

Cosmic Inflation as a Renormalization-Group Flow: the Running of Power Spectra in Quantum Gravity

Damiano Anselmi

Dipartimento di Fisica “Enrico Fermi”, Università di Pisa

Largo B. Pontecorvo 3, 56127 Pisa, Italy

and INFN, Sezione di Pisa,

Largo B. Pontecorvo 3, 56127 Pisa, Italy

damiano.anselmi@unipi.it

Abstract

We study the running of power spectra in inflationary cosmology by reformulating the slow roll expansion as a renormalization-group flow from the de Sitter fixed point. The beta function is provided by the equations of the background metric. The spectra of the scalar and tensor fluctuations obey RG evolution equations with vanishing anomalous dimensions in the superhorizon limit. By organizing the perturbative expansion in terms of leading and subleading logs, we calculate the spectral indices, their runnings, the runnings of the runnings, etc., to the next-to-leading log order in quantum gravity with fakeons (i.e., the theory $R + R^2 + C^2$ with the fakeon prescription/projection for C^2). We show that these quantities are related to the spectra in a universal way. We also compute the first correction to the relation $r = -8n_T$ and provide a number of quantum gravity predictions that can be hopefully tested in the forthcoming future.

1 Introduction

Gravity is the only interaction of nature that permeates the whole universe, from the smallest distances to the largest ones. Thanks to this, it may allow us to establish a connection between high-energy physics, specifically quantum field theory, and cosmology. In this paper we lay out an ingredient of this relation by reformulating the history of the early universe, starting from inflation [1, 2, 3, 4, 5, 6, 7, 8] and the primordial quantum fluctuations [9], as the evolution of a peculiar renormalization-group (RG) flow in quantum gravity, which we call “cosmic RG flow”. The reformulation offers an alternative way to view the slow roll expansion. Although the cosmic flow is not originated by the radiative corrections, but by the dependence on the background metric, it resembles the RG flow of quantum field theory in several respects. For example, the power spectra of the fluctuations satisfy a Callan-Symanzik equation in the superhorizon limit.

The effects of quantum gravity are expected to become important at energies that are too large for our laboratories. If we want to test predictions, a better idea is to use the universe itself as a laboratory. In this context, a consistent theory of quantum gravity relating the smallest and the largest scales of magnitude may be of great help. Because the universe was “small” at the beginning of time, what happened then may be calculable perturbatively. Since the primordial evolution of the universe left detectable remnants in the cosmic microwave background radiation as we see it today, by scrutinizing the sky we may have chances to put quantum gravity to a test.

Pursuing the idea of a connection between cosmology and high-energy physics, a sound proposal for quantum gravity should follow from principles similar to those that lead to the standard model of particle physics, which are locality, renormalizability and unitarity. The advantage of an approach like this is that, if it offers an answer, it gives a very constrained, basically unique one.

A proposal with the claimed features has indeed become available recently from quantum field theory [10]. The basic idea is the same that lead to the introduction of the intermediate W and Z bosons and gave birth to the standard model of elementary particles.

At that time, the goal was to remedy the problems of the Fermi theory of the weak interactions and gain renormalizability while preserving locality and unitarity. The Fermi theory can be schematically represented by means of four fermion (current-current) vertices, which are not renormalizable. Postulating the existence of suitable intermediate bosons, the four fermion vertices can be effectively generated through the exchange of vec-

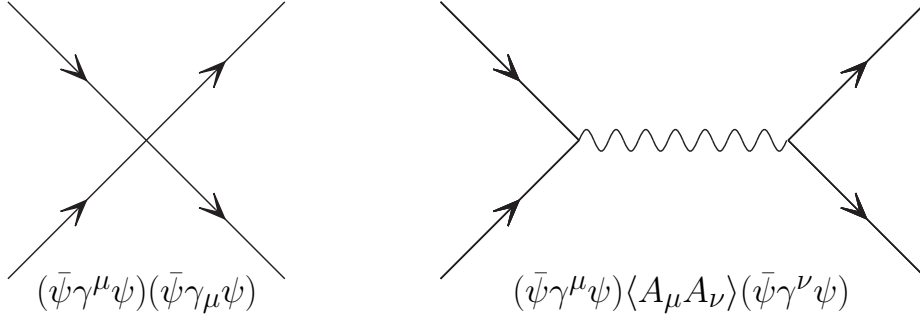


Figure 1: Fermi theory of the weak interactions vs standard model

tor fields, as shown in figure 1. In the case of quantum gravity, the left drawing is replaced by multi graviton vertices and the right drawing is replaced by trees where the graviton external legs exchange a new type of particle, the fakeon [11].

The difference between the standard model and quantum gravity is that the intermediate bosons that must be introduced to make sense of the latter are of a new type, which had to be uncovered anew. The fakeon is a purely virtual particle, which mediates interactions but does not belong to the spectrum of asymptotic states. It can be introduced by means of a new quantization prescription, alternative to the Feynman $i\epsilon$ one, combined with a projection that is consistent with it.

The prescription amounts to starting from the Euclidean theory and ending the Wick rotation nonanalytically by means of the average continuation, anytime a threshold involves fakeons [10]. The average continuation is the arithmetic average of the two analytic continuations that circumvent the threshold [12, 11]. The computations of renormalization constants, cross sections, widths and absorptive parts in quantum gravity with fakeons [13, 14] show that the renormalization is unaffected by the fakeon prescription [11] and so coincides with the one of the Euclidean version of the theory [15]; instead, the absorptive parts are crucially different, since unitarity depends on them.

The projection works to some extent like the one involved in the quantization of gauge theories (which is the one that allows us to consistently drop the gauge-trivial modes and the Faddeev-Popov ghosts from the spectrum), but does not follow from a symmetry principle. The combination of the two operations (prescription and projection) is consistent and returns a theory that is unitary at the fundamental level (see [11] for the proof to all orders and [16] for the analysis of the bubble and triangle diagrams).

In cosmology, new aspects play important roles. Because the loop corrections are negligible up to a high degree of precision [17], it is natural to try and quantize the

classical limit directly. However, when a theory contains fake particles, its classical limit is nontrivial. It is not described by the “classical” action one starts with to formulate the quantum field theory, because that action is unprojected [18]. Moreover, a purely virtual particle cannot be found by quantizing something classical [19]. The way out is to classicize quantum gravity as explained in [18]. Once the correct classical limit is found, it can be used as the starting point to study cosmology, as shown by Bianchi, Piva and the current author in [20].

In this paper we make a step forward towards a better definition of the relation between cosmology and high-energy physics. We describe the history of the early universe, starting from the slow roll expansion in inflationary cosmology, as an evolution that resembles the one of the running couplings in quantum field theory, although it is essentially different from it, because it originates from the dependence on the background metric rather than the radiative corrections. The formal similarities we find about the two types of flows are quite appealing and might lead to unforeseen developments. We use the resulting setup to derive new quantum gravity predictions about the spectra and their running behaviors.

Specifically, the cosmic RG flow starts from a fixed point, which is de Sitter space, and is described by a coupling “constant” α and its beta function $\beta_\alpha(\alpha)$, which follows from the equations satisfied by the background metric and can be expressed as a power series in α . Around $\alpha \sim 0$ the beta function is negative and proportional to α^2 . It further vanishes for $\alpha = 1$, which is however not a fixed point. The power spectra of the scalar and tensor fluctuations obey standard Callan-Symanzik RG evolution equations in the superhorizon limit, with vanishing anomalous dimensions.

The tools provided by high-energy physics allow us to gain insight into the structure of the power spectra, where we can distinguish a core part and an RG part. The core information is a power series in the coupling α with constant coefficients, but has no a priori relation to the beta function. Because of this, it is specific of the (scalar, tensor) spectrum we are considering. The RG part of the spectrum, on the other hand, is the one controlled by the RG equation. For this reason, it is universal (i.e., the same for every spectrum) and encoded into the beta function.

The running can be studied efficiently by organizing the perturbative expansion in terms of leading and subleading logs and resumming all the powers of $\alpha_* \ln(k_*/k)$, where k is the scale, k_* is the pivot scale and α_* is the value of the coupling at the pivot scale. In the next sections, we work out the scalar and tensor spectra and their runnings in quantum gravity to the next-to-leading log order. We also compute the first correction to the relation $r + 8n_T = 0$, where r is the tensor-to-scalar ratio and n_T is the tensor tilt.

These are quantities that can be hopefully measured in the future.

As already stressed, the cosmic RG flow is originated by the dependence on the background metric. In this sense, it is the intrinsic flow of inflation. The expression “running”, widely used for the dependence of the tilts on $\ln k$ [21], fits the terminology used here. Instead, the factor z that relates the curvature perturbation \mathcal{R} to the Mukhanov-Sasaki variable (see for example [22]) is not interpreted as a wave-function renormalization constant $Z^{1/2}$. If it were so, it would lead to an anomalous dimension, but we find that the anomalous dimensions vanish. Other concepts that can be found in the literature have meanings that differ from the ones we attribute to them. For example, the beta function β_α defined in this paper does not match the quantity called “beta function” by Binétry, Kiritsis, Mabillard, Pieroni and Rosset in [23] (which is proportional to the quantity α that we call “coupling”). We think that our interpretation of the formal relation between cosmology and high-energy physics is more to the point than the one proposed there, because it allows us to dissect the structure of the power spectra.

Although the leading log expansion we are talking about has nothing to do with the leading logs due to radiative corrections in quantum field theory, it is interesting to incorporate those effects, such as the running of masses and coupling constants induced by high-energy physics, in the investigation of primordial cosmology. This was done for example in ref. [24] by Myrzakulov, Odintsov and Sebastiani. It is also worth to mention the RG induced inflation defined by Márián, Defenu, Jentschura, Trombettoni and Nándori in ref. [25]. For various applications it might be useful to combine the different types of flows together.

In models without fakeons, the running of spectral indices has been calculated in various scenarios [26], including subleading corrections [27]. Besides upgrading the techniques of [26, 27] and including purely virtual quanta, the understanding offered in this paper allows us to appreciate important aspects of inflation and make calculations more efficiently.

Hopefully, primordial cosmology will provide an arena for precision tests of quantum gravity. With this in mind, we use the results of our calculations to derive a number of predictions that might be tested experimentally in the forthcoming years.

The paper is organized as follows. In section 2 we study inflation as an RG flow, define the coupling α and study its beta function. We also organize the perturbative expansion by resumming the leading logs and the subleading logs, and use these tools to compute the running coupling to various orders. In section 3 we study the scalar and tensor spectra in the limit of infinitely heavy fakeon, which returns the Starobinsky $R + R^2$ theory [2, 28]. In section 4 we upgrade the results of section 3 to quantum gravity, emphasizing the

dependence on the fakeon mass m_χ . In section 5 we summarize the predictions, while section 6 contains the conclusions.

2 The cosmic RG flow

In this section we define the cosmic RG flow and study the running coupling α and its beta function β_α . For later use, we organize and resum the perturbative expansion in terms of leading logs and subleading logs.

Quantum gravity with fakeons is described by a triplet made of the graviton, a massive scalar ϕ (the inflaton) and a massive spin-2 fakeon $\chi_{\mu\nu}$. The starting, unprojected classical action can be written in the form

$$S_{\text{QG}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R + \frac{1}{2m_\chi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) + \frac{1}{2} \int d^4x \sqrt{-g} (D_\mu \phi D^\mu \phi - 2V(\phi)), \quad (2.1)$$

where $V(\phi)$ is the Starobinsky potential

$$V(\phi) = \frac{m_\phi^2}{2\hat{\kappa}^2} (1 - e^{\hat{\kappa}\phi})^2, \quad (2.2)$$

while $\hat{\kappa} = \sqrt{16\pi G/3}$ and m_ϕ , m_χ are the masses of ϕ and $\chi_{\mu\nu}$, respectively. For convenience, we have switched off both the cosmological term and the matter sector.

The theory (2.1) is renormalizable. Indeed, up to a standard, nonderivative field redefinition, it is equivalent to the action

$$\tilde{S}_{\text{QG}}(g, \Phi) = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R + \frac{1}{2m_\chi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - \frac{R^2}{6m_\phi^2} \right), \quad (2.3)$$

which is renormalizable by power counting [29], once the cosmological term is reinstated. Both (2.1) and (2.3) contain the square C^2 of the Weyl tensor $C_{\mu\nu\rho\sigma}$. If all the poles of the free propagators (in the expansion of the metric around flat space) are quantized by means of the Feynman $i\epsilon$ prescription, Stelle's theory [29] is obtained, which contains a spin-2 ghost and violates unitarity.

The solution is to quantize the massive spin-2 pole by means of the fakeon prescription, which allows us to project the particle away from the physical spectrum at the fundamental level. The procedure works in cosmology under the consistency condition $m_\chi > m_\phi/4$, which puts a lower bound on the mass of the fakeon with respect to the mass of the inflaton [20].

Throughout this paper, we work in the “inflaton framework” and conform to the notation of [20]. The inflaton framework is the one where the scalar field ϕ is introduced explicitly and the action is (2.1). The action (2.3) defines the “geometric framework” of ref. [20]. It can be shown that the two frameworks lead to the same physical results, as expected, although several intermediate steps may look rather different. It is worth to stress that, even if we use (2.1), we do not have the freedom to change the potential V of (2.2), because if we did so we would destroy the renormalizability of the theory.

To define the cosmic RG flow, we start from the Friedmann equations, which are unaffected by the C^2 term and read

$$\dot{H} = -4\pi G\dot{\phi}^2, \quad H^2 = \frac{4\pi G}{3} \left(\dot{\phi}^2 + 2V(\phi) \right), \quad \ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \quad (2.4)$$

where $H = \dot{a}/a$ is the Hubble parameter. We study the evolution from the de Sitter limit (which is $\phi \rightarrow -\infty$). For the applications we have in mind it is sufficient to cover the region $\phi < 0$, so the equations we write below are valid in that domain, which means to the left of the minimum $\phi = 0$ of the potential (2.2). They can be extended to the other domains by flipping the signs in front of the square roots we are going to meet.

It is useful to define the “coupling”

$$\alpha = \sqrt{\frac{4\pi G}{3}} \frac{\dot{\phi}}{H} = \sqrt{-\frac{\dot{H}}{3H^2}} \quad (2.5)$$

and eliminate V and $\dot{\phi}$ by means of the first two equations of (2.4). So doing, we obtain

$$\dot{\phi} = \sqrt{\frac{3}{4\pi G}} \alpha H, \quad V = \frac{3}{8\pi G} (1 - \alpha^2) H^2. \quad (2.6)$$

Note that we must have $-1 \leq \alpha \leq 1$. Moreover, in the domain we are interested in we can take α to be nonnegative. Then the flow describes the evolution from $\phi = -\infty$ to $\phi = 0$.

We eliminate $\ddot{\phi}$ from the last equation of (2.4), by noting that the potential (2.2) satisfies

$$V' = 4V \sqrt{\frac{4\pi G}{3}} - m_\phi \sqrt{2V}. \quad (2.7)$$

Differentiating the first equation of (2.6) with respect to time and using the last equation (2.4), followed by (2.5), (2.7) and the second of (2.6), we find

$$\dot{\alpha} = m_\phi \sqrt{1 - \alpha^2} - H(2 + 3\alpha)(1 - \alpha^2). \quad (2.8)$$

This equation can be extended to the right of the V minimum ($\phi > 0$) by flipping the sign in front of the square root that appears on its right-hand side. It can be extended to $\alpha < 0$ (i.e., $\dot{\phi} < 0$), by flipping the sign in front of the square root of (2.5).

Equation (2.8) still contains H , which can be eliminated as follows. We first write the most general expansion

$$H = \sum_{n=0}^{\infty} h_n \alpha^n \quad (2.9)$$

in powers of α , where h_n are unknown constants. Then we differentiate this relation with respect to time. The left-hand side gives $\dot{H} = -3\alpha^2 H^2$, by (2.5), which is turned into a power series in α by using (2.9) again. The right-hand side is turned into another power series in α by means of (2.8) and (2.9). Comparing the two sides, we find

$$H = \frac{m_\phi}{2} \left[1 - \frac{3\alpha}{2} + \frac{7\alpha^2}{4} - \frac{47\alpha^3}{24} + \frac{293\alpha^4}{144} + \mathcal{O}(\alpha^5) \right]. \quad (2.10)$$

The formula can be pushed to arbitrarily high orders.

Thanks to (2.10), we can use (2.8) to express $\dot{\alpha}$ as a power series in α . However, the derivative is still with respect to time. It is convenient to introduce the conformal time

$$\tau = - \int_t^{+\infty} \frac{dt'}{a(t')} \quad (2.11)$$

and convert the derivative by means of

$$\frac{d}{dt} = \frac{H}{aH\tau} \frac{d}{d \ln|\tau|}.$$

For this purpose, we need the expansion

$$-aH\tau = 1 + 3\alpha^2 + 12\alpha^3 + 91\alpha^4 + \mathcal{O}(\alpha^5), \quad (2.12)$$

which can be derived with the method used for (2.10).

At this point, it is straightforward to convert (2.8) into the beta function

$$\beta_\alpha \equiv \frac{d\alpha}{d \ln|\tau|} = -2\alpha^2 f(\alpha), \quad (2.13)$$

where

$$f(\alpha) = 1 + \frac{5}{6}\alpha + \frac{25}{9}\alpha^2 + \frac{383}{27}\alpha^3 + \frac{8155}{81}\alpha^4 + \mathcal{O}(\alpha^5). \quad (2.14)$$

Note that β_α is negative for α small, so the cosmic RG flow is “asymptotically free”, which means that it gives the de Sitter metric in the infinite-past limit $t \rightarrow -\infty$, which is also $\tau \rightarrow -\infty$.

It is easy to show that $\alpha = 0$ is the only fixed point of the flow. Indeed, a fixed point has $\alpha = \text{constant}$. The right-hand side of equation (2.8) vanishes for

$$\frac{m_\phi}{2} = \pm H \left(1 + \frac{3}{2}\alpha \right) \sqrt{1 - \alpha^2} \quad (2.15)$$

(the \pm being inserted to cover the case $\phi > 0$) and for $\alpha = \pm 1$. We consider the two cases separately.

Equation (2.15) implies $\dot{H} = 0$. However, \dot{H} is also equal to $-3\alpha^2 H^2$, so we can either have $\alpha = 0$ or $H = 0$. The option $H = 0$ is not acceptable, because it is incompatible with (2.15). In the end, we obtain the de Sitter fixed point $\alpha = 0$, with $H = m_\phi/2$ (since H cannot be negative).

As far as the cases $\alpha = \pm 1$ are concerned, they are zeros of the beta function, but they are not true fixed points. Indeed, for $|\alpha| \lesssim 1$, (2.8) gives

$$\dot{\alpha} \sim \pm m_\phi \sqrt{1 - \alpha^2}.$$

This equation is solved by $\alpha \sim \cos(m_\phi(t - t_0))$, which takes us to the oscillating behavior of the reheating phase. Moreover, the spurious zeros at $\alpha = \pm 1$ disappear with a simple reparametrization, such as $\alpha = \sin \theta$.

To simplify some calculations, it is convenient to define the variable $\eta = -k\tau$, where k is just an arbitrary constant for the moment. We solve (2.13) by writing

$$d \ln \eta = -\frac{d\alpha}{2\alpha^2} \left(1 - \frac{5}{6}\alpha - \frac{25}{12}\alpha^2 - \frac{2189}{216}\alpha^3 + \mathcal{O}(\alpha^4) \right) \quad (2.16)$$

and integrating term by term. The solution can be organized by means of leading and subleading logs, which can be easily resummed.

Let α_0 denote the value of α at $\eta = 1$. The expansion in terms of leading and subleading logs is the expansion in powers of α_0 under the assumption that $\alpha_0 \ln \eta$ is of order unity. This means that the powers of $\alpha_0 \ln \eta$ must be resummed into exact expressions. Modulo overall factors α_0 , the leading logs are the powers $\alpha_0^n \ln^n \eta$, $n \geq 0$, the next-to-leading logs are the corrections proportional to $\alpha_0^{n+1} \ln^n \eta$, $n \geq 0$, the next-to-next-to-leading logs are the corrections $\alpha_0^{n+2} \ln^n \eta$, and so on.

Thanks to the renormalization group, the inclusion of each set of corrections requires to add just one term of the expansion in powers of α . At the leading log level, it is sufficient to keep the first term inside the parenthesis of (2.16), which gives the running coupling

$$\alpha(-\tau) = \frac{\alpha_0}{1 + 2\alpha_0 \ln \eta}. \quad (2.17)$$

The expansion in powers of $\alpha_0 \ln \eta$ is just a geometric series in this case. We can view the initial condition α_0 as implicitly k dependent, i.e., as the coupling

$$\alpha_0 = \alpha(1/k) \tag{2.18}$$

at $|\tau| = 1/k$. Then $\alpha(-\tau)$ is k independent, i.e., it is the coupling at conformal time τ .

More generally, the running coupling can be organized in the form

$$\alpha(-\tau) = \frac{\alpha_0}{\lambda} \prod_{n=1}^{\infty} (1 + \alpha_0^n \gamma_n(\lambda)), \quad \lambda \equiv 1 + 2\alpha_0 \ln \eta. \tag{2.19}$$

The first few functions $\gamma_n(\lambda)$ are

$$\begin{aligned} \gamma_1(\lambda) &= -\frac{5 \ln \lambda}{6\lambda}, & \gamma_2(\lambda) &= \frac{25}{12\lambda^2} \left[1 - \lambda - \frac{\ln \lambda}{3}(1 - \ln \lambda) \right], \\ \gamma_3(\lambda) &= \frac{1}{432\lambda^3} [(1 - \lambda)(2939 + 2189\lambda) - 125(14 - 6\lambda - 3 \ln \lambda) \ln \lambda]. \end{aligned} \tag{2.20}$$

To give an example, the running coupling to the next-to-leading log order can be found by keeping the first two terms inside the parenthesis of (2.16), which gives

$$\alpha(-\tau) = \frac{\alpha_0}{1 + 2\alpha_0 \ln \eta} \left(1 - \frac{5\alpha_0 \ln(1 + 2\alpha_0 \ln \eta)}{6(1 + 2\alpha_0 \ln \eta)} \right). \tag{2.21}$$

3 Limit of heavy fakeon ($m_\chi = \infty$, action $R + R^2$)

In this section, we study the cosmic RG flow in the limit of infinitely heavy fakeon, $m_\chi \rightarrow \infty$. The action (2.1) turns into

$$S_{\text{QG}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \frac{1}{2} \int d^4x \sqrt{-g} (D_\mu \phi D^\mu \phi - 2V(\phi)). \tag{3.1}$$

Recalling that the potential is (2.2), we obtain the Starobinsky $R + R^2$ theory in the inflaton approach, which provides a good arena to illustrate the strategy of our calculations before generalizing them to quantum gravity.

We show that the spectra of the tensor and scalar fluctuations obey RG evolution equations in the superhorizon limit, with vanishing anomalous dimensions. Due to this, the tilts and the running coefficients are related to the spectra in a universal way. The procedure we describe can be extended to arbitrarily high orders. We compute the first few orders explicitly.

For reviews that contain details on the parametrizations of the metric fluctuations and their transformations under diffeomorphisms, see [22, 30].

3.1 Tensor fluctuations

To study the tensor fluctuations, we parametrize the metric as

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2) - 2a^2 (u\delta_\mu^1\delta_\nu^1 - u\delta_\mu^2\delta_\nu^2 + v\delta_\mu^1\delta_\nu^2 + v\delta_\mu^2\delta_\nu^1), \quad (3.2)$$

where $u = u(t, z)$ and $v = v(t, z)$ denote the graviton modes. Denoting the Fourier transform of $u(t, z)$ with respect to the coordinate z by $u_{\mathbf{k}}(t)$, where \mathbf{k} is the space momentum, the quadratic Lagrangian obtained from (3.1) is

$$(8\pi G)\frac{\mathcal{L}_t}{a^3} = \dot{u}_{\mathbf{k}}\dot{u}_{-\mathbf{k}} - \frac{k^2}{a^2}u_{\mathbf{k}}u_{-\mathbf{k}}, \quad (3.3)$$

plus an identical contribution for $v_{\mathbf{k}}$, where $k = |\mathbf{k}|$. Defining

$$w_{\mathbf{k}} = au_{\mathbf{k}}\sqrt{\frac{k}{4\pi G}}, \quad (3.4)$$

and switching to the variable $\eta = -k\tau$, the action reads

$$S_t = \frac{1}{2} \int d\eta \left[w'^2 - w^2 + \left(\nu_t^2 - \frac{1}{4} \right) \frac{w^2}{\eta^2} \right], \quad (3.5)$$

where the prime denotes the derivative with respect to η and

$$\nu_t^2 - \frac{1}{4} = 2\tau^2 a^2 H^2 \left(1 - \frac{3}{2}\alpha^2 \right). \quad (3.6)$$

Using (2.12), we find

$$\nu_t = \frac{3}{2} + 3\alpha^2 + 16\alpha^3 + \frac{355}{3}\alpha^4 + \mathcal{O}(\alpha^5). \quad (3.7)$$

For a while, we drop the subscript \mathbf{k} , since no confusion is expected to arise. The equation derived from (3.5) is

$$w'' + w - \left(\nu_t^2 - \frac{1}{4} \right) \frac{w}{\eta^2} = 0 \quad (3.8)$$

and must be solved with the Bunch-Davies vacuum condition

$$w(\eta) \sim \frac{e^{i\eta}}{\sqrt{2}} \quad \text{for large } \eta. \quad (3.9)$$

The solution can be worked out by expanding in powers of $\alpha_0 = \alpha(1/k)$. Since there is no $\mathcal{O}(\alpha)$ term in (3.7), the first correction to w is of order α_0^2 . We have

$$w(\eta) = w_0(\eta) + \alpha_0^2 w_2(\eta) + \alpha_0^3 w_3(\eta) + \dots \quad (3.10)$$

Inserting (3.10) into (3.8), we obtain equations of the form

$$w_n'' + w_n - \frac{2w_n}{\eta^2} = \frac{g_n(\eta)}{\eta^2}. \quad (3.11)$$

where $g_0(\eta) = g_1(\eta) = 0$, while $g_n(\eta)$, $n > 1$, are functions that are determined recursively from w_j with $j < n$. Using the expansion (3.7), then expressing α , which is $\alpha(-\tau)$, as the running coupling of the previous section, which we can expand in powers of α_0 by means of formula (2.19), we find

$$g_2 = 9w_0, \quad g_3 = 12w_0(4 - 3 \ln \eta), \quad g_4 = 9w_2 + 2w_0(182 - 159 \ln \eta + 54 \ln^2 \eta), \quad (3.12)$$

etc.

The solution for $n = 0$ is

$$w_0 = \frac{(\eta + i)}{\eta\sqrt{2}} e^{i\eta}, \quad (3.13)$$

where the arbitrary constants are determined by the Bunch-Davies condition (3.9). The solutions for $n > 0$ are

$$w_n(\eta) = \int_{\eta}^{\infty} \frac{g_n(\eta') d\eta'}{\eta\eta'^3} [(\eta - \eta') \cos(\eta - \eta') - (1 + \eta\eta') \sin(\eta - \eta')]. \quad (3.14)$$

Again, the arbitrary constants are determined to preserve (3.9), which requires $w_n(\eta) \rightarrow 0$ for $\eta \rightarrow \infty$ for every $n > 1$.

For example, using (3.12) for g_2 and g_3 , we find

$$\begin{aligned} w_2(\eta) &= \frac{3}{\eta\sqrt{2}} [2ie^{i\eta} + (\eta - i)e^{-i\eta} (\text{Ei}(2i\eta) - i\pi)], \\ w_3(\eta) &= -\frac{12i\sqrt{2}e^{i\eta} \ln \eta}{\eta} + \frac{3\sqrt{2}(\eta - i)e^{-i\eta}}{\eta} \left(2i\pi - \frac{\pi^2}{12} - i\pi\gamma_M - 2(\ln \eta + 1)\text{Ei}(2i\eta) \right. \\ &\quad \left. + i\pi \ln \eta + (\ln \eta + \gamma_M)^2 + 4i\eta F_{2,2,2}^{1,1,1}(2i\eta) \right), \end{aligned} \quad (3.15)$$

where Ei is the exponential-integral function, $F_{b_1, \dots, b_q}^{a_1, \dots, a_p}(z)$ is the generalized hypergeometric function ${}_pF_q(\{a_1, \dots, a_p\}, \{b_1, \dots, b_q\}; z)$ and $\gamma_M \equiv \gamma_E + \ln 2$, γ_E being the Euler-Mascheroni constant. The combination γ_M is going to appear frequently from now on.

In the limit $\eta \rightarrow \infty$, the functions g_n and w_n , $n > 1$, satisfy

$$\lim_{\eta \rightarrow \infty} \eta^{1-\delta} g_n(\eta) = 0, \quad \lim_{\eta \rightarrow \infty} \eta^{1-\delta} w_n(\eta) = 0, \quad (3.16)$$

for every $\delta > 0$.

The power spectrum must be calculated in the superhorizon limit, which is $\eta \rightarrow 0$. There, we have

$$g_n(\eta) \underset{\eta \sim 0}{\sim} \frac{1}{\eta} P_{n-2}^{(g)}(\ln \eta) + \mathcal{O}(\ln^{n-2} \eta) \quad w_n(\eta) \underset{\eta \sim 0}{\sim} \frac{1}{\eta} P_{n-1}^{(w)}(\ln \eta) + \mathcal{O}(\ln^{n-1} \eta), \quad (3.17)$$

where $P_k^{(g)}$ and $P_k^{(w)}$ are polynomials of degree k . Moreover, the superhorizon limit allows us to drop the term w_n in equation (3.11), because it is dominated by $2w_n/\eta^2$. Once we do that, the solution (3.14) simplifies to

$$w_n(\eta) \underset{\eta \sim 0}{\sim} -\frac{1}{3\eta} \int_{\eta_1}^{\eta} g_n(\eta') d\eta' + \frac{\eta^2}{3} \int_{\eta_2}^{\eta} \frac{g_n(\eta') d\eta'}{\eta'^3}, \quad (3.18)$$

for $n > 1$, where η_1 and η_2 are arbitrary constants. In particular, if we take

$$g_n(\eta) \underset{\eta \sim 0}{\sim} \frac{c_n}{\eta} \ln^{n-2} \eta + \text{subleading}, \quad (3.19)$$

where c_n are constants, we obtain

$$w_n(\eta) \underset{\eta \sim 0}{\sim} -\frac{c_n}{3(n-1)} \frac{\ln^{n-1} \eta}{\eta} + \text{subleading}, \quad n > 1. \quad (3.20)$$

Now we use these results to resum the leading log corrections in the power spectrum. For this purpose it is enough to truncate (3.7) to order α^2 and replace α with (2.17), so equation (3.8) simplifies to

$$w'' + w - \frac{2w}{\eta^2} = \frac{9\alpha_0^2 w_0}{\eta^2 (1 + 2\alpha_0 \ln \eta)^2}.$$

Then we use the expansion (3.10) to read g_n and so the constants c_n of (3.19), which turn out to be

$$c_n = 9i(-1)^n (n-1) 2^{n-3} \sqrt{2}.$$

Using (3.20) and resumming (3.10), we find the first correction to $w(\eta)$:

$$w(\eta) \sim \frac{i}{\eta\sqrt{2}} \left(1 - \frac{3\alpha_0^2 \ln \eta}{1 + 2\alpha_0 \ln \eta} \right). \quad (3.21)$$

We quantize (3.5) as usual. Reinstating the subscript \mathbf{k} , the operator associated with the fluctuation $u_{\mathbf{k}}$ is

$$\hat{u}_{\mathbf{k}}(\tau) = u_{\mathbf{k}}(\tau) \hat{a}_{\mathbf{k}} + u_{-\mathbf{k}}^*(\tau) \hat{a}_{-\mathbf{k}}^\dagger,$$

where $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ are creation and annihilation operators, satisfying $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$. The power spectrum \mathcal{P}_u is defined by

$$\langle \hat{u}_{\mathbf{k}}(\tau) \hat{u}_{\mathbf{k}'}(\tau) \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_u, \quad \mathcal{P}_u = \frac{k^3}{2\pi^2} |u_{\mathbf{k}}|^2. \quad (3.22)$$

Summing over the tensor polarizations u and v and converting to the common normalization, the power spectrum of the tensor fluctuations is

$$\mathcal{P}_T(k) = 16\mathcal{P}_u(k). \quad (3.23)$$

Using (3.4) and (3.21), we find

$$\mathcal{P}_u = \frac{2Gk^2}{\pi a^2} |w_{\mathbf{k}}|^2 = \frac{GH^2}{\pi a^2 H^2 \tau^2} \left(1 - \frac{3\alpha_0^2 \ln \eta}{1 + 2\alpha_0 \ln \eta} \right)^2.$$

Formula (2.12) can be approximated to $-aH\tau = 1$ to the order we are interested in, while (2.10) can be truncated to $H = m_\phi(1 - 3(\alpha/2))/2$. In the end, we obtain

$$\mathcal{P}_T(k) = \frac{4Gm_\phi^2}{\pi} (1 - 3\alpha(-\tau)) \left(1 - \frac{6\alpha_0^2 \ln \eta}{1 + 2\alpha_0 \ln \eta} \right) = \frac{4Gm_\phi^2}{\pi} (1 - 3\alpha(1/k)), \quad (3.24)$$

having used (2.17) again and dropped higher orders.

Note that the η dependence has disappeared. This is a general fact, due to the RG evolution equation obeyed by the power spectrum (see below). The running coupling $\alpha(1/k)$ at $1/k$ can be written as the coupling α_* at a reference scale $1/k_*$ evolved to $1/k$ by means of the RG equation.

The spectral index is defined by

$$n_T(k) = \frac{d \ln \mathcal{P}_T(k)}{d \ln k}, \quad (3.25)$$

hence we have

$$\mathcal{P}_T(k) = \mathcal{P}_T(k_*) \exp \left(\int_{k_*}^k \frac{dk'}{k'} n_T(k') \right). \quad (3.26)$$

Viewing $\mathcal{P}_T(k)$ and $n_T(k)$ as functions $\tilde{\mathcal{P}}_T(\alpha)$ and $\tilde{n}_T(\alpha)$ of the coupling α , we can also write

$$\tilde{\mathcal{P}}_T(\alpha) = \tilde{\mathcal{P}}_T(\alpha_*) \exp \left(- \int_{\alpha_*}^{\alpha} \frac{\tilde{n}_T(\alpha') d\alpha'}{\beta_\alpha(\alpha')} \right).$$

Using (3.24) and the beta function (2.13), and dropping higher-order corrections, we find the first contributions to n_T and its running coefficients:

$$n_T = \frac{d \ln \mathcal{P}_T}{d \ln k} = 3\beta_\alpha = -6\alpha^2, \quad \frac{dn_T}{d \ln k} = 12\alpha\beta_\alpha = -24\alpha^3, \quad \frac{d^n n_T}{d \ln k^n} = -6 \cdot 2^n (n+1)! \alpha^{n+2},$$

where α now stands for $\alpha(1/k)$. All the coefficients are related to \mathcal{P}_T by the RG equations, which we derive now.

3.2 RG equation of the power spectrum

Now we derive the RG evolution equation obeyed by the power spectrum in the superhorizon limit. We have seen that the running coupling $\alpha(-\tau)$ is an expansion in powers $\alpha_0^{l+1}(\alpha_0 \ln |\tau|)^n$ with $l \geq 0$, $n \geq 0$. Using (2.10) we find that H is $m_\phi/2$ plus an expansion of the same type. By (2.12), $-aH\tau$ is equal to 1 plus an analogous expansion. Finally, integrating (2.11) we find the expansions

$$-\frac{m_\phi t}{2} = (1 + \alpha_0^{l+1}(\alpha_0 m_\phi t)^n) \ln \frac{m_\phi |\tau|}{2} = \left(1 + \alpha_0^{l+1} \left(\alpha_0 \ln \frac{m_\phi |\tau|}{2}\right)^n\right) \ln \frac{m_\phi |\tau|}{2},$$

$$a(t) = e^{m_\phi t/2} (1 + \alpha_0^l (\alpha_0 m_\phi t)^n), \quad \frac{m_\phi |\tau|}{2} = e^{-m_\phi t/2} (1 + \alpha_0^l (\alpha_0 m_\phi t)^n), \quad (3.27)$$

where products such as $\alpha_0^p (\alpha_0 m_\phi t)^q$ are symbolic and it is understood that $p + q > 0$. It is always possible to switch from an expansion in t to an expansion in $\ln |\tau|$.

The formulas just written allow us to express everything in terms of exponentials and powers of t . The exponential behaviors, which can be retrieved from the de Sitter limit, guide us through the superhorizon limit of the spectra.

Let us recall that the expansion in powers of α is, indeed, the expansion around the de Sitter limit. Once a contribution is exponentially suppressed in the superhorizon limit ($t \rightarrow +\infty$) of the de Sitter limit ($\alpha \rightarrow 0$), it cannot be resuscitated by turning on the expansion in powers of α , since the corrections are just powers of t , as shown above. The following arguments illustrate these facts more explicitly.

The equations of motion of (3.3) are

$$\frac{d}{dt} (a^3 \dot{u}_{\mathbf{k}}) + k^2 a u_{\mathbf{k}} = 0. \quad (3.28)$$

Let us first consider the de Sitter limit, where $a(t) = e^{m_\phi t/2}$. In the superhorizon limit $t \rightarrow +\infty$ ($\tau \rightarrow 0^-$), the second term on the left-hand side of (3.28) can be dropped, so we just have $a^3 \dot{u}_{\mathbf{k}} = \text{constant}$, which is solved by

$$u_{\mathbf{k}} = c \int \frac{dt'}{a(t')^3} + d = c' e^{-3m_\phi t/2} + d', \quad (3.29)$$

where c , c' , d and d' are integration constants. From this solution, we see that in the superhorizon limit we can also drop the contribution $c' e^{-3m_\phi t/2}$, since it is negligible with respect to the constant d' . We thus obtain $u_{\mathbf{k}} = \text{constant}$.

The conclusion holds throughout the RG flow, as we can prove by turning on α to move away from the de Sitter limit. Indeed, if we use the expansions given above, we see that

the exponential appearing in (3.29) gets multiplied by powers of t , which cannot overcome the exponential factor. Instead, the constant d' is unaffected, since the correction $k^2 a u_{\mathbf{k}}$ appearing in (3.28) is exponentially subleading, which ensures that $u_{\mathbf{k}} = \text{constant}$ solves (3.28) in the superhorizon limit even for nonzero α .

We conclude that u tends to a constant for $k|\tau| \rightarrow 0$. By formulas (3.22) and (3.23), so does $k^{-3}\mathcal{P}_T$. This gives the renormalization-group equation

$$\frac{d\mathcal{P}_T}{d \ln|\tau|} = 0. \quad (3.30)$$

By equation (3.8) and the Bunch-Davies condition (3.9), the solution w depends on τ only through η and $\alpha(-\tau)$. By (3.4), (3.22) and (3.23), so does the spectrum \mathcal{P}_T . In this respect, note that the factor a of (3.4) conspires with the factor $1/\eta$ of (3.17) to give $a\eta = (-aH\tau)k/H$, which is a power series in $\alpha(-\tau)$ by formulas (2.10) and (2.12). Thus, equation (3.30) can be rewritten as the RG evolution equation

$$\left(\frac{\partial}{\partial \ln|\tau|} + \beta_\alpha \frac{\partial}{\partial \alpha} \right) \mathcal{P}_T = 0. \quad (3.31)$$

The α on the right-hand side of this equation stands for $\alpha(-\tau)$. Equation (3.31) has the standard form of the Callan-Symanzik equation of quantum field theory, with zero anomalous dimension.

The RG equation tells us that \mathcal{P}_T is actually τ independent, so in the end it is just a function of $\alpha_0 = \alpha(1/k)$. Formula (3.24) provides an explicit check of the property and so does formula (3.52) below.

Summarizing, the superhorizon limit $k|\tau| \rightarrow 0$ kills the powers of $k|\tau|$ and the RG evolution equation completes the job by killing the logarithms of $k|\tau|$, so that, in the end, no dependence on τ survives.

3.3 Scalar fluctuations

Now we study the scalar fluctuations in the $R+R^2$ theory. We choose the comoving gauge, where the fluctuation $\delta\phi$ of the scalar field ϕ is identically zero. The metric is parametrized as

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2) + 2\text{diag}(\Phi, a^2\Psi, a^2\Psi, a^2\Psi) - \delta_\mu^0 \delta_\nu^i \partial_i B - \delta_\mu^i \delta_\nu^0 \partial_i B. \quad (3.32)$$

After Fourier transforming the space coordinates, (2.1) gives the quadratic Lagrangian

$$(8\pi G) \frac{\mathcal{L}_s}{a^3} = -3(\dot{\Psi} + H\Phi)^2 + 4\pi G \dot{\phi}^2 \Phi^2 + \frac{k^2}{a^2} \left[2B(\dot{\Psi} + H\Phi) + \Psi(\Psi - 2\Phi) \right],$$

omitting the subscripts \mathbf{k} and $-\mathbf{k}$.

Integrating B out, we obtain $\Phi = -\dot{\Psi}/H$. Inserting this solution back into the action, we find

$$(8\pi G) \frac{\mathcal{L}_s}{a^3} = 3\alpha^2 \left(\dot{\Psi}^2 - \frac{k^2}{a^2} \Psi^2 \right). \quad (3.33)$$

The equation of motion becomes in the superhorizon limit

$$\frac{d}{dt} \left(a^3 \alpha^2 \dot{\Psi} \right) = 0,$$

which is solved by

$$\dot{\Psi} = \frac{c}{a^3 \alpha^2},$$

where c is a constant. Using the expansions (3.27), it is easy to see that (in the superhorizon limit) $\dot{\Psi}$ tends exponentially to zero, so Ψ tends exponentially to a constant. The corrections proportional to k^2/a^2 in (3.33) are also exponentially suppressed. Recalling that Ψ coincides with the curvature perturbation \mathcal{R} in the gauge we are using, we obtain the RG equation

$$\frac{d\mathcal{P}_{\mathcal{R}}}{d \ln|\tau|} = 0 \quad (3.34)$$

for the spectrum $\mathcal{P}_{\mathcal{R}}$ of the \mathcal{R} fluctuations. Again, since the solution w of the Mukhanov-Sasaki equation depends on τ only through η and $\alpha(-\tau)$, and so does the spectrum $\mathcal{P}_{\mathcal{R}}$, equation (3.34) can be rewritten as

$$\left(\frac{\partial}{\partial \ln|\tau|} + \beta_\alpha \frac{\partial}{\partial \alpha} \right) \mathcal{P}_{\mathcal{R}} = 0. \quad (3.35)$$

The calculations we are going to perform will provide nontrivial checks of this formula. Like \mathcal{P}_T , the spectrum $\mathcal{P}_{\mathcal{R}}$ has a vanishing anomalous dimension.

Defining

$$w = \alpha a \Psi \sqrt{\frac{3k}{4\pi G}}, \quad (3.36)$$

the w action reads

$$S_s = \frac{1}{2} \int d\eta \left[w'^2 - w^2 + \left(\nu_s^2 - \frac{1}{4} \right) \frac{w^2}{\eta^2} \right], \quad (3.37)$$

where

$$\nu_s^2 - \frac{1}{4} = \left(\frac{\beta_\alpha}{\alpha} + aH\tau - 1 + \beta_\alpha \frac{d}{d\alpha} \right) \left(\frac{\beta_\alpha}{\alpha} + aH\tau \right).$$

Using (2.12), we find

$$\nu_s = \frac{3}{2} + 2\alpha + 6\alpha^2 + \frac{208}{9}\alpha^3 + \frac{1361}{9}\alpha^4 + \mathcal{O}(\alpha^5). \quad (3.38)$$

As before, the w equation

$$w'' + w - \left(\nu_s^2 - \frac{1}{4} \right) \frac{w}{\eta^2} = 0 \quad (3.39)$$

can be solved by expanding in powers of α , but this time we have to include a term proportional to α_0 :

$$w(\eta) = w_0(\eta) + \alpha_0 w_1(\eta) + \alpha_0^2 w_2(\eta) + \alpha_0^3 w_3(\eta) + \dots \quad (3.40)$$

We obtain equations of the form

$$w_n'' + w_n - \frac{2w_n}{\eta^2} = \frac{g_n(\eta)}{\eta^4}, \quad (3.41)$$

where the functions g_n are determined recursively from w_j , $j < n$, as explained in subsection 3.1. We find $g_0(\eta) = 0$ and

$$g_1 = 6w_0, \quad g_2 = 2(11 - 6 \ln \eta)w_0 + g_1^+, \quad g_3 = \left(\frac{280}{3} - 98 \ln \eta + 24 \ln^2 \eta \right) w_0 + g_2^+, \quad (3.42)$$

etc., where g_k^+ means g_k with every w_m replaced by w_{m+1} .

The function w_0 is still (3.13), while w_1 coincides with the w_2 of (3.15) apart from the overall factor:

$$w_1(\eta) = \frac{\sqrt{2}}{\eta} [2ie^{i\eta} + (\eta - i)e^{-i\eta} (\text{Ei}(2i\eta) - i\pi)]. \quad (3.43)$$

The functions w_k with $k > 1$ can be written by quadratures.

Now we study the power spectrum. In the limit $\eta \rightarrow \infty$, the functions g_n and w_n , $n > 0$, satisfy (3.16) for every $\delta > 0$. In the superhorizon limit $\eta \rightarrow 0$, we have

$$g_n(\eta) \underset{\eta \sim 0}{\sim} \frac{1}{\eta} P_{n-1}^{(g)}(\ln \eta) + \mathcal{O}(\ln^{n-1} \eta) \quad w_n(\eta) \underset{\eta \sim 0}{\sim} \frac{1}{\eta} P_n^{(w)}(\ln \eta) + \mathcal{O}(\ln^n \eta). \quad (3.44)$$

As in the case of the tensor fluctuations, we can drop the term w_n of equation (3.11) for $\eta \rightarrow 0$ and the solution (3.14) simplifies to the form (3.18).

To the leading log order, it is enough to truncate (3.38) to order α and replace α with (2.17). Equation (3.41) then simplifies to

$$w'' + w - \frac{2w}{\eta^2} = \frac{6\alpha_0 w}{\eta^2(1 + 2\alpha_0 \ln \eta)}.$$

Formulas (3.44) suggest to parametrize the leading behavior of w_n as

$$w_n(\eta) \underset{\eta \sim 0}{\sim} \frac{id_n}{\eta\sqrt{2}} (-2 \ln \eta)^n, \quad (3.45)$$

where d_n are constants. Expanding in powers of α_0 , we find equation (3.41) with

$$g_n(\eta) = \frac{3i\sqrt{2}}{\eta} (-2 \ln \eta)^{n-1} \sum_{k=0}^{n-1} d_k. \quad (3.46)$$

Using (3.18), we obtain

$$w_n(\eta) \underset{\eta \sim 0}{\sim} \frac{i}{\eta\sqrt{2}} (-2 \ln \eta)^n \frac{1}{n} \sum_{k=0}^{n-1} d_k. \quad (3.47)$$

Matching (3.47) with (3.45), we obtain a recursion relation for the constants d_k . Since $d_0 = 1$, all the d_k turn out to be equal to one.

Resumming (3.40) we find the leading log correction to $w(\eta)$, which is

$$w(\eta) \underset{\eta \sim 0}{\sim} \frac{i}{\eta\sqrt{2}} \frac{1}{1 + 2\alpha_0 \ln \eta}. \quad (3.48)$$

Now, using (3.36), the leading log behavior of the curvature perturbation turns out to be

$$\mathcal{R} = \Psi = \frac{w}{a\alpha} \sqrt{\frac{4\pi G}{3k}} \sim \sqrt{\frac{4\pi G}{3k}} \frac{i}{a\alpha\eta\sqrt{2}} \frac{1}{1 + 2\alpha_0 \ln \eta}. \quad (3.49)$$

The \mathcal{R} power spectrum is defined by

$$\langle \mathcal{R}_{\mathbf{k}}(\tau) \mathcal{R}_{\mathbf{k}'}(\tau) \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_{\mathcal{R}}, \quad \frac{2\pi^2}{k^3} \mathcal{P}_{\mathcal{R}} = |\Psi|^2, \quad (3.50)$$

so (3.49) gives

$$\mathcal{P}_{\mathcal{R}} = \frac{GH^2}{3\pi(aH\tau)^2} \frac{1}{\alpha(-\tau)^2(1 + 2\alpha_0 \ln \eta)^2}. \quad (3.51)$$

Using (2.10) and (2.12) with $\alpha(-\tau)$ given by (2.17), we find

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{Gm_\phi^2}{12\pi\alpha_0^2} = \frac{m_\phi^2 G}{12\pi\alpha(1/k)^2}, \quad (3.52)$$

which does satisfy the RG equation (3.35). As expected, the τ dependence has disappeared.

The spectral index is defined by

$$\frac{d \ln \mathcal{P}_{\mathcal{R}}(k)}{d \ln k} = n_{\mathcal{R}}(k) - 1.$$

We find

$$n_{\mathcal{R}} - 1 = -4\alpha, \quad \frac{dn_{\mathcal{R}}}{d \ln k} = 4\beta_\alpha = -8\alpha^2, \quad \frac{d^n n_{\mathcal{R}}}{d \ln k^n} = -2^{n+2} n! \alpha^{n+1}, \quad (3.53)$$

where α stands for $\alpha(1/k)$. Again, the leading log contributions to the running, the running of the running, etc., are all related to the leading log contribution to $n_{\mathcal{R}} - 1$, and ultimately $\mathcal{P}_{\mathcal{R}}$, by the RG equation.

3.4 Subleading log corrections

Now we study the subleading log corrections to the power spectra of the tensor and scalar fluctuations. It is convenient to expand the Mukhanov-Sasaki equation in a more systematic way. We write it as

$$w'' + w - 2\frac{w}{\eta^2} = \sigma\frac{w}{\eta^2}, \quad (3.54)$$

where $\sigma = \nu^2 - (9/4) = \mathcal{O}(\alpha)$ is a power series in α , while ν is ν_t or ν_s .

We start by studying some general properties of this equation. Following (3.17) and (3.44), we separate $\eta w(\eta)$ into a power series $Q(\ln \eta)$ in $\ln \eta$ plus the rest:

$$\eta w = Q(\ln \eta) + W(\eta), \quad (3.55)$$

where $W(\eta)$ is an expansion in powers of η and logarithms $\ln \eta$, such that $W(\eta) \rightarrow 0$ term-by-term for $\eta \rightarrow 0$. Then we observe that if the right-hand side of (3.54) is

$$\sigma\frac{w}{\eta^2} = \frac{c}{\eta^{3-m}} \ln^n \eta, \quad (3.56)$$

where c is a constant and $n, m \geq 0$ are integers, it is easy to show, using (3.14) or (3.18), that only $m = 0$ contributes to $Q(\ln \eta)$. Such contributions are equal to

$$Q(\ln \eta) = -\frac{c}{9} \sum_{k=-1}^{n-1} 3^{-k} \frac{d^k}{d \ln^k \eta} \ln^n \eta + \text{constant}, \quad (3.57)$$

where it is understood that the “ -1 derivative” ($k = 0$) is the integral from 0 to $\ln \eta$:

$$\frac{d^{-1}f(x)}{dx^{-1}} \equiv \int_0^x f(x') dx'.$$

The constant in (3.57) is left unspecified, because it does not provide useful information at this level, for the reasons we explain below. Summing the contributions due to (3.56) for $m = 0$, $n \geq 0$, we find the equation satisfied by Q , which is

$$Q = -\frac{1}{9} \sum_{n=-1}^{\infty} 3^{-n} \frac{d^n(\sigma Q)}{d \ln^n \eta} + \text{constant}. \quad (3.58)$$

Differentiating it once, we get

$$\frac{dQ}{d \ln \eta} = -\frac{\sigma}{3} Q - \frac{1}{3} \sum_{n=1}^{\infty} 3^{-n} \frac{d^n(\sigma Q)}{d \ln^n \eta}. \quad (3.59)$$

As for the leading logs, it is sufficient to truncate this equation to the first term on the right-hand side. Then the equation integrates to

$$Q(\ln \eta) = Q(0) \exp \left(-\frac{1}{3} \int_0^{\ln \eta} \sigma(\alpha(\eta'/k)) d \ln \eta' \right) = Q(0) \exp \left(-\frac{1}{3} \int_{\alpha_0}^{\alpha(-\tau)} \frac{\sigma(\alpha')}{\beta_\alpha(\alpha')} d\alpha' \right). \quad (3.60)$$

To verify this formula in the cases of the tensor and scalar fluctuations, it is sufficient to use formula (2.17) for the running coupling and truncate the beta function to its first contribution, which is $-2\alpha^2$. In the tensor case, formula (3.7) gives $\sigma = 9\alpha^2$ to the lowest order, so we obtain

$$Q(\ln \eta) = Q(0) \exp \left(-\frac{3\alpha_0^2 \ln \eta}{1 + 2\alpha_0 \ln \eta} \right) = Q(0) \left(1 - \frac{3\alpha_0^2 \ln \eta}{1 + 2\alpha_0 \ln \eta} + \text{subleading} \right),$$

which agrees with (3.21) for $Q(0) = i/\sqrt{2}$. In the scalar case, formula (3.38) gives $\sigma = 6\alpha$ to the lowest order, so we find

$$Q(\ln \eta) = \frac{Q(0)}{1 + 2\alpha_0 \ln \eta},$$

which agrees with (3.48) for $Q(0) = i/\sqrt{2}$ again.

The integration constant $Q(0)$ cannot be computed just from the behaviors of the functions for $\eta \rightarrow 0$. Indeed, it is not enough to know the integrand of (3.14) for $\eta \sim 0$ to calculate the contributions to the integral that have the form constant/ η . Yet, equation (3.59) is enough to perform a full check of the RG equations (3.31) and (3.35), since it encodes all the contributions $\sim (\ln^n \eta)/\eta$ with $n > 0$.

We first verify the validity of the evolution equations to the next-to-leading log order and later compute $Q(0)$ to the same order. The next-to-leading log corrections can be studied by keeping one term more on the right-hand-side of (3.59), as well as in the beta function, the running coupling and the expression of σ . Specifically, the beta function can be truncated to $\beta_\alpha = -2\alpha^2 - (5/3)\alpha^3$. As for the running coupling, we can use formula (2.21). The formulas of σ can instead be truncated to $\sigma = \nu_t^2 - (9/4) = 9\alpha^2 + 48\alpha^3$ and $\sigma = \nu_s^2 - (9/4) = 6\alpha + 22\alpha^2$ for tensors and scalars, respectively.

The truncated version of (3.59), which is

$$\frac{dQ}{d \ln \eta} = -\frac{\sigma}{3}Q - \frac{1}{9} \frac{d(\sigma Q)}{d \ln \eta},$$

is solved by

$$Q(\ln \eta) = Q(0) \frac{9 + \sigma(\alpha_0)}{9 + \sigma(\alpha(-\tau))} \exp \left(- \int_{\alpha_0}^{\alpha(-\tau)} \frac{3\sigma(\alpha') d\alpha'}{(9 + \sigma(\alpha'))\beta_\alpha(\alpha')} \right). \quad (3.61)$$

Moreover, the power spectra (3.23) and (3.50) require formulas (2.10) and (2.12) up to the order α^2 , so we use

$$H = \frac{m_\phi}{2} \left(1 - \frac{3\alpha}{2} + \frac{7\alpha^2}{4} \right), \quad -aH\tau = 1 + 3\alpha^2.$$

Putting all the ingredients together, we find

$$\begin{aligned} \mathcal{P}_T &= \frac{8Gm_\phi^2}{\pi} |Q^{(T)}(0)|^2 \left(1 - 3\alpha(1/k) - \frac{1}{4}\alpha(1/k)^2 \right), \\ \mathcal{P}_R &= \frac{Gm_\phi^2}{6\pi} \frac{|Q^{(R)}(0)|^2}{\alpha(1/k)^2} (1 - 3\alpha(1/k)). \end{aligned} \quad (3.62)$$

for tensors and scalars, respectively. We see that the τ dependence has disappeared again, in agreement with the RG equations.

Now we come to the integration constant $Q(0)$. In general, we have

$$Q(0) = \frac{i}{\sqrt{2}} h(\alpha_0), \quad (3.63)$$

where $h(\alpha_0) = 1 + \mathcal{O}(\alpha_0)$ is a power series in α_0 . As said, it is not necessary to know $Q(0)$ to check the RG equation. Yet, its knowledge is crucial to determine the subleading log corrections to the power spectra. The constants $Q^{(T)}(0)$ and $Q^{(R)}(0)$ of \mathcal{P}_T and \mathcal{P}_R can be calculated from the definition (3.55) by means of the complete expressions of the functions w_n . Using formulas (3.15) and (3.43), we find

$$\begin{aligned} Q^{(T)}(\ln \eta) &= \frac{i}{\sqrt{2}} \left(1 + \frac{3}{2}\alpha_0^2(4 - 2\gamma_M + i\pi - 2\ln \eta) \right) + \mathcal{O}(\alpha_0^3), \\ Q^{(R)}(\ln \eta) &= \frac{i}{\sqrt{2}} \left(1 + \alpha_0(4 - 2\gamma_M + i\pi - 2\ln \eta) \right) + \mathcal{O}(\alpha_0^2). \end{aligned} \quad (3.64)$$

In the end, we obtain

$$\begin{aligned} \mathcal{P}_T(k) &= \frac{4Gm_\phi^2}{\pi} \left[1 - 3\alpha(1/k) + \left(\frac{47}{4} - 6\gamma_M \right) \alpha(1/k)^2 \right], \\ \mathcal{P}_R(k) &= \frac{Gm_\phi^2}{12\pi\alpha(1/k)^2} \left(1 + (5 - 4\gamma_M)\alpha(1/k) \right). \end{aligned} \quad (3.65)$$

The results agree with those of ref. [20] in the limit $m_\chi \rightarrow \infty$. In addition, now we know how to use them to calculate the subleading log corrections to the spectral indices and

their running coefficients. For the tensor fluctuations, we find

$$\begin{aligned} n_T &= -\beta_\alpha(\alpha) \frac{\partial \ln P_T}{\partial \alpha} = -6\alpha^2 + 24\alpha^3(1 - \gamma_M), \\ \frac{dn_T}{d \ln k} &= -\beta_\alpha(\alpha) \frac{\partial n_T}{\partial \alpha} = -24\alpha^3 + 4\alpha^4(31 - 36\gamma_M), \\ \frac{d^2 n_T}{d \ln k^2} &= -\beta_\alpha(\alpha) \frac{\partial}{\partial \alpha} \left(-\beta_\alpha(\alpha) \frac{\partial n_T}{\partial \alpha} \right) = -144\alpha^4 + 8\alpha^5(109 - 144\gamma_M). \end{aligned} \quad (3.66)$$

etc., where now α stands for $\alpha(1/k)$. For the scalar fluctuations, we find

$$\begin{aligned} n_{\mathcal{R}} - 1 &= -4\alpha + \frac{4}{3}\alpha^2(5 - 6\gamma_M), \\ \frac{dn_{\mathcal{R}}}{d \ln k} &= -8\alpha^2 + 4\alpha^3(5 - 8\gamma_M), \quad \frac{d^2 n_{\mathcal{R}}}{d \ln k^2} = -32\alpha^3 + \frac{8}{3}\alpha^4(35 - 72\gamma_M), \end{aligned} \quad (3.67)$$

etc.

The results of this section agree with those available in the literature [26, 27] up to the resummations of the leading logs, which means once we expand $\alpha(1/k)$ in powers of the ‘‘pivot coupling’’ $\alpha_* = \alpha(1/k_*)$, i.e. the running coupling evaluated at the pivot scale k_* , and consider the product $\alpha_* \ln(k/k_*)$ of order one instead of order zero.

4 Quantum gravity

In this section we generalize the results to quantum gravity. We begin by proving that the RG equations still hold.

4.1 RG evolution equations of the power spectra

We prove that the power spectra obey the RG evolution equations (3.31) and (3.35), in the superhorizon limit. We refer to the notation and formulas of ref. [20].

In the case of the tensor fluctuations, we take the parametrization (3.2) of the metric and expand the action (2.1). The quadratic Lagrangian in the superhorizon limit reads

$$\mathcal{L}_t = \frac{a^3}{8\pi G} \left[1 + \frac{2H^2}{m_\chi^2} \left(1 - \frac{3}{2}\alpha^2 \right) \right] \dot{u}^2 - \frac{\ddot{u}^2}{m_\chi^2}. \quad (4.1)$$

We see that the field equations admit the obvious solution $u = \text{constant}$, since \mathcal{L}_t depends only on the derivatives of u . We want to show that this is the only solution that survives the superhorizon limit. Around the de Sitter space we can use the formulas (3.27), which

expand the solution in terms of exponentials multiplied by power series. The exponential factors can be derived from the de Sitter limit, where the Lagrangian becomes

$$\mathcal{L}_t = \frac{e^{3m_\phi t/2}}{8\pi G} \left[\dot{u}^2 \left(1 + \frac{m_\phi^2}{2m_\chi^2} \right) - \frac{\ddot{u}^2}{m_\chi^2} \right].$$

The most general solution of its equation of motion is

$$u(t) = c_1 + c_2 e^{-3m_\phi t/2} + e^{-3m_\phi t/4} (c_3 e^{ist} + c_4 e^{-ist}), \quad (4.2)$$

where $s = \sqrt{m_\chi^2 - (m_\phi^2/16)}$, which is real by the consistency condition $m_\chi > m_\phi/4$ of the fakeon projection [20]. We see that in the superhorizon limit $t \rightarrow +\infty$ only the constant c_1 survives. The conclusion does not change by moving away from the de Sitter limit, since the exponentials of (4.2) just get multiplied by powers of t .

The arguments just given apply to the unprojected action (2.1) and the whole space of solutions of its field equations. Since the relevant solution $u = \text{constant}$ belongs to the physical subspace, the conclusions extend to the projected action as well. Details on the fakeon projection are given below. Ultimately, by (3.22) and (3.23) the tensor power spectrum P_T satisfies equation (3.30), from which (3.31) follows.

In the case of the scalar fluctuations, we expand the action (2.1) to the quadratic order with the metric (3.32) (for the result of this operation, see [20]). Next, we remove Φ , which is an auxiliary field, by means of its own field equation. Third, we define

$$\Psi = U + f(t)B, \quad (4.3)$$

where $f(t)$ is a function, and express the Lagrangian in terms of B and the new field U . We choose f to remove the term proportional to \dot{B}^2 . After integrating by parts to eliminate \dot{B} altogether, B turns into an auxiliary field as well, so we remove it by means of its own field equation. Then we take the superhorizon limit $k/(aH) \rightarrow 0$ and see that \mathcal{L}_s and B depend only on the derivatives of U , but not on U itself, so $\Psi = \text{constant}$ is a solution. Explicitly, if we expand around the de Sitter metric, we obtain

$$(8\pi G) \frac{\mathcal{L}_s}{a^3} = 3\alpha^2 \left(\dot{U}^2 - \frac{2\ddot{U}^2}{2m_\chi^2 + m_\phi^2} \right), \quad \Psi = U + \frac{3m_\phi \dot{U} + 2\ddot{U}}{2m_\chi^2 + m_\phi^2},$$

up to higher powers of α . The expression of Ψ is obtained from (4.3) after substituting B with its solution. Using (3.27) again, the solution of the U field equation has the form (4.2) up to powers of t multiplying the exponential factors, so Ψ tends to a constant in

the superhorizon limit. The conclusion does not change moving away from the de Sitter fixed point. Moreover, it applies to the unprojected action and extends immediately to the projected one. This proves that the scalar power spectrum satisfies the RG evolution equations (3.34) and (3.35).

4.2 Tensor fluctuations

Parametrizing the metric as (3.2), the quadratic Lagrangian obtained from (2.1) is

$$(8\pi G) \frac{\mathcal{L}_t}{a^3} = \dot{u}^2 - \frac{k^2}{a^2} u^2 - \frac{1}{m_\chi^2} \left[\ddot{u}^2 - 2 \left(H^2 - \frac{3}{2} \alpha^2 H^2 + \frac{k^2}{a^2} \right) \dot{u}^2 + \frac{k^4}{a^4} u^2 \right], \quad (4.4)$$

plus an identical contribution for v . We can eliminate the higher derivatives by means of the procedure used in [20]. Specifically, we add an auxiliary field U and consider the extended Lagrangian

$$\mathcal{L}'_t = \mathcal{L}_t + \Delta \mathcal{L}_t, \quad (4.5)$$

where

$$(8\pi G m_\chi^2) \frac{\Delta \mathcal{L}_t}{a^3} = \left[m_\chi^2 \gamma U - \ddot{u} - \left(3H - \frac{12\alpha^2 H^3}{m_\chi^2 \gamma} \right) \dot{u} - \left(m_\chi^2 \gamma + \frac{k^2}{a^2} + \frac{3\alpha^2 H^2 (m_\chi^2 - 4H^2)}{m_\chi^2 \gamma} + \frac{24\alpha^3 H^4}{m_\chi^2 \gamma} \right) u \right]^2 \quad (4.6)$$

and

$$\gamma = 1 + 2 \frac{H^2}{m_\chi^2}. \quad (4.7)$$

It is immediate to show that \mathcal{L}'_t is equivalent to \mathcal{L}_t by replacing U , which appears algebraically, with the solution of its own field equation. Note that we have kept an additional term (the one of order α^3) with respect to the formulas of [20]. The reason is that, if we want to check the RG evolution equation to the order α^2 , it is necessary to make calculations to the order α^3 included.

We diagonalize \mathcal{L}'_t in the de Sitter limit by introducing a second field V such that

$$u = U + V. \quad (4.8)$$

To the order we need, the Lagrangian \mathcal{L}'_t takes the form

$$(8\pi G) \frac{\mathcal{L}'_t}{a^3 \gamma} = \dot{U}^2 - \frac{k^2}{a^2} \left(1 - \frac{12\alpha^2 H^2}{m_\chi^2 \gamma^2} - \frac{96\alpha^3 H^4}{m_\chi^4 \gamma^2} \right) U^2 + \frac{36\alpha^3 H^4}{m_\chi^4 \gamma^2} (m_\chi^2 - 4H^2) U^2 - \dot{V}^2 + \left(m_\chi^2 \gamma + \frac{k^2}{a^2} \right) V^2 + \frac{6\alpha^2}{m_\chi^2 \gamma} H^2 \left(m_\chi^2 - 4H^2 + \frac{4k^2}{\gamma a^2} \right) UV. \quad (4.9)$$

The fakeon projection amounts to remove the field V by replacing it with a special solution of its own field equations, which is defined by the fakeon Green function derived in [20]. For the moment, it is sufficient to know that the solution exists, because from (4.9) it is evident that the projection equates V to something of order α^2 . Once the solution is inserted back into (4.9), the second line turns out to be $\mathcal{O}(\alpha^4)$. Therefore, the projected action is given by the first line of (4.9) to the order α^3 included. We will need V later on, though, since it enters formula (4.8).

Defining

$$\begin{aligned} w &= \frac{a\sqrt{k\gamma}}{\sqrt{4\pi G}}U, & \sigma &= \frac{18m_\chi^2\alpha^2}{m_\phi^2 + 2m_\chi^2} + \frac{6m_\chi^2\alpha^3(32m_\chi^2 + 43m_\phi^2)}{(m_\phi^2 + 2m_\chi^2)^2} + \mathcal{O}(\alpha^4), \\ h &= 1 - \frac{12m_\chi^2m_\phi^2\alpha^2}{(m_\phi^2 + 2m_\chi^2)^2} - \frac{12m_\phi^2\alpha^3(2m_\phi^4 + 7m_\chi^2m_\phi^2 - 6m_\chi^4)}{(m_\phi^2 + 2m_\chi^2)^3} + \mathcal{O}(\alpha^4), \end{aligned} \quad (4.10)$$

then using (2.10) and (2.12) and switching to the conformal time (2.11), the projected w action to order α^3 reads

$$S_t^{\text{prj}} = \frac{1}{2} \int d\eta \left(w'^2 - hw^2 + 2\frac{w^2}{\eta^2} + \sigma\frac{w^2}{\eta^2} \right), \quad (4.11)$$

where the prime denotes the derivative with respect to $\eta = -k\tau$. Due to the function h in front of w^2 , this action is not of the form (3.5) and its equation of motion is not of the form (3.54). Although the term $-hw^2$ is negligible in the superhorizon limit, it is important in the opposite limit, through the Bunch-Davies vacuum condition.

The Bunch-Davies condition is necessary to calculate the constants $Q^{(T)}(0)$ and $Q^{(\mathcal{R})}(0)$ of formulas (3.62), but it is unnecessary to check RG invariance. For these reasons, we first check the RG evolution equation to the order α^2 , where we can ignore $-hw^2$, and then compute the constants $Q(0)$.

The first formula of (4.10) relates w to U , but formula (4.8) tells us that to work out the u spectrum we also need the relation between V and U . Such a relation is provided by the fakeon projection, which can be borrowed to the order we need from [20]. Here we just recall the basic steps that lead to the result.

The V equation of motion derived from (4.9) is

$$\left(\Sigma_0 + m_\chi^2 + \frac{m_\phi^2}{2} \right) V = -\frac{3\alpha^2 m_\phi^2}{2(m_\phi^2 + 2m_\chi^2)} \left(m_\chi^2 - m_\phi^2 + \frac{8m_\chi^2 k^2}{(m_\phi^2 + 2m_\chi^2)a^2} \right) U, \quad (4.12)$$

where

$$\Sigma_0 \equiv \frac{d^2}{dt^2} + \frac{3m_\phi}{2} \frac{d}{dt} + \frac{k^2}{a^2},$$

We can easily solve (4.12) in the superhorizon limit up to corrections of higher orders in α . In that limit, we can ignore the terms proportional to k^2/a^2 . Moreover, (2.8) gives $[\Sigma_0, \alpha] = \mathcal{O}(\alpha^2)$ and the U equation of motion gives $\Sigma_0 U = \mathcal{O}(\alpha)$. Basically, we can commute Σ_0 with α^2 and drop $\Sigma_0 U$, which leads to

$$V = -\frac{3m_\phi^2 (m_\chi^2 - m_\phi^2)}{m_\phi^2 + 2m_\chi^2} \frac{1}{2\Sigma_0 + 2m_\chi^2 + m_\phi^2} \Big|_f \alpha^2 U = -\frac{3\alpha^2 m_\phi^2 (m_\chi^2 - m_\phi^2)}{(m_\phi^2 + 2m_\chi^2)^2} U, \quad (4.13)$$

where the subscript f denotes the fakeon prescription. Its role is to ensure that the solution is the one written on the right-hand side, with no additions proportional to the solutions of the homogeneous equation.

At this point, we have all the ingredients we need to use formula (3.61), which gives

$$\mathcal{P}_T = \frac{16Gm_\chi^2 m_\phi^2}{\pi(m_\phi^2 + 2m_\chi^2)} |Q^{(T)}(0)|^2 \left[1 - \frac{6m_\chi^2 \alpha(1/k)}{m_\phi^2 + 2m_\chi^2} - \alpha(1/k)^2 \frac{m_\chi^2 (73m_\phi^2 + 2m_\chi^2)}{2(m_\phi^2 + 2m_\chi^2)^2} \right]. \quad (4.14)$$

As expected, the τ dependence disappears completely, in agreement with the RG evolution equation (3.30).

The final step is to calculate $Q^{(T)}(0)$, for which it is important to deal with the term $-hw^2$ of (4.11). We can rewrite (4.11) in the form studied so far by making a change of variables from η to $\tilde{\eta}(\eta)$, with $\tilde{\eta}'(\eta) = \sqrt{h(\eta)}$, and defining

$$\tilde{w}(\tilde{\eta}(\eta)) = h(\eta)^{1/4} w(\eta), \quad \tilde{\sigma} = \frac{\tilde{\eta}^2(\sigma + 2)}{\eta^2 h} + \frac{\tilde{\eta}^2}{16h^3} (4hh'' - 5h'^2) - 2. \quad (4.15)$$

So doing, (4.11) is recast as

$$\tilde{S}_t^{\text{prj}} = \frac{1}{2} \int d\tilde{\eta} \left(\tilde{w}'^2 - \tilde{w}^2 + \frac{2\tilde{w}^2}{\tilde{\eta}^2} + \tilde{\sigma} \frac{\tilde{w}^2}{\tilde{\eta}^2} \right), \quad (4.16)$$

where the prime on \tilde{w} denotes the derivative with respect to $\tilde{\eta}$.

We have to study the expansion in powers of α_0 . Observe that, using (4.10), $\tilde{\sigma}$ turns out to be $\mathcal{O}(\alpha_0^2)$, once we insert the expression (2.21) of the running coupling α . Using the Bunch-Davies condition (3.9) in the variable $\tilde{\eta}$, which reads

$$\tilde{w}(\tilde{\eta}) \sim \frac{e^{i\tilde{\eta}}}{\sqrt{2}} \quad \text{for } \tilde{\eta} \rightarrow \infty, \quad (4.17)$$

the solution of the \tilde{S}_t^{prj} equation of motion has the form

$$\tilde{w}(\tilde{\eta}) = \frac{(\tilde{\eta} + i)e^{i\tilde{\eta}}}{\sqrt{2}\tilde{\eta}} + \alpha_0^2 \Delta \tilde{w}(\tilde{\eta}), \quad \lim_{\tilde{\eta} \rightarrow \infty} \Delta \tilde{w}(\tilde{\eta}) = 0. \quad (4.18)$$

Indeed, we know that for $\alpha_0 = 0$ the exact solution is the written one, by (4.17). However, we also know that (4.17) must hold for every α_0 . This means that the corrections neglected in (4.18) must be $\mathcal{O}(\alpha_0^2)$ and disappear in the limit $\tilde{\eta} \rightarrow \infty$.

Switching back to the variable η by means of the formula

$$\tilde{\eta}(\eta) = \int_0^\eta \sqrt{h(\eta')} d\eta',$$

we can work out the vacuum condition for (4.11) from (4.18). Using the first equation of (4.15), we find

$$\tilde{\eta} \simeq \eta \left(1 - \frac{6\alpha_0^2 m_\chi^2 m_\phi^2}{(m_\phi^2 + 2m_\chi^2)^2} \right), \quad w(\eta) = \tilde{w}(\tilde{\eta}(\eta)) h(\eta)^{-1/4} \simeq \frac{e^{i\eta}}{\sqrt{2}} \left[1 + \frac{3(3 - 2i\eta)\alpha_0^2 m_\chi^2 m_\phi^2}{(m_\phi^2 + 2m_\chi^2)^2} \right], \quad (4.19)$$

up to terms that are of higher orders in α_0 or negligible for η large. If we want to push the calculations to sub-subleading log orders, other terms of the large $\tilde{\eta}$ expansion in (4.18) need to be kept.

By expanding w in powers of α_0 as in (3.10), we derive the differential equations obeyed by the functions w_n . Formula (4.19) gives the asymptotic conditions for the first two, which are

$$w_0(\eta) \simeq \frac{e^{i\eta}}{\sqrt{2}}, \quad w_2(\eta) \simeq \frac{3m_\chi^2 m_\phi^2 e^{i\eta} (3 - 2i\eta)}{\sqrt{2}(m_\phi^2 + 2m_\chi^2)^2}, \quad \text{for } \eta \text{ large.}$$

We find (3.13) and

$$w_2(\eta) = \frac{3\sqrt{2}m_\chi^2}{\eta(m_\phi^2 + 2m_\chi^2)} \left[2ie^{i\eta} \left(1 + \frac{m_\phi^2(3 - 3i\eta - 2\eta^2)}{4(m_\phi^2 + 2m_\chi^2)} \right) + (\eta - i)e^{-i\eta} (\text{Ei}(2i\eta) - i\pi) \right], \quad (4.20)$$

which upgrades the function w_2 of formula (3.15). Studying the $\eta \rightarrow 0$ behavior of $w(\eta)$ we can extract $Q(\ln \eta)$ by means of the decomposition (3.55). The outcome leads to

$$Q^{(T)}(0) = \frac{i}{\sqrt{2}} \left[1 + \frac{3\alpha_0^2 m_\chi^2 (7m_\phi^2 + 8m_\chi^2)}{(m_\phi^2 + 2m_\chi^2)^2} - \frac{3\alpha_0^2 m_\chi^2 (2\gamma_M - i\pi)}{m_\phi^2 + 2m_\chi^2} \right],$$

which upgrades the result encoded in the first line of (3.64). Finally, inserting $Q^{(T)}(0)$ into (4.14) we obtain

$$\mathcal{P}_T = \frac{8Gm_\chi^2 m_\phi^2}{\pi(m_\phi^2 + 2m_\chi^2)} \left[1 - \frac{6m_\chi^2 \alpha(1/k)}{m_\phi^2 + 2m_\chi^2} + \frac{36m_\chi^4 \alpha(1/k)^2}{(m_\phi^2 + 2m_\chi^2)^2} + \frac{\alpha(1/k)^2 m_\chi^2 (11 - 24\gamma_M)}{2(m_\phi^2 + 2m_\chi^2)} \right]. \quad (4.21)$$

which upgrades the first line of (3.65) and agrees with the result of [20].

The cosmic RG flow allows us to derive the spectral index to the next-to-leading log order (in [20] it was computed to the leading order), as well as the whole running of the spectrum to the next-to-leading log order. Using the beta function $\beta_\alpha = -2\alpha^2 - (5/3)\alpha^3$ to order α^3 , the right-hand sides of (3.66) are upgraded to

$$\begin{aligned}
 n_T &= -\beta_\alpha(\alpha) \frac{\partial \ln P_T}{\partial \alpha} = -\frac{12\alpha^2 m_\chi^2 (1 + 4\alpha\gamma_M)}{m_\phi^2 + 2m_\chi^2} + \frac{12\alpha^3 m_\chi^2 (m_\phi^2 + 8m_\chi^2)}{(m_\phi^2 + 2m_\chi^2)^2}, \\
 \frac{m_\phi^2 + 2m_\chi^2}{2^n m_\chi^2 \alpha^{n+2} (n+1)!} \frac{d^n n_T}{d \ln k^n} &= \frac{m_\phi^2 + 2m_\chi^2}{2^n m_\chi^2 \alpha^{n+2} (n+1)!} \left(-\beta_\alpha(\alpha) \frac{\partial}{\partial \alpha} \right)^n n_T \\
 &= -12 - 24(n+2)\alpha\gamma_M + (n+2)\alpha \left(3 \frac{26m_\chi^2 + 7m_\phi^2}{m_\phi^2 + 2m_\chi^2} - 10 \sum_{k=1}^{n+2} \frac{1}{k} \right).
 \end{aligned} \tag{4.22}$$

4.3 Scalar fluctuations and first correction to $r = -8n_T$

It was shown in ref. [20] that the fakeon $\chi_{\mu\nu}$ belonging to the gravitational triplet does not affect the quantum gravity predictions concerning the scalar fluctuations, up to the next-to-leading order included. The results of this paper promote the same conclusion to the next-to-leading log order, where they coincide with those of formulas (3.65) and (3.67). In particular,

$$\begin{aligned}
 \mathcal{P}_\mathcal{R}(k) &= \frac{Gm_\phi^2}{12\pi\alpha(1/k)^2} \left(1 + (5 - 4\gamma_M)\alpha(1/k) \right) \quad n_\mathcal{R} - 1 = -4\alpha + \frac{4}{3}\alpha^2(5 - 6\gamma_M), \\
 \frac{d^n (n_\mathcal{R} - 1)}{d \ln k^n} &= -2^{n+2} \alpha^{n+1} n! \left[1 - \frac{n+1}{6} \alpha \left(15 - 12\gamma_M - 5 \sum_{k=1}^{n+2} \frac{1}{k} \right) \right].
 \end{aligned} \tag{4.23}$$

Considering the ‘‘dynamical’’ tensor-to-scalar-ratio

$$r(k) = \frac{\mathcal{P}_T(k)}{\mathcal{P}_\mathcal{R}(k)}, \tag{4.24}$$

our results allow us to compute the first correction to the relation $r = -8n_T$. Using (4.21), (4.22) and (4.23), we find, to the order α^3 included,

$$r + 8n_T = \frac{\mathcal{P}_T}{\mathcal{P}_\mathcal{R}} + 8n_T = -\frac{384m_\chi^2 \alpha (1/k)^3}{m_\phi^2 + 2m_\chi^2}. \tag{4.25}$$

We recall that the theory does not predict other degrees of freedom (when the matter sector is switched off), because the fakeon projection eliminates any additional scalar and tensor perturbations, as well as the vector perturbations.

5 Testable predictions

In this section we work out a number of predictions that have a chance to be tested in the incoming years [31]. We express the results in terms of a “pivot” scale k_* and evolve $\alpha(1/k)$ from k_* to k by means of the RG evolution equations, using the next-to-leading log solution (2.21). So doing, the spectra become functions of $\alpha_* \equiv \alpha(1/k_*)$ and $\ln(k_*/k)$.

We take $k_* = 0.05 \text{ Mpc}^{-1}$ and plot the spectra in the range $10^{-4} \text{ Mpc}^{-1} \leq k \leq 1 \text{ Mpc}^{-1}$. The data reported in [21] give

$$\ln(10^{10}\mathcal{P}_{\mathcal{R}}^*) = 3.044 \pm 0.014, \quad n_{\mathcal{R}}^* = 0.9649 \pm 0.0042,$$

where the star superscript means that the quantity is evaluated at k_* . Using formulas (4.23) we find

$$\alpha_* = 0.0087 \pm 0.0010, \quad m_\phi = (2.99 \pm 0.37) \cdot 10^{13} \text{ GeV}.$$

Taking the mean values just listed, the logarithm of the scalar power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ given in formula (4.23) plots as shown in the images of figure 2, which give an idea of the precision we need to confirm the running experimentally. The blue line is the running spectrum, while the red line denotes the linearized truncation

$$\ln(10^{10}\mathcal{P}_{\mathcal{R}}(k))\Big|_{\text{linearized}} \equiv \ln(10^{10}\mathcal{P}_{\mathcal{R}}^*) + (n_{\mathcal{R}}^* - 1) \ln \frac{k}{k_*}. \quad (5.1)$$

The background of the left picture is an elaboration of the top-right image of Fig. 20 of ref. [21], which was obtained using Planck TT, TE, EE + lowE + lensing. The right picture magnifies the square that appears at the center of the left picture. The theoretical error is too small to be plotted (the contributions we are neglecting are of order $\alpha_*^2 \simeq 0.01\%$). We emphasize the effects of the running in fig. 3, by plotting the difference

$$\ln \mathcal{P}_{\mathcal{R}}(k) - \ln \mathcal{P}_{\mathcal{R}}(k)\Big|_{\text{linearized}}.$$

By construction, this difference vanishes at the pivot scale.

The running of the spectrum $\mathcal{P}_T(k)$ of the tensor fluctuations is shown in fig. 4, by plotting the difference

$$\ln \mathcal{P}_T(k) - \ln \mathcal{P}_T(k)\Big|_{\text{linearized}}, \quad \ln \mathcal{P}_T(k)\Big|_{\text{linearized}} \equiv \ln \mathcal{P}_T^* + n_T^* \ln \frac{k}{k_*}.$$

The mass m_χ of the fakeon $\chi_{\mu\nu}$ is constrained to lie in the range $m_\phi/4 < m_\chi < \infty$ by the consistency of the fakeon prescription/projection [20]. The blue border of fig. 4 is the spectrum for $m_\chi = m_\phi/4$, while the red border is the spectrum for $m_\chi = \infty$.

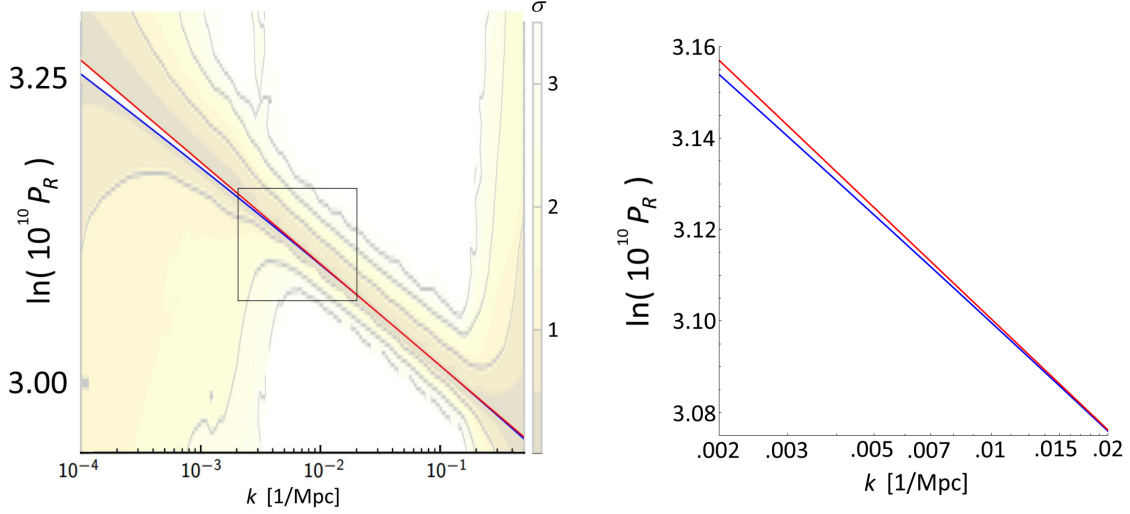


Figure 2: Running spectrum of the scalar fluctuations (in blue) compared to the nonrunning (linearized) truncation (in red). The left picture is superposed to the data reported in [21]. The right picture magnifies the square at the center of the left picture

Plotting the dynamical tensor-to-scalar ratio (4.24) as a function of k , we obtain fig. 5.

At the pivot scale, the correction (4.25) to the relation $r = -8n_T$ is predicted to be

$$1000 (r^* + 8n_T^*) = \begin{cases} -0.0146 \pm 0.0049 & \text{for } m_\chi = m_\phi/4, \\ -0.131 \pm 0.044 & \text{for } m_\chi = \infty. \end{cases}$$

We recall that the measurement of one quantity, such as r^* (or n_T^*), is enough to determine the only remaining free parameter of quantum gravity with fakeons, which is m_χ . After that, it is possible to make precision tests of the other predictions.

6 Conclusions

Quantum gravity with purely virtual particles allows us to anticipate the outcomes of various measurements in inflationary cosmology and maybe pave the way for precision tests. The constraints of quantum field theory overcome the lack of predictivity due to the arbitrariness of classical theories.

In this paper we laid out an important ingredient of the relation between cosmology and high-energy physics by reformulating the history of the early universe as the evolution of

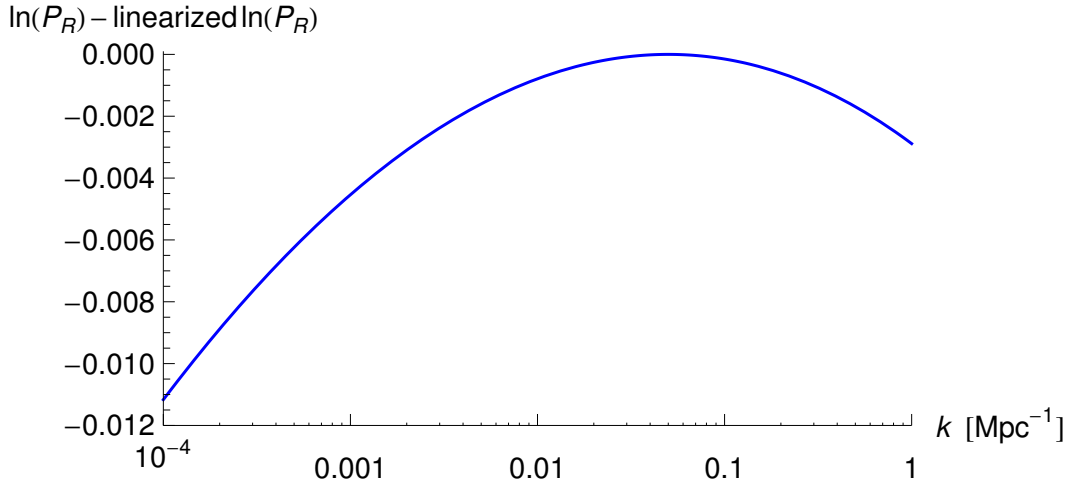


Figure 3: Running of the scalar spectrum

a flow that bears many resemblances with the RG flow we are familiar with from quantum field theory. Instead of being driven by the radiative corrections, it is due to the dependence on the background FLRW metric. Although the two flows are very different in nature, many properties of the slow roll expansion and the inflationary perturbation spectra in the superhorizon limit are similar to the ones encoded into the Callan-Symanzik equation satisfied by the amplitudes of quantum field theory, at least from the mathematical and formal points of view.

The cosmic RG flow starts from a fixed point, which is de Sitter space, around which the beta function behaves like the one of pure QCD. The flow does not end into a fixed point, but continues into the reheating phase, which we have not investigated here. Viewing inflation this way offers a better understanding of it and allows us to simplify various computations, upgrading the techniques that have been used earlier. It might also inspire new, unforeseen developments. To some extent, it might be possible to describe parts of the later evolution of the universe as RG flows with modified beta functions, to take into account the contributions of matter.

The RG flow of inflation is of a “pure” type, in the sense that it is solely governed by the beta function, since the power spectra have no anomalous dimensions. The core information of a spectrum is the constant $Q(0)$ of (3.63), which is a power series in the coupling α_0 with no a priori relation to the beta function. Because of this, $Q(0)$ is specific of the (scalar, tensor, etc.) spectrum we consider. The RG information is the part controlled

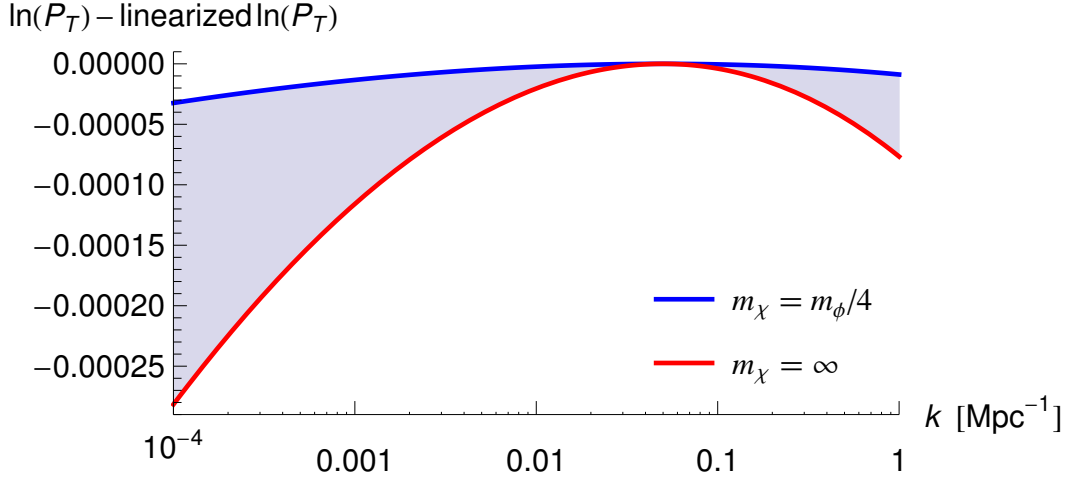


Figure 4: Running of the tensor spectrum

by the RG equation, so it is universal (i.e., the same for every spectrum) and encoded in the beta function. Recalling that in quantum field theory the core information of a correlation function is an extremely complicated nonlocal expression that can be calculated by means of Feynman diagrams, we can appreciate that the cosmic RG flow has the simplest, yet nontrivial structure we can hope for.

The running of the power spectra can be studied efficiently by organizing the perturbative expansion in terms of leading and subleading logs and resumming the powers of $\alpha_0 \ln \eta$ altogether. We have worked out the scalar and tensor spectra and their runnings in quantum gravity to the next-to-leading log order, together with the first correction to the relation $r + 8n_T = 0$.

Note that the coupling α of inflation has a value ($\sim 1/115$ at the pivot scale) that is close to the value of the QED fine structure constant. This makes the theoretical errors negligible, once we include a few orders of the expansion in powers of α , as we have done here. Depending on the experimental resolution that will be achieved in the future, cosmology might turn into an arena for precision tests of quantum gravity.

The procedures described in this paper can be pushed to higher orders. This requires to implement the fakeon projection beyond what we have done so far, by following the iteration procedure of ref. [32]. As shown there, the result is expected to be an asymptotic series. Given that the fine structure constant α_* of inflation is as small as $1/115$, this does not jeopardize the predictions for quite a while. Moreover, at some point high powers of α are outcompeted by the loop corrections we have been neglecting so far, which requires

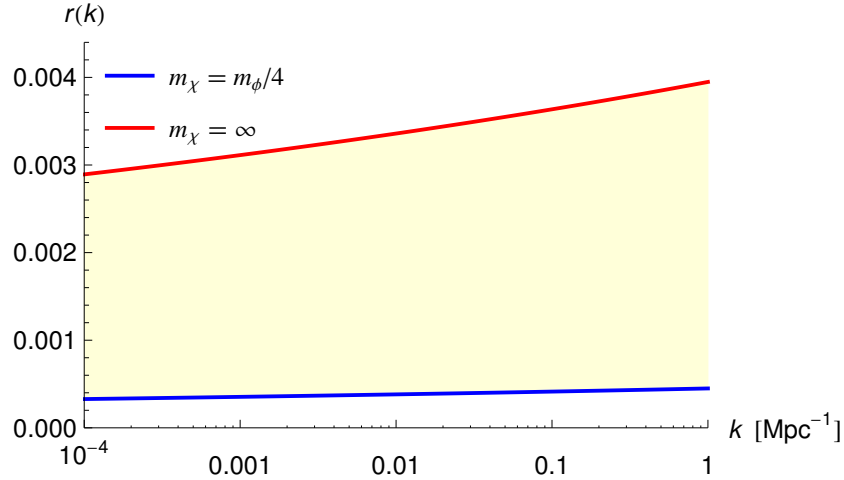


Figure 5: Running of the tensor-to-scalar ratio $r(k)$

to rethink the whole strategy anyway.

Acknowledgments

We are grateful to E. Bianchi, D. Comelli, F. Fruzza, G. Signorelli and M. Piva for helpful discussions.

References

- [1] R. Brout, F. Englert and E. Gunzig, The creation of the universe as a quantum phenomenon, *Annals Phys.* 115 (1978) 78.
- [2] A.A. Starobinsky, A new type of isotropic cosmological models without singularity, *Phys. Lett. B* 91 (1980) 99.
- [3] D. Kazanas, Dynamics of the universe and spontaneous symmetry breaking, *Astrophys. J.* 241 (1980) L59.
- [4] K. Sato, First-order phase transition of a vacuum and the expansion of the universe, *Monthly Notices of the Royal Astr. Soc.* 195 (1981) 467.
- [5] A.H. Guth, Inflationary universe: A possible solution to the horizon and flatness problems, *Phys. Rev. D* 23 (1981) 347.

- [6] A.D. Linde, A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems, *Phys. Lett.* B108 (1982) 389.
- [7] A. Albrecht and P.J. Steinhardt, Cosmology for grand unified theories with radiatively induced symmetry breaking, *Phys. Rev. Lett.* 48 (1982) 1220.
- [8] A.D. Linde, Chaotic inflation, *Phys. Lett.* B129 (1983) 177.
- [9] V.F. Mukhanov, G.V. Chibisov, Quantum fluctuations and a nonsingular universe, *JETP Lett.* 33 (1981) 532, *Pisma Zh. Eksp. Teor. Fiz.* 33 (1981) 549;
V.F. Mukhanov and G. Chibisov, The Vacuum energy and large scale structure of the universe, *Sov. Phys. JETP* 56 (1982) 258;
S. Hawking, The development of irregularities in a single bubble inflationary universe, *Phys. Lett.* B115 (1982) 295;
A.H. Guth and S. Pi, Fluctuations in the new inflationary universe, *Phys. Rev. Lett.* 49 (1982) 1110;
A.A. Starobinsky, Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations, *Phys. Lett.* B117 (1982) 175;
J.M. Bardeen, P.J. Steinhardt and M.S. Turner, Spontaneous creation of almost scale-free density perturbations in an inflationary universe, *Phys. Rev.* D28 (1983) 679;
V.F. Mukhanov, Gravitational instability of the universe filled with a scalar field, *JETP Lett.* 41 (1985) 493.
- [10] D. Anselmi, On the quantum field theory of the gravitational interactions, *J. High Energy Phys.* 06 (2017) 086, 17A3 Renormalization.com and arXiv:1704.07728 [hep-th].
- [11] D. Anselmi, Fakeons and Lee-Wick models, *J. High Energy Phys.* 02 (2018) 141, 18A1 Renormalization.com and arXiv:1801.00915 [hep-th].
- [12] D. Anselmi and M. Piva, A new formulation of Lee-Wick quantum field theory, *J. High Energy Phys.* 06 (2017) 066, 17A1 Renormalization.com and arXiv:1703.04584 [hep-th].

- [13] D. Anselmi and M. Piva, The ultraviolet behavior of quantum gravity, *J. High Energy Phys.* 05 (2018) 27, 18A2 Renormalization.com and arXiv:1803.07777 [hep-th].
- [14] D. Anselmi and M. Piva, Quantum gravity, fakeons and microcausality, *J. High Energy Phys.* 11 (2018) 21, 18A3 Renormalization.com and arXiv:1806.03605 [hep-th].
- [15] The one-loop beta functions were computed in
 I. G. Avramidi and A. O. Barvinsky, Asymptotic freedom in higher derivative quantum gravity, *Phys. Lett. B* 159 (1985) 269;
 for recent generalizations, see N. Ohta, R. Percacci and A.D. Pereira, Gauges and functional measures in quantum gravity II: Higher-derivative gravity, *Eur. Phys. J. C* 77 (2017) 611 and arXiv:1610.07991 [hep-th];
 for the beta functions in the presence of matter, see A. Salvio and A. Strumia, Agravity up to infinite energy, *Eur. Phys. C* 78 (2018) 124 and arXiv:1705.03896 [hep-th].
- [16] D. Anselmi and M. Piva, Perturbative unitarity in Lee-Wick quantum field theory, *Phys. Rev. D* 96 (2017) 045009 and 17A2 Renormalization.com and arXiv:1703.05563 [hep-th].
- [17] S. Weinberg, Quantum contributions to cosmological correlations, *Phys. Rev. D* 72 (2005) 043514 and arXiv:hep-th/0506236.
- [18] D. Anselmi, Fakeons, microcausality and the classical limit of quantum gravity, *Class. and Quantum Grav.* 36 (2019) 065010, 18A4 Renormalization.com and arXiv:1809.05037 [hep-th].
- [19] D. Anselmi, The quest for purely virtual quanta: fakeons versus Feynman-Wheeler particles, *J. High Energy Phys.* 03 (2020) 142, 20A1 Renormalization.com and arXiv:2001.01942 [hep-th].
- [20] D. Anselmi, E. Bianchi and M. Piva, Predictions of quantum gravity in inflationary cosmology: effects of the Weyl-squared term, *J. High Energy Phys.* 07 (2020) 211, 20A2 Renormalization.com and arXiv:2005.10293 [hep-th].
- [21] Planck collaboration, Planck 2018 results. X. Constraints on inflation, arXiv:1807.06211 [astro-ph.CO].

- [22] D. Baumann, TASI lectures on inflation, arXiv:0907.5424 [hep-th].
- [23] P. Binétruy, E. Kiritsis, J. Mabillard, M. Pieroni and C. Rosset, Universality classes for models of inflation, JCAP 04 (2015) 033 and arXiv:1407.0820 [astro-ph.CO];
see also P. Binétruy, J. Mabillard and M. Pieroni, Universality in generalized models of inflation, JCAP 03 (2017) 060 and arXiv:1611.07019 [gr-qc].
- [24] R. Myrzakulov, S. D. Odintsov and L. Sebastiani, Inflationary universe from higher-derivative quantum gravity, Phys. Rev. D 91 083529 (2015) 8 and arXiv:1412.1073 [gr-qc].
- [25] I.G. Mórián, N. Defenu, U.D. Jentschura, A. Trombettoni and I. Nándori, Renormalization-group running induced cosmic inflation, arXiv:1909.00580 [astro-ph.CO].
- [26] A. Kosowsky and M.S. Turner, CBR anisotropy and the running of the scalar spectral index, Phys. Rev. D 52 (1995) 1739 and arXiv:astro-ph/9504071;
D.J.H. Chung, G. Shiu and M. Trodden, Running of the scalar spectral index from inflationary models, Phys. Rev. D 68 (2003) 063501 and arXiv:astro-ph/0305193;
J.E. Lidsey and R. Tavakol, Running of the scalar spectral index and observational signatures of inflation, Phys. Lett. B 575 (2003) 157 and arXiv:astro-ph/0304113.
- [27] T.T. Nakamura and E.D. Stewart, The spectrum of cosmological perturbations produced by a multi-component inflaton to second order in the slow-roll approximation, Phys. Lett. B 381 (1996) 413 and arXiv:astro-ph/9604103;
E.D. Stewart, and J.-O. Gong, The density perturbation power spectrum to second-order corrections in the slow-roll expansion, Phys. Lett. B 510 (2001) 1 and arXiv:astro-ph/0101225;
W.H. Kinney, Inflation: flow, fixed points and observables to arbitrary order in slow roll, Phys. Rev. D 66 (2002) 083508 and arXiv:astro-ph/0206032;
T. Zhu, A. Wang, G. Cleaver, K. Kirsten and Q. Sheng, Gravitational quantum effects on power spectra and spectral indices with higher-order corrections, Phys. Rev. D 90 (2014) 063503 and arXiv:1405.5301 [astro-ph.CO];
M. Zarei, On the running of the spectral index to all orders: a new model dependent approach to constrain inflationary models, Class. Quantum Grav. 33 (2016) 115008 and arXiv:1408.6467 [astro-ph.CO];

- H. Motohashi and W. Hu, Generalized slow roll in the unified effective field theory of inflation, Phys. Rev. D 96 (2017) 023502 and arXiv:1704.01128 [hep-th];
- P. Chen, J. Matsumoto and R. Saitou, Inflationary spectral tilts as a result of the dilatation symmetry breaking, Phys. Rev. D 100 (2019) 023534 and arXiv:1812.08981 [hep-th];
- G.-H. Ding, J. Qiao, Q. Wu, T. Zhu and A. Wang, Inflationary perturbation spectra at next-to-leading slow-roll order in effective field theory of inflation, Eur. Phys. J. C 79 (2019) 976 and arXiv:1907.13108 [gr-qc].
- [28] A. Vilenkin, Classical and quantum cosmology of the Starobinsky inflationary model, Phys. Rev. D 32 (1985) 2511.
- [29] K.S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D 16 (1977) 953.
- [30] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, Phys. Rept. 215 (1992) 203; S. Weinberg, *Cosmology*, Oxford University Press, 2008; A. De Felice and S. Tsujikawa, $f(R)$ theories, Living Rev. Rel. 13 (2010) 3 and arXiv:1002.4928 [gr-qc].
- [31] K.N. Abazajian *et al.*, CMB-S4 Science Book, First Edition, arXiv:1610.02743 [astro-ph.CO].
- [32] D. Anselmi, Fakeons and the classicization of quantum gravity: the FLRW metric, J. High Energy Phys. 04 (2019) 61, 19A1 Renormalization.com and arXiv:1901.09273 [gr-qc].