Purely Virtual Extension of Quantum Field Theory for Gauge Invariant Fields: Quantum Gravity

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Abstract

Quantum field theory is extended to include purely virtual "cloud sectors", which allow us to define point-dependent observables, including a gauge invariant metric and gauge invariant matter fields, and calculate their off-shell correlation functions perturbatively in quantum gravity. Each extra sector is made of a cloud field, its anticommuting partner, a cloud function and a cloud Faddeev-Popov determinant. Thanks to certain cloud symmetries, the ordinary correlation functions and S matrix elements are unmodified. The clouds are rendered purely virtual, to ensure that they do not propagate unwanted degrees of freedom. So doing, the off-shell, diagrammatic version of the optical theorem holds and the extended theory is unitary. Every insertion in a correlation function can be dressed with its own cloud. The one-loop two-point functions of dressed scalars, vectors and gravitons are calculated. Their absorptive parts are positive, cloud independent and gauge independent, while they are unphysical if non purely virtual clouds are used. Renormalizability is proved to all orders by means of an extended Batalin-Vilkovisky formalism and its Zinn-Justin master equations. The purely virtual approach is compared to other approaches available in the literature.

1 Introduction

Defining point-dependent observables in general relativity is tricky, because the coordinates are not physical quantities, but just parametrizations of the location. A simple way out is available when the spacetime point is associated with a matter distribution. Consider, for definiteness, four scalar fields $\phi^i(x)$, $i = 1, \ldots 4$, and assume that they depend on the coordinates x^μ in such a way that it is possible to invert x^μ as functions $x^\mu(\phi)$ of ϕ^i. Then, every further field, say a fifth scalar $\varphi(x)$, can be written as a function of the reference fields ϕ^i: $\varphi(x) \rightarrow \varphi(x(\phi)) \equiv \tilde{\varphi}(\phi)$. The function $\tilde{\varphi}(\phi)$ is obviously invariant under general changes of coordinates. The basic idea is to go back to the physical location every time we change coordinates. The ϕ^i do not need to be new, independent fields, but can be functions of the metric itself.

This line of thinking has been pursued in the literature for a long time [1, 2, 3, 4, 5]. In refs. [2] Komar and Bergmann use functions of the metric. In refs. [3, 4] the fields ϕ^i describe physical matter. Donnelly and Giddings [5] view them as functions of the metric, for purposes similar to the Coulomb-Dirac dressing of QED [6], the Lavelle-McMullan dressing of non-Abelian gauge theories [7], the worldline dressing and the Wilson lines.

Ultimately, the presence of the observer, which is "matter", is what breaks general covariance, so we may want to view the reference fields as independent matter. In this spirit, we have to take into account that the fundamental theory is changed by the presence of the fields ϕ^i. In ref. [4] Rovelli questions the need to change the physical world for this purpose and proposes an improvement inspired by the GPS technology, based on a minimal amount of additional matter. Yet, one still needs to provide the physics of the additional matter and add it to the physics of the fundamental theory.

Viewing the reference fields as functions of the metric is more appealing from the conceptual point of view, since it does not force us to leave the realm of pure gravity. These problems are more challenging at the fundamental level, especially in quantum gravity, where they concern our understanding of the fundamental physics of nature.

In this paper we pursue a new strategy, which may shed a different light on the issue. We introduce purely virtual independent fields ζ^μ, which we call cloud fields, and their anticommuting partners H^μ. We extend quantum field theory to include such fields perturbatively in quantum gravity (and general relativity) without affecting the fundamental laws of physics.

The cloud fields play the roles of ϕ^i. Precisely, the fields ϕ^i should be imagined as the differences $x^\mu - \zeta^\mu(x)$. An arrangement of this type also appears in [5]. In our approach, however, the ζ^μ are neither additional matter, nor functions of the metric, but independent fields, with their own (higher-derivative) propagators, and their own interactions. Moreover, they must be accompanied by anticommuting partners H^μ in a suitable way, and rendered purely virtual, because only in that case the fundamental physics does not change, and no extra degrees of freedom (which could include ghosts) are propagated.

Because they are independent fields, ζ^μ and H^μ might be viewed as a sort of matter. Then, however, their purely virtual nature makes them "fake matter". Because they are introduced to be ultimately projected away (that is to say, integrated out), they might be understood as "functions of everything else", at least in some particular cases (like the classical limit). Nevertheless, they cannot be viewed as functions of the other fields beyond the tree level, since they keep circulating in loops. In view of these remarks, it is better to understand ζ^μ and H^μ as new entities, defined by the very same formalism we develop in the paper.

The cloud fields ζ^μ are used to surround the elementary fields of the theory, such as the metric tensor, with appropriate dressings, in order to render them invariant under infinitesimal changes of coordinates. The anticommuting partners H^μ are used to endow the extended theory with a certain cloud symmetry, to ensure that the fundamental interactions are unaffected by the presence of the cloud sectors. Specifically: *i*) the correlation functions of the undressed fields are unchanged, and *ii*) the S matrix amplitudes of the dressed fields coincide with the usual S matrix amplitudes of the undressed fields. Once these goals are achieved, we can view the usual (undressed) fields as mere integration and diagrammatic tools, and use dressed fields everywhere else. This way, gauge invariance and gauge independence become manifest in every operation we make.

Note that the dressed metric just propagates the two graviton helicities. In the approaches of refs. [3, 4] it may propagate six degrees of freedom (the additional ones coming from the four reference scalars).

We achieve the goals we have stated in a fully perturbative regime. By construction, the extended theory is local and unitary. Moreover, it is renormalizable (if the underlying gravity theory is renormalizable), up to the cloud sectors, which may be nonrenormalizable due to their arbitrariness. Although the observables that we define are invariant under infinitesimal changes of coordinates, they are not necessarily invariant under global changes of coordinates. From the conceptual and physical points of view, this is what we need: we break the global symmetry (our observations do that most of the times) without violating unitarity.

In a parallel paper [8], we explore similar issues in gauge theories.

The notion of pure virtuality relies on a new diagrammatics, which takes advantage of the possibility of splitting the usual optical theorem [9] into independent, algebraic spectral optical identities, associated with different (multi)thresholds [10]. It is possible to remove degrees of freedom at all energies, while preserving the optical theorem in a manifest way, by removing subsets of such identities. The removed degrees of freedom can be understood as fake particles, or “fakeons”. The main application of this idea is the formulation of a consistent theory of quantum gravity [11], which leads to observationally testable predictions in inflationary cosmology [12]. At the phenomenological level, fakeons evade common constraints that preclude the usage of normal particles [13, 14].

The idea of pure virtuality has other ramifications, which we do not recall here. The approach of this paper can offer a better understanding of the physics that lies beyond the realm of scattering processes in quantum field theory, and provides an anwser to the issue of finding a complete set of observables in quantum gravity.

Throughout the paper we work with the dimensional regularization [15], $\varepsilon = 4 - D$ denoting the difference between the physical dimension and the continued one.

The paper is organized as follows. In section 2 we introduce the fields we need to build the cloud sectors. In section 3 we recall the standard Batalin-Vilkovisky formalism for gravity and the Zinn-Justin master equation. In section 4 we define the cloud sector. In section 5 we show that the ordinary correlation functions are unaffected by the cloud sector, and so are the S matrix amplitudes. In section 6 we build the correlation functions of the dressed fields. In section 7 we prove that the gauge-trivial sector of the theory and the cloud sector are mirrored into one another by a certain duality relation. In section 8 we add several copies of the could sector and show that each insertion, in a correlation function, can be dressed with its own, independent cloud. In sections 9 and 10 we compute the one-loop two-point functions of the dressed scalars, vectors and gravitons. We do so in Einstein gravity and in quantum gravity with purely virtual particles. In section 9 we work with covariant clouds, and show that the absorptive parts are unphysical. In section 10 we turn to purely virtual clouds, and show that the absorptive parts are then physical. In section 11 we prove that the extended theory is renormalizable. Section contains 12 the conclusions.

2 The cloud field, its anticommuting partner, and the dressed fields

In this section we lay out the basic notions that are needed to build the cloud sectors. Let

$$\delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \xi^\rho + g_{\nu\rho}\partial_\mu \xi^\rho \tag{2.1}$$

denote the transformation of the metric tensor $g_{\mu\nu}$ under infinitesimal changes of coordinates $\delta x^\mu = -\xi^\mu(x)$. The closure relations read

$$[\delta_\xi, \delta_\eta] g_{\mu\nu} = \delta_{[\xi,\eta]} g_{\mu\nu}, \qquad [\xi, \eta]^\rho \equiv \eta^\sigma \partial_\sigma \xi^\rho - \xi^\sigma \partial_\sigma \eta^\rho. \tag{2.2}$$

We define the basic cloud field as an independent "vector" $\zeta^\mu(x)$ that transforms according to the rule

$$\delta_\xi \zeta^\mu(x) = \xi^\mu(x - \zeta(x)). \tag{2.3}$$

In the next sections we explain how to include ζ^μ into the action. For the moment, we just study its properties. A relation like (2.3) and similar ones below can be meant as expansions in powers of ζ. From now on, we understand that the argument of a function is x, whenever it is not specified.

It is easy to check that the definition (2.3) is meaningful, since it closes:

$$[\delta_\xi, \delta_\eta] \zeta^\mu = -\xi^\rho(x-\zeta)\eta^\mu_{,\rho}(x-\zeta) + \eta^\rho(x-\zeta)\xi^\mu_{,\rho}(x-\zeta) = \delta_{[\xi,\eta]}\zeta^\mu, \tag{2.4}$$

where $X_{,\mu} \equiv \partial_\mu X$. To avoid a certain confusion that may arise when the argument of a function is $x - \zeta(x)$, we need to pay attention to the notation. An expression line $\partial_\rho X^\mu(x-\zeta)$ is ambiguous, because the total derivative acts on the x dependence inside ζ^ν, while the partial derivative is not supposed to. We have

$$\partial_\rho(X^\mu(x-\zeta)) = X^\mu_{,\rho}(x-\zeta) - \zeta^\sigma_{,\rho} X^\mu_{,\sigma}(x-\zeta). \tag{2.5}$$

The point-dependent dressed fields are then

$$\begin{aligned} \varphi_{\rm d}(x) &= \varphi(x-\zeta(x)), \qquad A_{\mu{\rm d}}(x) = A_\mu(x-\zeta) - \zeta^\nu_{,\mu} A_\nu(x-\zeta), \\ g_{\mu\nu{\rm d}}(x) &= g_{\mu\nu}(x-\zeta) - \zeta^\rho_{,\mu} g_{\nu\rho}(x-\zeta) - \zeta^\rho_{,\nu} g_{\mu\rho}(x-\zeta) + \zeta^\rho_{,\mu}\zeta^\sigma_{,\nu} g_{\rho\sigma}(x-\zeta), \end{aligned} \tag{2.6}$$

for scalars, vectors and the metric, respectively. Indeed, using the Taylor expansion of $\varphi(x-\zeta)$, it is easy to check that $\delta_\xi \varphi = \xi^\rho \varphi_{,\rho}$ implies

$$\delta_\xi \varphi_{\rm d}(x) = \xi^\rho(x-\zeta)\varphi_{,\rho}(x-\zeta) - \delta_\xi \zeta^\rho \varphi_{,\rho}(x-\zeta) = 0.$$

Moreover, $\delta_\xi A_\mu = \xi^\rho A_{\mu,\rho} + A_\rho \xi^\rho_{,\mu}$ implies

$$\delta_\xi A_{\mu{\rm d}}(x) = A_\rho(x-\zeta)\xi^\rho_{,\mu}(x-\zeta) - \zeta^\nu_{,\mu} A_\rho(x-\zeta)\xi^\rho_{,\nu}(x-\zeta) - A_\rho(x-\zeta)\partial_\mu(\xi^\rho(x-\zeta)) = 0,$$

where we have used (2.5) in the last step with $X^\mu = \xi^\mu$. Similarly, (2.1) implies $\delta_\xi g_{\mu\nu{\rm d}}(x) = 0$.

Generically, if $T_{\mu_1\cdots\mu_n}(x)$ is a tensor, its gauge invariant, dressed version is

$$T_{\mu_1\cdots\mu_n\mathrm{d}}(x)=(\delta_{\mu_1}^{\nu_1}-\zeta_{,\mu_1}^{\nu_1})\cdots(\delta_{\mu_n}^{\nu_n}-\zeta_{,\mu_n}^{\nu_n})T_{\nu_1\cdots\nu_n}(x-\zeta). \tag{2.7}$$

We can also define dual fields, which allow us to raise and lower the indices and invert the definitions of the dressed fields given above. The dual cloud field $\tilde{\zeta}^{\mu}(x)$ is defined as the solution of the equation

$$\tilde{\zeta}^{\mu}(x)=-\zeta^{\mu}(x-\tilde{\zeta}(x)), \tag{2.8}$$

which can be worked out recursively by expanding in powers of ζ^{μ}:

$$\tilde{\zeta}^{\mu}(x)=-\zeta^{\mu}(x+\zeta(x+\zeta(x+\zeta(x+\cdots)))).$$

Differentiating (2.8), we find

$$\left[\delta_{\rho}^{\mu}-\zeta_{,\rho}^{\mu}(x-\tilde{\zeta}(x))\right]\left[\delta_{\nu}^{\rho}-\tilde{\zeta}_{,\nu}^{\rho}(x)\right]=\delta_{\nu}^{\mu}. \tag{2.9}$$

Using this identity and (2.3) we derive the infinitesimal transformation of $\tilde{\zeta}^{\mu}$, which reads

$$\delta_{\xi}\tilde{\zeta}^{\mu}(x)=-(\delta_{\nu}^{\mu}-\tilde{\zeta}_{,\nu}^{\mu}(x))\xi^{\nu}(x). \tag{2.10}$$

It is straightforward to check its closure.

The inverse relations are

$$\begin{aligned}\varphi(x)&=\varphi_{\mathrm{d}}(x-\tilde{\zeta}(x)),\qquad A_{\mu}(x)=A_{\mu\mathrm{d}}(x-\tilde{\zeta})-\tilde{\zeta}_{,\mu}^{\nu}A_{\nu\mathrm{d}}(x-\tilde{\zeta}),\\ g_{\mu\nu}(x)&=g_{\mu\nu\mathrm{d}}(x-\tilde{\zeta})-\tilde{\zeta}_{,\mu}^{\rho}g_{\nu\rho\mathrm{d}}(x-\tilde{\zeta})-\tilde{\zeta}_{,\nu}^{\rho}g_{\mu\rho\mathrm{d}}(x-\tilde{\zeta})+\tilde{\zeta}_{,\mu}^{\rho}\tilde{\zeta}_{,\nu}^{\sigma}g_{\rho\sigma\mathrm{d}}(x-\tilde{\zeta}),\\ T_{\mu_1\cdots\mu_n}(x)&=(\delta_{\mu_1}^{\nu_1}-\tilde{\zeta}_{,\mu_1}^{\nu_1})\cdots(\delta_{\mu_n}^{\nu_n}-\tilde{\zeta}_{,\mu_n}^{\nu_n})T_{\nu_1\cdots\nu_n\mathrm{d}}(x-\tilde{\zeta}).\end{aligned} \tag{2.11}$$

Observe that (2.8) implies

$$x^{\mu}-\zeta^{\mu}(x)|_{x\rightarrow x-\tilde{\zeta}}=x^{\mu}-\tilde{\zeta}^{\mu}(x)-\zeta^{\mu}(x-\tilde{\zeta}(x))=x^{\mu}.$$

Defining $y^{\mu}=x^{\mu}-\tilde{\zeta}^{\mu}(x)$ and relabelling $x\leftrightarrow y$, we find the dual identity

$$\zeta^{\mu}(x)=-\tilde{\zeta}^{\mu}(x-\zeta(x)). \tag{2.12}$$

Differentiating this relation, we also find

$$\left[\delta_{\rho}^{\mu}-\tilde{\zeta}_{,\rho}^{\mu}(x-\zeta)\right]\left[\delta_{\nu}^{\rho}-\zeta_{,\nu}^{\rho}(x)\right]=\delta_{\nu}^{\mu}. \tag{2.13}$$

Vectors and tensors with upper indices are dressed as follows:

$$\begin{aligned}A_{\mathrm{d}}^{\mu}(x)&=A^{\mu}(x-\zeta)-\tilde{\zeta}_{,\nu}^{\mu}(x-\zeta)A^{\nu}(x-\zeta),\\ g_{\mathrm{d}}^{\mu\nu}(x)&=(\delta_{\rho}^{\mu}-\tilde{\zeta}_{,\rho}^{\mu}(x-\zeta))(\delta_{\sigma}^{\nu}-\tilde{\zeta}_{,\sigma}^{\nu}(x-\zeta))g^{\rho\sigma}(x-\zeta),\\ T_{\mathrm{d}}^{\mu_1\cdots\mu_n}(x)&=(\delta_{\nu_1}^{\mu_1}-\tilde{\zeta}_{,\nu_1}^{\mu_1}(x-\zeta))\cdots(\delta_{\nu_n}^{\mu_n}-\tilde{\zeta}_{,\nu_n}^{\mu_n}(x-\zeta))T^{\nu_1\cdots\nu_n}(x-\zeta).\end{aligned} \tag{2.14}$$

Indeed, (2.9) with $x \to x - \zeta(x)$ ensures that

$$A^\mu_{\rm d}(x)A_{\mu{\rm d}}(x) = (\delta^\mu_\nu - \tilde{\zeta}^\mu_{,\nu}(x-\zeta))A^\nu(x-\zeta)(\delta^\rho_\mu - \zeta^\rho_{,\mu})A_\rho(x-\zeta) = A^\mu(x-\zeta)A_\mu(x-\zeta),$$

as required for a scalar. Similarly, $g^{\mu\nu}_{\rm d}(x)g_{\nu\rho{\rm d}}(x) = \delta^\mu_\rho$. Moreover, the behaviors of upper indices under infinitesimal transformations, as in $\delta_\xi A^\mu = \xi^\rho A^\mu_{,\rho} - A^\rho \xi^\mu_{,\rho}$, imply that the fields (2.14) are invariant. For example,

$$\delta_\xi A^\mu_{\rm d}(x) = -(\delta^\mu_\nu - \tilde{\zeta}^\mu_{,\nu}(x-\zeta))\xi^\nu_{,\rho}(x-\zeta)A^\rho(x-\zeta) + (\delta^\mu_\nu - \tilde{\zeta}^\mu_{,\nu}(x-\zeta))\xi^\nu_{,\rho}(x-\zeta)A^\rho(x-\zeta) = 0.$$

Here we have used

$$\delta_\xi \tilde{\zeta}^\mu_{,\rho}(x-\zeta) = -(\delta^\mu_\nu - \tilde{\zeta}^\mu_{,\nu}(x-\zeta))\xi^\nu_{,\rho}(x-\zeta),$$

which follows from (2.10).

It is also crucial to introduce anticommuting partners H^μ of ζ^μ, defined by the transformation law

$$\delta_\xi H^\mu = -H^\nu \xi^\mu_{,\nu}(x-\zeta). \tag{2.15}$$

The consistency of this transformation follows from its closure:

$$\begin{aligned}[\delta_\xi, \delta_\eta]H^\mu &= H^\rho \xi^\nu_{,\rho}(x-\zeta)\eta^\mu_{,\nu}(x-\zeta) + H^\nu \xi^\rho(x-\zeta)\eta^\mu_{,\nu\rho}(x-\zeta) - (\xi \leftrightarrow \eta) \\ &= -H^\rho(x)(\partial_\rho(\eta^\nu \xi^\mu_{,\nu} - \xi^\nu \eta^\mu_{,\nu}))(x-\zeta) = \delta_{[\xi,\eta]}H^\mu. \end{aligned} \tag{2.16}$$

The anticommuting partner $\tilde{H}^\mu$ of $\tilde{\zeta}^\mu$ is a field transforming exactly as H^μ.

We have achieved what we wanted, that is to say, define point-dependent observables in general relativity. However, we have done it at the cost of introducing new fields, the cloud fields (and their anticommuting partners). The next problem is to include the extra fields into the action, and ensure that the extension does not change the fundamental theory, and does not propagate unphysical degrees of freedom. First, we develop a formalism to ensure that the correlation functions of the undressed fields and the S matrix amplitudes are unmodified, despite the presence of new interactions. Then, we render the whole new sectors purely virtual.

We also want the whole construction to be perturbative (expanding the metric around flat space), diagrammatic and local. We do not require polynomiality, though, since in quantum gravity we have to renounce it anyway.

3 Batalin-Vilkovisky formalism and Zinn-Justin master equation

In this section we recall the standard Batalin-Vilkovisky formalism [16] for gravity, which is a convenient tool to study the Ward-Takahashi-Slavnov-Taylor identities [17] to all orders in a compact form.

The classical action $S_{\text{cl}}(g, A, \varphi)$ can be any action of classical gravity, possibly coupled to matter. For concreteness, we assume that the matter sector is made of an Abelian vector A_μ and a neutral scalar field φ. The specific form of S_{cl} is not important for the theoretical setup we are going to develop. However, particular forms of S_{cl} will be used in the computations.

We introduce the set of fields $\Phi^\alpha = (g_{\mu\nu}, C^\mu, \bar{C}^\mu, B^\mu, A_\mu, \varphi)$, where C^μ are the Faddeev-Popov ghosts [18], $\bar{C}^\mu$ are the antighosts and B^μ are the Nakanishi-Lautrup Lagrange multipliers [19]. The subscript α is collects all the indices. We couple sources $K^\alpha = (K_g^{\mu\nu}, K_\mu^C, K_\mu^{\bar{C}}, K_\mu^B, K_A^\mu, K_\varphi)$ to the field transformations by means of the functional

$$\begin{aligned} S_K(\Phi, K) = &- \int (C^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu C^\rho + g_{\nu\rho}\partial_\mu C^\rho) K_g^{\mu\nu} - \int (C^\rho \partial_\rho A_\mu + A_\rho \partial_\mu C^\rho) K_A^\mu \\ &- \int (C^\rho \partial_\rho \varphi) K_\varphi - \int C^\rho (\partial_\rho C^\mu) K_\mu^C - \int B^\mu K_\mu^{\bar{C}}. \end{aligned} \tag{3.1}$$

This way, the transformations of the fields can be written as

$$\delta_\xi \Phi^\alpha = \theta(S_K, \Phi^\alpha) = -\theta \frac{\delta_r S_K}{\delta K^\alpha}, \tag{3.2}$$

where $\xi^\mu = \theta C^\mu$, θ is a constant anticommuting (Grassmann) variable and

$$(X, Y) = \int \left(\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K^\alpha} - \frac{\delta_r X}{\delta K^\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} \right) \tag{3.3}$$

are the Batalin-Vilkovisky antiparentheses [16], the subscripts r and l denoting the right and left derivatives, respectively.

The closure of the algebra of the transformations is encoded in the identity

$$(S_K, S_K) = 0. \tag{3.4}$$

The Jacobi identity satisfied by the antiparentheses implies the nilpotence relation $(S_K, (S_K, X)) = 0$ for every X.

The gauge-fixed action reads

$$S_{\text{gf}}(\Phi) = S_{\text{cl}}(g, A, \varphi) + (S_K, \Psi(\Phi)), \tag{3.5}$$

where $\Psi(\Phi)$ is the "gauge fermion", that is to say, a local functional that is introduced to fix the gauge. For example, in a generic covariant gauge we may choose

$$\Psi(\Phi) = \int \sqrt{-g}\bar{C}^\mu \left(G_\mu(g) - \lambda g_{\mu\nu} B^\nu \right), \qquad G_\mu(g) = g^{\nu\rho}\partial_\rho g_{\mu\nu} - \frac{\lambda'}{2} g^{\nu\rho}\partial_\mu g_{\nu\rho}, \tag{3.6}$$

where λ, λ' are gauge-fixing parameters and $G_\nu(g)$ is the gauge-fixing function. We have

$$(S_K,\Psi)=\int\sqrt{-g}B^\mu\left(G_\mu(g)-\lambda g_{\mu\nu}B^\nu\right)+S_{\text{ghost}}\rightarrow\frac{1}{4\lambda}\int\sqrt{-g}G_\mu g^{\mu\nu}G_\nu+S_{\text{ghost}},$$
$$S_{\text{ghost}}=-\int\bar{C}^\mu\left(S_K,\sqrt{-g}G_\mu-\lambda\sqrt{-g}g_{\mu\nu}B^\nu\right),$$

where the arrow denotes the integration over B^μ and S_{ghost} is the ghost action. Other gauge choices will be considered in the paper.

The total action is

$$S(\Phi,K)=S_{\text{gf}}(\Phi)+S_K(\Phi,K) \tag{3.7}$$

and satisfies the Zinn-Justin equation [20]

$$(S,S)=0, \tag{3.8}$$

also known as master equation. This identity collects the gauge invariance of the classical action, the triviality of the gauge-fixing sector, as well as the closure of the algebra. We have the nilpotence relation $(S,(S,X))=0$ for every X.

4 Cloud sector

In this section we build the cloud sector. The idea is to add the cloud field ζ^μ to the action, but trivialize its introduction by means of a new symmetry (which we call cloud symmetry), built with the anticommuting partner H^μ, so as to keep the usual correlation functions and S matrix elements unchanged.

The goal is achieved as follows. First, we introduce a new set of fields $\tilde{\Phi}^\alpha=(\zeta^\mu, H^\mu, \bar{H}^\mu, E^\mu)$ and the sources $\tilde{K}^\alpha=(\tilde{K}_\mu^\zeta, \tilde{K}_\mu^H, \tilde{K}_\mu^{\bar{H}}, \tilde{K}_\mu^E)$ coupled to their transformations, where H^μ can be seen as the "Faddeev-Popov ghosts" of the cloud, $\bar{H}^\mu$ are the antighosts and E^μ are new Lagrange multipliers. Second, we extend the definition (3.3) of antiparentheses to include the new sector:

$$(X,Y)=\int\left(\frac{\delta_r X}{\delta\Phi^\alpha}\frac{\delta_l Y}{\delta K^\alpha}-\frac{\delta_r X}{\delta K^\alpha}\frac{\delta_l Y}{\delta\Phi^\alpha}+\frac{\delta_r X}{\delta\tilde{\Phi}^\alpha}\frac{\delta_l Y}{\delta\tilde{K}^\alpha}-\frac{\delta_r X}{\delta\tilde{K}^\alpha}\frac{\delta_l Y}{\delta\tilde{\Phi}^\alpha}\right).$$

Third, we collect the gauge transformations (2.3) and (2.15) of the new fields ζ^μ and H^μ, and the cloud transformations, into the functionals

$$S_K^{\text{gauge}}=S_K-\int C^\mu(x-\zeta)\tilde{K}_\mu^\zeta-\int H^\nu C^\mu{}_{,\nu}(x-\zeta)\tilde{K}_\mu^H,$$
$$S_K^{\text{cloud}}=\int H^\mu\tilde{K}_\mu^\zeta-\int E^\mu\tilde{K}_\mu^{\bar{H}},\qquad S_K^{\text{tot}}=S_K^{\text{gauge}}+S_K^{\text{cloud}}.$$

The cloud transformations, which are encoded into the second functional, are just the most general shifts of ζ^μ and $\bar{H}$. For example, the total (gauge plus cloud) transformation of ζ^μ reads

$$\delta_{\xi,\mathcal{H}}\zeta^\mu = \theta\left(S_K^{\text{tot}},\zeta^\mu\right) = -\theta H^\mu(x) + \theta C^\mu(x-\zeta(x)) = -\mathcal{H}^\mu(x) + \xi^\mu(x-\zeta(x)),$$

where $\mathcal{H} = \theta H$.

It is easy to check the identities

$$(S_K^{\text{gauge}}, S_K^{\text{gauge}}) = (S_K^{\text{tot}}, S_K^{\text{tot}}) = 0, \tag{4.1}$$

which express the closures of both types of transformations. The first identity follows from (2.2), (2.16) and (3.4).

We also have:

$$S_K^{\text{gauge}} - S_K = -\left(S_K^{\text{cloud}}, \int C^\mu(x-\zeta)\tilde{K}_\mu^H\right), \qquad S_K^{\text{cloud}} = -\left(S_K^{\text{cloud}}, \int \zeta^\mu \tilde{K}_\mu^\zeta + \int \bar{H}^\mu \tilde{K}_\mu^{\bar{H}}\right). \tag{4.2}$$

These formulas show that the functionals $S_K^{\text{gauge}} - S_K$ and S_K^{cloud} are cohomologically exact under the cloud symmetry, i.e., they have the form (S_K^{cloud}, local functional). Together with $(S_K, S_K^{\text{cloud}}) = 0$ (which is trivial), they imply the further identity

$$(S_K^{\text{gauge}}, S_K^{\text{cloud}}) = 0, \tag{4.3}$$

which gives, together with (4.1),

$$(S_K^{\text{tot}}, S_K^{\text{tot}}) = 0. \tag{4.4}$$

4.1 The cloud and the total action

We fix the cloud by adding $(S_K^{\text{tot}}, \tilde{\Psi})$ to the action, where $\tilde{\Psi}(\Phi, \tilde{\Phi})$ is the "cloud fermion". A typical form of it is

$$\tilde{\Psi}(\Phi, \tilde{\Phi}) = \int \sqrt{-g_{\text{d}}}\bar{H}^\mu \left(V_\mu - \tilde{\lambda} g_{\mu\nu\text{d}} E^\nu\right), \tag{4.5}$$

where g_{d} is the determinant of $g_{\mu\nu\text{d}}$ and V_μ denotes the "cloud function", i.e., the function that specifies the cloud. We assume that V_μ is gauge invariant:

$$(S_K^{\text{gauge}}, V_\mu) = 0, \qquad (S_K^{\text{gauge}}, \tilde{\Psi}) = 0. \tag{4.6}$$

Basically, we can view V_μ as a function of the dressed metric $g_{\mu\nu\text{d}}$.

We find

$$(S_K^{\text{tot}},\tilde{\Psi})=(S_K^{\text{cloud}},\tilde{\Psi})=\int\sqrt{-g_{\text{d}}}E^{\mu}\left(V_{\mu}-\tilde{\lambda}g_{\mu\nu\text{d}}E^{\nu}\right)$$
$$+\int\bar{H}^{\mu}\frac{\delta}{\delta\zeta^{\rho}}\left[\sqrt{-g_{\text{d}}}\left(V_{\mu}-\tilde{\lambda}g_{\mu\nu\text{d}}E^{\nu}\right)\right]H^{\rho}. \tag{4.7}$$

The last term gives a "Faddeev-Popov determinant" for the cloud, which is crucial for the diagrammatic properties that we derive in the next sections.

The total action of the extended theory is then

$$S_{\text{tot}}(\Phi,K,\tilde{\Phi},\tilde{K})=S_{\text{cl}}+(S_K^{\text{tot}},\Psi+\tilde{\Psi})+S_K^{\text{tot}} \tag{4.8}$$

and satisfies its own master equation

$$(S_{\text{tot}},S_{\text{tot}})=0. \tag{4.9}$$

Note that S_{tot} is gauge invariant, since equations (4.1), (4.4) and (4.6) imply

$$(S_K^{\text{gauge}},S_{\text{tot}})=(S_K^{\text{cloud}},S_{\text{tot}})=0. \tag{4.10}$$

We can also write

$$S_{\text{tot}}(\Phi,K,\tilde{\Phi},\tilde{K})=S(\Phi,K)+(S_K^{\text{cloud}},\Theta),\qquad \Theta=\tilde{\Psi}-\int C^{\mu}(x-\zeta)\tilde{K}_{\mu}^{H}-\int\zeta^{\mu}\tilde{K}_{\mu}^{\zeta}-\int\bar{H}^{\mu}\tilde{K}_{\mu}^{\bar{H}},$$

which shows that the difference between the dressed action and the ordinary action is cohomologically exact with respect to the cloud symmetry.

4.2 Covariant cloud

To make explicit calculations, we need to choose the could function V_{μ} in (4.5). The large arbitrariness in this choice does not change the fundamental physics, as we show in the next section. A convenient starting point is the covariant cloud function

$$V_{\mu}(g,\zeta)=g_{\text{d}}^{\nu\rho}\partial_{\rho}g_{\mu\nu\text{d}}-\frac{\tilde{\lambda}'}{2}g_{\text{d}}^{\nu\rho}\partial_{\mu}g_{\nu\rho\text{d}}, \tag{4.11}$$

where $\tilde{\lambda}'$ is a further cloud parameter. Other choices will be considered in the paper.

Note that we are allowed to use second metrics, inside the clouds (as well as inside the gauge-fixing function). For example, we can use the flat-space metric $\eta_{\mu\nu}$ to raise the indices of ∂_{μ}. Sometimes, however, it may be convenient to use a unique metric everywhere, for a better control on the renormalization properties of the dressed theory.

5 Cloud independence of the non-cloud sector

In this section we prove that the clouds do not affect the non-cloud sector. Specifically, we show that the ordinary correlation functions of elementary and composite fields are unmodified. This also ensures that the vertices and diagrams of the cloud sector do not affect the renormalization of the non-cloud sector of the theory. Moreover, we show that the S matrix amplitudes of the dressed fields are cloud independent and coincide with the usual S matrix amplitudes of the undressed fields.

The generating functional of the correlation functions is

$$Z(J,K,\tilde{J},\tilde{K})=\int[\mathrm{d}\Phi\mathrm{d}\tilde{\Phi}]\exp\left(iS_{\mathrm{tot}}(\Phi,K,\tilde{\Phi},\tilde{K})+i\int\Phi^{\alpha}J_{\alpha}+i\int\tilde{\Phi}^{\alpha}\tilde{J}_{\alpha}\right) \tag{5.1}$$

and $W(J,K,\tilde{J},\tilde{K})=-i\ln Z(J,K,\tilde{J},\tilde{K})$ is the generating functional of the connected ones. The ordinary correlation functions are the functional derivatives with respect to the sources J^{α}, calculated at $\tilde{J}=\tilde{K}=0$. They are collected in

$$\begin{aligned}Z(J,K,0,0)&=\int[\mathrm{d}\Phi\mathrm{d}\tilde{\Phi}]\exp\left(iS(\Phi,K)+i(S_{K}^{\mathrm{tot}},\tilde{\Psi})+i\int\Phi^{\alpha}J_{\alpha}\right)\\&=\int[\mathrm{d}\Phi]\exp\left(iS(\Phi,K)+i\int\Phi^{\alpha}J_{\alpha}\right)\int[\mathrm{d}\tilde{\Phi}]\mathrm{e}^{i(S_{K}^{\mathrm{cloud}},\tilde{\Psi})}.\end{aligned} \tag{5.2}$$

We want to prove that this expression coincides with the ordinary generating functional, thanks to the identity

$$\int[\mathrm{d}\tilde{\Phi}]\mathrm{e}^{i(S_{K}^{\mathrm{cloud}},\tilde{\Psi})}=1. \tag{5.3}$$

Since $\tilde{\Psi}$ depends on both Φ and $\tilde{\Phi}$, the left-hand side of (5.3) is in principle a functional of Φ. To show that it is actually a constant, we consider arbitrary infinitesimal deformations of the fields Φ. Let $\delta\tilde{\Psi}$ denote the variation of $\tilde{\Psi}$ due to them. The variation of the integral is then

$$\delta\int[\mathrm{d}\tilde{\Phi}]\mathrm{e}^{i(S_{K}^{\mathrm{cloud}},\tilde{\Psi})}=i\int[\mathrm{d}\tilde{\Phi}](S_{K}^{\mathrm{cloud}},\delta\tilde{\Psi})\mathrm{e}^{i(S_{K}^{\mathrm{cloud}},\tilde{\Psi})}, \tag{5.4}$$

Performing the change of field variables $\tilde{\Phi}^{\alpha}\rightarrow\tilde{\Phi}^{\alpha}+\theta(S_{K}^{\mathrm{cloud}},\tilde{\Phi}^{\alpha})$ in the integral

$$\int[\mathrm{d}\tilde{\Phi}]\delta\tilde{\Psi}\mathrm{e}^{i(S_{K}^{\mathrm{cloud}},\tilde{\Psi})},$$

we obtain

$$\int[\mathrm{d}\tilde{\Phi}]\delta\tilde{\Psi}\mathrm{e}^{i(S_{K}^{\mathrm{cloud}},\tilde{\Psi})}=\int[\mathrm{d}\tilde{\Phi}]\left[\delta\tilde{\Psi}+\theta(S_{K}^{\mathrm{cloud}},\delta\tilde{\Psi})\right]\mathrm{e}^{i(S_{K}^{\mathrm{cloud}},\tilde{\Psi})}. \tag{5.5}$$

We have used the fact that $(S_{K}^{\mathrm{cloud}},\tilde{\Psi})$ is independent of the sources, so $(S_{K}^{\mathrm{cloud}},\tilde{\Psi})\rightarrow(S_{K}^{\mathrm{cloud}},\tilde{\Psi})+\theta(S_{K}^{\mathrm{cloud}},(S_{K}^{\mathrm{cloud}},\tilde{\Psi}))=(S_{K}^{\mathrm{cloud}},\tilde{\Psi})$. The equality (5.5) shows that the right-hand side of (5.4) vanishes, as we wished to prove.

5.1 Cloud independence of the S matrix amplitudes

Now we prove that the scattering amplitudes of the dressed fields coincide with the usual scattering amplitudes (of undressed fields). Specifically, the clouds have no effect on shell, when the polarizations are attached to the amputated external legs.

In momentum space, we have

$$\langle\prod_{i=1}^{n}k_i^2\varepsilon_i^{\mu_i\nu_i}(k_i)g_{\mu_i\nu_i\mathrm{d}}(k_i)\prod_{a=1}^{j}p_a^2\varepsilon_a^{\rho_a}(p_a)A_{\rho_a\mathrm{d}}(p_a)\prod_{b=1}^{l}(q_b^2-m_b^2)\varphi_{\mathrm{d}}(q_b)\rangle_{\text{on-shell}}$$

$$=\langle\prod_{i=1}^{n}k_i^2\varepsilon_i^{\mu_i\nu_i}(k_i)g_{\mu_i\nu_i}(k_i)\prod_{a=1}^{j}p_a^2\varepsilon_a^{\rho_a}(p_a)A_{\rho_a}(p_a)\prod_{b=1}^{l}(q_b^2-m_b^2)\varphi(q_b)\rangle_{\text{on-shell}}, \quad (5.6)$$

where $\varepsilon^{\mu\nu}(k)$ and $\varepsilon^{\mu}(p)$ are the polarizations of the gravitons and the vector fields, respectively, and satisfy $k_\mu\varepsilon^{\mu\nu}(k)=p_\mu\varepsilon^{\mu}(p)=0$. With an abuse of notation, we use the same symbols for the fields and their Fourier transforms, since the meaning is clear from the context. By the theorem proved in the first part of this section, the right-hand side of (5.6) is cloud independent and coincides with the usual S matrix amplitude, once appropriate factors $\sqrt{Z}$ are included.

To prove the identity (5.6), we start by considering the difference

$$\lim_{k^2\to 0}k^2\varepsilon^{\mu\nu}(k)\left[g_{\mu\nu\mathrm{d}}(k)-g_{\mu\nu}(k)\right]. \quad (5.7)$$

Formulas (2.6) show that the expansion of $g_{\mu\nu\mathrm{d}}$ in powers of ζ^ρ, combined with the expansion of $g_{\mu\nu}$ around the flat-space metric $\eta_{\mu\nu}$, contains a linear contribution $-\zeta_{\mu,\nu}-\zeta_{\nu,\mu}$, besides $g_{\mu\nu}$ itself, plus nonlinear terms (which have to be regarded as composite fields), where $\zeta_\mu=\eta_{\mu\nu}\zeta^\nu$. The linear contribution is killed by the polarization $\varepsilon^{\mu\nu}(k)$, after Fourier transform. This means that (5.7) is just a composite field. A correlation function that contains a composite-field insertion cannot provide the factor $1/k^2$ that is necessary to simplify the multiplication by k^2. This means that the insertion of (5.7) is killed by the on-shell limit $k^2\to 0$.

Ultimately, the insertion of

$$\lim_{k^2\to 0}k^2\varepsilon^{\mu\nu}(k)\langle g_{\mu\nu}(k)\cdots\rangle$$

in a correlation function (the limit being taken after the average) is gauge invariant (and, therefore, gauge independent, for the arguments given below) and does not need any dressing. More precisely, its dressing is trivial:

$$\lim_{k^2\to 0}k^2\varepsilon^{\mu\nu}(k)\langle g_{\mu\nu\mathrm{d}}(k)\cdots\rangle=\lim_{k^2\to 0}k^2\varepsilon^{\mu\nu}(k)\langle g_{\mu\nu}(k)\cdots\rangle.$$

The result also holds for the insertions of vectors and scalars (and fermions, if present). In the case of unstable particles, like the muon, it is understood that the self-energies attached to the external legs are resummed into effective propagators, which are then amputated by means of complex masses m, to take care of their nonvanishing widths.

Specializing to the two-point functions, formula (5.6) ensures that the factors $\sqrt{Z}$ are the same for dressed and undressed fields.

The identity (5.6) proves that the ordinary theory of scattering can be understood as a theory of scattering of dressed fields. We can even forget about the undressed fields altogether, and always work with the dressed fields. So doing, gauge invariance and gauge independence become manifest. In particular, the S-matrix amplitudes are automatically ensured to be gauge independent.

6 Dressed correlation functions

In this section we study the correlation functions that contain insertions of dressed fields. It is possible to study them systematically by coupling new sources to them and extending the generating functionals again. We replace the action S_{tot} inside (5.1) by

$$S_{\text{tot}}^{\text{ext}} = S_{\text{tot}} + \int \left(J_{\text{d}}^{\mu\nu} g_{\mu\nu\text{d}} + J_{\text{d}}^{\mu} A_{\mu\text{d}} + J_{\text{d}} \varphi_{\text{d}} \right), \tag{6.1}$$

and denote the extended functionals by $Z_{\text{tot}}^{\text{ext}}(J, K, \tilde{J}, \tilde{K}, J_{\text{d}}) = \exp(iW_{\text{tot}}^{\text{ext}}(J, K, \tilde{J}, \tilde{K}, J_{\text{d}}))$. The insertions of dressed fields can be studied by taking the functional derivatives with respect to the new sources $J_{\text{d}}^{\alpha} = (J_{\text{d}}^{\mu\nu}, J_{\text{d}}^{\mu}, J_{\text{d}})$. The extended action is gauge invariant, since (4.10) implies

$$(S_K^{\text{gauge}}, S_{\text{tot}}^{\text{ext}}) = 0. \tag{6.2}$$

Clearly, $S_{\text{tot}}^{\text{ext}}$ is not cloud invariant.

It is straightforward to prove that the correlation functions of the dressed fields, collected in the functional $Z_{\text{tot}}^{\text{ext}}(J_{\text{d}}) = \exp(iW_{\text{tot}}^{\text{ext}}(J_{\text{d}})) \equiv Z_{\text{tot}}^{\text{ext}}(0, 0, 0, 0, J_{\text{d}}) = \exp(iW_{\text{tot}}^{\text{ext}}(0, 0, 0, 0, J_{\text{d}}))$, are gauge independent. The argument is identical to the one of subsection 7.1 of [21], so we do not repeat it here. Gauge independence will be verified explicitly in the computations.

7 Gauge/cloud duality

In this section we show that the gauge-trivial sector and the cloud sector are dual to each other. We call this property *gauge/cloud duality*. Sometimes, it can be used to simplify the computations.

We begin by noting that the transformation law (2.3) and the definitions (2.6) imply that the could transformation of the dressed metric is just an infinitesimal diffeomorphism (2.1) with parameters H_{d}^{μ} such that

$$H^{\mu} = (\delta_{\nu}^{\mu} - \zeta_{,\nu}^{\mu}) H_{\mathrm{d}}^{\nu}. \tag{7.1}$$

Precisely,

$$(S_K^{\mathrm{tot}}, g_{\mu\nu\mathrm{d}}) = (S_K^{\mathrm{cloud}}, g_{\mu\nu\mathrm{d}}) = H_{\mathrm{d}}^{\rho}\partial_{\rho} g_{\mu\nu\mathrm{d}} + g_{\mu\rho\mathrm{d}}\partial_{\nu} H_{\mathrm{d}}^{\rho} + g_{\nu\rho\mathrm{d}}\partial_{\mu} H_{\mathrm{d}}^{\rho} = \delta_{H_{\mathrm{d}}}^{\mathrm{diff}} g_{\mu\nu\mathrm{d}}. \tag{7.2}$$

Similarly, for scalars and vectors we have

$$(S_K^{\mathrm{cloud}}, \varphi_{\mathrm{d}}) = H_{\mathrm{d}}^{\rho}\partial_{\rho}\varphi_{\mathrm{d}}, \qquad (S_K^{\mathrm{cloud}}, A_{\mu\mathrm{d}}) = H_{\mathrm{d}}^{\rho}\partial_{\rho} A_{\mu\mathrm{d}} + A_{\rho\mathrm{d}}\partial_{\mu} H_{\mathrm{d}}^{\rho}. \tag{7.3}$$

It is easy to check that H_{d}^{μ} is indeed gauge invariant $(S_K^{\mathrm{gauge}}, H_{\mathrm{d}}^{\mu}) = 0$. Moreover, the cloud transformation of H_{d}^{μ} mimics the gauge transformation of the ghosts C^{μ}:

$$(S_K^{\mathrm{cloud}}, H_{\mathrm{d}}^{\mu}) = H_{\mathrm{d}}^{\rho}\partial_{\rho} H_{\mathrm{d}}^{\mu}. \tag{7.4}$$

We define the dressed cloud field ζ_{d}^{μ} as the dual field of formula (2.8):

$$\zeta_{\mathrm{d}}^{\mu} = \tilde{\zeta}^{\mu}. \tag{7.5}$$

Using (7.1), it is easy to derive the cloud transformation of ζ_{d}^{μ}, which reads

$$(S_K^{\mathrm{cloud}}, \zeta_{\mathrm{d}}^{\mu}) = H_{\mathrm{d}}^{\mu}(x - \zeta_{\mathrm{d}}(x)). \tag{7.6}$$

Note that ζ_{d}^{μ} is not gauge invariant. Using (2.10), its gauge transformation can be used to define the dressed Faddeev-Popov ghosts

$$C_{\mathrm{d}}^{\mu} \equiv (\delta_{\nu}^{\mu} - \zeta_{\mathrm{d},\nu}^{\mu}) C^{\nu} = -(S_K^{\mathrm{gauge}}, \zeta_{\mathrm{d}}^{\mu}), \tag{7.7}$$

which, instead, are gauge invariant by construction. Their cloud transformations read

$$(S_K^{\mathrm{cloud}}, C_{\mathrm{d}}^{\mu}) = (S_K^{\mathrm{gauge}}, (S_K^{\mathrm{cloud}}, \zeta_{\mathrm{d}}^{\mu})) = (S_K^{\mathrm{gauge}}, H_{\mathrm{d}}^{\mu}(x - \zeta_{\mathrm{d}}(x))) = C_{\mathrm{d}}^{\rho} H_{,\rho\mathrm{d}}^{\mu}(x - \zeta_{\mathrm{d}}), \tag{7.8}$$

having used (4.3) and $(S_K^{\mathrm{gauge}}, H_{\mathrm{d}}^{\mu}) = 0$.

Now, collecting (2.6), (7.1), (7.5) and (7.7), we define the change of field variables

$$\Phi, \tilde{\Phi} \to \Phi_{\mathrm{d}}, \tilde{\Phi}_{\mathrm{d}} \tag{7.9}$$

from undressed fields to dressed fields, leaving all the other fields unchanged: $\bar{C}_{\mathrm{d}}^{\mu}=\bar{C}^{\mu}$, $B_{\mathrm{d}}^{\mu}=B^{\mu}$, $\bar{H}_{\mathrm{d}}^{\mu}=\bar{H}^{\mu}$ and $E_{\mathrm{d}}=E$. The transformations are perturbatively local, which means that when we use them as changes of field variables in the functional integral, the Jacobian determinant is equal to one, using the dimensional regularization.

To ensure that the antiparentheses are preserved, so that all the properties derived till now continue to hold, we embed (7.9) into a canonical transformation

$$\Phi,\tilde{\Phi},K,\tilde{K}\rightarrow\Phi_{\mathrm{d}},\tilde{\Phi}_{\mathrm{d}},K_{\mathrm{d}},\tilde{K}_{\mathrm{d}}, \tag{7.10}$$

of the Batalin-Vilkovisky type. Its generating functional is

$$F(\Phi,\tilde{\Phi},K_{\mathrm{d}},\tilde{K}_{\mathrm{d}})=\int\Phi_{\mathrm{d}}(\Phi,\tilde{\Phi})K_{\mathrm{d}}+\int\tilde{\Phi}_{\mathrm{d}}(\Phi,\tilde{\Phi})\tilde{K}_{\mathrm{d}}.$$

At the practical level, the whole operation amounts to work out the transformations of the dressed fields, which we have already done, and couple them to the dressed sources. Collecting the gauge transformations (7.7) and the cloud transformations (7.2), (7.3), (7.4), (7.6) and (7.8), we find

$$\begin{aligned}
S_K^{\text{gauge}} &= \int C_{\mathrm{d}}^{\mu}\tilde{K}_{\mu\mathrm{d}}^{\zeta}-\int B_{\mathrm{d}}^{\mu}K_{\mu\mathrm{d}}^{\bar{C}},\\
S_K^{\text{cloud}} &= -\int(H_{\mathrm{d}}^{\rho}\partial_{\rho}g_{\mu\nu\mathrm{d}}+g_{\mu\rho\mathrm{d}}\partial_{\nu}H_{\mathrm{d}}^{\rho}+g_{\nu\rho\mathrm{d}}\partial_{\mu}H_{\mathrm{d}}^{\rho})K_{g\mathrm{d}}^{\mu\nu}-\int H_{\mathrm{d}}^{\rho}(\partial_{\rho}H_{\mathrm{d}}^{\mu})K_{\mu\mathrm{d}}^{H}\\
&\quad-\int H_{\mathrm{d}}^{\rho}(\partial_{\rho}\varphi_{\mathrm{d}})K_{\varphi\mathrm{d}}-\int(H_{\mathrm{d}}^{\rho}\partial_{\rho}A_{\mathrm{d}\mu}+A_{\rho\mathrm{d}}\partial_{\mu}H_{\mathrm{d}}^{\rho})K_{\mu\mathrm{d}}^{A}\\
&\quad-\int E_{\mathrm{d}}^{\mu}\tilde{K}_{\mu\mathrm{d}}^{\bar{H}}-\int H_{\mathrm{d}}^{\mu}(x-\zeta_{\mathrm{d}})\tilde{K}_{\mu\mathrm{d}}^{\zeta}-\int C_{\mathrm{d}}^{\nu}H_{,\nu\mathrm{d}}^{\mu}(x-\zeta_{\mathrm{d}})\tilde{K}_{\mu\mathrm{d}}^{C}.
\end{aligned}$$

We see that the canonical transformation (7.10) switches the gauge transformations and the cloud transformations. Similarly, it exchanges the roles of the gauge-fixing function G_{μ} and the cloud function V_{μ}: $G_{\mu}(g(\zeta,g_{\mathrm{d}}))\leftrightarrow V_{\mu}(g_{\mathrm{d}})$. It also exchanges the quantization prescription of the gauge-trivial sector with the one of the cloud sector (see below).

The correlation functions of the dressed fields coincide with the ones of the undressed fields in a specific gauge. For example, choosing the covariant gauge (3.6) and the covariant cloud (4.5), (4.11), we have

$$\begin{aligned}
&\langle g_{\mu_1\nu_1\mathrm{d}}(x_1)\cdots g_{\mu_n\nu_n\mathrm{d}}(x_n)\varphi_{\mathrm{d}}(y_1)\cdots\varphi_{\mathrm{d}}(y_j)A_{\rho_1\mathrm{d}}(z_1)\cdots A_{\rho_k\mathrm{d}}(z_k)\rangle\\
&\qquad=\langle g_{\mu_1\nu_1}(x_1)\cdots g_{\mu_n\nu_n}(x_n)\varphi(y_1)\cdots\varphi(y_j)A_{\rho_1}(z_1)\cdots A_{\rho_k}(z_k)\rangle_{\lambda\rightarrow\tilde{\lambda},\lambda'\rightarrow\tilde{\lambda}'}.
\end{aligned} \tag{7.11}$$

Combined with the cloud independence of the right-hand side, proved in section 5, this property ensures that the dressed correlation function can be calculated by replacing λ

with $\tilde{\lambda}$ and λ' with $\tilde{\lambda}'$ in a usual correlation function. The left-hand side of (7.11) normally includes a huge number of diagrams. However, (7.11) implies that most contributions cancel out in the end.

8 Multiclouds

In this section we show how to equip each insertion with its own, independent dressing. To do so, we extend the formalism of the previous sections by adding several copies of the could sector.

We introduce many cloud fields $\zeta^{\mu i}$, where i labels the copies. Then we add copies of their anticommuting partners $H^{\mu i}$ (the cloud ghosts), the antighosts $\bar{H}^{\mu i}$ and the Lagrange multipliers $E^{\mu i}$. We collect them in $\tilde{\Phi}^{\alpha i} = (\zeta^{\mu i}, H^{\mu i}, \bar{H}^{\mu i}, E^{\mu i})$. We also couple sources $\tilde{K}^{\alpha i}$ to their transformations. Next, we extend the definition (3.3) of antiparentheses to include all the copies:

$$(X,Y) = \int \left[\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K^\alpha} - \frac{\delta_r X}{\delta K^\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} + \sum_i \left(\frac{\delta_r X}{\delta \tilde{\Phi}^{\alpha i}} \frac{\delta_l Y}{\delta \tilde{K}^{\alpha i}} - \frac{\delta_r X}{\delta \tilde{K}^{\alpha i}} \frac{\delta_l Y}{\delta \tilde{\Phi}^{\alpha i}} \right) \right]. \tag{8.1}$$

Finally, we extend the gauge transformations and introduce cloud transformations for each copy:

$$\begin{aligned}
S_K^{\text{gauge}} &= S_K - \sum_i \int C^\mu (x - \zeta^i) \tilde{K}_\mu^{\zeta i} - \sum_i \int H^{\nu i} C^{\mu}_{\ ,\nu} (x - \zeta^i) \tilde{K}_\mu^{H i}, \\
S_K^{\text{cloud}\, i} &= \int (H^{\mu i} \tilde{K}_\mu^{\zeta i} - E^{\mu i} \tilde{K}_\mu^{\bar{H} i}), \qquad S_K^{\text{cloud}} = \sum_i S_K^{\text{cloud}\, i}, \qquad S_K^{\text{tot}} = S_K^{\text{gauge}} + S_K^{\text{cloud}}.
\end{aligned} \tag{8.2}$$

It is easy to check that the identities (4.1) and (4.4) continue to hold.

The simplest cloud fermion is just the sum of the cloud fermions of each copy:

$$\tilde{\Psi}(\Phi, \tilde{\Phi}) = \sum_i \int \sqrt{-g_{\text{d}}^i} \bar{H}^{\mu i} \left(V_\mu^i + \frac{\tilde{\lambda}_i}{2} g_{\mu\nu\text{d}}^i E^{\nu i} \right), \tag{8.3}$$

where $g_{\mu\nu\text{d}}^i$ is the dressed metric tensor built with the ith cloud field $\zeta^{\mu i}$, and V_μ^i is the ith cloud function, assumed to be gauge invariant, $(S_K^{\text{gauge}}, V_\mu^i) = 0$. For simplicity, we also assume that each V_μ^i depends only on the ith cloud field $\zeta^{\mu i}$ (besides $g_{\mu\nu}$), i.e., different cloud sectors are not mixed. We can just take each V_μ^i to be a function of $g_{\mu\nu\text{d}}^i$.

The total action of the extended theory is still (4.8), and satisfies the master equations (4.9) and (4.10). Moreover,

$$(S_K^{\text{cloud}\, i}, S_{\text{tot}}) = 0 \tag{8.4}$$

for every i.

It is always possible to build gauge invariant functions with two cloud fields. For example, the functions

$$\zeta_{1i}^{\mu}(x) \equiv \zeta^{\mu 1}(x) + \tilde{\zeta}^{\mu i}(x - \zeta^{1}(x)) \tag{8.5}$$

are gauge invariant, since (2.3) and (2.10) imply

$$\delta_{\xi}\zeta_{1i}^{\mu}(x) \equiv \xi^{\mu}(x-\zeta^{1}) - \tilde{\zeta}_{,\nu}^{\mu i}(x-\zeta^{1})\xi^{\mu}(x-\zeta^{1}) - (\delta_{\nu}^{\mu} - \tilde{\zeta}_{,\nu}^{\mu i}(x-\zeta^{1}))\xi^{\nu}(x-\zeta^{1}) = 0.$$

We do not have control on such functions, when they are turned on. For this reason, it may be important to prove, when possible, that the operations we make preserve the unmixing stated above.

The insertions of dressed fields can be studied by means of the extended action

$$S_{\text{tot}}^{\text{ext}} = S_{\text{tot}} + \sum_{i} \int \left(J_{\text{d}}^{\mu\nu i} g_{\mu\nu\text{d}}^{i} + J_{\text{d}}^{\mu i} A_{\mu\text{d}}^{i} + J_{\text{d}}^{i} \varphi_{\text{d}}^{i} \right). \tag{8.6}$$

The correlation functions that do not contain insertions belonging to some cloud sector are independent of that cloud sector. Indeed, the proof of (5.3) can be repeated for every cloud sector separately.

The gauge/cloud duality is less powerful in the presence of many clouds. It can be used to eliminate one cloud, or a combination of clouds, but not all of them. For example, a correlation function

$$\langle g_{\mu_1\nu_1\text{d}}^{(1)}(x_1)\cdots g_{\mu_n\nu_n\text{d}}^{(n)}(x_n) A_{\rho_1\text{d}}^{(n+1)}(y_1)\cdots A_{\rho_j\text{d}}^{(n+j)}(y_j) \varphi_{\text{d}}^{(n+j+1)}(z_1)\cdots \varphi_{\text{d}}^{(n+j+k)}(z_k)\rangle, \tag{8.7}$$

with different clouds for every field, can be simplified to

$$\langle g_{\mu_1\nu_1}^{(1)}(x_1) g_{\mu_2\nu_2\text{d}}^{(2)\prime}(x_1)\cdots g_{\mu_n\nu_n\text{d}}^{(n)\prime}(x_n) A_{\rho_1\text{d}}^{(n+1)\prime}(y_1)\cdots A_{\rho_j\text{d}}^{(n+j)\prime}(y_j) \varphi_{\text{d}}^{(n+j+1)\prime}(z_1)\cdots \varphi_{\text{d}}^{(n+j+k)\prime}(z_k)\rangle,$$

by means of a field redefinition that exchanges the first dressed field with its undressed version. The primes mean that the clouds of the other fields must be redefined as a consequence.

These operations preserve the unmixing, after further redefinitions of the cloud fields. For example, the transformation (7.10) leads to

$$\varphi_{\text{d}}^{(i)}(x) = \varphi_{\text{d}}^{(1)}(x - \zeta^{i\prime}(x)), \qquad i > n+j,$$

where $\varphi_{\text{d}}^{(1)}$ is the scalar field dressed with the first cloud (which does not even appear in (8.7), but this does not matter for what we are saying) and $\zeta^{i\prime}(z_i)$ is the solution of

$$\zeta^{\mu i\prime}(x) = \zeta^{\mu i}(x) - \zeta^{\mu 1}(x - \zeta^{i\prime}(x)).$$

To restore the unmixing, it is sufficient to define the new ith cloud field as $\zeta^{\mu i\prime}$, $i>1$, after relabelling $\varphi_{\mathrm{d}}^{(1)}$ as φ. The same can be done for the other insertions of (8.7).

In the explicit calculations of this paper we work with a unique cloud, for simplicity.

9 One-loop two-point functions

In this section we calculate the one-loop two-point functions of the basic dressed fields in the covariant gauge, with a covariant cloud. We show that the absorptive parts are in general unphysical. In the next section we turn to purely virtual clouds, and show that the absorptive parts then become physical.

We define the expansion around flat space by writing $g_{\mu\nu}=\eta_{\mu\nu}+2\kappa h_{\mu\nu}$ and $g_{\mu\nu\mathrm{d}}=\eta_{\mu\nu}+2\kappa h_{\mu\nu\mathrm{d}}$, where $\kappa=\sqrt{8\pi G}$ and G is Newton's constant. It is convenient to make the replacements

$$\begin{array}{lllll} C^{\mu}\rightarrow\kappa C^{\mu}, & B^{\mu}\rightarrow\kappa B^{\mu}, & \bar{C}^{\mu}\rightarrow\kappa\bar{C}^{\mu}, & \Psi\rightarrow\kappa^{-2}\Psi, & \\ \zeta^{\mu}\rightarrow\kappa\zeta^{\mu}, & H^{\mu}\rightarrow\kappa H^{\mu}, & E^{\mu}\rightarrow\kappa E^{\mu}, & \bar{H}^{\mu}\rightarrow\kappa\bar{H}^{\mu}, & \tilde{\Psi}\rightarrow\kappa^{-2}\tilde{\Psi}, \end{array}$$

so that the loop expansion coincides with the expansion in powers of κ.

The two-point functions can be calculated by expanding the dressed fields (2.6) to the first order in κ, where we find

$$\begin{aligned} \varphi_{\mathrm{d}} &= \varphi-\kappa\zeta^{\mu}\varphi_{,\mu}, \qquad A_{\mu\mathrm{d}}=A_{\mu}-\kappa\zeta^{\rho}A_{\mu,\rho}-\kappa\zeta_{,\mu}^{\rho}A_{\rho}, \\ h_{\mu\nu\mathrm{d}} &= h_{\mu\nu}-\frac{1}{2}(\zeta_{\mu,\nu}+\zeta_{\nu,\mu})-\kappa\zeta^{\rho}h_{\mu\nu,\rho}-\kappa\zeta_{,\mu}^{\rho}h_{\nu\rho}-\kappa\zeta_{,\nu}^{\rho}h_{\mu\rho}+\frac{\kappa}{2}\zeta_{,\mu}^{\rho}\eta_{\rho\sigma}\zeta_{,\nu}^{\sigma}, \end{aligned}$$

where $\zeta_{\mu}=\eta_{\mu\nu}\zeta^{\nu}$. The higher-order corrections can be neglected in our calculations, since they give only tadpoles.

We start from Einstein gravity minimally coupled to a massless scalar field φ and a vector field A_{μ}. The action is

$$-\frac{1}{16\pi G}\int\mathrm{d}^{4}x\sqrt{-g}R+\frac{1}{2}\int\mathrm{d}^{4}x\sqrt{-g}g^{\mu\nu}(\partial_{\mu}\varphi)(\partial_{\nu}\varphi)-\frac{1}{4}\int\mathrm{d}^{4}x\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}.$$

The two-point function of the dressed scalar field reads

$$\langle\varphi_{\mathrm{d}}|\varphi_{\mathrm{d}}\rangle=\langle\varphi|\varphi\rangle-\kappa\langle\zeta^{\mu}\varphi_{,\mu}|\varphi\rangle-\kappa\langle\varphi|\zeta^{\mu}\varphi_{,\mu}\rangle+\kappa^{2}\langle\zeta^{\mu}\varphi_{,\mu}|\zeta^{\nu}\varphi_{,\nu}\rangle+\mathcal{O}(\kappa^{3}),$$

to the quadratic order in κ. A vertical bar separates the (elementary or composite) field of momentum p (to the left) from the one of momentum $-p$ (to the right). The diagrams are shown in fig. 1.

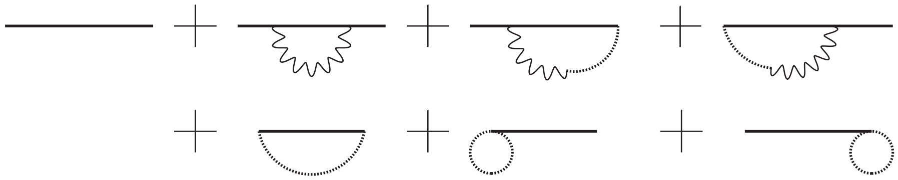

Figure 1: Two-point function of the dressed scalar field to order κ^2

The covariant gauge defined by (3.6) gives the ordinary scalar self-energy

$$\langle\varphi|\varphi\rangle = \frac{i}{p^2+i\epsilon} + \frac{3i\kappa^2(\lambda\lambda' - 3\lambda - \lambda' + 5)(\lambda' - 1)}{16\pi^2\varepsilon(\lambda' - 2)^2}(-p^2 - i\epsilon)^{-\varepsilon/2} + \mathcal{O}(\kappa^3).$$

Using the covariant cloud (4.11), the remaining diagrams of fig. 1 give the total

$$\langle\varphi_{\mathrm{d}}|\varphi_{\mathrm{d}}\rangle = \frac{i}{p^2+i\epsilon} + \frac{3i\kappa^2(\tilde\lambda\tilde\lambda' - 3\tilde\lambda - \tilde\lambda' + 5)(\tilde\lambda' - 1)}{16\pi^2\varepsilon(\tilde\lambda' - 2)^2}(-p^2 - i\epsilon)^{-\varepsilon/2} + \mathcal{O}(\kappa^3). \qquad (9.1)$$

The dependence on the gauge-fixing parameters λ and λ' has disappeared, as expected. The result depends on the choice of the cloud, through the parameters $\tilde\lambda$ and $\tilde\lambda'$, and satisfies formula (7.11), due to the gauge/cloud duality.

The off-shell absorptive part of the two-point function is defined by amputating the external legs and taking the real part, multiplied by minus 2 (see [8] for details). We find

$$\begin{aligned}\mathrm{Abso}[\langle\varphi_{\mathrm{d}}|\varphi_{\mathrm{d}}\rangle] &= -2\mathrm{Re}[(ip^2)\langle\varphi_{\mathrm{d}}|\varphi_{\mathrm{d}}\rangle(ip^2)] \\ &= -\frac{3\kappa^2(\tilde\lambda\tilde\lambda' - 3\tilde\lambda - \tilde\lambda' + 5)(\tilde\lambda' - 1)}{16\pi(\tilde\lambda' - 2)^2}(p^2)^2\theta(p^2) + \mathcal{O}(\kappa^3).\end{aligned}$$

The sign of the lowest-order contribution is positive or negative, depending on the cloud parameters $\tilde\lambda$ and $\tilde\lambda'$, so $\mathrm{Abso}[\langle\varphi_{\mathrm{d}}|\varphi_{\mathrm{d}}\rangle]$ is not physical.

In the case of the vector field, the undressed two-point function reads

$$\langle A_\mu|A_\nu\rangle = \langle A_\mu|A_\nu\rangle_0 + \frac{i\kappa^2\left(3\lambda - (1-2\lambda')^2\right)}{24\pi^2\varepsilon(\lambda' - 2)^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)(-p^2 - i\epsilon)^{-\varepsilon/2},$$

where $\langle A_\mu|A_\nu\rangle_0$ is the free propagator. After the gravitational dressing, we find

$$\langle A_{\mu\mathrm{d}}|A_{\nu\mathrm{d}}\rangle = \langle A_\mu|A_\nu\rangle_0 + \frac{i\kappa^2\left(3\tilde\lambda - (1-2\tilde\lambda')^2\right)}{24\pi^2\varepsilon(\tilde\lambda' - 2)^2}\left[\eta_{\mu\nu} + f(\lambda,\lambda',\tilde\lambda,\tilde\lambda',\lambda_A)\frac{p_\mu p_\nu}{p^2}\right](-p^2 - i\epsilon)^{-\varepsilon/2},$$

where f is a function that we do not report here, while λ_A is the gauge-fixing parameter of the A_μ propagator. To get rid of λ_A, we must include a gauge dressing for A_μ (besides the gravitational dressings we have already included). This operation is straightforward, since it is sufficient to consider the two-point function $\langle F_{\mu\nu\mathrm{d}}|F_{\rho\sigma\mathrm{d}}\rangle$ of the field strength. We obtain $\langle F_{\mu\nu\mathrm{d}}|F_{\rho\sigma\mathrm{d}}\rangle = \langle F_{\mu\nu}|F_{\rho\sigma}\rangle_{\lambda\rightarrow\tilde{\lambda},\lambda'\rightarrow\tilde{\lambda}'}$, in agreement with (7.11). The absorptive part reads

$$\begin{aligned}\mathrm{Abso}[\langle F_{\mu\nu\mathrm{d}}|F_{\rho\sigma\mathrm{d}}\rangle] &= -2\mathrm{Re}[(ip^2)\langle F_{\mu\nu\mathrm{d}}|F_{\rho\sigma\mathrm{d}}\rangle(ip^2)] \\ &= -\frac{\kappa^2\left(3\tilde{\lambda}-(1-2\tilde{\lambda}')^2\right)}{24\pi(\tilde{\lambda}'-2)^2}(p^2)^2\theta(p^2)(\eta_{\mu\rho}p_\nu p_\sigma-\eta_{\mu\sigma}p_\nu p_\rho-\eta_{\nu\rho}p_\mu p_\sigma+\eta_{\nu\sigma}p_\mu p_\rho).\end{aligned}$$

Again, it is not physical.

10 Purely virtual clouds and physical absorptive parts

To find physical absorptive parts, we must turn to purely virtual clouds. This is achieved as follows. The free propagators mix the graviton field $h_{\mu\nu}$ and the cloud field ζ^μ. Inside the propagators, we can distinguish three types of poles in p^2: the physical poles, the gauge-trivial poles and the cloud poles. The cloud poles are those introduced by the cloud, and appear in $\langle h_{\mu\nu}|\zeta^\rho\rangle_0$ and $\langle\zeta^\rho|\zeta^\sigma\rangle_0$. The gauge-trivial poles are those involving the unphysical components of $h_{\mu\nu}$, which are h_{00}, h_{0i}, the longitudinal components $p^j h_{ij}(p)$ and the trace h_{ii} (in some reference frame), where i,j are space indices. The physical poles are the remaining ones.

The three classes of poles can be clearly distinguished in the special gauge of ref. [22], which can be extended to the could sector straightforwardly. The gauge fermion (3.6) is replaced by

$$\begin{aligned}\Psi(\Phi) &= \int\bar{C}^0\left(G_0(g)-\frac{\lambda+3}{4}B_0\right)+\int\bar{C}^i\left(G_i(g)-\lambda\frac{\lambda+3}{4}B_i\right), \\ G_0(g) &= \lambda\partial_0 h_{00}+\partial_0 h_{ii}-2\partial_i h_{0i}, \\ G_i(g) &= 2\lambda\partial_0 h_{0i}-\lambda\partial_i h_{00}-\frac{\lambda+3}{2}\partial_j h_{ij}+\frac{\lambda+1}{2}\partial_i h_{jj}.\end{aligned}$$

The cloud fermion (4.5) is replaced by an analogous formula, with $\lambda\rightarrow\tilde{\lambda}$, $\bar{C}^\mu\rightarrow\bar{H}^\mu$, $B^\mu\rightarrow E^\mu$, $G_\mu\rightarrow V_\mu$, $h_{\mu\nu}\rightarrow h_{\mu\nu\mathrm{d}}=h_{\mu\nu}-(\partial_\mu\zeta_\nu+\partial_\nu\zeta_\mu)/2+\mathcal{O}(\kappa)$.

We do not report the free propagators explicitly, because they are quite lengthy and not strictly necessary for our calculations (see [8] for their expressions in gauge theories). We just report that they contain only single poles (which is what makes the special gauge "special"), and that the gauge-trivial poles are located at $\lambda E^2-\mathbf{p}^2=0$ and $4\lambda E^2-\mathbf{p}^2(3+\lambda)=0$, the cloud poles are located at $\tilde{\lambda}E^2-\mathbf{p}^2=0$ and $4\tilde{\lambda}E^2-\mathbf{p}^2(3+\tilde{\lambda})=0$, and the physical poles are obviously located at $E^2 - \mathbf{p}^2 = 0$, where $p^\mu = (E, \mathbf{p})$ is the propagator momentum. What is important is that a unique gauge-fixing parameter, λ, and a unique cloud parameter, $\tilde{\lambda}$, are sufficient to distinghuish the three classes of poles in a manifest way.

The cloud poles must be quantized as purely virtual [10]. This means that, after performing the threshold decomposition of a diagram as explained in ref. [10], the (multi) thresholds receiving contributions from those poles must be removed. The gauge-trivial poles can be quantized in the way we want (because the correlation functions of the dressed fields are gauge independent). The physical poles must be quantized by means of the Feynman $i\epsilon$ prescription.

It is convenient to quantize the gauge-trivial poles as purely virtual as well, like the cloud poles. Since the purely virtual poles do not contribute to the absorptive parts of the two-point functions at one loop, we can just ignore all of them.

At the end, each calculation amounts to just one diagram, the usual self-energy diagram (second drawing of fig. 1), with a caveat: we must replace the internal graviton and vector propagators with their physical parts, which are

$$\begin{aligned} \langle A^{\mu a}|A^{\nu b}\rangle_{0\text{phys}} &= i\frac{\delta^{ab}\delta_i^\mu\delta_j^\nu\Pi^{ij}}{p^2+i\epsilon}, \\ \langle h_{\mu\nu}|h_{\rho\sigma}\rangle_{0\text{phys}} &= \frac{i}{2}\frac{\delta_i^\mu\delta_j^\nu\delta_k^\rho\delta_l^\sigma}{p^2+i\epsilon}(\Pi^{ik}\Pi^{jl}+\Pi^{il}\Pi^{jk}-\Pi^{ij}\Pi^{kl}), \end{aligned} \tag{10.1}$$

where $\Pi^{ij} = \delta^{ij} - (p^i p^j/\mathbf{p}^2)$.

In the scalar case, the absorptive part of $\langle\varphi_\text{d}|\varphi_\text{d}\rangle$ turns out to be zero. We can partially understand this result by noting that the optical theorem relates it to the cross section of a process (graviton emission by a scalar field), which cannot occur on shell. Nevertheless, the correlation function we are studying is not on shell. Yet, the result is still zero, due to the graviton polarizations, which are implicit in (10.1).

In the case of the vectors, we find, at rest,

$$\text{Abso}[\langle A_{i\text{d}}|A_{j\text{d}}\rangle] = -2\text{Re}[(ip^2)\langle A_{i\text{d}}|A_{j\text{d}}\rangle(ip^2)] = \frac{\kappa^2\delta_{ij}}{12\pi}(p^2)^2\theta(p^2) + \mathcal{O}(\kappa^3), \tag{10.2}$$

which is positive, as expected.

Finally, the absorptive part of the dressed graviton two-point function (in pure gravity, at rest) is

$$\begin{aligned} \text{Abso}[\langle h_{ij\text{d}}|h_{kl\text{d}}\rangle] &= -2\text{Re}[(ip^2)\langle h_{ij\text{d}}|h_{kl\text{d}}\rangle(ip^2)] \\ &= \frac{\kappa^2(3\delta_{ik}\delta_{jl}+3\delta_{il}\delta_{jk}-2\delta_{ij}\delta_{kl})}{80\pi}(p^2)^2\theta(p^2) + \mathcal{O}(\kappa^3), \end{aligned} \tag{10.3}$$

which is again positive definite.

Now we switch to the theory of quantum gravity with purely virtual particles [11]. It is convenient to formulate it in the variables of ref. [21], to gain an explicit distinction among the graviton, the inflaton ϕ and the massive purely virtual spin-2 particle $\chi_{\mu\nu}$. We can actually ignore $\chi_{\mu\nu}$, because it does not contribute to the absorptive parts that we want to compute. Neglecting the cosmological constant, the relevant terms of the action are

$$-\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R+\frac{1}{2}\int\sqrt{-g}\left[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)-\frac{3m_\phi^2}{2\kappa^2}\left(1-\mathrm{e}^{\kappa\phi\sqrt{2/3}}\right)^2\right]$$
$$+\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}\mathrm{e}^{\kappa\phi\sqrt{2/3}}(\partial_\mu\varphi)(\partial_\nu\varphi)-\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}.$$

The absorptive part (10.2) of the vector two-point function does not change. The one of the scalar two-point function is no longer zero, because it receives a contribution from the inflaton. In the high-energy limit (where we can neglect the mass m_ϕ), we find

$$\mathrm{Abso}[\langle\varphi_{\mathrm{d}}|\varphi_{\mathrm{d}}\rangle]=\frac{\kappa^2}{48\pi}(p^2)^2\theta(p^2)+\mathcal{O}(\kappa^3).$$

Switching off the matter sector, the absorptive part (10.3) of the dressed graviton two-point function also receives a correction from the inflaton ϕ, and the final result is (10.3) multiplied by 19/18.

11 Renormalization

In this section we study the renormalization of the extended theory. We assume that the starting theory of quantum gravity is renormalizable by power counting, like the theory based on purely virtual particles of ref. [11]. To have better power-counting behaviors, it may be convenient to use a higher-derivative gauge-fixing, as in [23], and higher-derivative clouds as well. The cloud fields $\zeta^{\mu i}$ have dimension minus one in units of mass, so non-polynomial functions of them are turned on by renormalization.

The theory of [11] is unitary. If the clouds are purely virtual, the complete dressed theory is unitary as well. However, the arguments of this section do not rely on unitarity, so the results we obtain also apply to nonunitary clouds, and even nonunitary theories, such as the Stelle theory [24], where the Feynman prescription is used for the quantization of the every field (and so $\chi_{\mu\nu}$ is a ghost).

When the arguments work for gravity exactly as they do for gauge theories, we skip the details of the proofs. The reader should refer to [8] for the missing derivations.

First, the master equation (4.9) satisfied by S_{tot} implies an analogous master equation

$$(\Gamma_{\text{tot}}, \Gamma_{\text{tot}}) = 0 \tag{11.1}$$

for the generating functional $\Gamma_{\text{tot}} = W_{\text{tot}}(J, \tilde{J}, K, \tilde{K}) - \int \Phi^\alpha J_\alpha - \sum_i \int \tilde{\Phi}^{\alpha i} \tilde{J}^i_\alpha$ of the one-particle irreducible (1PI) Green functions, where $\Phi^\alpha = \delta_r W_{\text{tot}}/\delta J^\alpha$, $\tilde{\Phi}^{\alpha i} = \delta_r W_{\text{tot}}/\delta \tilde{J}^{\alpha i}$. Second, the ith cloud invariance (4.10) of the total action S_{tot}, which is the identity $(S_K^{\text{cloud}\, i}, S_{\text{tot}}) = 0$, implies the ith cloud invariance

$$(S_K^{\text{cloud}\, i}, \Gamma_{\text{tot}}) = 0 \tag{11.2}$$

of the Γ functional.

Proceeding inductively, we can show that the total renormalized action $S_{R\text{tot}}$ satisfies the renormalized master equations

$$(S_{R\text{tot}}, S_{R\text{tot}}) = 0, \qquad (S_K^{\text{cloud}\, i}, S_{R\text{tot}}) = 0. \tag{11.3}$$

Since the cloud symmetry is the most general shift of the cloud fields, the second equation ensures that the total renormalized action is the sum of a cloud-independent renormalized action S_R and some cloud-exact rest. Separating S_K^{cloud} itself, which is non-renormalized, we can write

$$S_{R\text{tot}} = S_R + (S_K^{\text{cloud}}, \Upsilon_R) + S_K^{\text{cloud}}$$

for some local functional Υ_R. It is possible to extend the proof of gauge independence of section 5 to $S_{R\text{tot}}$ and show that S_R is cloud independent and coincides with the usual renormalized action.

Moreover, the dependence on the gauge-fixing parameters and the dependences on the cloud parameters go through renormalization as canonical transformations. In particular, the beta functions of the physical parameters are gauge independent and cloud independent.

Some simplification comes from the introduction of "cloud numbers", besides the usual ghost number. The usual ghost number is defined to be equal to 1 for C^μ, minus 1 for $\bar{C}^\mu$, K_μ^B, $K_g^{\mu\nu}$, K_A^μ, K_φ, $\tilde{K}_\mu^{\zeta i}$ and $\tilde{K}_\mu^{Hi}$, minus 2 for K_μ^C, and 0 for every other field and source. The ith cloud number is defined to be equal to one for $H^{\mu i}$, minus one for $\bar{H}^{\mu i}$, $\tilde{K}_\mu^{Hi}$ and $\tilde{K}_\mu^{Ei}$, and zero in all the other cases.

Every term of the action S_{tot} is neutral with respect to the ghost and cloud numbers just defined, with the exception of the source terms $\int H^{\mu i} \tilde{K}_\mu^{\zeta i}$. Since, however, such terms cannot be used in nontrivial 1PI diagrams, all the counterterms are neutral. This ensures that each cloud number is separately conserved in 1PI diagrams beyond the tree level.

Power counting is not very helpful in the cloud sectors, since the cloud fields $\zeta^{\mu i}$ have negative dimensions. A simplification can be achieved by combining the background-field method with the Batalin-Vilkovisky formalism, as shown in ref. [25]. So doing, the gauge and cloud transformations are not renormalized. Yet, the cloud sectors are nonrenormalizable, strictly speaking, since infinitely many counterterms are allowed by power counting and the symmetry constraints. For example, we can always build gauge invariant candidate counterterms that depend nontrivially on the fields of each cloud sector and are exact under every cloud symmetry. Examples are

$$(S_K^{\mathrm{cloud}\,1},(S_K^{\mathrm{cloud}\,2},\ldots(S_K^{\mathrm{cloud}\,N},\tilde{\Upsilon}))),$$

where N denotes the number of clouds, and $\tilde{\Upsilon}$ is a gauge invariant local functional, built with gauge invariant combinations of cloud fields, such as (8.5). We cannot exclude that different cloud sectors mix under renormalization. Nevertheless, we can prove that the counterterms that do not contain fields and sources of some cloud sector are the same as if that sector were absent.

The renormalization in every non-background-field approach can be reached by means of a (renormalized) canonical transformation. Details on this can be found in ref. [25].

Equipped with the renormalized action and the renormalized gauge transformations, we can build dressed fields that are gauge-invariant with respect to the latter. The correlation functions of the renormalized dressed fields are gauge independent.

Finally, the arguments that lead to the identity (5.6) continue to hold after renormalization. We have an identity analogous to (5.6), where the dressed and undressed fields are replaced by their renormalized versions. In particular, the S matrix amplitudes of the renormalized dressed fields are cloud independent and coincide with the usual S matrix amplitudes of the renormalized undressed fields. Since the former are gauge independent by construction, the latter are gauge independent as well.

12 Conclusions

We have extended quantum field theory to include purely virtual cloud sectors, to study point-dependent physical observables in general relativity and quantum gravity, with particular emphasis on gauge invariant versions of the metric and the matter fields. The cloud diagrammatics and its Feynman rules are derived from a local action, which is built by means of cloud fields $\zeta^{\mu i}$ and their anticommuting partners $H^{\mu i}$. It incorporates the choices of clouds, the cloud Faddeev-Popov determinants and the cloud symmetries.

The formalism allows us to define physical, off-shell correlation functions of point-dependent observables, and calculate them within the realm of perturbative quantum field theory. Every field insertion can be equipped with its own, independent cloud. We may eventually replace the elementary fields with the dressed ones everywhere, to work in a manifestly gauge independent environment.

The extension does not change the fundamental physics, in the sense that the ordinary correlation functions and the S matrix amplitudes are unmodified. If the clouds are quantized as purely virtual, the extended theory is unitary. In particular, the correlation functions of the dressed fields obey the off-shell, diagrammatic version of the optical theorem. No unwanted degrees of freedom propagate.

A Batalin-Vilkovisky formalism and its Zinn-Justin master equations allow us to study renormalizability and the WTST identities to all orders in the perturbative expansion. A gauge/cloud duality shows that the usual gauge-fixing is nothing but a particular cloud, provided it is rendered purely virtual. A purely virtual gauge-fixing is a natural upgrade of the so-called physical gauges [26].

We have illustrated the key properties of our approach by computing the one-loop two-point functions of dressed scalars, vectors and gravitons, and comparing purely virtual clouds to non-purely virtual clouds, in Einstein gravity as well as in quantum gravity with purely virtual particles. If purely virtual clouds are used, the absorptive parts are positive, cloud independent and gauge independent. This suggests that they are properties of the fundamental theory. The absorptive parts are not positive, in general, if non-purely virtual clouds are used.

Pure virtuality could be a natural environment to extract physical information from off-shell correlation functions. Among the other things, it allows us to break global invariances without breaking the local ones, which may have undesirable consequences on unitarity.

We did not introduce true matter to define the metric as a physical observable. Instead, we used purely virtual dressings. In this sense, our approach provides the identitication of a complete set of observables in quantum gravity. Yet, it raises new issues. First, pure virtuality is an instrinsically quantum notion, so the classical limit of what we have done here deserves an investigation of its own. Although we may view the completeness of observables in general relativity as inherited from the one of quantum gravity, it may be redundant to do so, since we should not need quantum gravity to understand what occurs around us. In particular, it would be interesting to clarify the relation between the Komar-Bergmann classical approach [2] and the one formulated here, as well as investigate the nonlocal nature of the algebra of commutators (check [5] for this aspect in the Donnelly-Giddings approach). In this respect, it is important to recall that pure virtuality at the operatorial level is still awaiting to be understood (the formulation we have today being mainly diagrammatic [10]), so we may not be ready to use the completeness mentioned above for a canonical analysis and a Hamiltonian quantization.

The ideas of this paper may raise more questions than they provide answers to. Certainly, they make the whole story about point-dependent observables in general relativity and quantum gravity more interesting.

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References

[1] J. Géhéniau and R. Debever, Les quatorze invariants de courbure de l'espace Riemannien a quatre dimensions, Helv. Phys. Acta Suppl. 4 (1956) 101;

[2] A.B. Komar, Construction of a complete set of independent observables in the general theory of relativity, Phys. Rev. 111 (1958) 1182;

P. Bergmann and A. Komar, Poisson brackets between locally defined observables in general relativity, Phys. Rev. Lett. 4 (1960) 432;

P.G. Bergmann, Conservation laws in general relativity as the generators of coordinate transformations, Phys. Rev. 112 (1958) 287;

P.G. Bergmann, Observables in general relativity, Rev. Mod. Phys. 33 (1961) 510.

[3] B. De Witt, in "Gravitation: An Introduction to Current Research", L. Witten ed. (Wiley, New York,1962);

B. de Witt, Quantum theory of gravity. I. The canonical theory, Phys. Rev. 160 (1967) 1113;

J. Earman and J. Norton, What price spacetime substantivalism? The Hole Story, British Journal for the Philosophy of Science 38 (1987) 515;

J. Earman, "World enough and space-time: Absolute versus Relational Theories of Spacetime", (MIT Press, Cambridge 1989);

C. Rovelli, What is observable in classical and quantum gravity?, Class. Quant. Grav. 8 (1991) 297;

J.D. Brown and D. Marolf, Relativistic material reference systems, Phys. Rev. D 53 (1996) 1835.

[4] C. Rovelli, GPS observables in general relativity, Phys. Rev. D 65 (2002) 044017 and arXiv:gr-qc/0110003.

[5] W. Donnelly and S.B. Giddings, Diffeomorphism-invariant observables and their non-local algebra, Phys. Rev. D 93 (2016) 024030 and arXiv:1507.07921 [hep-th];

W. Donnelly and S.B. Giddings, Observables, gravitational dressing, and obstructions to locality and subsystems Phys. Rev. D 94 (2016) 104038 and arXiv:1607.01025 [hep-th].

[6] P. A. M. Dirac, Gauge invariant formulation of quantum electrodynamics, Can. J. Phys. 33 (1955) 650.

[7] M. Lavelle and D. McMullan, Observables and gauge fixing in spontaneously broken gauge theories, Phys. Lett. B347 (1995) 89 and arXiv:hep-ph/9412145;

M. Lavelle and D. McMullan, The color of quarks, Phys. Lett. B 371 (1996) 83 and arXiv:hep-ph/9509343.

M. Lavelle and D. McMullan, Constituent quarks from QCD, Phys. Rept. 279 (1997) 1 and arXiv:hep-ph/9509344.

[8] D. Anselmi, Purely virtual extension of quantum field theory for gauge invariant fields: Yang-Mills theory, 22A3 Renorm and arXiv:2207.11271 [hep-ph].

[9] R.E. Cutkosky, Singularities and discontinuities of Feynman amplitudes, J. Math. Phys. 1 (1960) 429;

M. Veltman, Unitarity and causality in a renormalizable field theory with unstable particles, Physica 29 (1963) 186;

G. 't Hooft, Renormalization of massless Yang-Mills fields, Nucl. Phys. B 33 (1971) 173;

G. 't Hooft, Renormalizable Lagrangians for massive Yang-Mills fields, Nucl. Phys. B 35 (1971) 167;

G. 't Hooft and M. Veltman, *Diagrammar*, CERN report CERN-73-09;

M. Veltman, *Diagrammatica. The path to Feynman rules* (Cambridge University Press, New York, 1994).

[10] D. Anselmi, Diagrammar of physical and fake particles and spectral optical theorem, J. High Energy Phys. 11 (2021) 030, 21A5 Renormalization.com and arXiv: 2109.06889 [hep-th].

[11] D. Anselmi, On the quantum field theory of the gravitational interactions, J. High Energy Phys. 06 (2017) 086, 17A3 Renormalization.com and arXiv: 1704.07728 [hep-th].

[12] D. Anselmi, E. Bianchi and M. Piva, Predictions of quantum gravity in inflationary cosmology: effects of the Weyl-squared term, J. High Energy Phys. 07 (2020) 211, 20A2 Renormalization.com and arXiv:2005.10293 [hep-th].

[13] D. Anselmi, K. Kannike, C. Marzo, L. Marzola, A. Melis, K. Müürsepp, M. Piva and M. Raidal, Phenomenology of a fake inert doublet model, J. High Energy Phys. 10 (2021) 132, 21A3 Renormalization.com and arXiv:2104.02071 [hep-ph].

[14] D. Anselmi, K. Kannike, C. Marzo, L. Marzola, A. Melis, K. Müürsepp, M. Piva and M. Raidal, A fake doublet solution to the muon anomalous magnetic moment, Phys. Rev. D 104 (2021) 035009, 21A4 Renormalization.com and arXiv:2104.03249 [hep-ph].

[15] C.G. Bollini and J.J. Giambiagi, Lowest order divergent graphs in ν-dimensional space, Phys. Lett. B40 (1972) 566;

G.t Hooft and M.Veltman, Regularization and renormalization of gauge fields, Nucl. Phys. B 44 (1972) 189;

G.M. Cicuta and E. Montaldi, Analytic renormalization via continuous space dimension, Lett. Nuovo Cim. 4 (1972) 329.

[16] I.A. Batalin and G.A. Vilkovisky, Gauge algebra and quantization, Phys. Lett. B 102 (1981) 27;

I.A. Batalin and G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generators, Phys. Rev. D 28 (1983) 2567, Erratum-ibid. D 30 (1984) 508.

[17] J.C. Ward, An identity in quantum electrodynamics, Phys. Rev. 78, (1950) 182;

Y. Takahashi, On the generalized Ward identity, Nuovo Cimento, 6 (1957) 371;

A.A. Slavnov, Ward identities in gauge theories, Theor. Math. Phys. 10 (1972) 99;

J.C. Taylor, Ward identities and charge renormalization of Yang-Mills field, Nucl. Phys. B33 (1971) 436.

[18] L.D. Faddeev and V. Popov, Feynman diagrams for the Yang-Mills field, Phys. Lett. B 25 (1967) 29.

[19] N. Nakanishi, Covariant quantization of the electromagnetic field in the Landau gauge, Prog. Teor. Phys. 35 (1966) 1111;

B. Lautrup, Canonical quantum electrodynamics in covariant gauges, Kgl. Dan. Vid. Se. Mat. Fys. Medd. 35 (11) (1967) 1.

[20] J. Zinn-Justin, Renormalization of gauge theories, Bonn lectures 1974, published in *Trends in Elementary Particle Physics*, Lecture Notes in Physics 37 (1975) 1, H. Rollnik and K. Dietz eds., Springer Verlag, Berlin.

[21] D. Anselmi and M. Piva, Quantum gravity, fakeons and microcausality, J. High Energy Phys. 11 (2018) 21, 18A3 Renormalization.com and arXiv:1806.03605 [hep-th].

[22] D. Anselmi, Aspects of perturbative unitarity, Phys. Rev. D 94 (2016) 025028, 16A1 Renorm and arXiv:1606.06348 [hep-th].

[23] D. Anselmi and M. Piva, The ultraviolet behavior of quantum gravity, J. High Energ. Phys. 05 (2018) 27, 18A2 Renormalization.com and arXiv:1803.07777 [hep-th].

[24] K.S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D 16 (1977) 953.

[25] D. Anselmi, Background field method and the cohomology of renormalization, Phys. Rev. D 93 (2016) 065034 and arXiv:1511.01244 [hep-th].

[26] P. Gaigg, W. Kummer and M. Schweda (eds.), *Physical and Nonstandard Gauges*, Lecture Notes in Physics 361, Springer Verlag, Heidelberg, 1990.