

# Fakeons versus Lee-Wick Ghosts: Physical Pauli-Villars Fields, Finite QED and Quantum Gravity

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## **Abstract**

We reconsider the Lee-Wick (LW) models and compare their properties to the properties of the models that contain purely virtual particles. We argue against the LW premise that unstable particles can be removed from the sets of incoming and outgoing states in scattering processes. The removal leads to a non-Hermitian classical limit, besides clashing with the observation of the muon. If, on the other hand, all the states are included, the LW models have an unbounded Hamiltonian or negative norms. Purely virtual particles, on the contrary, lead to a Hermitian classical limit and are absent from the sets of incoming and outgoing states without implications on the observation of long-lived unstable particles. We give a vademecum to summarize the properties of most options. We study a method to remove the LW ghosts only partially, by saving the physical particles they contain. Specifically, we replace a LW ghost with a certain superposition of a fakeon and an observable particle, and drop only the former from the sets of the external states. The trick can be used to make the Pauli-Villars fields consistent and observable, without sending their masses to infinity, or build a finite QED, by tweaking the original Lee-Wick construction. However, it has issues with general covariance, so it cannot be applied as is to quantum gravity, where a manifestly covariant decomposition requires the introduction of a massive spin 2 multiplet.

# 1 Introduction

In the late 1960's Lee and Wick proposed a way to give sense to models that contain fields with negative kinetic terms [1, 2]. A key point of their idea is that “abnormal” particles do not belong to the spectrum of asymptotic states, as long as they are unstable. In their approach, it is sufficient that all the stable particle states have positive square lengths. The purpose of Lee and Wick was to provide a unitary  $S$  matrix in the subspace of stable states, by extending the previous results on unitarity [3, 4, 5, 6, 7]. In this paper, we reconsider the Lee-Wick (LW) models, concentrating on the treatment of unstable particles.

The muons are unstable elementary particles that can be observed directly before they decay. Tauon traces can also be observed in special situations. Composite long-lived particles are more common, but their relatively long lifetimes can be due to their compositeness. The other elementary particles are stable, short-lived, or confined. The resonances can in principle be boosted enough to be detected as particles before they decay. In light of these remarks, it does not seem so justified to remove a particle from the set of asymptotic states just because it is unstable.

In the context of the Lee-Wick models, it is actually sufficient to remove the abnormal particles from the sets of the external states and keep the physical particles as usual. The advantage of this option is that it does not clash with the observation of the muon. Nevertheless, it leads to an unacceptable classical limit.

The classical limit is defined by the tree diagrams that have physical particles in the external legs and no physical particles in the internal legs. We can also define a “reduced” action, which is the effective action obtained by integrating out the abnormal particles. It collects all the diagrams that have physical particles in the external legs and no physical particles in the internal legs, and includes the loops of abnormal particles as effective vertices. If we want to define a fundamental theory by removing particles, the reduced action should be seen as the “classical” action of that fundamental theory.

The abnormal particles propagating in the internal legs generate nonlocal, acausal, non-Hermitian effective self-interactions among the physical particles. Violations of locality and causality in the classical action are not excluded by the requirements of internal consistency, as long as they are microscopic and compatible with data, which occurs (for example) if the masses of the abnormal particles are sufficiently large. On the other hand, a classical Lagrangian with a nonvanishing imaginary part is troubling. The simplest explanation for such an instance is that something has been provisionally integrated out, which is precisely what is going on in the case we are considering. If we reinstate the missing entity (which

is the Lee-Wick abnormal particle) as an independent degree of freedom, we remove the imaginary part of the classical Lagrangian, but go back to the initial problem of negative kinetic terms (unbounded free Hamiltonian). This either/or situation is the trouble with the Lee-Wick models.

For most purposes, the muon can be treated as a stable particle, since its width is very small (around  $10^{-19}\text{GeV}$ ). If we resum the muon self-energies into the dressed propagator, as is commonly done for resonances, we find that the theory predicts no muon observation [8]. The reason is that we cannot observe an unstable particle with infinite resolving power on the energy, because this occurrence violates the energy-time uncertainty relation. Once the energy resolution  $\Delta E$  of the experimental setup is inserted explicitly, the problem disappears [8]. We can argue in a similar way for every resonance, if we imagine to boost it enough to make it detectable as a particle. Indeed, the resonances and the muon just differ by the magnitudes of their widths. We conclude that in a sound theoretical framework, unstable particles should be included among the external states.

In this paper, we compare the Lee-Wick idea to several other options, including physical particles, ordinary ghosts and purely virtual particles. We also consider the effects of the removals of those from the sets of incoming and outgoing states. By “ghost” we mean a degree of freedom that appears with a negative kinetic term in the classical Lagrangian.

Purely virtual particles, or fake particles, or “fakeons”, are based on a new diagrammatics [9]. They allow us to formulate a consistent theory of quantum gravity [10], which is experimentally testable due to its sharp prediction of the tensor-to-scalar ratio in inflationary cosmology [11]. They can also be used to search for new physics beyond the standard model, by evading common constraints in collider phenomenology [12] and offering new possibilities to solve discrepancies with data [13]. The only requirement is that fakeons are massive and nontachyonic. Their diagrammatics can be implemented in softwares like FeynCalc, FormCalc, LoopTools and Package-X [14].

Unlike the LW abnormal particles (which we call “LW ghosts” from now on), fakeons lead to a Hermitian classical limit and a Hermitian reduced action. Their absence from the sets of incoming and outgoing states has no implications on the observation of long-lived unstable particles. The reason is that the fakeons are purely virtual. Instead, the LW ghosts are not purely virtual, which is why they leave an imaginary remnant in the classical limit, once they are removed.

After reconsidering the Lee-Wick construction, we formulate a procedure that is as close as possible to the idea of removing the LW ghost from the sets of external states only

partially and save the physical degree of freedom it contains. Specifically, we switch to a theory of particles and fakeons by replacing the LW ghost with a certain superposition of a fakeon and an observable particle, and remove only the former. The trick works with neutral matter fields and can be used to make the Pauli-Villars fields consistent, and observable, without sending their masses to infinity. It also allows us to build a finite QED, by overcoming the difficulties of the original Lee-Wick construction. In quantum gravity, the method could lead to an extra (observable) massive spin 2 particle. However, a number of unresolved issues with general covariance (and gauge invariance) show that it cannot be applied to gravity as is. A covariant decomposition can be achieved by adding a massive spin 2 multiplet (which can be done in a unitary and renormalizable way as explained in [15]). However, this procedure just gives the theory of [10] coupled to matter in a peculiar way.

We do not cover all the proposals available in the literature about ghosts. Among the missing ones, we mention the PT symmetric approach of Berends and Manheim [16].

The removal of degrees of freedom from the incoming and outgoing states is consistent only if it is compatible with unitarity, in which case we call it “projection” and call the action “projected action”. The fakeon projection is compatible with unitarity order by order (and diagram by diagram) in the perturbative expansion (see for example [9]). The removal of unstable particles (which we call Veltman’s projection, see below) is compatible with unitarity in a semi-perturbative approach, because the self-energies of unstable particles must be resummed into their dressed propagators. After this diagrammatic reorganization, it is also valid diagram by diagram.

Since our approach is perturbative, we require the Hamiltonian to be bounded from below in the free-field limit (in flat space), both classically and at the quantum level. Once a particle is projected away, it is no longer relevant to the issue, because it disappears from the free-field limit. We have no way to say whether the Hamiltonian is bounded or not in the complete theory. Moreover, the nonlocalities generated by the projection, which survive the classical limit, normally affect only high orders, where they compete with the quantum corrections, which are nonlocal anyway. For example, in [17] it is shown that, in primordial cosmology, the fakeon projection leaves the theory practically local for various orders of the perturbative expansion.

The paper is organized as follows. In section 2 we discuss Veltman’s projection and the issue of unitarity with unstable particles. In section 3 we compare various options for the quantization of fields with negative kinetic terms. In section 4 we briefly recall how such

fields are treated inside the loop diagrams. In section 5 we present the trick that makes the Pauli-Villars physical by means of fakeons. In section 6 we apply it to build a finite QED. In section 7 we discuss the obstacles we meet when we apply the same method to quantum gravity. Section 8 contains the conclusions.

## 2 Veltman's projection

A result due to Veltman states that the  $S$  matrix constructed with the dressed propagators and connecting stable particle states only is unitary [4]. Because of this, unstable particles can be consistently dropped from the sets of incoming and outgoing states of the scattering processes. We call this removal Veltman's projection. The  $S$  matrix obtained from it is called reduced (or projected)  $S$  matrix and denoted by  $S_r$ .

Veltman's result  $S_r^\dagger S_r = 1$  follows from the common proofs of perturbative unitarity by means of cut diagrams [4, 3, 5, 6, 7]. When an unstable particle is projected away by means of Veltman's projection, it generates effective nonlocal, non-Hermitian interactions among the other particles (see, for example, formula (3.13) below). In general, non-Hermitian interactions are problematic for unitarity, but in the case of Veltman's projection they are precisely what makes the unitarity equation  $S_r^\dagger S_r = 1$  hold true. At the end of this section, we briefly recall how this happens and also show that Veltman's projection preserves CPT invariance.

Veltman considered stable and unstable particles. To apply Veltman's projection to ghosts, we should first ensure that Veltman's results extend to them. There are various options to treat ghosts at the quantum level.

The simplest possibility, which we call  $i\epsilon$  ghost, is the standard quantization by means of the Feynman  $i\epsilon$  prescription, which means that we choose the free propagator

$$-\frac{i}{p^2 - M^2 + i\epsilon} \quad (2.1)$$

and integrate on Minkowski spacetime as usual. In ref. [8] it has been shown that the dressed propagators of the  $i\epsilon$  ghosts do not make sense close to the peaks, because the resummation of the perturbative expansion does not exist there. Having no knowledge about the nonperturbative sector of the theory, Veltman's projection cannot be applied to the  $i\epsilon$  ghosts. This is not a big deal, since the  $i\epsilon$  ghosts violate unitarity.

Other options to quantize ghosts have different properties. The second possibility, which we call  $-i\epsilon$  ghost, is to choose the free propagator

$$-\frac{i}{p^2 - M^2 - i\epsilon}. \quad (2.2)$$

Then we cannot integrate on Minkowski spacetime, because if we do so we run into the consistency problems described in ref. [18], which means nonlocal divergent parts, exchanges of roles between the usual thresholds and the pseudothresholds, instabilities, violations of unitarity, etc. Lee and Wick proposed a different set of rules for handling (2.2) in Feynman diagrams, which must be combined with the Cutkosky *et al.* (CLOP) prescription [19] and possibly other rules, to solve the ambiguities mentioned in [19] (see also [20]). For the purposes of this paper, we can just assume that a complete set of rules does exist. At the end, the  $-i\epsilon$  ghost turns into a new type of object, which we call “LW ghost”.

More importantly, the idea of Lee and Wick is to arrange the model so that the interactions make the ghost unstable, to apply Veltman’s projection to it. To this purpose, we note that the dressed propagator makes sense, even close to the peak, where it reads<sup>1</sup>

$$-\frac{iZ}{p^2 - M_{\text{gh}}^2 - i(\epsilon + M_{\text{gh}}\Gamma)}. \quad (2.3)$$

Here,  $\Gamma$  is a positive width,  $M_{\text{gh}}^2 = M^2 + \Delta M^2$  is the “physical” mass squared and  $Z$  is the normalization factor. The  $\epsilon$  prescription is there to show that the corrections proportional to  $\Gamma$  have the same sign. According to the arguments of [8], it is correct to extend the resummation of the self energies from the convergence region to the peak region by means of analyticity.

Recapitulating, Lee and Wick get rid of ghosts by turning them into LW ghosts and arranging the model so that they are unstable or become so dynamically, to build a unitary reduced  $S$  matrix  $S_r$  à la Veltman.

Although Veltman’s result is correct, it does not suggest that we *should* drop unstable particles from the physical spectrum<sup>2</sup>. It simply proves that if we drop them, we get a unitary reduced scattering matrix  $S_r$ . The problem with  $S_r$  is that it turns a blind eye to the experimental observation of the muon.

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<sup>1</sup>Formula (2.3) is correct as is for legs that disconnect the diagram once they are broken. Inside loops we must use the rules mentioned previously.

<sup>2</sup>Actually, Veltman seems to suggest precisely that in [4], by saying that it is an undesirable feature of perturbation theory to have unstable particles among the asymptotic states. Our position, instead, is that a theory of scattering where processes end at the end of time is not satisfactory.

Normally, the incoming and outgoing states of a scattering process are assumed to be at  $t = -\infty$  and  $t = +\infty$ , respectively. This is a nonrealistic simplification, useful to derive general formulas. A more realistic assumption is  $\bar{\Delta}t \ll \Delta t < \infty$ , where  $\Delta t$  denotes the time separation between the incoming and outgoing states and  $\bar{\Delta}t$  is the duration of the interactions. This leaves room for including long-lived unstable particles, by assuming that  $\Delta t$  is smaller than the lifetimes of some of them. Once the scattering process ends and the outgoing particles fly away, there is no reason why we should wait till they decay, if we can catch them on the fly.

To establish an unambiguous terminology, we talk about “physical” spectra when we include everything we can physically observe, in practice and/or in principle. Clearly, the muon is included, among the other unstable particles. With the word “asymptotic”, we mean the same, i.e.,  $\Delta t \gg \bar{\Delta}t$ . Thus, the muon is also included in the set of “asymptotic states” and is part of the asymptotic spectra. Incoming and outgoing states that are literally taken at  $t = -\infty$  and  $t = +\infty$  will be called “strictly asymptotic states”.

Given that we never see resonances like the  $Z$  boson, one could ask why we should include them in the physical spectra. The reason is that a fundamental theory should be able to cover all the situations, including the ones that are nowadays out of reach experimentally.

Although Veltman’s projection is not acceptable for physical particles, because it forces us to drop the muon from the physical spectrum of the standard model, we could accept a restricted form of it, by applying it to the LW ghosts only. The restricted option is compatible with the observations of long-lived unstable particles. Nevertheless, Veltman’s projection has another problem, which concerns the classical limit.

Every unstable particle becomes stable in the classical limit, by definition. If we ignore unstable particles as asymptotic states at the quantum level, the classical limit cannot resuscitate them. This means that the reduced  $S$  matrix  $S_r$  does not correspond to an acceptable classical Lagrangian, typically because the latter turns out to be non-Hermitian.

Thus, even if there existed no muon in nature, or we applied Veltman’s projection to the LW ghosts only, the model would still not be good enough to define a fundamental quantum field theory, although it could be acceptable in the realm of effective field theories.

The theories with fakeons do not have these problems, because they are defined in a radically different way. In particular, fakeons are purely virtual particles, so they do not need to be unstable and decay to be removed from the physical spectrum, which they never enter in the first place. There is no implication on the observation of unstable long-lived particles like the muon. The classical limit is described by a Hermitian Lagrangian, which

collects, after the projection, anti-Hermitian effective vertices. Moreover, the diagrammatic analysis of [9] shows that all the effective vertices given by the 1PI diagrams with no fakeons in the external legs and no physical particles in the internal legs are anti-Hermitian, even if they close loops. Thus, the reduced action is Hermitian and the CPT theorem holds after the projection.

The reason why the LW ghosts leave an imaginary remnant in the classical limit is that they are not purely virtual. What Lee and Wick suggest, i.e., assume that they are unstable and drop them from the physical spectrum, does not remove them completely.

## 2.1 Unitarity, Hermiticity and CPT invariance

Normally, the unitarity equation  $S^\dagger S = 1$  is proved by means of “cutting equations”, which are identities

$$G + \bar{G} + \sum_c G_c = 0, \quad (2.4)$$

among cut and uncut diagrams. Specifically, one rewrites  $S^\dagger S = 1$  as the optical theorem  $iT - iT^\dagger + T^\dagger T = 0$ , where  $S = 1 + iT$ . Then,  $G$  is the uncut diagram and stands for  $iT$ ,  $\bar{G}$  is its complex conjugate and stands for  $-iT^\dagger$ , while  $G_c$  are the cut diagrams, which are obtained by cutting internal lines and stand for  $T^\dagger T$ . The cut propagators encode the on-shell content of the full propagators.

Let us see how the unitarity equation  $S_r^\dagger S_r = 1$  works after Veltman’s projection. We can build  $S_r$  in two ways. The straightforward method is to quantize the classical, unprojected Lagrangian  $\mathcal{L}$  as usual, build the (unprojected)  $S$  matrix from it and perform Veltman’s projection at the very end. The second method is to work out the projected Lagrangian  $\mathcal{L}^V$  right away and then derive  $S_r$  from  $\mathcal{L}^V$ . Then, however,  $\mathcal{L}^V$  contains effective non-Hermitian interactions due to the removal of the unstable particles. We show that these effective interactions make unitarity work as desired.

The projection  $\mathcal{L} \rightarrow \mathcal{L}^V$  is obtained in two steps. First, one builds the effective vertices, which are given by the one-particle irreducible (1PI) diagrams (generated by  $\mathcal{L}$ ) that have stable particles in the external legs and no stable particles in the internal legs. Second, the self-energies of the unstable particles are resummed into their dressed propagators. In the end,  $\mathcal{L}^V$  is made of “dressed effective vertices”. In some sense, it is semi non-perturbative. The point is that its vertices are not Hermitian, in general. So, how can the  $S$  matrix  $S_r$  obtained from  $\mathcal{L}^V$  be unitary?

In the algebraic approach of ref. [9], it is simple to prove that a non-Hermitian classical

Lagrangian  $\mathcal{L}^V$  leads to a generalized version of the cutting equations of the form

$$G + \bar{G} + \sum_c G_c + \sum_{c'} G_{c'} = 0, \quad (2.5)$$

where  $G$  and  $\bar{G}$  are as above,  $G_c$  are the cut diagrams obtained by cutting internal (stable-particle) lines and  $G_{c'}$  are additional cut diagrams, obtained by cutting the non-Hermitian vertices as well. Normally, the extra terms  $G_{c'}$  quantify the violations of unitarity, because they have no interpretation in the unitary equation  $S^\dagger S = 1$ . However, in the case of Veltman's projection, they are precisely what is needed to interpret the identities (2.5) as the correct diagrammatic versions of  $S_r^\dagger S_r = 1$ . The reason is that the cut vertices of  $G_{c'}$  describe the decays of the unstable particles that have been projected away, which are not included in the diagrams  $G_c$ . More details and the diagrammatic analysis of the extra terms can be found in Veltman's paper [4].

Another issue is the CPT theorem after Veltman's projection. If the unprojected theory is CPT invariant, the projected theory described by  $S_r$  should be CPT invariant as well. The trouble is, again, that  $\mathcal{L}^V$  is not Hermitian. If we want to have CPT invariance after the projection, we must treat the effective vertices of  $\mathcal{L}^V$  in a particular way under that symmetry.

Specifically, let  $S(\mathcal{L}, \epsilon)$  denote the  $S$  matrix built from a local Lagrangian  $\mathcal{L}$  with Feynman's  $i\epsilon$  prescription. Then,  $S(-\mathcal{L}^\dagger, -\epsilon)$  is the conjugate matrix  $S^\dagger$ . CPT invariance is the statement that  $S(\mathcal{L}^\dagger, \epsilon) = S$ , or  $S(-\mathcal{L}, -\epsilon) = S^\dagger$ , which is true if  $\mathcal{L} = \mathcal{L}^\dagger$ . If we take  $\mathcal{L}^V$  as the Lagrangian, we have  $S(\mathcal{L}^V, \epsilon) = S_r$ , where  $\epsilon$  refers to the stable particles only. The point is that  $\mathcal{L}^V$  also depends on the Feynman prescription (for the unstable particles projected away). So,  $S(-\mathcal{L}^V(\epsilon), -\epsilon) \neq S_r^\dagger$ . Nevertheless, we have the identities  $(\mathcal{L}^V(\epsilon))^\dagger = \mathcal{L}^V(-\epsilon)$  and  $S(-\mathcal{L}^V(-\epsilon), -\epsilon) = S_r^\dagger$ , which can be interpreted as the CPT theorem for  $S_r$ .

### 3 Basic quantization options

In this section we compare the quantizations of physical particles, ghosts and purely virtual particles and emphasize their basic properties, also in relation with Veltman's projection, when it applies. We concentrate on the tree diagrams, the classical limit and the dressed propagators. In the next section we consider the loop diagrams.

We start from the Lagrangian

$$\mathcal{L}_{\text{cl}} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 + \mathcal{L}_\phi - g\varphi^2\phi - \Lambda\phi \quad (3.1)$$

in four spacetime dimensions, which couples a physical particle  $\varphi$  to some other type of particle  $\phi$ , to be defined below, with free Lagrangian

$$\mathcal{L}_\phi = \frac{\rho}{2} [(\partial_\mu \phi)(\partial^\mu \phi) - M^2 \phi^2]. \quad (3.2)$$

For the time being, we assume  $M > 2m$  and  $\rho = \pm 1$ . The last term of (3.1) can be removed by translating  $\phi$  and redefining the other parameters, so we ignore it from now on. The theory is super-renormalizable and the field  $\phi$  gets a nonvanishing width.

The quantization of  $\varphi$  proceeds as usual, so we concentrate on  $\phi$ , starting from the free-field limit. The presence of  $\varphi$  lets us study the effects of interactions.

The  $\phi$  momentum and its commutation relations read

$$\pi_\phi = \rho \partial_0 \phi, \quad [\pi_\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (3.3)$$

The free classical Hamiltonian is

$$H_\phi = \frac{\rho}{2} \int d^3\mathbf{x} [(\partial_0 \phi)^2 + (\nabla \phi)^2 + M^2 \phi^2]. \quad (3.4)$$

To study more possibilities at once, we expand the field operator  $\hat{\phi}$  as

$$\hat{\phi}(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} [(-1)^\eta a_{\mathbf{k}} e^{-i\sigma kx} + (-1)^{\eta'} a_{\mathbf{k}}^\dagger e^{i\sigma kx}], \quad (3.5)$$

in terms of creation and annihilation operators  $a_{\mathbf{k}}^\dagger$  and  $a_{\mathbf{k}}$ , where  $kx = \omega t - \mathbf{k} \cdot \mathbf{x}$  and  $\omega = \sqrt{\mathbf{k}^2 + M^2}$ . The parameters  $\eta$  and  $\eta'$  can have values 0 or 1, while  $\sigma$  can have values  $\pm 1$ . To have agreement with (3.3), the commutation relations of  $a_{\mathbf{k}}^\dagger$  and  $a_{\mathbf{k}}$  must be

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = 2\rho\sigma(-1)^{\eta+\eta'}\omega(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad [a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger] = [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0. \quad (3.6)$$

We define the vacuum  $|0\rangle$  to be annihilated by  $a_{\mathbf{k}}$  and the states to be created by  $a_{\mathbf{k}}^\dagger$ :

$$a_{\mathbf{k}}|0\rangle = 0, \quad |n\rangle = \frac{1}{\sqrt{n!A}} \int \left( \prod_{i=1}^n \frac{d^3\mathbf{k}_i}{(2\pi)^3 2\omega_i} \right) f(\mathbf{k}_1, \dots, \mathbf{k}_n) a_{\mathbf{k}_1}^\dagger \cdots a_{\mathbf{k}_n}^\dagger |0\rangle, \quad (3.7)$$

where

$$A = \int \left( \prod_{i=1}^n \frac{d^3\mathbf{k}_i}{(2\pi)^3 2\omega_i} \right) |f(\mathbf{k}_1, \dots, \mathbf{k}_n)|^2.$$

From (3.4) we derive the Hamiltonian operator

$$\hat{H}_\phi = \frac{\rho}{2} (-1)^{\eta+\eta'} \int \frac{d^3\mathbf{k}}{(2\pi)^3} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \quad (3.8)$$

(neglecting an infinite additive constant). From (3.6) we find the norms

$$\langle n|n\rangle = \rho^n \sigma^n (-1)^{n(\eta+\eta')}. \quad (3.9)$$

The  $\hat{H}_\phi$  eigenvalues are

$$\hat{H}_\phi|n\rangle = h_n|n\rangle, \quad h_n = \sigma \sum_{i=1}^n \bar{\omega}_i, \quad (3.10)$$

for  $f(\mathbf{k}_1, \dots, \mathbf{k}_n) = \prod_{i=1}^n \delta^{(3)}(\mathbf{k}_i - \bar{\mathbf{k}}_i)$  ( $A$  being replaced by an arbitrary finite constant), where  $\bar{\mathbf{k}}_i$  are given momenta.

The free (T ordered)  $\hat{\phi}$  propagator is

$$\begin{aligned} \langle 0|T\hat{\phi}(x)\hat{\phi}(y)|0\rangle &= \rho\sigma \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} [\theta(x^0 - y^0)e^{-i\sigma k(x-y)} + \theta(y^0 - x^0)e^{-i\sigma k(y-x)}] \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i\rho e^{-ip(x-y)}}{p^2 - M^2 + i\sigma\epsilon}. \end{aligned} \quad (3.11)$$

The physical particles have  $\rho = \sigma = 1$  and  $\eta = \eta' = 0$ . Then the Hamiltonians  $H_\phi$  and  $\hat{H}_\phi$  are bounded from below and the norms are positive.

Now we consider the options with  $\rho = -1$ .

### 3.1 $i\epsilon$ ghost

The first possibility is to perform the  $\phi$  quantization as usual, which means choose (3.5) with  $\sigma = 1$ ,  $\eta = \eta' = 0$ . Then, formula (3.9) shows that there are states with positive norms and states with negative norms. From (3.10), we see that the Hamiltonian  $\hat{H}_\phi$  is bounded from below. Formula (3.11) shows that the propagator is equal to (2.1), that is to say, the opposite of a physical particle. In particular, the  $i\epsilon$  prescription is the usual one ( $M^2 \rightarrow M^2 - i\epsilon$ ). This is just the ordinary ghost, which has positive energy, but indefinite metric. We call it “ $i\epsilon$  ghost”.

The  $i\epsilon$  ghosts violate unitarity. Nevertheless, they satisfy a pseudounitary equation (see [6, 7]), which holds perturbatively diagram by diagram.

Since we are assuming  $M > 2m$ , the interaction equips  $\phi$  with a positive width  $\Gamma_\phi$ . The  $\phi$  dressed propagator formally reads

$$-\frac{iZ}{p^2 - M_{\text{gh}}^2 + i(\epsilon - M_{\text{gh}}\Gamma_\phi)} \quad (3.12)$$

around the peak. The minus sign between  $\epsilon$  and  $M_{\text{gh}}\Gamma_\phi$  signals that the resummation cannot be trusted close to the peak, as shown in [8], so we have a peak uncertainty. We

cannot apply Veltman's projection, because we do not know what dressed propagator we should use inside bigger diagrams.

On the other hand, we cannot remove the ghosts from the external states order by order in the perturbative expansion, since this kind of removal is not a projection, because it is not compatible with the pseudounitariness equation. Without projections, the classical limit is just (3.1). Formula (3.4) shows that the free  $\phi$  classical Hamiltonian  $H_\phi$  is not bounded from below, although, as we have seen above, the quantum one  $\hat{H}_\phi$  is.

A way to overcome these obstacles is to *define* the dressed propagator of a ghost as the one of formula (3.12) at  $\epsilon = 0$  and start over from there. Then we obtain the same options as with the  $-i\epsilon$  ghost discussed below.

### 3.2 $-i\epsilon$ ghost

Now we define  $\phi$  in (3.5) with  $\sigma = -1$  and  $\eta = \eta' = 0$ . These choices give positive norms in (3.9), but formula (3.10) shows that the quantum Hamiltonian  $\hat{H}_\phi$  is not bounded from below. The propagator (3.11) becomes (2.2) and acquires an unusual prescription ( $M^2 \rightarrow M^2 + i\epsilon$ ).

This option is not equivalent to the previous one, because the roles of the annihilation and creation operators are interchanged inside  $\phi$ , but not in the definitions (3.7) of vacuum state and occupied states.

The dressed propagator can be resummed straightforwardly, including the region around the peak, where we find (2.3). Since there is no peak uncertainty, Veltman's projection can be applied. Once we remove  $\phi$  from the set of asymptotic states, because of its finite lifetime,  $\phi$  is no longer a degree of freedom in the classical limit. This means that it is "frozen", integrated out by means of its own propagator (calculated at  $\hbar \rightarrow 0$ ).

The classical limit is obtained by collecting the tree diagrams. Veltman's projection reduces the set of such diagrams to those that do not have  $\phi$  external legs. The  $\phi$  internal legs build nonlocal interactions among the physical fields  $\varphi$ . At the end, the true classical Lagrangian  $\mathcal{L}_{\text{cl}}^{\text{V}}$  is the projected version of (3.1), obtained by integrating out  $\phi$  with the ghost propagator (2.2), which is the classical limit of (2.3). The result is

$$\mathcal{L}_{\text{cl}}^{\text{V}} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - \frac{g^2}{2}\varphi^2 \frac{1}{\square + M^2 + i\epsilon}\varphi^2. \quad (3.13)$$

As predicted, it contains a micro nonlocal  $\varphi$  self-interaction, which is also micro acausal. If  $M$  is large enough to have agreement with the experimental data available today, micro

nonlocalities and micro acausalities are not problematic. What makes  $\mathcal{L}_{\text{cl}}^{\text{V}}$  not acceptable is that it is not Hermitian. The imaginary part of the projected classical action is

$$\int d^4x \text{Im} [\mathcal{L}_{\text{cl}}^{\text{V}}] = \frac{\pi g^2}{2} \int d^4x \varphi^2 \delta(-\square - M^2) \varphi^2 = \frac{\pi g^2}{2} \int \frac{d^4p}{(2\pi)^4} \widetilde{\varphi}^2(-p) \delta(p^2 - M^2) \widetilde{\varphi}^2(p),$$

where  $\widetilde{\varphi}^2$  is the Fourier transform of  $\varphi^2$ .

If we choose not to advocate Veltman's projection, the classical Lagrangian we obtain is (3.1): the denominator  $\square + M^2 + i\epsilon$  is moved to the numerator, sandwiched in between two fields  $\phi$ , so the contribution of  $-i\epsilon$  becomes negligible. In that case,  $\phi$  is not integrated out, but an independent degree of freedom, with its own field equations and boundary conditions. The classical limit is Hermitian, but still unacceptable, in the realm of perturbation theory, because the classical free Hamiltonian  $H_\phi$  is not bounded from below.

The Lee-Wick ghosts are  $-i\epsilon$  ghosts equipped with appropriate rules to treat them inside the loop diagrams (see section 4).

### 3.3 Non-Hermitian ghost

We mention a third option, because it is the one preferred by Lee and Wick in their papers, although it is equivalent to the  $-i\epsilon$  ghost just described. We choose  $\rho = 1$  and expand  $\phi$  with  $\sigma = 1$ ,  $\eta = 0$ ,  $\eta' = 1$ . So doing, we understand that  $\phi$  is anti-Hermitian and the coupling  $g$  is purely imaginary. The metric is indefinite and the Hamiltonians  $H_\phi$  and  $\hat{H}_\phi$  are bounded from below. The dressed propagator is fine, so there is no peak uncertainty and Veltman's projection can be applied. The  $-i\epsilon$  ghost can be obtained from this type of ghost, which we call non-Hermitian (nH) ghost, by turning  $\phi$  into  $i\phi$  and  $g$  into  $-ig$ .

There is also a variant with  $\rho = 1$ ,  $\sigma = -1$ ,  $\eta = 0$ ,  $\eta' = 1$ . Then, the norms are positive, but  $\hat{H}_\phi$  is not bounded from below. The dressed propagator cannot be resummed in the peak region, so there is a peak uncertainty and Veltman's projection cannot be used.

### 3.4 Fakeon

In the case of purely virtual particles, we can take  $\rho = \pm 1$ . Doubling the set of creation and annihilation operators, we write

$$\hat{\phi}(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} \left[ \frac{a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger}{\sqrt{2}} e^{-ikx} + \frac{a_{\mathbf{k}}^\dagger + b_{\mathbf{k}}}{\sqrt{2}} e^{ikx} \right]$$

and assume

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = 2\rho\omega(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad [b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = -2\rho\omega(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}'),$$

all the other commutators being identically zero.

Inside the loop diagrams, the fakeon projection amounts to integrate  $\phi$  out with the appropriate diagrammatic rules (see [9] for explicit formulas). In the classical limit, we must integrate it out with the propagator

$$\mathcal{P} \frac{i\rho}{p^2 - M^2}, \quad (3.14)$$

which coincides with the Fourier transform of  $\langle 0|T\hat{\phi}(x)\hat{\phi}(y)|0\rangle$ , where  $\mathcal{P}$  is the Cauchy principal value. The Lagrangian describing the classical limit, which reads

$$\mathcal{L}_{\text{cl}}^{\text{f}} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 + \rho\frac{g^2}{2}\varphi^2\mathcal{P}\left(\frac{1}{\square + M^2}\right)\varphi^2, \quad (3.15)$$

is the sum of a standard kinetic term plus a micro nonlocal Hermitian self-interaction.

### 3.5 Summary

We summarize the various options considered so far and their main properties in table 1, where “l” stands for local, “nl” means nonlocal, “+V” and “-V” mean with and without Veltman’s projection, respectively, and “f $^\pm$ ” denotes the fakeons with  $\rho = \pm 1$ . Finally, “Re” means Hermitian and “Im” means non-Hermitian.

## 4 Loops and unstable particles

We briefly recall how the various options listed in the previous section are treated inside the loop diagrams, referring to the literature for more details.

The propagator of an  $i\epsilon$  ghost coincides with the one of a physical particle, apart from its overall sign, so its diagrammatics is straightforward. The propagator of a  $-i\epsilon$  ghost, on the other hand, is defined by the opposite prescription. If we integrate the loop diagrams on real energies and real momenta, the  $-i\epsilon$  prescription cannot coexist with the usual  $i\epsilon$  one [18], because it switches the roles of the thresholds with those of the pseudothresholds, causing instabilities, violations of unitarity, as well as violations of the locality and Hermiticity of counterterms. To avoid this type of problems, it is necessary to formulate better integration prescriptions or give alternative diagrammatic rules.

	phys. part.	$i\epsilon gh$	$-i\epsilon gh+V$	$-i\epsilon gh-V$	nH+V	nH-V	nH $^{\sigma=-1}$	$f^{\pm}$
$\rho$	1	-1	-1	-1	1	1	1	$\pm 1$
$\sigma$	1	1	-1	-1	1	1	-1	$\pm 1$
$\eta$	0	0	0	0	0	0	0	0
$\eta'$	0	0	0	0	1	1	1	0
$\hat{\phi}^\dagger$	$\hat{\phi}$	$\hat{\phi}$	$\hat{\phi}$	$\hat{\phi}$	$-\hat{\phi}$	$-\hat{\phi}$	$-\hat{\phi}$	$\hat{\phi}$
norms	+	$\pm$	$\times$	+	$\times$	$\pm$	+	$\times$
$H_\phi$	$\geq 0$	$\leq 0$	$\times$	$\leq 0$	$\times$	$\geq 0$	$\geq 0$	$\times$
$\hat{H}_\phi$	$\geq 0$	$\geq 0$	$\times$	$\leq 0$	$\times$	$\geq 0$	$\leq 0$	$\times$
$\mathcal{L}_{cl}$	1	1	nl	1	nl	1	1	nl
	Re	Re	Im	Re	Im	Re	Re	Re
peak uncert.	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\checkmark$

Table 1: Main properties of the options for quantization

The LW ghost is obtained from the  $-i\epsilon$  ghost by adopting the Lee-Wick integration prescription on the loop energies [1], combined with the CLOP prescription [19] and any other rules that might be necessary for the internal consistency. Here, we do not need to prove that they exist, so we just assume that they do. The LW ghosts must be unstable, dynamically or not, to apply Veltman's projection to them. The procedure is semi-perturbative, because it requires to use the dressed propagators inside diagrams and reorganize the diagrammatic rules accordingly.

In the model (3.1), it is enough to turn on the vertex  $-g\varphi^2\phi$  and assume the inequality  $M > 2m$ . Then, the decay  $\phi \rightarrow \varphi\varphi$  gives  $\phi$  a nonvanishing width  $\Gamma_\phi$ . The  $-i\epsilon$  prescription guarantees that it is possible to resum the self-energies into the  $\phi$  dressed propagators with no peak uncertainty, so Veltman's projection can be applied to  $\phi$ , to build a unitary reduced  $S$  matrix  $S_r$  on the strictly asymptotic states.

Fakeons inside loops are defined by means of a different diagrammatics [9], which works as a mathematical tool to surgically eradicate the potential degree of freedom at all

energies, turning it *de facto* into a fake degree of freedom. The mathematics of the fakeon projection does not have a direct physical interpretation, such as a decay. We may expect that if the removal of a degree of freedom (or its impossibility to be observed in nature) is due to the physics, it is either nonperturbative (as in the cases of quarks and gluons) or not a complete removal (as in the case of the LW ghost).

In particular, a purely virtual particle does not need to have a nonvanishing width  $\Gamma$ . The assumption  $M > 2m$  is unnecessary to make the model (3.1) work with  $\phi = \text{fakeon}$ . We can even replace the vertex  $-g\varphi^2\phi$  with an interaction like  $-g\varphi\phi^2$ , which makes the fakeon width identically zero. Phenomenological models with fakeons of vanishing widths are studied in ref. [12].

## 5 Pauli-Villars fields made physical

In this section we use fakeons to make the Pauli-Villars fields consistent and observable without sending their masses to infinity.

We first recall the main properties of the Pauli-Villars fields [21, 6]. Consider the Lagrangian

$$\mathcal{L}_{\text{PV}} = \frac{1}{2} \sum_{j=1}^N [(\partial_\mu \varphi_j)(\partial^\mu \varphi_j) - m_j^2 \varphi_j^2] - \frac{1}{2} \sum_{j=1}^{N'} [(\partial_\mu \phi_j)(\partial^\mu \phi_j) - M_j^2 \phi_j^2] - V(\check{\varphi}, \phi), \quad (5.1)$$

where

$$\phi \equiv \sum_{j=1}^N c_j \varphi_j + \sum_{j=1}^{N'} d_j \phi_j,$$

$c_j, d_j$  are real constants and  $\check{\varphi}$  are the fields  $\varphi_j$  that do not appear inside  $\phi$  (because they have  $c_j = 0$ ).  $V$  is a potential, or, more generally, the interaction part (if it depends on the derivatives of the fields).

It is possible to organize the diagrammatics so that each non- $\check{\varphi}$  internal leg of the diagrams propagates the whole combination  $\phi$ , with free propagator

$$\langle \phi(p)\phi(-p) \rangle_{\text{free}} = \sum_{j=1}^N \frac{ic_j^2}{p^2 - m_j^2 + i\epsilon} - \sum_{j=1}^{N'} \frac{id_j^2}{p^2 - M_j^2 + i\epsilon}. \quad (5.2)$$

For the moment, we use the standard  $i\epsilon$  prescription for the PV fields  $\phi_j$ . We examine different options later.

If we choose  $c_j, d_j$  such that

$$\sum_{j=1}^N c_j^2 (m_j^2)^k = \sum_{j=1}^{N'} d_j^2 (M_j^2)^k, \quad (5.3)$$

$k = 0, 1, 2, \dots, \bar{n}$ , the propagator (5.2) falls off as  $1/(p^2)^{2+\bar{n}}$  for large  $|p^2|$ . So doing, we can improve the power counting and in some cases render the theory (5.1) completely finite.

If we plan to send the masses  $M_j$  to infinity, we can use the PV fields as regulators. The Pauli-Villars regularization technique is obtained by adding PV fields so as to make the theory completely finite at finite masses  $M_j$ .

An important property of the PV regularization technique is that it is not gauge invariant, nor general covariant, because it does not treat the quadratic and interaction parts of the Lagrangian on an equal footing. Indeed, the interaction part of (5.1) depends only on the physical fields  $\check{\varphi}$  and the linear combination  $\phi$ , while the quadratic terms cannot be expressed by means of  $\check{\varphi}$  and  $\phi$ . Gauge invariance and general covariance can be recovered in the limit  $M_j \rightarrow \infty$  (provided they are not anomalous) by subtracting local counterterms.

If we want to give physical significance to the PV fields without sending their masses to infinity, we must restrict to neutral matter fields. We study the main options we have in this context.

The first option is the one already considered, i.e., quantize the PV fields as  $i\epsilon$  ghosts. Then, it is not possible to ignore them from the incoming and outgoing states, because the dressed propagators cannot be resummed around the peaks. The classical Lagrangian is not acceptable, because it has negative kinetic terms.

The second option is what Lee and Wick do, i.e., quantize the PV fields as  $-i\epsilon$  ghosts, treat them as LW ghosts inside the loop diagrams, ensure that they are unstable, build their dressed propagators and apply Veltman's projection. The problem with this option is that the classical limit is not Hermitian.

The standard option with fakeons is to quantize  $\phi_j$  as purely virtual particles, which removes them completely. There is a new possibility, though, which emerges by combining the PV, LW and fakeon ideas in a certain way. It amounts to removing the  $\phi_j$  only partially, to overcome the difficulties described above and gain extra (observable) physical particles. The trick works with neutral matter fields and is fully perturbative.

We first describe the new option in the model (3.1), with  $\Lambda = 0$ , which is a particular case of (5.1), then we generalize it to (5.1). In the next sections we apply it to finite QED and quantum gravity.

We decompose  $\phi$  as the combination of a physical field  $\Phi$  and an additional field  $Q$ . Specifically, we turn the classical Lagrangian (3.1) into

$$\begin{aligned} \mathcal{L}_{\text{cl}} = & \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 + \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{M^2}{2}\Phi^2 \\ & - \frac{1}{2} [(\partial_\mu Q)(\partial^\mu Q) - M^2 Q^2] - g\varphi^2(\Phi + \sqrt{2}Q) \end{aligned} \quad (5.4)$$

and view  $\Phi$  as a standard physical particle and  $Q$  as a fakeon. The combined  $\phi$  propagator becomes

$$-\frac{2i}{p^2 - M^2} \Big|_{\text{f}} + \frac{i}{p^2 - M^2 + i\epsilon}, \quad (5.5)$$

where the subscript “f” means “fakeon prescription”. At the tree level, (3.14) (with  $\rho \rightarrow -2$ ) gives

$$-\frac{i}{p^2 - M^2 - i\epsilon} - \frac{i}{p^2 - M^2 + i\epsilon} + \frac{i}{p^2 - M^2 + i\epsilon} = -\frac{i}{p^2 - M^2 - i\epsilon},$$

which is the propagator of a  $-i\epsilon$  ghost. Resumming the self-energies, the dressed propagator reads

$$-\frac{iZ}{p^2 - M_{\text{ph}}^2 - i(\epsilon + \Gamma)} \rightarrow -\frac{iZ}{p^2 - M_{\text{ph}}^2 - i\Gamma}$$

around the peak, where  $\Gamma$  is nonnegative. Although the  $\phi$  two-point function has no peak uncertainty, this is not crucial now, because we do not need Veltman’s projection, having abandoned the LW approach to adopt the fakeon one.

Since  $\Phi$  is a fakeon, it does not belong to the set of asymptotic states, by definition. Instead,  $Q$  does: unstable or not, it is an extra, physically observable particle, originated by a PV field. The decomposition of  $\phi$  as  $\Phi + \sqrt{2}Q$  and the properties of fakeons allow us to treat  $\Phi$  and  $Q$  differently, which is crucial to have a Hermitian classical limit. That limit, obtained by keeping  $\Phi$  and projecting  $Q$  away in (5.4), is given by the Lagrangian

$$\mathcal{L}_{\text{cl}}^{\text{f}} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 + \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{M^2}{2}\Phi^2 - g\varphi^2\Phi - g^2\varphi^2\mathcal{P}\frac{1}{\square + M^2}\varphi^2,$$

which describes two physical particles,  $\varphi$  and  $\Phi$ , with a nonstandard (micro nonlocal and micro acausal)  $\varphi$  self-interaction.

The decomposition also specifies how to treat  $\phi$  inside the loop diagrams. We must proceed as in [9], distinguishing the contributions due to  $\Phi$  from those due to  $Q$ , since  $\Phi$  is a physical particle, while  $Q$  is a fakeon. Note that  $\Phi$  and  $Q$  have the same mass, so there are many coinciding thresholds, which must be treated as limits of distinct ones. The

counterterms of (5.4) just depend on  $\varphi$  and  $\phi$ . The  $\phi$  propagator renormalizes exactly as for the theory (3.1). The  $\Phi$  and  $Q$  two-point functions can be derived from it. They may separately have peak uncertainties, but, again, this is not of our concern.

We can generalize the trick by writing (5.1) as

$$\begin{aligned} \mathcal{L}_{\text{PV}} = & \frac{1}{2} \sum_{j=1}^N [(\partial_\mu \varphi_j)(\partial^\mu \varphi_j) - m_j^2 \varphi_j^2] + \frac{1}{2} \sum_{j=1}^{N'} [(\partial_\mu \Phi_j)(\partial^\mu \Phi_j) - M_j^2 \Phi_j^2] \\ & - \frac{1}{2} \sum_{j=1}^{N'} [(\partial_\mu Q_j)(\partial^\mu Q_j) - M_j^2 Q_j^2] - V(\check{\varphi}, \phi), \end{aligned} \quad (5.6)$$

where

$$\phi = \sum_{j=1}^N c_j \varphi_j + \sum_{j=1}^{N'} d_j (\Phi_j + \sqrt{2} Q_j),$$

and interpreting  $\Phi_j$  as additional physical particles and  $Q_j$  as fakeons. The  $\phi$  propagator is

$$\sum_{j=1}^N \frac{ic_j^2}{p^2 - m_j^2 + i\epsilon} + \sum_{j=1}^{N'} \frac{id_j^2}{p^2 - M_j^2 + i\epsilon} - \sum_{j=1}^{N'} \frac{2id_j^2}{p^2 - M_j^2} \Big|_f.$$

Thanks to conditions such as (5.3), we can make it fall off as fast as we want for large  $|p^2|$ . The classical limit is a theory of  $N + N'$  physical particles with certain micro nonlocal self-interactions.

## 6 Finite QED

In this section we use the trick explained in the previous one to build a finite QED, by tweaking the original Lee-Wick construction [2]. In the next section we investigate the possibility of generalizing the trick to quantum gravity.

We start from the classical Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{QED}} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{M^2}{2} B_\mu B^\mu + \sum_{j=1}^2 \bar{\psi}_j [i\gamma^\mu (\partial_\mu + ieA_\mu + ieB_\mu) - m_j] \psi_j \\ & + \bar{\Psi} \sigma_1 [i\gamma^\mu (\partial_\mu + e\sigma_2 A_\mu + e\sigma_2 B_\mu) - M_\Psi] \Psi, \end{aligned} \quad (6.1)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ,  $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ ,  $\sigma_1$  and  $\sigma_2$  are the first two Pauli matrices,  $\psi_1$  and  $\psi_2$  denote the electron and the muon, respectively,  $\Psi$  is an extra fermion doublet

and  $M_\Psi$  denotes the  $\Psi$  mass matrix. The Lagrangian is Hermitian and gauge invariant, the gauge transformations being

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad B_\mu \rightarrow B_\mu, \quad \psi \rightarrow e^{-ie\Lambda} \psi, \quad \bar{\psi} \rightarrow e^{ie\Lambda} \bar{\psi}, \quad \Psi \rightarrow e^{-e\sigma_2 \Lambda} \Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi} e^{-e\sigma_2 \Lambda}.$$

## 6.1 Lee-Wick QED

If we follow Lee and Wick, the vector  $B_\mu$  is a  $-i\epsilon$  ghost at the tree level, to be treated as a LW ghost inside the loops. Since the interactions contain only the combination  $A_\mu + B_\mu$ , what matters, for power counting, is the combined propagator of  $A_\mu + B_\mu$ , which reads

$$-i\eta_{\mu\nu} \left( \frac{1}{p^2 + i\epsilon} - \frac{1}{p^2 - M^2 - i\epsilon} \right) + p_\mu p_\nu (\dots), \quad (6.2)$$

where  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  is the flat-space metric. The transverse part, proportional to  $\eta_{\mu\nu}$ , is gauge independent and falls off like  $1/(p^2)^2$  for large momenta, à la Pauli-Villars. The longitudinal part does not fall off rapidly. Nevertheless, it is gauge dependent and does not affect the physical quantities.

The behavior of (6.2) is enough to ensure that every diagram but one is convergent, up to gauge-dependent contributions. The exception is the one-loop photon self-energy. Its convergence is provided by the doublet  $\Psi$ , introduced to obtain a completely finite theory.

At one loop the photon self-energy receives contributions from the bubble diagrams with circulating electrons, muons and  $\Psi$  fields. The diagram with circulating  $\Psi$  fields has an extra  $-2$  factor with respect to the electron and muon bubble diagrams, because of the trace

$$\text{tr}[\sigma_1(-i\sigma_1\sigma_2)\sigma_1(-i\sigma_1\sigma_2)] = -2.$$

A  $\sigma_1$  is brought by each  $\Psi$  propagator and a  $-i\sigma_1\sigma_2$  is brought by each vertex. The factor  $-2$  is precisely what is needed to compensate the logarithmic divergences due to electrons and muons.

The dressed propagator of  $A_\mu + B_\mu$  can be resummed in the transverse sector. We do not repeat the calculation of Lee and Wick here, but just recall that  $B_\mu$  acquires a nonvanishing width and becomes unstable. It is then removed from the set of strictly asymptotic states à la Veltman.

Lee and Wick need to make  $\Psi$  decay as well. Since  $\Psi$  does not become unstable dynamically, they equip it with a nonvanishing width at the classical level, by choosing a mass matrix of the form

$$M_\Psi = m_\Psi + \frac{i}{2}\sigma_2\gamma_\Psi, \quad (6.3)$$

where  $m_\Psi$  and  $\gamma_\Psi$  are real numbers. The Lagrangian (6.1) remains Hermitian.

Once Veltman's projection is advocated for the unstable particles  $B_\mu$  and  $\Psi$  (the muon being stable here), the reduced  $S$  matrix  $S_r$  is unitary. As expected, the classical limit, which reads

$$\begin{aligned} \mathcal{L}_{\text{cl}}^{\text{LW}} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{j=1}^2 \bar{\psi}_j [i\gamma^\mu(\partial_\mu + ieA_\mu) - m_j] \psi_j \\ & + \frac{e^2}{2M^2} \left( \sum_{j=1}^2 \bar{\psi}_j \gamma^\mu \psi_j \right) \frac{\eta_{\mu\nu}M^2 + \partial_\mu \partial_\nu}{\square + M^2 + i\epsilon} \left( \sum_{l=1}^2 \bar{\psi}_l \gamma^\nu \psi_l \right), \end{aligned} \quad (6.4)$$

contains a non-Hermitian self interaction.

## 6.2 Standard option with fakeons

The easiest way to solve the problems of the Lee-Wick construction is to treat  $B_\mu$  and  $\Psi$  as fakeons. The theory remains finite. The dressed  $B_\mu$  propagator cannot be resummed around its peak, so  $B_\mu$  has a peak uncertainty, equal to its width divided by 2 [8]. As far as  $\Psi$  is concerned, we can just leave  $\gamma_\Psi = 0$  in formula (6.3), since  $\Psi$  is out of the physical spectrum without advocating its instability.

The classical limit becomes

$$\begin{aligned} \mathcal{L}_{\text{cl}}^{\text{f}} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{j=1}^2 \bar{\psi}_j [i\gamma^\mu(\partial_\mu + ieA_\mu) - m_j] \psi_j \\ & + \frac{e^2}{2M^2} \left( \sum_{j=1}^2 \bar{\psi}_j \gamma^\mu \psi_j \right) \mathcal{P} \frac{\eta_{\mu\nu}M^2 + \partial_\mu \partial_\nu}{\square + M^2} \left( \sum_{l=1}^2 \bar{\psi}_l \gamma^\nu \psi_l \right), \end{aligned} \quad (6.5)$$

which is the standard QED Lagrangian with a non standard four fermion Hermitian, micro nonlocal self interaction. The field  $\Psi$  disappears in the limit  $\hbar \rightarrow 0$ , but contributes to the reduced action, through effective vertices generated by its loops.

## 6.3 New option with fakeons

The new option, instead, amounts to interpreting  $B_\mu$  as a superposition  $\tilde{B}_\mu + \sqrt{2}Q_\mu$  of a physical vector  $\tilde{B}_\mu$  and a different fakeon  $Q_\mu$ , while the doublet  $\Psi$  is still seen as a fakeon.

We obtain the Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{QED}} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{M^2}{2}\tilde{B}_\mu\tilde{B}^\mu + \frac{1}{4}Q_{\mu\nu}Q^{\mu\nu} - \frac{M^2}{2}Q_\mu Q^\mu \\
& + \sum_{j=1}^2 \bar{\psi}_j \left[ i\gamma^\mu (\partial_\mu + ieA_\mu + ie\tilde{B}_\mu + ie\sqrt{2}Q_\mu) - m_j \right] \psi_j \\
& + \bar{\Psi}\sigma_1 \left[ i\gamma^\mu (\partial_\mu + e\sigma_2 A_\mu + e\sigma_2 \tilde{B}_\mu + e\sqrt{2}\sigma_2 Q_\mu) - M_\Psi \right] \Psi,
\end{aligned} \tag{6.6}$$

where  $\tilde{F}_{\mu\nu} = \partial_\mu\tilde{B}_\nu - \partial_\nu\tilde{B}_\mu$  and  $Q_{\mu\nu} = \partial_\mu Q_\nu - \partial_\nu Q_\mu$ . The theory remains finite, because the combined propagator of  $A_\mu + \tilde{B}_\mu + \sqrt{2}Q_\mu$  behaves like (6.2) for large  $|p^2|$ , although it is defined by a different prescription at finite momenta. The classical limit reads

$$\begin{aligned}
\mathcal{L}_{\text{cl}}^f = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{M^2}{2}\tilde{B}_\mu\tilde{B}^\mu + \sum_{j=1}^2 \bar{\psi}_j \left[ i\gamma^\mu (\partial_\mu + ieA_\mu + ie\tilde{B}_\mu) - m_j \right] \psi_j \\
& + \frac{e^2}{M^2} \left( \sum_{j=1}^2 \bar{\psi}_j \gamma^\mu \psi_j \right) \mathcal{P} \frac{\eta_{\mu\nu}M^2 + \partial_\mu\partial_\nu}{\square + M^2} \left( \sum_{l=1}^2 \bar{\psi}_l \gamma^\nu \psi_l \right).
\end{aligned}$$

This is the standard QED Lagrangian with an extra Proca vector  $\tilde{B}_\mu$  and a peculiar Hermitian, micro nonlocal four fermion self interaction.

## 7 Quantum gravity

In this section we discuss the possibility of applying the trick to quantum gravity and stress the difficulties that arise with general covariance. Then we explain how a fully covariant decomposition can be achieved by adding a massive spin 2 multiplet.

Consider the classical action

$$S_{\text{QG}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( 2\Lambda + R + \frac{\lambda}{2m_\chi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - \frac{R^2}{6m_\phi^2} \right), \tag{7.1}$$

where  $\lambda = m_\chi^2(3m_\phi^2 + 4\Lambda)/(m_\phi^2(3m_\chi^2 - 2\Lambda))$  is a parameter very close to 1. The theory includes the square  $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$  of the Weyl tensor  $C_{\mu\nu\rho\sigma}$  and is renormalizable by power counting [22]. It propagates the graviton, a scalar field  $\phi_{\text{infl}}$  of mass  $m_\phi$  (which can be interpreted as the inflaton) and a spin 2 field  $\chi_{\mu\nu}$  of mass  $m_\chi$ , which has a kinetic term multiplied by the wrong sign. The three can be made explicit with the help of auxiliary fields, as shown in [23].

If  $\chi_{\mu\nu}$  is interpreted as an  $i\epsilon$  ghost, we obtain the Stelle theory [22], which is not unitary. Since  $\chi_{\mu\nu}$  dynamically acquires a nonvanishing width  $\Gamma_\chi$ , it is interesting to consider the  $\chi_{\mu\nu}$  dressed propagator. The results of [8] show that the resummation of the self-energies does not make sense around the peak, because we formally obtain (3.12). Thus, we cannot project  $\chi_{\mu\nu}$  away à la Veltman. The classical limit of this theory is exactly (7.1), which is not acceptable.

The LW option, studied by Donoghue and Menezes in ref. [24], is to interpret  $\chi_{\mu\nu}$  as a  $-i\epsilon$  ghost at the tree level and as a LW ghost inside the loops. In this case, it is meaningful to resum the self energies into the  $\chi_{\mu\nu}$  dressed propagator, which has the form (2.3). It is possible to project  $\chi_{\mu\nu}$  away à la Veltman and focus on the reduced  $S$  matrix  $S_r$ . However, the classical limit is not Hermitian, like (3.13) and (6.4).

If we tweak the Lee-Wick proposal by removing Veltman's projection, the classical limit is still (7.1). If we remove Veltman's projection just for the physical massive spin-2 particle singled out by the narrow-width approximation (the real part of the propagator (2.2)), we break general covariance, for arguments similar to the ones we explain below.

Other Lee-Wick approaches to quantum gravity, starting from different classical actions, have been considered in the literature [25].

Now we examine the options we have with fakeons. The standard option is to interpret  $\chi_{\mu\nu}$  as a fakeon, which gives the quantum gravity theory of [10]. Then,  $\chi_{\mu\nu}$  does not belong to the sets of initial and final states, because it is purely virtual, and has a peak uncertainty, quantified by its width  $\Gamma_\chi$  divided by two. The classical limit is Hermitian, like (6.5) (see [26]).

The new option is to pursue the strategy of the previous two sections, as in the extensions (5.4), (5.6) and (6.6). We wish to interpret  $\chi_{\mu\nu}$  as the superposition  $\tilde{\chi}_{\mu\nu} + \sqrt{2}\chi'_{\mu\nu}$  of an extra, observable massive spin 2 particle  $\tilde{\chi}_{\mu\nu}$  and a new fakeon  $\chi'_{\mu\nu}$ .

We recall, from [23], that the  $\chi_{\mu\nu}$  action  $S_\chi(g, \phi_{\text{infl}}, \chi)$ , which can be obtained from (7.1) by means of auxiliary fields, is the sum

$$S_\chi(g, \phi_{\text{infl}}, \chi) = -\frac{\lambda}{8\pi G} S_{\text{PF}}(g, \chi) + S_{\chi\text{int}}(g, \phi_{\text{infl}}, \chi) \quad (7.2)$$

of a term proportional to the covariantized Pauli-Fierz action

$$S_{\text{PF}}(g, \chi) = \frac{1}{2} \int d^4x \sqrt{-g} [D_\rho \chi_{\mu\nu} D^\rho \chi^{\mu\nu} - D_\rho \chi D^\rho \chi + 2D_\mu \chi^{\mu\nu} D_\nu \chi - 2D_\mu \chi^{\rho\nu} D_\rho \chi_\nu^\mu - m_\chi^2 (\chi_{\mu\nu} \chi^{\mu\nu} - \chi^2) + R^{\mu\nu} (\chi \chi_{\mu\nu} - 2\chi_{\mu\rho} \chi_\nu^\rho)], \quad (7.3)$$

with nonminimal terms, plus further interactions  $S_{\chi\text{int}}(g, \phi_{\text{infl}}, \chi)$ .

The decomposition of  $\chi_{\mu\nu}$  in terms of  $\tilde{\chi}_{\mu\nu}$  and  $\chi'_{\mu\nu}$  requires that we treat the quadratic parts of  $\tilde{\chi}_{\mu\nu}$  and  $\chi'_{\mu\nu}$  differently from their interactions. We expand the metric tensor  $g_{\mu\nu}$  around the flat-space metric  $\eta_{\mu\nu}$  and write

$$S_{\chi}(g, \phi_{\text{infl}}, \chi) \equiv -\frac{\lambda}{8\pi G} S_{\text{PF}}(\eta, \chi) + \Delta S_{\chi}(\eta, g, \phi_{\text{infl}}, \chi).$$

Then we modify the theory according to the strategy of section 5. The interaction part remains the same and contains the combination  $\chi_{\mu\nu} = \tilde{\chi}_{\mu\nu} + \sqrt{2}\chi'_{\mu\nu}$ . Instead, the quadratic part is turned into the sum of the quadratic parts of  $\chi'_{\mu\nu}$  and  $\tilde{\chi}_{\mu\nu}$ . Mimicking (5.4), the replacement reads

$$S_{\chi}(g, \phi_{\text{infl}}, \chi) \rightarrow \frac{\lambda}{8\pi G} S_{\text{PF}}(\eta, \tilde{\chi}) - \frac{\lambda}{8\pi G} S_{\text{PF}}(\eta, \chi') + \Delta S_{\chi}(\eta, g, \phi_{\text{infl}}, \tilde{\chi} + \sqrt{2}\chi'), \quad (7.4)$$

so  $\tilde{\chi}_{\mu\nu}$  becomes a physically observable massive spin 2 particle, while  $\chi'_{\mu\nu}$  must be treated as a fakeon.

The right-hand side of (7.4) breaks general covariance, because it depends on both metrics  $g_{\mu\nu}$  and  $\eta_{\mu\nu}$ . The fields  $\tilde{\chi}_{\mu\nu}$  and  $\chi'_{\mu\nu}$  are defined by different prescriptions and physically distinguished:  $\tilde{\chi}_{\mu\nu}$ , which is a physical particle, must be included in the set of incoming and outgoing states;  $\chi'_{\mu\nu}$ , as a fakeon, does not belong there. In these circumstances, it is not obvious how to recover general covariance. Below we study the issue in more detail.

We remarked in section 5 that this problem is a well-known aspect of the Pauli-Villars approach, which treats the interactions differently from the quadratic parts.

To conclude, the new option, which works well in QED, cannot be used as is in quantum gravity. This is unfortunate, because the resulting theory would contain an additional, observable massive spin 2 particle  $\tilde{\chi}_{\mu\nu}$  with respect to the theory of [10] (as well as a different spin 2 fakeon  $\chi'_{\mu\nu}$ ).

## 7.1 General covariance and PV fields

The breaking of general covariance due to the decomposition of  $\chi_{\mu\nu}$  into the fields  $\tilde{\chi}_{\mu\nu}$  and  $\chi'_{\mu\nu}$  is entirely due to the quantization prescriptions. For this reason, the issue deserves a careful analysis.

We begin by describing an alternative procedure to apply the trick of converting a ghost into the superposition of a physical particle plus a fakeon, which helps us keep the symmetries under control in a more transparent way. Let

$$S(\phi, g) = -\frac{1}{2} \int d^4x [(\partial_{\mu}\phi)(\partial^{\mu}\phi) - M^2\phi^2] + S_{\text{int}}(\phi, g)$$

denote the action of a field  $\phi$  with negative kinetic term coupled to gravity. Adding a decoupled free field  $\Omega$  with the same mass, we obtain

$$S'(\phi, g, \Omega) = S(\phi, g) + \frac{1}{2} \int d^4x [(\partial_\mu \Omega)(\partial^\mu \Omega) - M^2 \Omega^2]. \quad (7.5)$$

The total action is still invariant under general coordinate transformations, provided  $\Omega$  does not transform. At the infinitesimal level, the transformations read

$$\delta\phi = \xi^\rho \partial_\rho \phi, \quad \delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho, \quad \delta\Omega = 0. \quad (7.6)$$

The action (7.5) gives an invariant theory at the quantum level as long as the quantization prescriptions are compatible with the symmetry (7.6). If so, the Ward-Takahashi-Slavnov-Taylor (WTST) identities [27] can be derived in the usual fashion, by means of a change of field variables dictated by (7.6) in the functional integral, after introducing the source term  $\int d^4x (J_\phi \phi + J_\Omega \Omega + J^{\mu\nu} g_{\mu\nu})$ . If we focus on the matter sector and treat the metric as an external field, the identities read

$$\int d^4x [J_\phi \langle \delta\phi \rangle_J + J_\Omega \langle \delta\Omega \rangle_J + J^{\mu\nu} \delta g_{\mu\nu}] = 0, \quad (7.7)$$

where  $\langle \dots \rangle_J$  denotes the connected Green functions at nonvanishing sources. The term  $\langle \delta\Omega \rangle_J$  vanishes by (7.6). We specialize to  $g_{\mu\nu} = \eta_{\mu\nu}$  and  $\xi^\rho = \text{constant}$  (translations), so that the term  $\delta g_{\mu\nu}$  vanishes as well. Differentiating the resulting equation (7.7) with respect to  $J_\phi$  and  $J_\Omega$  and setting  $J_\phi = J_\Omega = 0$  afterwards, we find

$$\langle \delta\phi(x) \Omega(y) \rangle = \xi^\rho \langle \partial_\rho \phi(x) \Omega(y) \rangle = 0. \quad (7.8)$$

In particular, the quantization prescription should not mix  $\phi$  with  $\Omega$ : a quadratic contribution like

$$\int d^4x d^4y J_\phi(x) G_{\text{mix}}(x, y) J_\Omega(y) \quad (7.9)$$

to the generating functional of the connected Green functions is not compatible with general covariance.

Now, observe that the free-field action

$$-\frac{1}{2} \int d^4x [(\partial_\mu \phi)(\partial^\mu \phi) - M^2 \phi^2] + \frac{1}{2} \int d^4x [(\partial_\mu \Omega)(\partial^\mu \Omega) - M^2 \Omega^2]$$

is invariant under the hyperbolic rotation

$$\phi = \Phi + \sqrt{2}Q, \quad \Omega = \sqrt{2}\Phi + Q. \quad (7.10)$$

The rotated action

$$\begin{aligned}
 S''(\Phi, g, Q) &\equiv S'(\Phi + \sqrt{2}Q, g, \sqrt{2}\Phi + Q) = \frac{1}{2} \int d^4x [(\partial_\mu \Phi)(\partial^\mu \Phi) - M^2 \Phi^2] \\
 &\quad - \frac{1}{2} \int d^4x [(\partial_\mu Q)(\partial^\mu Q) - M^2 Q^2] + S_{\text{int}}(\Phi + \sqrt{2}Q, g)
 \end{aligned} \tag{7.11}$$

matches the actions (5.4), (5.6), (6.6) and (7.4). It is precisely what we need to decompose the field  $\phi$  into a physical particle  $\Phi$  plus a fakeon  $Q$ .

We can read the symmetries of  $S''(\Phi, g, Q)$  from (7.6). They are

$$\delta\Phi = -\xi^\rho \partial_\rho \Phi - \sqrt{2}\xi^\rho \partial_\rho Q, \quad \delta Q = -\sqrt{2}\delta\Phi.$$

The free-field propagators we want can be derived from (5.5). They are, in momentum space,

$$\langle \Phi(p)\Phi(-p) \rangle_0 = \frac{i}{p^2 - M^2 + i\epsilon}, \quad \langle \Phi(p)Q(-p) \rangle_0 = 0, \quad \langle Q(p)Q(-p) \rangle_0 = -\frac{i}{p^2 - M^2} \Big|_f. \tag{7.12}$$

Switching to the field variables  $\phi$  and  $\Omega$  by means of (7.10), we find, for legs that disconnect the diagrams,

$$\begin{aligned}
 \langle \phi(p)\phi(-p) \rangle_0 &= -\frac{i}{p^2 - M^2 - i\epsilon}, & \langle \phi(p)\Omega(-p) \rangle_0 &= \sqrt{2}\pi\delta(p^2 - M^2), \\
 \langle \Omega(p)\Omega(-p) \rangle_0 &= \frac{i}{p^2 - M^2 + i\epsilon} + \pi\delta(p^2 - M^2).
 \end{aligned} \tag{7.13}$$

We see that, although  $\langle \phi(p)\phi(-p) \rangle_0$  is the desired one, i.e. (2.2), we cannot fulfil (7.8) and make  $G_{\text{mix}}$  vanish. This is inconsistent with the WTST identities. Moreover, the  $Q$  projection amounts to set  $J_Q = 0$ , which does not kill the contributions like (7.9). For these reasons, we cannot ensure that general covariance can be recovered.

The case of gravity is obtained by means of the substitutions  $\phi \rightarrow \chi_{\mu\nu}$ ,  $\Phi \rightarrow \tilde{\chi}_{\mu\nu}$ ,  $Q \rightarrow \chi'_{\mu\nu}$ , and adapting the formulas where necessary. The role of  $\Omega$  is then played by a free Pauli-Fierz spin 2 particle  $\Omega_{\mu\nu}$  of mass  $m_\chi$ . Renormalizability is ensured by the very fact that  $\Omega_{\mu\nu}$  decouples from the rest (apart from the quantization prescription, which does not affect the renormalizability). The conclusions do not change.

## 7.2 Manifestly covariant decomposition by means of a massive spin 2 multiplet

A way to perform the decomposition in a manifestly covariant way is to include a Pauli-Fierz spin 2 particle  $\Omega_{\mu\nu}$  of mass  $m_\chi$ , coupled to gravity as required by general covariance,

and then rotate the degenerate pair  $\chi_{\mu\nu}, \Omega_{\mu\nu}$ , so as to single out the physically observable spin 2 particle  $\tilde{\chi}_{\mu\nu}$  and the fakeon  $\chi'_{\mu\nu}$ . However, such a theory is not renormalizable, because the Pauli-Fierz propagator does not fall off as required by power counting at large momenta.

It is possible to have renormalizability (and unitarity) if we replace  $\Omega_{\mu\nu}$  with a whole massive spin 2 multiplet  $\Upsilon_{\mu\nu}$ , of the type studied in ref. [15]. In that case  $\Upsilon_{\mu\nu}$  is a symmetric, traceless tensor and contains a triplet: the spin 2 particle  $\Omega_{\mu\nu}$ , a spin 1 fakeon  $\Omega_\mu$  and a massive scalar  $\Omega$ . If we choose the mass of  $\Omega_{\mu\nu}$  to be equal to  $m_\chi$ , to have degeneracy with  $\chi_{\mu\nu}$ , the masses  $m_1$  and  $m_0$  of  $\Omega_\mu$  and  $\Omega$  are related to  $m_\chi$  by a certain formula that can be found in [15]. Then, we assume that  $\chi_{\mu\nu}$  has the free propagator of a LW ghost and complete the set of free propagators as in (7.13), so that, after rotating the degenerate pair  $\chi_{\mu\nu}, \Omega_{\mu\nu}$ , we can identify the physically observable spin 2 particle  $\tilde{\chi}_{\mu\nu}$  and the fakeon  $\chi'_{\mu\nu}$ , with free propagators of the form (7.12).

So doing, we manage to extend the original LW concept to gravity in a general covariant way, under the requirement that the projected classical action be Hermitian. However, what we obtain is just the theory of [10] coupled to matter in a peculiar way.

## 8 Conclusions

The Lee-Wick models rely on the premise that a unitary reduced  $S$  matrix  $S_r$  can be built by removing the LW ghosts, which are unstable, from the sets of asymptotic states. However, a finite lifetime is not a sufficient reason to ignore a particle from the physical spectrum. If we just drop the LW ghosts, saving the muon and the resonances, the models have non-Hermitian classical limits.

A proper classical limit is important to develop a meaningful cosmology. Although it is legitimate to ignore heavy massive particles at low energies in particle physics, in primordial cosmology the understanding of high energies (subhorizon scales) is necessary to make predictions about the low energies (superhorizon scales). The Bunch-Davies condition [28, 29], for example, specifies the vacuum in the subhorizon region. We cannot make realistic assumptions about that region, which is experimentally and observationally inaccessible, if the theory has ghosts or non-Hermitian interactions. The ABP bound  $m_\chi > m_\phi/4$  of [11], crucial for the prediction of the tensor-to-scalar ratio  $r$ , also follows from the interpolation between the subhorizon and the superhorizon scales.

We have shown that a non purely virtual particle cannot be completely removed, within

the realm of perturbation theory. Barring nonperturbative mechanisms, unacceptable remnants emerge one way or another, like a non-Hermitian self interaction, an indefinite metric or an unbounded Hamiltonian.

Fakeons, on the other hand, are purely virtual, so it is not necessary to worry about making them decay. For this reason, they avoid the problems of the other approaches, without using semi-perturbative approaches or advocating nonperturbative effects. Besides being fully perturbative, the models with fakeons have a Hermitian classical limit and a Hermitian reduced action. In quantum gravity, they lead to a predictive primordial cosmology. The fakeon width  $\Gamma$  is not interpreted as a lifetime, but as (twice) the magnitude of the peak uncertainty, for processes that probe energies close to the fakeon mass.

The investigation carried out in this paper suggests a way to remove a LW ghost only partially, after converting it into a superposition of a fakeon and an observable physical particle. Under certain assumptions, this trick makes the Pauli-Villars fields consistent without sending their masses to infinity. It also allows us to build a finite QED. Nevertheless, it works only with neutral matter fields, in the absence of gravity, because it clashes with general covariance and gauge invariance. A manifestly covariant decomposition can be obtained by adding a massive spin 2 multiplet, which in the end just gives quantum gravity coupled to matter in a peculiar way.

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