High-Order Corrections to Inflationary Perturbation Spectra in Quantum Gravity

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20A5 Renorm

Abstract

We compute the inflationary perturbation spectra and the quantity $r + 8n_T$ to the next-to-next-to-leading log order in quantum gravity with purely virtual particles (which means the theory $R + R^2 + C^2$ with the fakeon prescription/projection for C^2). The spectra are functions of the inflationary running coupling $\alpha(1/k)$ and satisfy the cosmic renormalization-group flow equations, which determine the tilts and the running coefficients. The tensor fluctuations receive contributions from the spin-2 fakeon $\chi_{\mu\nu}$ at every order of the expansion in powers of $\alpha \sim 1/115$. The dependence of the scalar spectrum on the $\chi_{\mu\nu}$ mass m_χ, on the other hand, starts from the α^2 corrections, which are handled perturbatively in the ratio m_ϕ/m_χ, where m_ϕ is the inflaton mass. The predictions have theoretical errors ranging from $\alpha^4 \sim 10^{-8}$ to $\alpha^3 \sim 10^{-6}$. Nontrivial issues concerning the fakeon projection at higher orders are addressed.

1 Introduction

Quantum field theory, through the constraints of locality, renormalizability and unitarity, is equipped with powerful tools to overcome the arbitrariness of classical theories. Following a line of reasoning similar to the one that lead to the standard model of particle physics, a basically unique quantum field theory of gravity emerges from the constraints just mentioned [1]. It is rooted into the concept of purely virtual particle, or fakeon, which can be introduced by adopting a new quantization prescription for the poles of a free propagator, alternative to the Feynman $i\epsilon$ one. The degrees of freedom associated with those poles can then be projected away from the physical spectrum consistently with unitarity.

When the matter sector is switched off, the quantum field theory of gravity propagates a triplet made of the graviton, a massive scalar field ϕ (the inflaton) and a massive spin-2 fakeon $\chi_{\mu\nu}$. The best chances to test its predictions, possibly within years, are offered by primordial cosmology. Applying the idea of purely virtual particle to inflation, the mass m_χ of $\chi_{\mu\nu}$ is bound to be larger than $m_\phi/4$, where m_ϕ is the ϕ mass, and the tensor-to-scalar ratio r is predicted within less than one order of magnitude [2]. The measurement of r, which is expected to be achieved in the next future, will fix m_χ and determine the theory completely. With some technical developments, primordial cosmology might turn into an arena for precision tests of quantum gravity.

Calculations involving purely virtual particles do not require much more effort than usual (see [3, 4] for renormalization constants, widths and absorptive parts, [2, 5] for inflationary cosmology). Actually, various similarities between primordial cosmology and high-energy physics allow us to import techniques from quantum field theory to boost the computations of higher-order corrections. In this context, it is convenient to formulate inflation as a "cosmic" renormalization-group (RG) flow, with a "fine structure constant" $\alpha \sim 1/115$ [5]. As proved in [5], the power spectra satisfy RG flow equations in the superhorizon limit. The computations of tilts, running coefficients and higher-order corrections can be simplified by resumming the leading and subleading logs and expressing the spectra $\mathcal{P}(k)$ as functions of the running coupling $\alpha(1/k)$.

In this paper, we extend the results of [5] by one order of magnitude. Specifically, we compute the RG improved power spectra of the curvature perturbation $\mathcal{R}$ and the tensor fluctuations to the next-to-next-to-leading log (NNLL) order in the superhorizon limit. Tilts and running coefficients follow straightforwardly from the flow equations. We also compute the quantity $r + 8n_T$, where n_T denotes the tensor tilt. A coincidence makes the first contribution to $r + 8n_T$ vanish in quantum gravity with fakeons [2], as well as in several other models explored in the literature [6]. However, no argument suggests that it should vanish identically. Our results show that it is a factor α smaller than naively expected, but definitely not zero.

The outcomes highlight the effects of the fakeon $\chi_{\mu\nu}$ on the spectra. While the spectrum of the tensor fluctuations depends on m_χ at every order of the expansion in powers of α, the spectrum of the scalar fluctuations is affected by $\chi_{\mu\nu}$ only from the NNLL order onwards. The calculation of the NNLL correction to the scalar spectrum involves nontrivial aspects of the fakeon projection, which lead in general to a nonlocal Mukhanov-Sasaki action. A way to avoid these difficulties is to expand in powers of the ratio m_ϕ/m_χ. So doing, the result contains a function of $\xi = m_\phi^2/m_\chi^2$, which we calculate to order ξ^9. The first nine contributions suggest that the expansion is asymptotic. Nevertheless, it gives precise predictions for $m_\chi^2 > 2m_\phi^2$ and fair predictions for $2m_\phi^2 > m_\chi^2 > m_\phi^2$. It is not very helpful in the rest of the ξ range, which is $m_\phi > m_\chi > m_\phi/4$.

In the case of the tensor spectrum, instead, no expansion in ξ is necessary, since ξ can be handled exactly to the NNLL order included. All in all, the relative theoretical errors of the physical predictions we obtain in this paper range from $\alpha^4 \sim 10^{-8}$ to $\alpha^3 \sim 10^{-6}$. We also perform a number of independent checks of the RG equations.

Earlier calculations of the running of spectral indices in various scenarios (in models without fakeons) can be found in [7] and calculations of subleading corrections can be found in [8]. The cosmic RG flow offers a way to upgrade the techniques used previously and provides a better insight into the structures of the spectra of primordial fluctuations.

The paper is organized as follows. In section 2 we briefly review quantum gravity with fakeons and the cosmic RG flow, its beta function and the evolution equations satisfied by the RG improved spectra. In section 3 we study the scalar and tensor spectra to the NNLL order in the limit $m_\chi \to \infty$, where the theory tends to the Starobinsky $R + R^2$ model [9, 10]. In section 4 we calculate the spectrum of the tensor fluctuations to the NNLL order in quantum gravity, while in section 5 we do the same for the spectrum of the curvature perturbation $\mathcal{R}$. In section 6 we collect the physical predictions and estimate their theoretical errors. Section 7 contains the conclusions, while appendix A collects some reference formulas that are useful for the calculations.

2 Inflationary beta function and cosmic RG flow

Quantum gravity with fakeons [1] is described by a triplet made of the graviton, a massive scalar ϕ (the inflaton) and a massive spin-2 fakeon $\chi_{\mu\nu}$. It can be formulated starting from the classical action

$$S_{\rm QG}=-\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(R+\frac{1}{2m_\chi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right)+\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(D_\mu\phi D^\mu\phi-2V(\phi)\right), \tag{2.1}$$

where

$$V(\phi)=\frac{3m_\phi^2}{32\pi G}\left(1-\mathrm{e}^{\phi\sqrt{16\pi G/3}}\right)^2 \tag{2.2}$$

is the Starobinsky potential and m_ϕ and m_χ are the masses of ϕ and $\chi_{\mu\nu}$, respectively. For convenience, the cosmological term and the matter sector are switched off. The action (2.1) does not contain the fakeon $\chi_{\mu\nu}$ explicitly. We identify it from the appropriate pole of the two-point function of the metric fluctuation around flat space.

The theory is renormalizable, because (2.1) is equivalent to the higher-derivative action

$$S_{\rm geom}(g,\Phi)=-\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(R+\frac{1}{2m_\chi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}-\frac{R^2}{6m_\phi^2}\right) \tag{2.3}$$

up to a standard, nonderivative field redefinition. Once the cosmological term is reinstated (2.3) is manifestly renormalizable by power counting [11].

If all the degrees of freedom, defined by expanding the metric around flat space, are quantized by means of the Feynman $i\epsilon$ prescription, Stelle's theory [11] is obtained, which violates unitarity. The reason is that the $\chi_{\mu\nu}$ propagator is multiplied by the wrong sign, so the Feynman prescription generates a ghost. The problem can be avoided by adopting the Feynman prescription just for the graviton and the inflaton, while quantizing $\chi_{\mu\nu}$ by means of the fakeon prescription [1]. So doing, $\chi_{\mu\nu}$ becomes a spin-2 purely virtual particle [12] and can be projected away from the physical spectrum consistently with unitarity. The informations we need on the fakeon projection are given in sections 4 and 5 (for a more detailed review, see [13]). Important things to know are that, once we make the choices just stated:

1) the theory is unitary (see [14, 15] for the analysis of bubble and triangle diagrams in related models, [16] for the proof to all orders, [3, 4] for the absorptive parts in quantum gravity);

2) the theory remains renormalizable by power counting [1, 16]; its beta functions and renormalization constants coincide with those of the Euclidean version [17];

3) the true classical limit is not described by either (2.1) or (2.3), which are unprojected; it is obtained by "classicizing" quantum gravity as explained in [13, 18];

4) the classicization is more challenging when the metric is expanded around nontrivial backgrounds rather than flat space; yet, a lucky coincidence, crucial for cosmology, makes

the degrees of freedom decouple from one another at the quadratic level when the background is the FLRW metric in the de Sitter limit [2]; this fact allows us to hook the fakeon projection on a curved background to the flat-space one and proceed perturbatively from there.

It can be shown [2] that the procedure described in point 4) works under the consistency condition $m_\chi > m_\phi/4$, which puts a lower bound on the mass of the fakeon $\chi_{\mu\nu}$ with respect to the mass of the inflaton ϕ. Note that a bound of this type is nonperturbative, if viewed from the expansion around flat space. Combining the constraints coming from high-energy physics (that is to say, the requirements of locality, renormalizability and unitarity, which make (2.3) essentially unique [1]) with those coming from cosmology, a very predictive theory emerges, to the extent that the tensor-to-scalar ratio r is determined within less than an order of magnitude, even before knowing the value of m_χ [2].

Another important property is that the quantum field theory of gravity does not predict other degrees of freedom besides the curvature perturbation $\mathcal{R}$ and the tensor fluctuations, when the matter sector is switched off. Indeed, the fakeon projection eliminates the possibility of having additional scalar and tensor perturbations, as well as the vector perturbations.

In the rest of this section we recall the main features of the cosmic RG flow introduced in [5] and the RG equations satisfied by the spectra. Given the Friedmann equations and the ϕ equation

$$\dot{H} = -4\pi G\dot{\phi}^2, \qquad H^2 = \frac{4\pi G}{3}\left(\dot{\phi}^2 + 2V(\phi)\right), \qquad \ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \tag{2.4}$$

where $H = \dot{a}/a$ is the Hubble parameter, we define the coupling[1]

$$\alpha = \sqrt{\frac{4\pi G}{3}}\frac{\dot{\phi}}{H} = \sqrt{-\frac{\dot{H}}{3H^2}}. \tag{2.5}$$

Eliminating V and $\dot{\phi}$ by means of the first two equations of (2.4) and $\ddot{\phi}$ from the last equation, it is easy to show that α satisfies

$$\dot{\alpha} = m_\phi\sqrt{1-\alpha^2} - H(2+3\alpha)\left(1-\alpha^2\right). \tag{2.6}$$

Introducing the conformal time

$$\tau = -\int_t^{+\infty}\frac{\mathrm{d}t'}{a(t')}, \tag{2.7}$$

[1]For the purposes of this paper, we can assume $\dot{\phi} > 0$. When $\dot{\phi} < 0$, α becomes negative and the last equality of (2.5) must be replaced by $\alpha = -\sqrt{-\dot{H}}/\sqrt{3H^2}$.

equation (2.6) can be converted into the beta function $\beta_\alpha \equiv \mathrm{d}\alpha/\mathrm{d}\ln|\tau|$ of the cosmic RG flow, which can be easily worked out to arbitrarily high orders in α. For example, to order α^6 we find

$$\beta_\alpha = -2\alpha^2\left[1+\frac{5}{6}\alpha+\frac{25}{9}\alpha^2+\frac{383}{27}\alpha^3+\frac{8155}{81}\alpha^4+\frac{72206}{81}\alpha^5+\frac{2367907}{243}\alpha^6+\mathcal{O}(\alpha^7)\right]. \tag{2.8}$$

The running coupling $\alpha(x)$ is the solution of

$$\ln\frac{\tau}{\tau'} = \int_{\alpha(-\tau')}^{\alpha(-\tau)} \frac{\mathrm{d}\alpha}{\beta_\alpha(\alpha)}.$$

Throughout this paper we use the notations α for $\alpha(-\tau)$ and α_k for $\alpha(1/k)$, where k is just a constant for now. Later on k will denote the absolute value of the space momentum of the fluctuations. We have

$$\ln(-k\tau) = \int_{\alpha_k}^{\alpha} \frac{\mathrm{d}\alpha'}{\beta_\alpha(\alpha')}.$$

For example, the leading-log running coupling is

$$\alpha = \frac{\alpha_k}{1+2\alpha_k\ln(-k\tau)}. \tag{2.9}$$

The expression of the running coupling to the NNLL order can be found in [5] or appendix A.

It can be proved [5] that the spectra $\mathcal{P}_T$ and $\mathcal{P}_\mathcal{R}$ of the tensor and scalar fluctuations satisfy RG evolution equations in the superhorizon limit with vanishing anomalous dimensions. Here we summarize various versions of the equations. Viewing the spectra as functions of τ and α, they satisfy

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\ln|\tau|} = \left(\frac{\partial}{\partial\ln|\tau|}+\beta_\alpha(\alpha)\frac{\partial}{\partial\alpha}\right)\mathcal{P} = 0. \tag{2.10}$$

Viewing them as functions of α and α_k, the RG equations imply that the dependence on α actually drops out,

$$\mathcal{P} = \tilde{\mathcal{P}}(\alpha_k), \qquad \frac{\mathrm{d}\tilde{\mathcal{P}}(\alpha_k)}{\mathrm{d}\ln k} = -\beta_\alpha(\alpha_k)\frac{\mathrm{d}\tilde{\mathcal{P}}(\alpha_k)}{\mathrm{d}\alpha_k}, \tag{2.11}$$

so the spectra depend on the momentum k only through the running coupling α_k. Finally, viewing the spectra as functions of k/k_* and $\alpha_* = \alpha(1/k_*)$, where k_* is the pivot scale and α_* is the "pivot coupling", the spectra satisfy

$$\left(\frac{\partial}{\partial\ln k}+\beta_\alpha(\alpha_*)\frac{\partial}{\partial\alpha_*}\right)\mathcal{P}(k/k_*,\alpha_*) = 0. \tag{2.12}$$

As said, the RG techniques allow us to calculate RG improved power spectra. This means that $\mathcal{P}_T$ and $\mathcal{P}_\mathcal{R}$ are expanded in powers of α_*, but the product $\alpha_*\ln(k/k_*)$ is considered of order zero and treated exactly.

3 Limit $m_\chi \to \infty$ of infinitely heavy fakeon

In this section we derive the running power spectra to the NNLL order in the limit of infinitely heavy fakeon $m_\chi \to \infty$, which returns the Starobinsky $R + R^2$ theory [9, 10]. The action is

$$S = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R + \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(D_\mu\phi D^\mu\phi - 2V(\phi)\right), \tag{3.1}$$

with the potential (2.2). We parametrize the metric as

$$\begin{aligned} g_{\mu\nu} = {} & \mathrm{diag}(1, -a^2, -a^2, -a^2) - 2a^2\left(u\delta_\mu^1\delta_\nu^1 - u\delta_\mu^2\delta_\nu^2 + v\delta_\mu^1\delta_\nu^2 + v\delta_\mu^2\delta_\nu^1\right), \\ & +2\mathrm{diag}(\Phi, a^2\Psi, a^2\Psi, a^2\Psi) - \delta_\mu^0\delta_\nu^i\partial_i B - \delta_\mu^i\delta_\nu^0\partial_i B. \end{aligned} \tag{3.2}$$

Without loss of generality, the coordinate dependencies of the graviton modes $u = u(t, z)$ and $v = v(t, z)$ are chosen to have a space momentum oriented along the z axis after Fourier transform. As far as the scalar modes are concerned, we work in the comoving gauge, where the ϕ fluctuation $\delta\phi$ vanishes and the curvature perturbation $\mathcal{R}$ coincides with Ψ. For reviews that contain details on the parametrizations of the metric fluctuations and their transformations under diffeomorphisms, see [19, 20].

The tensor fluctuations are studied by setting $\Phi = \Psi = B = 0$. Denoting the Fourier transform of $u(t, z)$ with respect to the coordinate z by $u_{\mathbf{k}}(t)$, where $\mathbf{k}$ is the space momentum, the quadratic Lagrangian obtained from (3.1) is

$$(8\pi G)\frac{\mathcal{L}_\mathrm{t}}{a^3} = \dot{u}_{\mathbf{k}}\dot{u}_{-\mathbf{k}} - \frac{k^2}{a^2}u_{\mathbf{k}}u_{-\mathbf{k}}, \tag{3.3}$$

plus an identical contribution for $v_{\mathbf{k}}$, where $k = |\mathbf{k}|$. We often drop the subscripts $\mathbf{k}$ and $-\mathbf{k}$, when no confusion is expected to arise.

The scalar fluctuations are studied by setting $u = v = 0$. After Fourier transforming the space coordinates to momentum space, (2.1) gives the quadratic Lagrangian

$$(8\pi G)\frac{\mathcal{L}_\mathrm{s}}{a^3} = -3(\dot{\Psi} + H\Phi)^2 + 4\pi G\dot{\phi}^2\Phi^2 + \frac{k^2}{a^2}\left[2B(\dot{\Psi} + H\Phi) + \Psi(\Psi - 2\Phi)\right],$$

omitting the subscripts $\mathbf{k}$ and $-\mathbf{k}$. Integrating B out, we obtain $\Phi = -\dot{\Psi}/H$. Inserting this solution back into the action, we find

$$(8\pi G)\frac{\mathcal{L}_\mathrm{s}}{a^3} = 3\alpha^2\left(\dot{\Psi}^2 - \frac{k^2}{a^2}\Psi^2\right). \tag{3.4}$$

Defining

$$w = au\sqrt{\frac{k}{4\pi G}}, \qquad w = \alpha a\Psi\sqrt{\frac{3k}{4\pi G}}, \tag{3.5}$$

for tensors and scalars, respectively, and switching to the variable $\eta = -k\tau$, the Lagrangians (3.3) and (3.4) give the actions

$$S_{\mathrm{t,s}} = \frac{1}{2}\int \mathrm{d}\eta \left[w'^2 - w^2 + (2+\sigma_{\mathrm{t,s}})\frac{w^2}{\eta^2} \right], \tag{3.6}$$

where the prime denotes the derivative with respect to η and

$$\sigma_{\mathrm{t}} = 9\alpha^2 + 48\alpha^3 + 364\alpha^4 + \mathcal{O}(\alpha^5), \qquad \sigma_{\mathrm{s}} = 6\alpha + 22\alpha^2 + \frac{280}{3}\alpha^3 + \mathcal{O}(\alpha^4). \tag{3.7}$$

We quantize (3.6) as usual, by introducing the operator

$$\hat{w}_{\mathbf{k}}(\eta) = w_{\mathbf{k}}(\eta)\hat{a}_{\mathbf{k}} + w^*_{-\mathbf{k}}(\eta)\hat{a}^\dagger_{-\mathbf{k}},$$

where $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ are creation and annihilation operators satisfying $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$. Summing over the tensor polarizations u and v and recalling that $\mathcal{R} = \Psi$, the power spectra $\mathcal{P}_T$ and $\mathcal{P}_{\mathcal{R}}$ of the tensor and scalar fluctuations are defined by the two-point functions

$$\langle \hat{u}_{\mathbf{k}}(\tau)\hat{u}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\frac{\pi^2}{8k^3}\mathcal{P}_T, \qquad \mathcal{P}_T = \frac{8k^3}{\pi^2}|u_{\mathbf{k}}|^2, \tag{3.8}$$

$$\langle \mathcal{R}_{\mathbf{k}}(\tau)\mathcal{R}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}, \qquad \mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2}|\Psi_{\mathbf{k}}|^2. \tag{3.9}$$

The calculations of $\mathcal{P}_T$ and $\mathcal{P}_{\mathcal{R}}$ are divided in two steps. In the first step we isolate the time dependence, which disappears in the superhorizon limit by the RG equation (2.10). In the second step we work out the overall constants (which are functions of α_k).

We start from the Mukhanov-Sasaki equation derived from the action (3.6), which reads

$$w'' + w - 2\frac{w}{\eta^2} = \sigma_{\mathrm{t,s}}\frac{w}{\eta^2} \tag{3.10}$$

and must be solved with the Bunch-Davies vacuum condition

$$w(\eta) \sim \frac{\mathrm{e}^{i\eta}}{\sqrt{2}} \qquad \text{for large } \eta. \tag{3.11}$$

To study the time dependence in the superhorizon limit, it is convenient to decompose $\eta w(\eta)$ as the sum of a power series $Q(\ln\eta)$ in $\ln\eta$ plus a power series $W(\eta)$ in η and $\ln\eta$, such that $W(\eta) \to 0$ term-by-term for $\eta \to 0$:

$$\eta w = Q(\ln\eta) + W(\eta). \tag{3.12}$$

It was shown in ref. [5] that the w equation (3.10) leads to the Q equation

$$\frac{\mathrm{d}Q}{\mathrm{d}\ln\eta} = -\frac{\sigma}{3}Q - \frac{1}{3}\sum_{n=1}^{\infty} 3^{-n}\frac{\mathrm{d}^n(\sigma Q)}{\mathrm{d}\ln^n\eta}, \tag{3.13}$$

where σ stands for either $\sigma_{\rm t}$ or $\sigma_{\rm s}$ and the higher-derivative terms on the right-hand side have to be handled perturbatively in α_k. We can also view $Q(\ln\eta)$ as a function $\tilde{Q}(\alpha,\alpha_k)$ of α and α_k, satisfying

$$\beta_\alpha\frac{\partial\tilde{Q}}{\partial\alpha}=-\frac{\sigma\tilde{Q}}{3}-\frac{1}{3}\sum_{n=1}^{\infty}3^{-n}\left(\beta_\alpha\frac{\partial}{\partial\alpha}\right)^n(\sigma\tilde{Q}). \tag{3.14}$$

Since the RG equations hold only in the superhorizon limit, where the Bunch-Davies vacuum condition is ineffective, equation (3.13) does not give the normalization of the spectrum. The initial condition $Q(0)$, which remains arbitrary at this stage, is determined in the second step of the calculation, by working directly on w and the equation (3.10).

Formulas (3.7) are enough to work out the NNLL corrections. For that purpose, we can truncate (3.14) to

$$\beta_\alpha\frac{\partial\tilde{Q}}{\partial\alpha}=-\frac{\sigma\tilde{Q}}{3}-\frac{\beta_\alpha}{9}\frac{\partial(\sigma\tilde{Q})}{\partial\alpha}-\frac{\beta_\alpha}{27}\frac{\partial}{\partial\alpha}\left(\beta_\alpha\frac{\partial(\sigma\tilde{Q})}{\partial\alpha}\right) \tag{3.15}$$

and the beta function to

$$\beta_\alpha=-2\alpha^2-\frac{5}{3}\alpha^3-\frac{50}{9}\alpha^4+\mathcal{O}(\alpha^5),$$

We solve (3.15) by searching for solutions of the form

$$\tilde{Q}_{\rm t}(\alpha,\alpha_k)=\tilde{Q}_{\rm t}(\alpha_k)\frac{1+\sum_{j=1}^{\infty}c_j\alpha^j}{1+\sum_{j=1}^{\infty}c_j\alpha_k^j},\qquad \tilde{Q}_{\rm s}(\alpha,\alpha_k)=\tilde{Q}_{\rm s}(\alpha_k)\frac{\alpha}{\alpha_k}\frac{1+\sum_{j=1}^{\infty}c'_j\alpha^j}{1+\sum_{j=1}^{\infty}c'_j\alpha_k^j},$$

for tensors and scalars, respectively, where c_j and c'_j are coefficients to be determined. Inserting these expressions into (3.15), we easily find

$$\begin{aligned}\tilde{Q}_{\rm t}(\alpha,\alpha_k)&=\tilde{Q}_{\rm t}(\alpha_k)\frac{2+3\alpha+7\alpha^2+\frac{199}{6}\alpha^3+\mathcal{O}(\alpha^4)}{2+3\alpha_k+7\alpha_k^2+\frac{199}{6}\alpha_k^3+\mathcal{O}(\alpha_k^4)},\\ \tilde{Q}_{\rm s}(\alpha,\alpha_k)&=\tilde{Q}_{\rm s}(\alpha_k)\frac{\alpha}{\alpha_k}\frac{2+3\alpha+7\alpha^2+\mathcal{O}(\alpha^3)}{2+3\alpha_k+7\alpha_k^2+\mathcal{O}(\alpha_k^3)}.\end{aligned} \tag{3.16}$$

As said, the constants $\tilde{Q}_{\rm t,s}(\alpha_k)=\tilde{Q}_{\rm t,s}(\alpha_k,\alpha_k)=Q_{\rm t,s}(0)$ are determined separately.

Formula (3.12) gives $\eta w\sim Q(\ln\eta)$ in the superhorizon limit, while formulas (3.5) give u and Ψ. Together with formulas (A.2) and (A.3) to order α^3, it is easy to show that the spectra (3.8) and (3.9) are equal to

$$\begin{aligned}\mathcal{P}_T(k)&=\frac{8Gm_\phi^2}{\pi}|\tilde{Q}_{\rm t}(\alpha_k)|^2\left[1-3\alpha_k-\frac{\alpha_k^2}{4}-\frac{91}{6}\alpha_k^3+\mathcal{O}(\alpha_k^4)\right],\\ \mathcal{P}_{\mathcal{R}}(k)&=\frac{Gm_\phi^2}{6\pi\alpha_k^2}|\tilde{Q}_{\rm s}(\alpha_k)|^2\left[1-3\alpha_k-\frac{\alpha_k^2}{4}+\mathcal{O}(\alpha_k^3)\right],\end{aligned} \tag{3.17}$$

respectively, in the superhorizon limit. We see that the α dependence disappears, in agreement with the RG equations (2.10) and (2.11). The dependence on k is encoded into the running coupling $\alpha_k = \alpha(1/k)$. The similarity between the two square brackets of (3.17) will not survive the extension to $m_\chi < \infty$.

Now we calculate the integration constants $\tilde{Q}_{\mathrm{t,s}}(\alpha_k) = Q_{\mathrm{t,s}}(0)$. We expand w in powers of α_k by writing

$$w(\eta) = w_0(\eta) + \sum_{n=1}^{\infty} \alpha_k^n w_n(\eta), \tag{3.18}$$

and insert the expansion into (3.10). We obtain equations of the form

$$w_n'' + w_n - \frac{2w_n}{\eta^2} = \frac{g_n(\eta)}{\eta^2}, \tag{3.19}$$

where the functions g_n are determined recursively from w_m, $m < n$. The Bunch-Davies vacuum condition (3.11) gives $w_0(\eta) \sim \mathrm{e}^{i\eta}/\sqrt{2}$ and $w_m(\eta) \to 0$, $m > 0$, for large η.

We need the solutions w_i with $i = 0, 1, 2, 3$ for tensors and $i = 0, 1, 2$ for scalars, which are reported in the appendix. The behaviors of $w_i(\eta)$ in the superhorizon limit $\eta \sim 0$, also given in appendix A, allow us to derive $w(\eta)$ for $\eta \sim 0$, once the solutions w_i are inserted into (3.18). Formula (3.12) tells us that $\eta w_{\mathrm{t,s}}(\eta) \sim Q_{\mathrm{t,s}}(\ln\eta)$ in the same limit. From that, we can read $Q_{\mathrm{t,s}}(\ln\eta)$ and hence $Q_{\mathrm{t,s}}(0)$. After these calculations, we find

$$\begin{aligned}
\tilde{Q}_{\mathrm{t}}(\alpha_k) &= \frac{i}{\sqrt{2}}\left[1 + 3(2-\tilde{\gamma}_M)\alpha_k^2 + \left(12\tilde{\gamma}_M - 6\tilde{\gamma}_M^2 - \pi^2\right)\alpha_k^3 + \mathcal{O}(\alpha_k^4)\right], \\
\tilde{Q}_{\mathrm{s}}(\alpha_k) &= \frac{i}{\sqrt{2}}\left[1 + 2(2-\tilde{\gamma}_M)\alpha_k + \frac{2}{3}(2 - 7\tilde{\gamma}_M + \pi^2)\alpha_k^2 + \mathcal{O}(\alpha_k^3)\right],
\end{aligned} \tag{3.20}$$

where $\tilde{\gamma}_M = \gamma_M - (i\pi/2)$, $\gamma_M = \gamma_E + \ln 2$, γ_E being the Euler-Mascheroni constant.

A further check of the RG equations can be made by noting that at $\ln\eta \neq 0$, the expressions of $Q_{\mathrm{t}}(\ln\eta)$ and $Q_{\mathrm{s}}(\ln\eta)$ just obtained must coincide with (3.16) – equipped with (3.20) – up to the RG improvements. This means that, once α is replaced with the running coupling given in the appendix, which is a function of $\ln\eta$ and α_k, $Q_{\mathrm{t}}(\ln\eta)$ must agree with $\tilde{Q}_{\mathrm{t}}(\alpha, \alpha_k)$ to order α_k^3, while $Q_{\mathrm{s}}(\ln\eta)$ must agree with $\tilde{Q}_{\mathrm{s}}(\alpha, \alpha_k)$ to order α_k^2. It is easy to verify that both properties are satisfied. We recall that the enhancement contained in (3.16) is the RG improvement, i.e. the resummation of the leading and subleading logs to order α^3.

Finally, inserting the normalizations (3.20) into (3.17) the spectra are

$$\begin{aligned}
\mathcal{P}_T(k) &= \frac{4Gm_\phi^2}{\pi}\left[1 - 3\alpha_k + (47 - 24\gamma_M)\frac{\alpha_k^2}{4} - \left(\frac{307}{6} + 12\gamma_M^2 - 42\gamma_M - \pi^2\right)\alpha_k^3 + \mathcal{O}(\alpha_k^4)\right], \\
\mathcal{P}_{\mathcal{R}}(k) &= \frac{Gm_\phi^2}{12\pi\alpha_k^2}\left[1 + (5 - 4\gamma_M)\alpha_k - \frac{67}{12}\alpha_k^2 + (12\gamma_M^2 - 40\gamma_M + 7\pi^2)\frac{\alpha_k^2}{3} + \mathcal{O}(\alpha_k^3)\right].
\end{aligned} \tag{3.21}$$

The tilts and the running coefficients can be found by differentiating with respect to $\ln k$, using the RG equations and the expression of the beta function. We postpone this part and proceed to compute the spectra of quantum gravity.

4 Quantum gravity: tensor fluctuations

In this section we compute the spectrum of the tensor fluctuations to the NNLL order in quantum gravity.

Parametrizing the metric as (3.2) with $\Phi = \Psi = B = 0$, the quadratic Lagrangian obtained from (2.1) is

$$(8\pi G)\frac{\mathcal{L}_{\mathrm{t}}}{a^3} = \dot{u}^2 - \frac{k^2}{a^2}u^2 - \frac{1}{m_\chi^2}\left[\ddot{u}^2 - 2\left(H^2 - \frac{3}{2}\alpha^2 H^2 + \frac{k^2}{a^2}\right)\dot{u}^2 + \frac{k^4}{a^4}u^2\right], \tag{4.22}$$

plus an identical contribution for v. The calculation proceeds as follows. First, we eliminate the higher derivatives from $\mathcal{L}_{\mathrm{t}}$ by introducing an extra field. Then, we diagonalize the new Lagrangian and perform the fakeon projection. As expected, we obtain a projected Lagrangian that depends on just one field and has no higher derivatives. Third, we apply a number of field redefinitions and time reparametrizations to cast the action into the standard form (3.6). Then we solve the w equation with the Bunch-Davies vacuum condition. Finally, we undo all the transformations and work out the u two-point function and the tensor spectrum in the superhorizon limit.

To calculate the spectra to the NNLL order we need the projected Lagrangian to order α^3. We also plan to make an independent check the RG evolution equations. For that purpose, the k-independent corrections to the Lagrangian are needed to order α^4.

The higher derivatives of $\mathcal{L}_{\mathrm{t}}$ can be eliminated with the procedure outlined in [2]. Specifically, we add an auxiliary field U and consider the extended Lagrangian

$$\mathcal{L}_{\mathrm{t}}' = \mathcal{L}_{\mathrm{t}} + \frac{a^3}{8\pi G m_\chi^2}\left[m_\chi^2\sqrt{\gamma}U - \ddot{u} - 3H\left(1 - \frac{4\alpha^2 H^2}{m_\chi^2\gamma}\right)\dot{u} - fu\right]^2, \tag{4.23}$$

where[2]

$$f = m_\chi^2\gamma + \frac{k^2}{a^2} + \frac{\alpha^2 H^2}{m_\chi^2\gamma}\left(3m_\chi^2 - 12H^2 + 24\alpha H^2 - \frac{2\alpha^2 H^2(17m_\chi^2 - 38H^2)}{m_\chi^2\gamma}\right)$$

and

$$\gamma = 1 + 2\frac{H^2}{m_\chi^2}. \tag{4.24}$$

[2]Note some rearrangements with respect to the parametrizations of [2] and [5], for a better inclusion of the extra order we need.

It is straightforward to show that $\mathcal{L}'_{\mathrm{t}}$ is equivalent to $\mathcal{L}_{\mathrm{t}}$ by replacing U, which appears algebraically, with the solution of its own field equation. As said, the reason why we keep the order α^4 is that we need it to check the RG equation to order α^3. If we are happy with just using the RG equation, which we know to hold since it was proved on general grounds in [5], the Lagrangian to order α^3 is sufficient for our purposes.

We diagonalize $\mathcal{L}'_{\mathrm{t}}$ in the de Sitter limit by making the field redefinition

$$u = \frac{U+V}{\sqrt{\gamma}}, \tag{4.25}$$

where V is a new field. To the order we need, the "de-Sitter-diagonal" $\mathcal{L}'_{\mathrm{t}}$ reads

$$\begin{aligned}(8\pi G)\frac{\mathcal{L}'_{\mathrm{t}}}{a^3} = {} & \dot{U}^2 - \frac{hk^2}{a^2}U^2 - \frac{9}{8}m_\phi^2\xi\zeta\alpha^2\left[1-\frac{\zeta\alpha}{6}(40-7\xi)+\frac{\zeta^2\alpha^2}{144}(2800-3806\xi-497\xi^2)\right]U^2 \\ & -\dot{V}^2+\left[m_\chi^2+\frac{m_\phi^2}{2}(1-3\alpha)+\frac{k^2}{a^2}\right]V^2+\frac{3m_\phi^2\zeta\alpha^2}{2}\left(1-\xi+\frac{4\xi\zeta k^2}{m_\phi^2 a^2}\right)UV \\ & -\frac{3m_\phi^2\zeta^2\alpha^3}{4}\left[6-22\xi+\xi^2+(6-7\xi-2\xi^2)\frac{4\xi\zeta k^2}{m_\phi^2 a^2}\right]UV,\end{aligned} \tag{4.26}$$

where

$$\xi = \frac{m_\phi^2}{m_\chi^2}, \qquad \zeta = \left(1+\frac{\xi}{2}\right)^{-1}, \qquad h = 1-3\xi\zeta^2\alpha^2+\frac{3\xi\zeta^3\alpha^3}{2}(6-7\xi-2\xi^2)+\mathcal{O}(\alpha^4). \tag{4.27}$$

It will be clear in a moment that it is enough to keep the terms proportional to V^2 to order α and the terms proportional to UV, as well as those proportional to k^2U^2, to order α^3, as done in (4.26) and (4.27).

4.1 The fakeon projection

The fakeon projection amounts to integrating V out by replacing it with a particular solution $V(U)$ of its own field equations, defined by the fakeon Green function [2][3]. From the expression of (4.26) we see that the projection equates V to something of order α^2. Once the solution is inserted back into (4.26), the second and third lines of (4.26) turn out to be $\mathcal{O}(\alpha^4)$. This means that to the order α^3 included, which is enough for our present purposes, the projected U action is unaffected by $V(U)$. Specifically, it is given by the first line of (4.26) and can be solved with the standard Bunch-Davies vacuum condition[4].

[3] See also [13, 18].

[4] At higher orders, instead, the projected U action receives contributions from $V(U)$ and depends on k in a nontrivial way. Then the Bunch-Davies vacuum condition for large k needs to be reconsidered and possibly replaced by a different condition.

This does not mean we can forget about V altogether, however. Indeed, formula (4.25) tells us that we need $V(U)$ to calculate the spectrum. Since we want the spectrum to the NNLL order, we must compute $V(U)$ to order α^3, for which (4.26) is enough. A lot of effort can be saved by noting that it is sufficient to obtain $V(U)$ in the superhorizon limit $k/(m_\phi a) \to 0$. Moreover, we can drop several contributions by making use of the U field equations.

With this in mind, the projected V can be easily found by inserting the ansatz

$$V = \alpha^2 (v_1 + v_2\alpha) U + v_3\alpha^3\dot{U} + \mathcal{O}(\alpha^4)$$

into the V field equation derived from (4.26), where v_i, $i = 1, 2, 3$, are constants. Then we take the superhorizon limit by dropping the k-dependent terms. Third, we use the U field equation to turn the higher-derivatives $\ddot{U}$ and $U^{(3)}$ into corrections proportional to U and $\dot{U}$. At the end, we obtain a linear combination of $\alpha^2 U$, $\alpha^3 U$ and $\alpha^3\dot{U}$. Equating the coefficients of such three terms to zero, we determine the constants v_i. The result gives

$$V = -\frac{3\xi\zeta^2\alpha^2}{4}\left[1 - \xi - (6 - 19\xi - 2\xi^2)\frac{\zeta\alpha}{2}\right] U + \frac{3(1-\xi)\xi^2\zeta^3\alpha^3}{m_\phi}\dot{U} + \mathcal{O}(\alpha^4). \tag{4.28}$$

4.2 w action and RG equation

We first calculate the spectrum to the NNLL order by means of the RG equation. In a second moment, we check the RG equation and the spectrum independently. As said, the tensor analogue of the Mukhanov-Sasaki action to order α^3 is given by the first line of (4.26). We define

$$w = \frac{aU\sqrt{k}}{\sqrt{4\pi G}}, \qquad \sigma_{\mathrm{t}} = 9\zeta\alpha^2 + \frac{3\zeta^2\alpha^3}{2}(32 + 43\xi) + \zeta^3\alpha^4 F_{\mathrm{t}}(\xi) + \mathcal{O}(\alpha^5), \tag{4.29}$$

where the function $F_{\mathrm{t}}(\xi)$ parametrizes the order α^4 of σ_{t}, for the moment unknown. Using (A.2) and (A.3) and switching to conformal time, the projected w action to the order we need can be cast into the form

$$S_{\mathrm{t}}^{\mathrm{prj}} = \frac{1}{2}\int \mathrm{d}\eta \left(w'^2 - hw^2 + 2\frac{w^2}{\eta^2} + \sigma_{\mathrm{t}}\frac{w^2}{\eta^2}\right). \tag{4.30}$$

The unusual feature of this expression is the function h in front of w^2, which is a sort of running squared mass.

Having parametrized the order α^4 of σ_{t} as shown in (4.29), we first determine $F_{\mathrm{t}}(\xi)$ by means of the RG equation and then work out the spectrum to the NNLL order. Later on, in subsection 4.5, we make an independent check of the RG equation by calculating

$\sigma_{\rm t}$ directly to order α^4, with the help of an expansion in powers of ξ, and rederive the spectrum without using the RG equation.

It is important to recall that the RG equation holds in the superhorizon limit, where the Bunch-Davies condition is unnecessary and we can ignore the term $-hw^2$ of (4.30). In particular, once we decompose w as shown in (3.12), the equations (3.13) and (3.14) still hold, so we can calculate the function $Q_{\rm t}(\ln\eta) = \tilde{Q}_{\rm t}(\alpha, \alpha_k)$ by solving (3.15) with the $\sigma_{\rm t}$ of (4.29) and repeating the steps from (3.12) to (3.16). We find

$$\tilde{Q}_{\rm t}(\alpha, \alpha_k) = \frac{J_{\rm t}(\alpha)}{J_{\rm t}(\alpha_k)}\tilde{Q}_{\rm t}(\alpha_k), \tag{4.31}$$

with

$$J_{\rm t}(\alpha) = 1 + \frac{3\zeta\alpha}{2} + \frac{56+73\xi}{16}\zeta^2\alpha^2 + \left(F_{\rm t}(\xi) - \frac{131}{2} - \frac{2029\xi}{16} - \frac{1441\xi^2}{16}\right)\frac{\zeta^3\alpha^3}{18} + \mathcal{O}(\alpha^4).$$

At this point, we go back to u as follows. We first use (3.12) to compute w in the superhorizon limit, where $\eta w \sim Q(\ln\eta) = \tilde{Q}(\alpha, \alpha_k)$. Then we use the first formula of (4.29) to compute U, (4.28) to compute V and finally (4.25) to compute u.

The RG equation (2.10) implies that u is time independent in the superhorizon limit. In the parametrization we are using, which is the one of (2.11), u can depend on α_k, but not on α. It is easy to check that, indeed, no α dependence appears to order α^3. As far as the contributions proportional to α^4 are concerned, they disappear by setting

$$F_{\rm t}(\xi) = 364 + \frac{4037}{8}\xi + \frac{6145}{16}\xi^2 + \frac{81}{2}\xi^3. \tag{4.32}$$

This is how the RG equation fixes $F_{\rm t}(\xi)$. Now we are ready to use formula (3.8), which gives

$$\mathcal{P}_T(k) = \frac{8m_\phi^2\zeta G}{\pi}|\tilde{Q}_{\rm t}(\alpha_k)|^2\left[1 - 3\zeta\alpha_k - \frac{\zeta^2\alpha_k^2}{8}(2+73\xi)\right.$$
$$\left. - \frac{\zeta^3\alpha_k^3}{12}(182 + 11\xi + 392\xi^2 + 54\xi^3) + \mathcal{O}(\alpha_k^4)\right]. \tag{4.33}$$

4.3 w action and Bunch-Davies vacuum condition

For the purpose of calculating $\tilde{Q}_{\rm t}(\alpha_k)$, it is important to deal with the running squared mass encoded in the term $-hw^2$ of (4.30). Following [5], we change variables from η to $\tilde{\eta}(\eta)$, such that $\tilde{\eta}'(\eta) = \sqrt{h(\eta)}$, $\tilde{\eta}(0) = 0$, and rewrite (4.30) in the more standard form

$$\tilde{S}_{\rm t}^{\rm prj} = \frac{1}{2}\int {\rm d}\tilde{\eta}\left(\tilde{w}'^2 - \tilde{w}^2 + \frac{2\tilde{w}^2}{\tilde{\eta}^2} + \tilde{\sigma}_{\rm t}\frac{\tilde{w}^2}{\tilde{\eta}^2}\right), \tag{4.34}$$

where

$$\tilde{w}(\tilde{\eta}(\eta)) = h(\eta)^{1/4} w(\eta), \qquad \tilde{\sigma}_{\mathrm{t}} = \frac{\tilde{\eta}^2(\sigma_{\mathrm{t}}+2)}{\eta^2 h} + \frac{\tilde{\eta}^2}{16h^3}\left(4hh'' - 5h'^2\right) - 2. \tag{4.35}$$

The Bunch-Davies vacuum condition for (4.34) is straightforward and reads

$$\tilde{w}(\tilde{\eta}) \simeq \frac{\mathrm{e}^{i\tilde{\eta}}}{\sqrt{2}} \qquad \text{for } \tilde{\eta} \to \infty, \tag{4.36}$$

as usual. Now we prove that it implies

$$\tilde{w}(\tilde{\eta}) = \frac{(\tilde{\eta}+i)\mathrm{e}^{i\tilde{\eta}}}{\sqrt{2}\tilde{\eta}} + \alpha_k^2 \Delta\tilde{w}(\tilde{\eta}), \qquad \lim_{\tilde{\eta}\to\infty} \Delta\tilde{w}(\tilde{\eta}) = 0. \tag{4.37}$$

Using (4.27), (4.29) and the expression (A.1) of the running coupling α, $\tilde{\sigma}_{\mathrm{t}}$ turns out to be $\mathcal{O}\left(\alpha_k^2\right)$. The solution of the $\tilde{w}$ equation of motion derived from (4.34) obviously agrees with (4.37) for $\alpha_k = 0$, by (4.36). This means that we can parametrize $\tilde{w}(\tilde{\eta})$ as shown in the first equation of (4.37), for a suitable $\Delta\tilde{w}(\tilde{\eta})$. Moreover, the asymptotic behavior (4.36) must hold for every α_k. This implies that $\Delta\tilde{w}(\tilde{\eta})$ must tend to zero for $\tilde{\eta} \to \infty$.

Using (4.27), we find

$$\tilde{\eta} = \eta\left[1 - \frac{3\xi\zeta^2\alpha_k^2}{2}(1 - 4\alpha_k \ln\eta) - \frac{3\xi\zeta^3\alpha_k^3}{4}(2 + 11\xi + 2\xi^2) + \mathcal{O}(\alpha_k^4)\right].$$

Then, $\alpha_k^2\Delta\tilde{w}(\tilde{\eta}) = \alpha_k^2\Delta\tilde{w}(\eta) + \mathcal{O}(\alpha_k^4)$, which allows us to conclude, using the first formula of (4.35), that $w(\eta)$ has the form

$$w(\eta) = \frac{(\tilde{\eta}(\eta)+i)\mathrm{e}^{i\tilde{\eta}(\eta)}}{\sqrt{2}\tilde{\eta}(\eta)h(\eta)^{1/4}} + \alpha_k^2\frac{\Delta\tilde{w}(\eta)}{h(\eta)^{1/4}} + \mathcal{O}(\alpha_k^4), \qquad \lim_{\eta\to\infty}\frac{\Delta\tilde{w}(\eta)}{h(\eta)^{1/4}} = 0.$$

The Bunch-Davies conditions for $w(\eta)$ to order α_k^3 can be read from the first term on the right-hand side of the first equation, no correction coming from the rest. Referring to the expansion (3.18), we obtain $w_1(\eta) = 0$ and, for η large,

$$\begin{aligned} w_0(\eta) &\simeq \frac{\mathrm{e}^{i\eta}}{\sqrt{2}}, \qquad w_2(\eta) \simeq \frac{3\xi\zeta^2\mathrm{e}^{i\eta}(3-2i\eta)}{4\sqrt{2}}, \\ w_3(\eta) &\simeq -\frac{3\xi\zeta^2\mathrm{e}^{i\eta}}{\sqrt{2}}(3-2i\eta)\ln\eta - \frac{3\xi\zeta^3\mathrm{e}^{i\eta}}{8\sqrt{2}}[2 - 29\xi - 6\xi^2 + 2i\eta(2+11\xi+2\xi^2)]. \end{aligned} \tag{4.38}$$

4.4 The spectrum

Inserting the expansion (3.18) into the equation of motion derived from the action (4.30), we obtain the w_n equations

$$w_n'' + w_n - 2\frac{w_n}{\eta^2} = \sum_{j=0}^{n-2}\sigma_j\frac{w_{n-2-j}}{\eta^2} - \sum_{j=0}^{n-2}h_j w_{n-2-j}, \tag{4.39}$$

where

$$h = 1 + \alpha_k^2 \sum_{j=0}^{\infty} h_j \alpha_k^j, \qquad \sigma_{\mathrm{t}} = \alpha_k^2 \sum_{j=0}^{\infty} \sigma_j \alpha_k^j,$$

h_j and σ_j being functions of η. The asymptotic conditions (4.38) determine the solutions.

Although (4.39) and (4.38) are somewhat different form the equations and conditions met in the previous section, they can be solved with similar techniques. The results involve the same types of functions, listed in appendix A. Formulas (A.6) contain the relevant right-hand sides of (4.39), while formulas (A.7) contain the solutions. Studying the $\eta \sim 0$ limit of $\eta w(\eta)$ with the help of the asymptotic behaviors (A.8), we extract $Q_{\mathrm{t}}(\ln \eta)$ by means of the decomposition (3.12) and from that we obtain $\tilde{Q}_{\mathrm{t}}(\alpha_k) = Q_{\mathrm{t}}(0)$. The outcome is

$$\tilde{Q}_{\mathrm{t}}(\alpha_k) = \frac{i}{\sqrt{2}} \left[1 + \frac{3\zeta^2\alpha_k^2}{4}(8+7\xi) - 3\zeta\alpha_k^2\tilde{\gamma}_M - \zeta\alpha_k^3(6\tilde{\gamma}_M^2 + \pi^2) \right. \\ \left. + \frac{3\zeta^2\alpha_k^3}{2}\tilde{\gamma}_M(8+\xi) + \frac{3\xi\zeta^3\alpha_k^3}{8}(22 + 41\xi + 6\xi^2) + \mathcal{O}(\alpha_k^4) \right]. \qquad (4.40)$$

Inserting this result into (4.33) we finally obtain the spectrum to the NNLL order, which reads

$$\mathcal{P}_T(k) = \frac{4m_\phi^2\zeta G}{\pi} \left[1 - 3\zeta\alpha_k \left(1 + 2\alpha_k\gamma_M + 4\gamma_M^2\alpha_k^2 - \frac{\pi^2\alpha_k^2}{3} \right) + \frac{\zeta^2\alpha_k^2}{8}(94 + 11\xi) \right. \\ \left. + 3\gamma_M\zeta^2\alpha_k^3(14+\xi) - \frac{\zeta^3\alpha_k^3}{12}(614 + 191\xi + 23\xi^2) + \mathcal{O}(\alpha_k^4) \right]. \qquad (4.41)$$

4.5 Checks of the RG equation and the spectrum

Now we perform two checks of the RG equation and one of the spectrum. The first check of the RG equation is straightforward. Indeed, the expression of $Q_{\mathrm{t}}(\ln \eta)$ just found by solving (4.39) must coincide with the RG improved one – given by formula (4.31) with the overall constant (4.40) –, once we ignore the RG improvement (that is to say, to order α_k^3 and considering $\alpha_k \ln \eta$ of order one instead of order zero). Using the expansions (A.8) of the appendix and the relations (A.7), we easily verify that it is so.

We wish to make a more invasive, independent check of the RG equation to the NNLL order. To achieve this goal, which also provides a check of the spectrum, we need the projected U Lagrangian to order α^4, which we do not have exactly. Since the projection $V(U)$ is obtained by integrating out V with the fakeon prescription, the solution $V(U) = \mathcal{O}(\alpha^2)$ is nonlocal, in general. This feature of $V(U)$ becomes visible beyond the superhorizon limit and before using the U field equations. The scale of nonlocality is the fakeon mass m_χ and is also the scale of the violation of microcausality [13, 18]. The second and third lines of

(4.26) show that the projected U Lagrangian is nonlocal starting from order α^4. Yet, we know that these issues do not affect the RG improved spectrum to the NNLL order. One way to bypass the difficulty is to take m_χ large by expanding in powers of ξ and realizing that the lowest orders of the expansion give the exact result. In the derivation below, we manage to check the RG equation and the spectrum to order ξ^5.

Moving the subleading terms to the right-hand side, the V field equation can be arranged into the form

$$V = \xi\alpha^2 \left(v_4 + v_5\alpha + \frac{(v_6 + v_7\alpha)\xi k^2}{m_\phi^2 a^2} \right) U - \frac{\xi}{2} \left(1 - 3\alpha + \frac{2k^2}{m_\phi^2 a^2} \right) V - \frac{\xi \ddot{V}}{m_\phi^2} - \frac{3\xi H \dot{V}}{m_\phi^2} \tag{4.42}$$

and solved perturbatively in ξ, where v_i, $i = 4, \ldots 7$, are constants. The solution $V(U)$ is then inserted back into $\mathcal{L}'_{\mathrm{t}}(U, V)$, to obtain the projected Lagrangian $\mathcal{L}''_{\mathrm{t}}(U) = \mathcal{L}'_{\mathrm{t}}(U, V(U))$. Due to the expansion in powers of ξ, $\mathcal{L}''_{\mathrm{t}}$ is local, but contains higher derivatives of U and high powers of $k^2/(m_\phi^2 a^2)$. We eliminate both by means of a field redefinition

$$U(W) = W + \xi^2\alpha^4 \sum_{j\geqslant 0} \tilde{h}_j(\xi) W^{(j)}, \tag{4.43}$$

such that $\mathcal{L}'''_{\mathrm{t}}(W) = \mathcal{L}''_{\mathrm{t}}(U(W))$ is cast into the standard form

$$(8\pi G)\frac{\mathcal{L}'''_{\mathrm{t}}(W)}{a^3} = \dot{W}^2 - \frac{k^2}{a^2}W^2 + \xi\alpha^2 \left[h_1(\alpha, \xi) + \frac{k^2}{a^2} h_2(\alpha, \xi) \right] W^2, \tag{4.44}$$

where $h_{1,2}(\alpha, \xi)$ are power series in α and ξ and $\tilde{h}_j(\xi)$ are power series in ξ.

Finally, defining, $w = a\sqrt{k}W/\sqrt{4\pi G}$, formula (4.44) gives the projected w action, which turns out to have the form (4.30) with the same h (to order α^3), but a different σ_{t}. Note that various field redefinitions are involved, so only the final results (the spectra $\mathcal{P}$) should match. The ingredients of the intermediate steps do not need to coincide. This means that we must repeat the derivation to the very end with the new σ_{t}: first, we work out (4.31) from (3.15); second, we extract $\tilde{Q}_{\mathrm{t}}(\alpha_k) = Q_{\mathrm{t}}(0)$ from the solution to the w equation; third, we set $\eta w = Q_{\mathrm{t}}(\ln \eta) = \tilde{Q}_{\mathrm{t}}(\alpha, \alpha_k)$ in the superhorizon limit, according to (3.12); then we have $W = \sqrt{4\pi G}w/(a\sqrt{k})$, which allows us to obtain U from (4.43), V from the recursive solution to (4.42), u from (4.25) and finally the spectrum from (3.8). At the end, we correctly find the expansion of (4.41) in powers of ξ, which we push to order ξ^5.

The steps described above allow us to calculate the RG improved spectrum directly, while the arguments of the previous subsections give F_{t}, and hence $\mathcal{P}_T$, *from* the RG equation. The approximation to order ξ^5 is also able to determine F_{t} exactly, because it outperforms F_{t}, which is a polynomial of degree three in ξ. The results we have just obtained provide the desired checks of the RG equation and the spectrum.

5 Quantum gravity: scalar fluctuations

In this section we derive the running scalar spectrum to the NNLL order, which is the first order affected by the fakeon $\chi_{\mu\nu}$ in quantum gravity. We do not calculate the NNLL corrections exactly in ξ, because they lead to a nonlocal projected action. Instead, we expand in powers of ξ. The result contains an asymptotic series that we work out to order ξ^9.

We expand the Lagrangian (2.1) by means of (3.2) (with $u = v = 0$) to the quadratic order in the fluctuations. Then we eliminate the auxiliary field Φ by means of its own field equation. Finally, we make the field redefinitions

$$\Psi = \frac{U}{\sqrt{3}\alpha\sqrt{1+\alpha^2}}, \qquad B = \frac{a^2}{k^2}V + \frac{U}{\sqrt{3}\alpha H(1-\alpha^2)\sqrt{1+\alpha^2}}. \tag{5.45}$$

So doing, we obtain a Lagrangian $\mathcal{L}'_{\mathrm{s}}(U, V)$ that admits a regular expansion in powers of k and α. We do not report its expression here, because it is rather involved.

The next step is to determine the fakeon projection $V(U)$, which starts from order α. For the purposes of this paper, it is sufficient to work out the projected Lagrangian $\mathcal{L}''_{\mathrm{s}}(U) = \mathcal{L}'_{\mathrm{s}}(U, V(U))$ to order α^3 included, which requires to study $V(U)$ to order α^2.

As said, we perform the projection by expanding in powers of $\xi = m_\phi^2/m_\chi^2$ and solving the V equation of motion iteratively in ξ. We first illustrate the procedure by writing the key formulas to the first order. Later we describe how to handle the higher orders.

To order ξ, we find

$$\begin{aligned} V(U) = &-\frac{\sqrt{3}}{2}m_\phi \left[(2-\xi)\alpha + \frac{2}{3}(2+3\alpha+4\xi\alpha)\frac{k^2}{m_\phi^2 a^2} - \frac{2\xi(2+3\alpha)}{3}\frac{k^4}{m_\phi^4 a^4}\right]\alpha U \\ &+\frac{\sqrt{3}}{2}\left(2-\xi+3\xi\alpha-\frac{4\xi}{3}\frac{k^2}{m_\phi^2 a^2}\right)\alpha\dot{U} \\ &-\frac{\sqrt{3}}{2}\left(3-\frac{5}{2}\alpha-\frac{2}{3}(2+3\alpha)\frac{k^2}{m_\phi^2 a^2}\right)\frac{\xi\alpha}{m_\phi}\ddot{U} - \frac{\sqrt{3}\xi\alpha U^{(3)}}{m_\phi^2} + \mathcal{O}(\alpha^3). \end{aligned} \tag{5.46}$$

Inserting the solution back into $\mathcal{L}'_{\mathrm{s}}(U, V)$, we obtain the projected Lagrangian $\mathcal{L}''_{\mathrm{s}}(U)$, which reads

$$\begin{aligned} (8\pi G)\frac{\mathcal{L}''_{\mathrm{s}}}{a^3} = &\left[1-\alpha^2+\frac{\xi}{2}\alpha^2(1-3\alpha)+\frac{2\xi\alpha^2 k^2}{m_\phi^2 a^2}\right]\dot{U}^2 - \frac{k^2}{a^2}\left(1-\alpha^2+\frac{\xi\alpha^2 k^2}{m_\phi^2 a^2}\right)U^2 \\ &+\frac{3m_\phi^2\alpha}{2}\left(1-\frac{5}{6}\alpha+\frac{29+18\xi}{36}\alpha^2\right)U^2 - \frac{\xi\alpha^2\ddot{U}^2}{m_\phi^2} + \mathcal{O}(\alpha^4). \end{aligned} \tag{5.47}$$

The higher-derivatives can be eliminated by means of the change of variables

$$U = u\left(1+\frac{\alpha^2}{2}\right) - \frac{\xi\alpha^2}{4}\left(1+6\alpha+\frac{2k^2}{m_\phi^2 a^2}\right)u - \frac{\xi\alpha^2(6+23\alpha)\dot{u}}{8m_\phi} - \frac{\xi\alpha^2\ddot{u}}{2m_\phi^2} + \mathcal{O}(\alpha^4), \quad (5.48)$$

which turns (5.47) into the standard form

$$(8\pi G)\frac{\mathcal{L}_{\mathrm{s}}'''(u)}{a^3} = \dot{u}^2 - \frac{k^2}{a^2}u^2 + \frac{3m_\phi^2\alpha}{2}\left(1-\frac{5}{6}\alpha+\frac{29}{36}\alpha^2+\frac{\xi\alpha^2}{2}\right)u^2. \quad (5.49)$$

If we want to include higher orders, the procedure does not change by much. Once $V(U)$ and $\mathcal{L}_{\mathrm{s}}''(U)$ are obtained, the key point is to find the change of variables

$$U = \left[1+\frac{\alpha^2}{2}-\frac{\alpha^2\xi}{4}F_{\mathrm{s}}(\xi)\right]u + \alpha^2\mathcal{O}(k^2|\tau|^2, \dot{u}, \ddot{u}, \cdots) + \mathcal{O}(\alpha^3) \quad (5.50)$$

that turns the projected action $\mathcal{L}_{\mathrm{s}}'''(u)$ into the form

$$(8\pi G)\frac{\mathcal{L}_{\mathrm{s}}'''(u)}{a^3} = \dot{u}^2 - \frac{k^2}{a^2}u^2 + m^2(\alpha)u^2, \quad (5.51)$$

for a suitable $m^2(\alpha) = \mathcal{O}(\alpha)$.

For the power spectrum, it is sufficient to report the function $F_{\mathrm{s}}(\xi)$ of (5.50). To the eighth order included, we find

$$F_{\mathrm{s}}(\xi) = 1 + \frac{\xi}{4} + \frac{\xi^2}{8} + \frac{\xi^3}{8} + \frac{7\xi^4}{32} + \frac{19}{32}\xi^5 + \frac{295}{128}\xi^6 + \frac{1549}{128}\xi^7 + \frac{42271}{512}\xi^8 + \mathcal{O}(\xi^9), \quad (5.52)$$

which suggests that the series expansion of $F_{\mathrm{s}}(\xi)$ is asymptotic. It is possible to extend the calculation to arbitrarily high orders of ξ.

Finally, the Lagrangian (5.51) becomes

$$(8\pi G)\frac{\mathcal{L}_{\mathrm{s}}'''(u)}{a^3} = \dot{u}^2 - \frac{k^2}{a^2}u^2 + \frac{3m_\phi^2\alpha}{2}\left(1-\frac{5}{6}\alpha+\frac{29}{36}\alpha^2+\frac{\alpha^2\xi}{2}F_{\mathrm{s}}(\xi)\right)u^2 + \mathcal{O}(\alpha^4). \quad (5.53)$$

The terms $\mathcal{O}(k^2, \dot{u}, \ddot{u}, \cdots)$ and $\mathcal{O}(\alpha^3)$ of (5.50) are necessary to derive $F_{\mathrm{s}}(\xi)$ and this expression. However, they are rather involved and can be ignored from this point onwards, so we do not report them here. In particular, we can drop them when we derive the spectrum to order α^2 in the superhorizon limit $k/(m_\phi a) \ll 1$, i.e., $\eta \ll 1$. In this respect, note that the Lagrangian (5.51) ensures that at $k=0$, $\alpha=0$ the solution is $u = $ constant, which implies $\dot{u} = \mathcal{O}(\alpha)$ for $k/(m_\phi a) \to 0$.

As far as the Mukhanov-Sasaki action is concerned, we find (3.6) with

$$w = \frac{a\sqrt{k}}{\sqrt{4\pi G}}u, \qquad \sigma_{\mathrm{s}} = 6\alpha + 22\alpha^2 + \left(\frac{280}{3}+3\xi F_{\mathrm{s}}(\xi)\right)\alpha^3 + \mathcal{O}(\alpha^4). \quad (5.54)$$

Solving (3.15), we obtain

$$Q^{(\mathcal{R})}(\ln \eta) = \tilde{Q}_{\rm s}(\alpha_k)\frac{\alpha}{\alpha_k}\frac{J_{\rm s}(\alpha)}{J_{\rm s}(\alpha_k)}, \qquad J_{\rm s}(\alpha) = 1 + \frac{3\alpha}{2} + \left(\frac{7}{2} + \frac{\xi}{4}F_{\rm s}(\xi)\right)\alpha^2 + \mathcal{O}(\alpha^3).$$

We go back to Ψ as follows. Formula (3.12) tells us that w is equal to Q/η plus corrections that are negligible in the superhorizon limit. Then (5.54) gives u, (5.50) gives U and (5.45) leads to Ψ. At last, we use (3.9) and obtain the spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{Gm_\phi^2}{6\pi\alpha^2}|\tilde{Q}_{\rm s}(\alpha_k)|^2\left[1 - 3\alpha_k - \frac{\alpha_k^2}{4}\left(1 + 2\xi F_{\rm s}(\xi)\right) + \mathcal{O}(\alpha_k^3)\right]. \tag{5.55}$$

As expected, the α dependence disappears, in agreement with the RG equation (2.11).

The last step is to compute $\tilde{Q}_{\rm s}(\alpha_k)$, which is straightforward, since it coincides with the result of (3.20). The reason is that the action (5.53) depends on m_χ only from order α^3, so $\tilde{Q}_{\rm s}(\alpha_k)$ is unaffected by m_χ to order α_k^2. At the end, we get

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{Gm_\phi^2}{12\pi\alpha_k^2}\left[1 + (5 - 4\gamma_M)\alpha_k + \left(4\gamma_M^2 - \frac{40}{3}\gamma_M + \frac{7}{3}\pi^2 - \frac{67}{12} - \frac{\xi}{2}F_{\rm s}(\xi)\right)\alpha_k^2 + \mathcal{O}(\alpha_k^3)\right]. \tag{5.56}$$

6 Predictions

If primordial cosmology turns into an arena for precision tests of quantum gravity, as we hope and deem realistic, the predictions of this paper have a chance to be tested in the incoming years [21]. In this section we summarize them and comment on their validity. Besides the power spectra (4.41) and (5.56), a number of other quantities can be calculated straightforwardly from them. We mention the running ("dynamical") tensor-to-scalar ratio

$$r(k) = \frac{\mathcal{P}_T(k)}{\mathcal{P}_{\mathcal{R}}(k)} \tag{6.1}$$

as a function of k, as well as the tilts

$$n_T = -\beta_\alpha(\alpha_k)\frac{\partial \ln \mathcal{P}_T}{\partial \alpha_k}, \qquad n_{\mathcal{R}} - 1 = -\beta_\alpha(\alpha_k)\frac{\partial \ln \mathcal{P}_{\mathcal{R}}}{\partial \alpha_k},$$

and the running coefficients

$$\frac{\mathrm{d}^n n_T}{\mathrm{d}\ln k^n} = \left(-\beta_\alpha(\alpha_k)\frac{\partial}{\partial \alpha_k}\right)^n n_T, \qquad \frac{\mathrm{d}^n n_{\mathcal{R}}}{\mathrm{d}\ln k^n} = \left(-\beta_\alpha(\alpha_k)\frac{\partial}{\partial \alpha_k}\right)^n n_{\mathcal{R}}.$$

Using (4.41) and (5.56) we find

$$
\begin{aligned}
n_T &= -6\left[1+4\gamma_M\alpha_k+(12\gamma_M^2-\pi^2)\alpha_k^2\right]\zeta\alpha_k^2+\left[24+3\xi+4(31+2\xi)\gamma_M\alpha_k\right]\zeta^2\alpha_k^3 \\
&\qquad -\frac{1}{8}(1136+566\xi+107\xi^2)\zeta^3\alpha_k^4+\mathcal{O}(\alpha_k^5), \qquad (6.2)\\
n_{\mathcal{R}}-1 &= -4\alpha_k+\frac{4\alpha_k^2}{3}(5-6\gamma_M)-\frac{2\alpha_k^3}{9}(338-90\gamma_M+72\gamma_M^2-42\pi^2+9\xi F_{\mathrm{s}})+\mathcal{O}(\alpha_k^4).
\end{aligned}
$$

The running coefficients follow immediately form the RG equation and the beta function. The first two corrections to the relation $r+8n_T=0$ can be derived from (4.41), (5.56) and (6.2):

$$
r+8n_T=-192\zeta\alpha_k^3+8(202\zeta+65\xi\zeta-144\gamma_M-8\pi^2+3\xi F_{\mathrm{s}})\zeta\alpha_k^4+\mathcal{O}(\alpha_k^5). \qquad (6.3)
$$

We do not give the expression of r separately, since it can be obtained straightforwardly by combining (6.2) and (6.3).

To discuss the validity of the predictions, we express the results in terms of a pivot scale k_* and evolve $\alpha(1/k)$ from k_* to k by means of the RG evolution equations, using the NNLL running coupling (A.1). So doing, the spectra become functions of $\ln(k_*/k)$ and the pivot coupling $\alpha_*\equiv\alpha(1/k_*)$. With $k_*=0.05$ Mpc^{-1} and (for definiteness) $\xi\sim F_{\mathrm{s}}\sim 1$, the data reported in [22] give $\ln(10^{10}\mathcal{P}_{\mathcal{R}}^*)=3.044\pm 0.014$ and $n_{\mathcal{R}}^*=0.9649\pm 0.0042$, where the star superscript means that the quantity is evaluated at the pivot scale. Formulas (5.56) and (6.2) then give the values $\alpha_*=0.0087\pm 0.0010$ and $m_\phi=(2.99\pm 0.37)\cdot 10^{13}$GeV for the fine structure constant α_* and the inflaton mass, respectively.

The value of m_χ will be known as soon as the tensor-to-scalar ratio r will be measured. We recall that the mass m_χ of the fakeon $\chi_{\mu\nu}$ is constrained to lie in the range $m_\phi/4<m_\chi<\infty$, which means $0<\xi<16$, by the consistency of the fakeon prescription/projection [2]. This restricts the window of allowed values of r to $4\cdot 10^{-4}\lesssim r\lesssim 3.5\cdot 10^{-3}$ at the pivot scale.

Formula (4.41) predicts the tensor spectrum $\mathcal{P}_T$ with a relative theoretical error equal to $\alpha_*^4\sim 10^{-8}$, while the relative error on the tensor tilt n_T is $\alpha_*^3\sim 10^{-6}$. As far as the quantities involving the scalar fluctuations are concerned, we have to take into account that the function $F_{\mathrm{s}}(\xi)$ is only partially known. To find when the expansion (5.52) of $F_{\mathrm{s}}(\xi)$ can be trusted, and estimate its errors, we proceed as follows.

Typically, an asymptotic series is a sum of corrections that decrease up to a certain point and then blow up. The expansion must be truncated right at that point, because it cannot be trusted further. The last good term, multiplied by $\pm 1/2$, can be used to give an estimate of the error. From (5.52) we see that for $4<\xi<16$, we can trust no correction at all, so we can say $F_{\mathrm{s}}(\xi)=1\pm 1$ for $\xi>4$. For $2<\xi<4$ the second term is smaller than the first one, but also smaller than the third one, so $F_{\rm s}(\xi) = 1 + \xi/4 \pm \xi/8$. For $1 < \xi < 2$ the third term is smaller than the second one as well as the fourth one, so $F_{\rm s}(\xi) = 1 + \xi/4 + \xi^2/8 \pm \xi^2/16$. And so on:

$$
\begin{aligned}
F_{\rm s}(\xi) &= 1 + \frac{\xi}{4} + \frac{\xi^2}{8} + \frac{\xi^3}{8} \pm \frac{\xi^3}{16} \qquad \text{for } \frac{4}{7} < \xi < 1, \\
F_{\rm s}(\xi) &= 1 + \frac{\xi}{4} + \frac{\xi^2}{8} + \frac{\xi^3}{8} + \frac{7\xi^4}{32} \pm \frac{7\xi^4}{64} \qquad \text{for } \frac{7}{19} < \xi < \frac{4}{7},
\end{aligned}
$$

etc. For example, we have $F_{\rm s}(1/2) = 1.186 \pm 0.007$. An error around 1% is comparable with the error we are making anyway by neglecting the NNNLL order. This means that the NNLL prediction on the scalar spectrum, obtained by expanding in powers of ξ, is good for every $\xi < 1/2$. Even for $1/2 < \xi < 1$ (where the error is below 6%) it is a fair improvement with respect to the NLL prediction.

Summarizing, the scalar spectrum $\mathcal{P}_{\mathcal{R}}$ and the scalar tilt $n_{\mathcal{R}} - 1$ have a relative theoretical error that can be estimated around $\alpha_*^3 \sim 10^{-6}$ for $\xi < 1/2$, where the expansion in powers of ξ is most successful. In the remaining interval of values of ξ, the relative error ranges from 10^{-5} (for $1/2 < \xi < 1$) to 10^{-4} (for $1 < \xi < 16$).

7 Conclusions

Primordial cosmology gives us the chance to test quantum gravity in the forthcoming future. The results may trigger a virtuous circle and open a season of precision tests, which can hopefully lead to the same level of success experienced by the standard model of particle physics. To make this possible, it is crucial to overcome the lack of predictivity due to the arbitrariness of classical theories. The constraints of quantum field theory allow us to achieve this goal, by singling out a theory that contains very few parameters. Additional constraints coming from cosmology make its predictions quite sharp, starting from the tensor-to-scalar ratio r.

Various similarities with high-energy physics allow us to import techniques from quantum field theory into cosmology and boost the computations of higher-order corrections. The cosmic RG flow, for example, allows us to enhance the spectra of primordial fluctuations and understand their structure better. In this paper, we have computed the RG improved spectra of the curvature perturbation $\mathcal{R}$ and the tensor fluctuations, together with the quantity $r + 8n_T$, to the NNLL order in the superhorizon limit. Tilts and running coefficients follow straightforwardly from the flow. The relative theoretical errors range from $\alpha^4 \sim 10^{-8}$ to $\alpha^3 \sim 10^{-6}$.

We have made independent checks of the cosmic RG equations. The results show that the tensor spectrum is affected by the fakeon $\chi_{\mu\nu}$ at every order of the expansion in α. As far as the scalar spectrum is concerned, the first corrections that depend on m_χ are the NNLL ones, which we have computed by expanding perturbatively in m_ϕ/m_χ, due to nontrivial issues related with the fakeon projection at high orders. This approach gives precise results for $m_\chi^2 > 2m_\phi^2$ and fairly precise ones for $2m_\phi^2 > m_\chi^2 > m_\phi^2$.

So far, we have been able to treat the fakeon projection on nontrivial backgrounds relatively easily. It was far from obvious that we could do so to such high orders. From the practical point of view, it may be unnecessary to push the calculations further, at least for the moment. Yet, it is conceptually interesting to understand how to handle the corrections systematically to arbitrary orders. Without entering into details, a preliminary analysis allows us to anticipate that the fakeon projection *per se*, which can be defined algorithmically, does not pose the major challenge. What appears to be more difficult is to properly generalize the Bunch-Davies vacuum condition. A deeper investigation into these issues may clarify the missing points.

Acknowledgments

We thank M. Piva for helpful discussions and NICPB, Tallin, Estonia, for hospitality during the final stage of this work.

A Appendix. Reference formulas

In this appendix we collect some formulas used in the paper. We start from the running coupling to the NNLL order, from [5], which reads

$$\alpha = \frac{\alpha_k}{\lambda}\left(1 - \frac{5\alpha_k}{6\lambda}\ln\lambda\right)\left[1 + \frac{25\alpha_k^2}{12\lambda^2}\left(1 - \lambda - \frac{\ln\lambda}{3}(1 - \ln\lambda)\right)\right], \tag{A.1}$$

where $\lambda \equiv 1 + 2\alpha_k \ln\eta$, $\eta = -k\tau$. We also have the expansions

$$H = \frac{m_\phi}{2}\left[1 - \frac{3\alpha}{2} + \frac{7\alpha^2}{4} - \frac{47\alpha^3}{24} + \frac{293\alpha^4}{144} - \frac{1645}{864}\alpha^5 + \frac{5489}{5184}\alpha^6 + \mathcal{O}(\alpha^7)\right], \tag{A.2}$$

$$-aH\tau = 1 + 3\alpha^2 + 12\alpha^3 + 91\alpha^4 + \frac{2464}{3}\alpha^5 + \frac{81659}{9}\alpha^6 + \mathcal{O}(\alpha^7). \tag{A.3}$$

Next, we give the functions $g_n(\eta)$ of equations (3.19) and their solutions $w_n(\eta)$, determined by the Bunch-Davies condition (3.11). For the tensor perturbations in the limit of infinitely heavy fakeon, we have $g_0^{\mathrm{t}} = g_1^{\mathrm{t}} = 0$, $g_2^{\mathrm{t}} = 9w_0^{\mathrm{t}}$, $g_3^{\mathrm{t}} = 12w_0^{\mathrm{t}}(4 - 3\ln\eta)$, with solutions

$$w_0^{\mathrm{t}} = \frac{i(1-i\eta)}{\eta\sqrt{2}}\mathrm{e}^{i\eta}, \qquad w_1^{\mathrm{t}} = 0, \qquad w_2^{\mathrm{t}} = \frac{6w_0^{\mathrm{t}}}{1-i\eta} - 3\left(i\pi - \mathrm{Ei}(2i\eta)\right) w_0^{\mathrm{t}*},$$
$$w_3^{\mathrm{t}} = \left[6(\ln\eta + \tilde{\gamma}_M)^2 + 24i\eta F_{2,2,2}^{1,1,1}\left(2i\eta\right) + \pi^2\right] w_0^{\mathrm{t}*} + \frac{24w_0^{\mathrm{t}}}{1-i\eta} - 4(\ln\eta + 1)w_2^{\mathrm{t}}, \quad \text{(A.4)}$$

where Ei denotes the exponential-integral function and $F_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}(z)$ denotes the generalized hypergeometric function ${}_pF_q(\{a_1,\cdots,a_p\},\{b_1,\cdots,b_q\};z)$.

We can use the functions (A.4) to express the solutions in other situations. Using self-explanatory superscripts to distinguish different cases, the scalar fluctuations at $m_\chi = \infty$ have $g_0^{\mathrm{s}} = 0$, $g_1^{\mathrm{s}} = 6w_0^{\mathrm{s}}$, $g_2^{\mathrm{s}} = 2(11 - 6\ln\eta)w_0^{\mathrm{s}} + 6w_1^{\mathrm{s}}$, with solutions $w_0^{\mathrm{s}} = w_0^{\mathrm{t}}$ and

$$w_1^{\mathrm{s}} = \frac{2}{3}w_2^{\mathrm{t}}, \qquad w_2^{\mathrm{s}} = -\frac{16w_0^{\mathrm{t}}}{1+\eta^2} + \frac{2(13+i\eta)w_2^{\mathrm{t}}}{9(1+i\eta)} + \frac{w_3^{\mathrm{t}}}{3} + 4G_{2,3}^{3,1}\left(-2i\eta\,\middle|\,{}_{0,0,0}^{0,1}\right) w_0^{\mathrm{t}},$$

where $G_{p,q}^{m,n}$ denotes the Meijer G function.

When m_χ is finite, the solutions do not change in the case of the scalar perturbations. They do change in the case of the tensor perturbations, where the equations acquire the form (4.39), instead of (3.19). It is convenient to organize them as

$$w_n'' + w_n - \frac{2w_n}{\eta^2} = \frac{g_n(\eta)}{\eta^2} + \tilde{g}_n(\eta), \quad \text{(A.5)}$$

where the functions g_n and $\tilde{g}_n$ are determined recursively from w_m, $m < n$. We find $g_0^{\mathrm{t},m_\chi} = g_1^{\mathrm{t},m_\chi} = \tilde{g}_0^{\mathrm{t},m_\chi} = \tilde{g}_1^{\mathrm{t},m_\chi} = 0$ and

$$g_2^{\mathrm{t},m_\chi} = 9\zeta w_0^{\mathrm{t},m_\chi}, \qquad \tilde{g}_2^{\mathrm{t},m_\chi} = 3\xi\zeta^2 w_0^{\mathrm{t},m_\chi}, \qquad g_3^{\mathrm{t},m_\chi} = \frac{3}{2}(32\zeta + 43\xi\zeta - 24\ln\eta)\zeta w_0^{\mathrm{t},m_\chi},$$
$$\tilde{g}_3^{\mathrm{t},m_\chi} = -\frac{3}{2}(6\zeta - 7\xi\zeta - 2\xi^2\zeta + 8\ln\eta)\xi\zeta^2 w_0^{\mathrm{t},m_\chi}, \quad \text{(A.6)}$$

with solutions $w_0^{\mathrm{t},m_\chi} = w_0^{\mathrm{t}}$ and

$$w_2^{\mathrm{t},m_\chi} = \zeta w_2^{\mathrm{t}} + \frac{3\xi\zeta^2}{4}\frac{3-3i\eta-2\eta^2}{1-i\eta}w_0^{\mathrm{t}}, \qquad w_3^{\mathrm{t},m_\chi} = \zeta w_3^{\mathrm{t}} + \frac{3\xi\zeta^2}{2}w_2^{\mathrm{t}} + \frac{3\xi\zeta^2 F_3 w_0^{\mathrm{t}}}{4(1-i\eta)},$$
$$F_3 \equiv (3-3i\eta-2\eta^2)(\xi^2\zeta - 4\ln\eta) - \zeta(1+i\eta)(1-2i\eta) + \frac{\xi\zeta}{2}(29 - 29i\eta - 22\eta^2). \quad \text{(A.7)}$$

Now we give details on the asymptotic behaviors. The functions $w_n(\eta)$ with $n > 0$ tend to zero for $\eta \to \infty$. The relevant behaviors in the superhorizon limit $\eta \sim 0$ are

$$\eta w_2^{\mathrm{t}} \sim \frac{3i}{\sqrt{2}}\left(2 - \tilde{\gamma}_M - \ln\eta\right), \qquad \eta w_2^{\mathrm{s}} \sim -8i\sqrt{2} + \frac{26}{9}\eta w_2^{\mathrm{t}} + \frac{\eta w_3^{\mathrm{t}}}{3} + i\sqrt{2}(\ln\eta + \tilde{\gamma}_M)^2 + \frac{i\pi^2}{\sqrt{2}},$$
$$\eta w_3^{\mathrm{t}} \sim -3i\sqrt{2}(\ln\eta + \tilde{\gamma}_M)^2 - \frac{i\pi^2}{\sqrt{2}} + 12i\sqrt{2} - 4(\ln\eta + 1)\eta w_2^{\mathrm{t}}. \quad \text{(A.8)}$$

The other behaviors can be found from these ones and the relations given above.

For convenience, we conclude with a list of notations frequently used in the paper, which are

$$\xi = \frac{m_\phi^2}{m_\chi^2}, \qquad \zeta = \left(1 + \frac{\xi}{2}\right)^{-1}, \qquad \tilde{\gamma}_M = \gamma_M - \frac{i\pi}{2}, \qquad \gamma_M = \gamma_E + \ln 2,$$

γ_E being the Euler-Mascheroni constant.

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