$

where

$$\tilde{\omega}_{\text{ph}} = \sqrt{\omega_{\text{ph}}^2 - im_{\text{ph}}\Gamma}, \quad \omega_{\text{ph}} = \sqrt{\mathbf{k}^2 + m_{\text{ph}}^2}, \quad x_{\pm\text{ph}} \equiv (e \mp \tilde{\omega}_{\text{ph}})\tau.$$

We see that we just need to make the replacements

$$\pm ix_\pm \rightarrow \pm ix_{\pm\text{ph}} = \pm ix_\pm + a + ib, \quad a + ib = i\tau(\omega - \tilde{\omega}_{\text{ph}}),$$

inside the functions f , with $a, b \in \mathbb{R}$. While b is a simple translation of x_\pm , the quantity

$$a = \tau\text{Im}[\tilde{\omega}_{\text{ph}}] < 0 \quad (3.7)$$

measures the displacement of the plot profile into the physical half plane. Assuming $\Gamma \ll m_{\text{ph}}$, we have $a \simeq -\tau m_{\text{ph}}\Gamma/(2\omega_{\text{ph}})$. Moreover, $a \simeq -\tau\Gamma/2$ in the static limit.

The total propagator $\tilde{G}_{\tau d}(e, e')$ is described by the function

$$g(x, y, a) = \frac{f(i(x - y) + a) + f(-i(x + y) + a)}{2(y + ia)}, \quad y > -a > 0, \quad (3.8)$$

where $x = (x_+ + x_-)/2 = \tau e$, $y = (x_- - x_+ - 2b)/2 = \omega\tau - b = \tau\text{Re}[\tilde{\omega}_{\text{ph}}]$. The quantity $2y$ is a measure of the separation between the particle peak and the antiparticle peak. It is convenient to choose y large enough, if we want to avoid the superposition of the two. The parameter $-a$ is the extra width due to the radiative corrections. The inequality

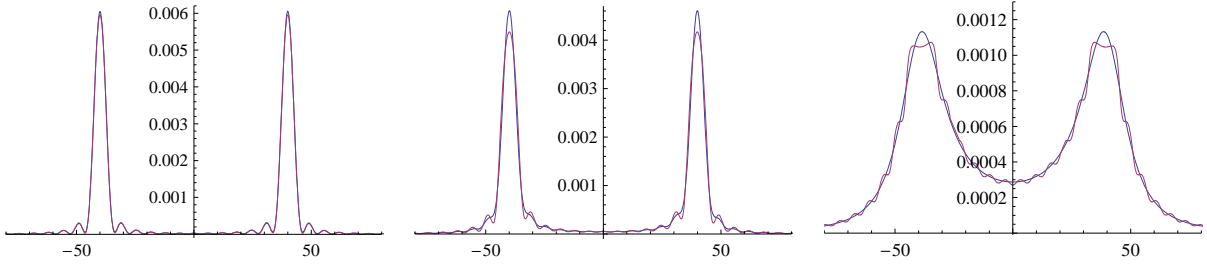


Figure 3: Plots of $\text{Re}[g(x, y, a)]$ (in blue) and $\text{Re}[g'(x, y, a)]$ (in red) for $y = 40$, $a = -1/10$ (left), $a = -1$ (middle) and $a = -10$ (right).

$y + a > 0$ holds because $\tilde{\omega}_{\text{ph}}^2$ lies in the fourth quadrant of the complex plane, so $\text{Re}[\tilde{\omega}_{\text{ph}}] + \text{Im}[\tilde{\omega}_{\text{ph}}] > 0$. It is easy to check that the real part of $g(x, y, a)$ is positive, in agreement with unitarity (see fig. 3).

The second way of resumming the self-energies amounts to working directly on the Fourier transforms, by means of (2.12) and (3.3). For simplicity, we assume $\Delta m^2 = 0$ from now on, since the mass redefinition is not crucial for what we are going to say. We take care of the energy conservation by approximating the factor $(2/(e + e')) \sin((e + e')\tau/2)$ to $(2\pi)\delta(e + e')$ everywhere in the sum, and switching back to the original factor only in the final formula. Then we get a straightforward geometric series, which sums to

$$\frac{(e + e')\tilde{G}'_{\tau\text{d}}(e, e')}{2\tau^2 \sin\left(\frac{e+e'}{2}\tau\right)} \simeq h(x_+, x_-) \sum_{n=0}^{\infty} (-m\Gamma\tau^2 h(x_+, x_-))^n \equiv g'(x, y, a). \quad (3.9)$$

We find

$$g'(x, y, a) = \frac{f(i(x - \bar{y})) + f(-i(x + \bar{y}))}{2\bar{y}(1 + \gamma f(i(x - \bar{y})) + \gamma f(-i(x + \bar{y})))}, \quad (3.10)$$

where

$$x = \tau e, \quad \gamma = \frac{\tau m \Gamma}{2\omega} = -\frac{ay}{\bar{y}}, \quad \bar{y} = \tau\omega = \sqrt{y^2 - a^2}.$$

In fig. 3 we compare $\text{Re}[g(x, y, a)]$ to $\text{Re}[g'(x, y, a)]$ for $y = 40$, $a = -1/10$, -1 and -10 . We see that the approximation (3.9) captures the effects of the restriction to finite τ much better when $|a|$ is large, while the approximation (3.8) tends to smear them out. It is also easy to show that the real parts are not positive when a is positive.

We can estimate the total effective width Γ_{tot} of the dressed propagator by means of twin BW approximations, obtained by replacing f with the function f_{BW} of (2.13) inside (3.8) or (3.10). We assume that y and \bar{y} are large, to avoid superpositions between the particle and antiparticle peaks. Making the replacement $f \rightarrow f_{\text{BW}}$ in $g(x, y, a)$, the shift

$x \rightarrow x - ia$ in (2.14) gives

$$\Gamma_{\text{tot}} \simeq \Gamma_{\text{eff}} - \frac{2a}{\tau} = \frac{16}{3\tau} - 2\text{Im}[\tilde{\omega}_{\text{ph}}] \simeq \frac{16}{3\tau} + \frac{m_{\text{ph}}\Gamma}{\omega_{\text{ph}}} \simeq \frac{16}{3\tau} + \Gamma, \quad (3.11)$$

the last-but-one approximation being for $\Gamma \ll m_{\text{ph}}$, and the last one being at rest.

These results prove that the radiative corrections generate a shift into the physical half plane. The effective width Γ_{eff} of the free propagator, due to the restriction to finite τ , is enlarged to the total width Γ_{tot} of the dressed propagator by an amount proportional to the usual width Γ at $\tau = \infty$.

4 Purely virtual particles

In this section we study purely virtual particles χ , taking $\Delta m^2 = 0$ again for simplicity. The diagrams we are considering do not have χ legs inside loops (the self-energy Σ being treated as a whole), so the χ free propagator is (see, for example, [2])

$$G_{\text{pv}}^+(t, t') = \frac{\theta(t - t') - \theta(t' - t)}{2} \frac{e^{-i\omega(t-t')}}{2\omega},$$

In Fourier transforms, it coincides with the principal value of the first pole of (2.3). Working out the Fourier transform $\tilde{G}_{\tau\text{pv}}^+(e, e')$ of $G_{\tau\text{pv}}^+(t, t') \equiv \Pi_\tau(t)G_{\text{pv}}^+(t, t')\Pi_\tau(t')$, defined as in formula (2.7), with $t_f = -t_i = \tau/2$, we find the result (2.9) with $f(z)$ replaced by

$$f_{\text{pv}}(z) = \frac{\sinh(z) - z}{z^2}.$$

Hence, by (2.12), the total propagator reads

$$\tilde{G}_{\tau\text{pv}}(e, e') \simeq \frac{2\tau^2 \sin\left(\frac{e+e'}{2}\tau\right)}{e+e'} h_{\text{pv}}(x_+, x_-), \quad h_{\text{pv}}(x_+, x_-) \equiv \frac{f_{\text{pv}}(ix_+) + f_{\text{pv}}(-ix_-)}{x_- - x_+}.$$

The key function is now

$$f_{\text{pv}}(ix) = i \frac{x - \sin(x)}{x^2},$$

which satisfies the important bound

$$|f_{\text{pv}}(ix)| \leq \frac{1}{\pi}. \quad (4.1)$$

As in (3.9), the dressed propagator is a geometric series

$$\frac{(e+e')\tilde{G}_{\tau\text{d}}^{\text{pv}}(e, e')}{2\tau^2 \sin\left(\frac{e+e'}{2}\tau\right)} \simeq h_{\text{pv}}(x_+, x_-) \sum_{n=0}^{\infty} (-m\Gamma\tau^2 h_{\text{pv}}(x_+, x_-))^n \equiv g'_{\text{pv}}(x, y, a), \quad (4.2)$$

but we cannot resum it without checking its actual convergence. The reason is that the prescription for purely virtual particles is not analytic [9, 10], so we cannot advocate analyticity to justify the continuation of the sum from its convergence domain to the rest of the complex plane, as we normally do for physical particles.

The bound (4.1) ensures that there is a situation where the series is always convergent (on the real axis). It occurs when the quantity raised to the power n in the sum of (4.2) has a modulus that is always smaller than 1. In turn, this requires $\gamma = \tau m \Gamma / (2\omega) < \pi/2$, which is true for every energy e and every frequency ω , if $\tau \Gamma < \pi$. Thus, it is sufficient to assume

$$\Delta E \equiv \frac{\pi}{2\tau} > \frac{\Gamma}{2}, \quad (4.3)$$

to obtain

$$g'_{\text{pv}}(x, y, a) = \frac{f_{\text{pv}}(i(x - \bar{y})) + f_{\text{pv}}(-i(x + \bar{y}))}{2\bar{y}(1 + \gamma f_{\text{pv}}(i(x - \bar{y})) + \gamma f_{\text{pv}}(-i(x + \bar{y})))}. \quad (4.4)$$

When \bar{y} is large, the “particle” and “antiparticle” contributions separate well enough, and we can write

$$g'_{\text{pv}}(x, y, a) \simeq \frac{\tilde{f}_{\text{pv}}(x - \bar{y}, \gamma) + \tilde{f}_{\text{pv}}(x + \bar{y}, \gamma)}{2\bar{y}}, \quad \tilde{f}_{\text{pv}}(x, \gamma) = \frac{f_{\text{pv}}(ix)}{1 + \gamma f_{\text{pv}}(ix)}.$$

It is easy to prove that the twin peaks of $\text{Re}[\tilde{f}_{\text{pv}}(x, \gamma)]$ occur at $x = \pm\pi$, and have $\text{Re}[\tilde{f}_{\text{pv}}(\pm\pi, \gamma)] = \gamma/(\pi^2 + \gamma^2)$. Thus, γ (hence Γ) is related to the heights of the peaks, while their positions are universal. The stationary points of $\text{Re}[\tilde{f}_{\text{pv}}(x, \gamma)]$ are at $x = n\pi$, $n = \text{odd}$, and have heights $\gamma/(n^2\pi^2 + \gamma^2)$.

In fig. 4 we compare the properties of physical and purely virtual particles through the functions $\text{Re}[g'(x, y, a)]$ and

$$\text{Re}[g'_{\text{pv}}(x, y, a)] = g_{\text{pv}}^*(x, y, a)(m\tau^2\Gamma)g'_{\text{pv}}(x, y, a), \quad (4.5)$$

taking $\bar{y} = 30$ and $\gamma = 3/2$. The right-hand side of expression (4.5) looks like the decay rate of the purely virtual particle χ , because it is the product of the propagator, times (minus the real part of) the bubble diagram, times the conjugate propagator (times a further factor τ^2 , introduced for convenience). Since χ does not exist on the mass shell, the expression “decay rate” just refers to the existence of a channel mediated by it.

Fig. 4 includes the plot of the convolution

$$\int_{-\infty}^{+\infty} du \frac{4}{(4u)^2 + \pi^2} \text{Re}[g'_{\text{pv}}(x - u, y, a)] \quad (4.6)$$

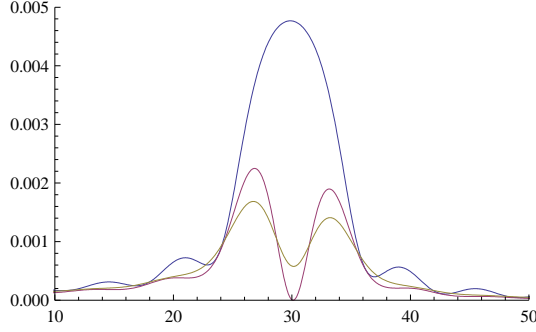


Figure 4: Comparison between the dressed propagators $\text{Re}[g'(x, y, a)]$ and $\text{Re}[g'_{\text{pv}}(x, y, a)]$ of physical (in blue) and purely virtual (in red and green) particles in a situation ($\gamma = 3/2$) where we can trust the resummation for both. We have taken $\bar{y} = 30$. The green plot is the convolution (4.6)

with a Lorentzian function of width $\pi/2 = \Delta x = \tau \Delta E$. The convolution is useful if we want to interpret ΔE as the resolving power of our instrumentation on the energy.

Since we are working in a finite interval of time τ , we are implying that the resolving power Δt on time itself is better than that: $\Delta t < \tau$. Then, the energy-time uncertainty relation $\Delta t \Delta E \gtrsim 1$ tells us that the uncertainty ΔE on the energy is bigger than $\sim 1/\tau$. The best situation is when that uncertainty is close to its minimum value, which is approximately equal to the ΔE defined in formula (4.3). Thus, we can view the condition (4.3) as a condition on the resolving power on the energy. If we resolve the energies too well, we cannot have enough precision in time to claim that we are working in a finite interval τ . The convoluted profile of fig. 4 is probably closer to what we can see experimentally.

The plots show rather different phenomenological behaviors: while physical particles exhibit the usual peak, purely virtual particles show two smaller humps. What is important is that the difference between the two cases is experimentally testable, at least in principle.

Qualitatively, we may expect similar differences at $\tau = \infty$. However we cannot make this statement rigorous, because when τ grows we eventually violate (4.3) and enter a nonperturbative region, where we cannot trust the resummation for purely virtual particles. It may also be possible that the condition (4.3) turns into an uncertainty relation of new type [1], a “peak uncertainty” $\Delta E > \Gamma/2$, telling us that, when we approach the peak $k^2 = m^2$ of a purely virtual particle too much, identical experiments may give different results.

5 The problem of the muon (unstable particles vs resonances)

In this section we study the problem of describing the muon decay in quantum field theory. Since the muon is unstable, the right framework is not the one at $\tau = \infty$, because a too large τ gives the muon enough time to decay and, strictly speaking, makes it unobservable. If we ignore this fact and insist on describing the muon decay at $\tau = \infty$, quantum field theory retaliates by generating mathematical inconsistencies [1].

The point is that we are demanding something that violates the uncertainty principle: as stressed before, if we want to resolve a finite time (the muon lifetime in this case), we must have a finite time uncertainty Δt , which needs a nontrivial uncertainty ΔE on the energy. There are no such things in quantum field theory at $\tau = \infty$. On the other hand, a finite τ implies a finite time uncertainty $\Delta t < \tau$, so quantum field theory on a finite time interval is better equipped to address the problem we are considering. Moreover, if we want to be able to see the muon, we must have $\tau < \omega\tau_\mu/m$, where $\tau_\mu = 1/\Gamma_\mu$ is the muon lifetime at rest and ω/m is the boost factor, which is crucial to make the muon live longer.

It is important that we separate the observation of the particle before it decays, from the observation of its decay products. This can be done as explained in ref. [1], and amounts to writing the real part of

$$g'(x, y, a) = \frac{h(x_+, x_-)}{1 + 2\gamma\bar{y}h(x_+, x_-)}, \quad (5.1)$$

from formula (3.10), as the sum $\text{Re}[g'] = (\Omega_{\text{particle}} + \Omega_{\text{decay}})/\tau^2$ of two terms, one capturing the particle observation, and the other one capturing its decay. The factor $1/\tau^2$ takes care of the analogous factor appearing on the left-hand side of (3.9). We find

$$\frac{\Omega_{\text{particle}}}{\tau^2} = \frac{h_1}{|1 + 2\gamma\bar{y}h|^2}, \quad \frac{\Omega_{\text{decay}}}{\tau^2} = \frac{2\gamma\bar{y}|h|^2}{|1 + 2\gamma\bar{y}h|^2} = g'^*(m\Gamma)g'. \quad (5.2)$$

where h stands for $h(x_+, x_-)$ and h_1 is its real part. The structure of the last expression matches the one of (4.5), while the first expression has no analogue in the case of purely virtual particles (which admit no particle observation, by definition, but just a “decay” channel).

We want to study Ω_{particle} and Ω_{decay} in two limiting situations of physical interest: unstable particles and resonances.

Although we just need to take τ smaller than the boosted muon lifetime $\omega\tau_\mu/m$, it is convenient to take $\tau \ll \omega\tau_\mu/m$, both because it is realistic to do so, but also because it simplifies the results. Furthermore, in all the colliders built, or planned, so far, the muon mass m_μ is much larger than the resolving power on the energy, so we may assume

$$m_\mu \gg \frac{1}{\tau} \gg \frac{m}{\omega\tau_\mu} = \frac{m\Gamma_\mu}{\omega}. \quad (5.3)$$

It is easy to prove the inequality $|h(x_+, x_-)| \leq 1/(2\bar{y})$, which implies that under the assumptions (5.3), the denominator of $g'(x, y, a)$ in (5.1) can be approximated to one, and the function $g'(x, y, a)$ can be approximated to its free value $h(x_+, x_-)$ (which implies $\Omega_{\text{decay}} \rightarrow 0$).

Furthermore, the conditions (5.3) imply $\bar{y} \gg 1$. Then it is easy to prove⁴, from the second limit of (2.10), that

$$\Omega_{\text{particle}} \simeq \pi\delta(e^2 - \omega^2). \quad (5.4)$$

In other words, Ω_{particle} tends to the delta function that describes the muon observation, while Ω_{decay} tends to zero.

Note that we have not taken τ to infinity to prove this result. Actually, it is impossible to obtain it by working at $\tau = \infty$ [1], because in that case (2.11) tells us that

$$\tau^2 h(x_+, x_-) \rightarrow \frac{i}{e^2 - \omega^2 + i(\epsilon + m_\mu\Gamma_\mu)},$$

which means

$$\Omega_{\text{particle}} \rightarrow \frac{\epsilon}{(e^2 - \omega^2)^2 + (\epsilon + m_\mu\Gamma_\mu)^2}, \quad \Omega_{\text{decay}} \rightarrow \frac{m_\mu\Gamma_\mu}{(e^2 - \omega^2)^2 + (\epsilon + m_\mu\Gamma_\mu)^2}. \quad (5.5)$$

Since Γ_μ is nonzero and ϵ is a mathematical artifact, we get $\Omega_{\text{particle}} \rightarrow 0$, while Ω_{decay} tends to the Breit-Wigner function of a resonance. Normally, people confuse Ω_{particle} and Ω_{decay} , and say that, because the muon width Γ_μ is very small, one can let it tend to zero in $\Omega_{\text{decay}}|_{\epsilon=0}$, which gives $\pi\delta(e^2 - \omega^2)$. However, the desired delta function should not come from Ω_{decay} (it would be like resuscitating the muon by making it eternal after its decay): the correct $\pi\delta(e^2 - \omega^2)$ must come from Ω_{particle} . This can happen only at $\tau < \infty$, as in (5.4).

⁴We need to take $|\tau\epsilon|$ large and comparable to $\bar{y} = \tau\omega$, otherwise we miss the delta function support. Basically, we are rescaling e and ω by a common factor, and letting it tend to infinity. At the same time, we keep τ fixed.

In the case of a resonance, like the Z boson, there is no reason why we should keep τ finite, since in all the experiments of collider physics, so far, the Z lifetime τ_Z is much larger than the interval τ separating the incoming particles from the outgoing ones (we are very far from observing the Z boson directly):

$$m_Z > \Gamma_Z = \frac{1}{\tau_Z} \gg \frac{1}{\tau}. \quad (5.6)$$

This means that we can use the formulas (5.5) with $\mu \rightarrow Z$, where Ω_{particle} correctly gives zero, while Ω_{decay} tends to the right Breit-Wigner formula.

6 Conclusions

We have studied the propagators of physical and purely virtual particles in quantum field theory in a finite interval of time τ , and on a compact manifold Ω . In the free-field limit, the typical pole $1/z$ is replaced by the entire function $f(z) = (e^z - 1 - z)/z^2$. The shape of the latter on the real axis $z = ix$ reminds the one of a Breit-Wigner function, with an effective width equal to $16/(3\tau)$. The two functions are very different in the rest of the complex plane.

When we include the radiative corrections, the key function remains $f(z)$, but it is shifted into the physical half plane. The width is enlarged by an amount equal to Γ (the usual width at $\tau = \infty$). The real part of the propagator is always positive, in agreement with unitarity.

We have studied the case of purely virtual particles, and showed that, for τ small enough ($\tau < \pi/\Gamma$), there is an arrangement where the geometric series of the self-energies is always convergent. The key reason is that the function $f(z)$ is bounded on the real axis and on the physical half plane. In that situation, it is possible to rigorously resum the series into the dressed propagator, and compare the result with what we find in the case of physical particles. The plots differ in ways that can in principle be tested: physical particles are characterized by the usual, single peak; instead, purely virtual particles are characterized by two twin peaks, which are separated from one another in a universal way, and have heights that depend on the width of the particle.

Finally, we have investigated the effects of the restriction to finite τ on the problem “muon vs Z boson” (i.e., unstable particles vs resonances). It is crucial to work at $\tau < \infty$, if we want to properly explain the observation of an unstable particle. Once we do that, the muon observation emerges naturally from the right physical process. In particular,

there is no need to confuse the observation of a particle with the observation of its decay products, and pretend that the particle resuscitates after its decay (which is basically how one normally adjusts the matter by sticking to $\tau = \infty$). The results confirm those argued in ref. [1] on general grounds.

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