

# Inconsistency Of Minkowski Higher-Derivative Theories

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## Abstract

We show that Minkowski higher-derivative quantum field theories are generically inconsistent, because they generate nonlocal, non-Hermitian ultraviolet divergences, which cannot be removed by means of standard renormalization procedures. By “Minkowski theories” we mean theories that are defined directly in Minkowski spacetime. The problems occur when the propagators have complex poles, so that the correlation functions cannot be obtained as the analytic continuations of their Euclidean versions. The usual power counting rules fail and are replaced by much weaker ones. Self-energies generate complex divergences proportional to inverse powers of D’Alembertians. Three-point functions give more involved nonlocal divergences, which couple to infrared effects. We illustrate the violations of the locality and Hermiticity of counterterms in scalar models and higher-derivative gravity.

# 1 Introduction

The ultraviolet structure of quantum field theories is notoriously a fundamental problem in high-energy physics. Nowadays, with the Large Hadron Collider (LHC) currently running at a center-of-mass energy of 13TeV, the standard model is experimentally verified at the TeV scale. On the theoretical side, the renormalizability of the standard model, together with the relatively small value found for the Higgs mass,  $m_H \simeq 125\text{GeV}$ , imply that the model could be valid at energies much higher than the ones investigated so far. The interest in a high-energy modification of the standard model is, therefore, rather limited, on the practical side. As far as quantum gravity is concerned, the situation is different. The negative mass dimension of the coupling (compared with the dimensionless gauge couplings of the standard model) makes the Hilbert-Einstein action nonrenormalizable [1]. This fact, together with the difficulties to build up a phenomenology, render the investigation of alternative high-energy structures of quantum gravity more attractive.

An interesting class of higher-derivative quantum field theories are those whose propagators have complex poles. In that case, the Euclidean and Minkowski versions of the theories are not related to each other by the analytic continuation. In this paper, we concentrate on the Minkowski formulation of such theories, that is to say we integrate the loop energies along the real axis. We show that such theories are generically inconsistent, because they violate both the locality and Hermiticity of counterterms. For example, the one-loop bubble diagram  $\Sigma(p)$  of massless higher-derivative scalar fields in six spacetime dimensions has an ultraviolet divergence of the form:

$$\Sigma(p) = -\frac{M^4}{2(4\pi)^3} \left[ \frac{M^2}{(p^2)^2} - \frac{i}{p^2} \right] \ln \left( \frac{\Lambda_{UV}^2}{M^2} \right) + \dots, \quad (1.1)$$

where  $\Lambda_{UV}$  is a hard ultraviolet cutoff on the space momenta,  $M$  is the scale associated with the higher-derivative terms and the dots denote convergent terms. Similar results occur in four dimensions, when vertices carry derivatives. More involved nonlocal structures appear in triangle diagrams. Moreover, gauge symmetries are unable to protect the locality and Hermiticity of counterterms. We prove this fact by extending the calculations to a model of higher-derivative quantum gravity [2].

The rules of power counting obeyed by Minkowski higher-derivative theories are much weaker than the standard ones, because a propagator calculated on the pole of another propagator falls off half less rapidly than expected. This property implies that the higher-derivative terms often have an ‘‘antiregulating’’ effect, in the sense that they enhance divergences rather

than suppressing them. The divergences (1.1) can also be related to specific pinch singularities occurring for  $p^2 \rightarrow 0$ , which have no direct analog in standard field theories.

It is well-known that, when the free propagators just have poles infinitesimally close to the real axis, in the second and forth quadrants, Minkowski higher-derivative theories are physically unacceptable, because they violate perturbative unitarity. Our results show that when the free propagators also contain poles that are located at finite distances from the real axis, in the first and third quadrants, the theories are in general unacceptable from the mathematical point of view, because they violate both the locality and Hermiticity of counterterms.

The problems we have found do not occur in Euclidean theories, Lee-Wick models [3] or Minkowski theories that are analytically equivalent to their Euclidean versions. In those cases, nonlocal divergences may appear in some intermediate steps of the calculations (examples being the residues of the integrals on the energies, if taken separately), but they cancel out at the end.

The paper is organized as follows. In sec. 2 we study some key aspects of the higher-derivative Minkowski propagator. In sec. 3, we calculate the nonlocal divergent part of the bubble diagram in six dimensions and generalize the calculation to the bubble diagram with nontrivial numerators in four dimensions. In sec. 4, we study the one-loop triangle diagram and provide an interpretation for its nonlocal divergent part. In sec. 5 we investigate the modified power counting of Minkowski quantum field theories. In sec. 6, we study a higher-derivative version of quantum gravity in four dimensions and prove that gauge symmetries fail to protect the locality and Hermiticity of counterterms. Finally, in sec. 7 we draw our conclusions.

In appendix A we show that the dimensional regularization (of the integrals on the space momenta) allows us to apply the residue theorem on the energy integrals, even when they are divergent. In appendix B we discuss the gauge fixing of Minkowski higher-derivative gravity and show that the Ward-Takahashi-Slavnov-Taylor (WTST) identities [4] are also plagued with nonlocal divergences.

## 2 Higher-derivative propagator

The standard propagator of a spinless particle reads

$$\Delta(p, m) = \frac{1}{p^2 - m^2 + i\epsilon}. \quad (2.1)$$

To increase the convergence of the loop integrals for large virtualities,  $|p^2| \gg m^2$ , it is natural to introduce additional powers of  $p^2$  in the denominator to obtain, for example, a modified propagator of the form

$$S(p, m) = \frac{1}{p^2 - m^2 + i\epsilon} \frac{M^4}{(p^2)^2 + M^4}. \quad (2.2)$$

For small virtualities,  $|p^2| \ll M^2$ , the modified propagator approaches the standard one,  $S(p, m) \simeq \Delta(p, m)$ , while in the asymptotic region

$$|p^2| \gg M^2, \quad (2.3)$$

it decays quite faster:

$$S(p, m) \simeq \frac{M^4}{(p^2)^3}.$$

By means of a partial fractioning in  $p^2$ , the higher-derivative propagator (2.2) can also be written as

$$\begin{aligned} S(p, m) = & \frac{M^4}{M^4 + m^4} \frac{1}{2\omega_\epsilon(p_s)} \left[ \frac{1}{p_0 - \omega_\epsilon(p_s)} - \frac{1}{p_0 + \omega_\epsilon(p_s)} \right] \\ & - \frac{M^2}{M^2 - im^2} \frac{1}{4\Omega(p_s)} \left[ \frac{1}{p_0 - \Omega(p_s)} - \frac{1}{p_0 + \Omega(p_s)} \right] + \\ & - \frac{M^2}{M^2 + im^2} \frac{1}{4\bar{\Omega}(p_s)} \left[ \frac{1}{p_0 - \bar{\Omega}(p_s)} - \frac{1}{p_0 + \bar{\Omega}(p_s)} \right], \end{aligned} \quad (2.4)$$

where  $\omega_\epsilon(p_s) \equiv \sqrt{p_s^2 + m^2 - i\epsilon}$ ,  $\Omega(p_s) \equiv \sqrt{p_s^2 - iM^2}$ ,  $p^\mu = (p_0, \mathbf{p})$  and  $p_s = |\mathbf{p}|$ . The poles are located at  $p_0 = \pm\omega_\epsilon(p_s)$ ,  $p_0 = \pm\Omega(p_s)$  and  $p_0 = \pm\bar{\Omega}(p_s)$ , where the bar denotes the complex conjugation. Since poles are present in every quadrant, the Euclidean theory and the Minkowski theory are not related in a simple way. In this paper we concentrate on the Minkowski theory, that is to say we assume that the loop integral is defined by integrating the energy along the real axis.

In general, the factor

$$\frac{M^4}{(p^2)^2 + M^4} \leq 1 \quad (2.5)$$

is expected to have an ultraviolet regulating effect, by suppressing the states with  $|p^2| \gg M^2$ . We show that it is not the case in Minkowski theories. Actually, often the factor (2.5) roughly has an opposite, ‘‘anti-regulating’’ effect.

For large space momenta,  $p_s \gg M$ , the complex poles come close to the real axis, since

$$\Omega(p_s) \equiv \sqrt{p_s^2 - iM^2} \simeq p_s - i\eta, \quad \bar{\Omega}(p_s) \simeq p_s + i\eta,$$

where  $\eta$  is the small positive quantity

$$\eta = \eta(p_s) \equiv \frac{M^2}{2p_s}.$$

Therefore, in the asymptotic region  $p_s \gg M$  the terms involving  $M$  effectively act as  $\pm i\eta$  prescriptions for the propagation of exotic, high-energy excitations on the light cone, with the large lifetimes

$$\tau = \tau(p_s) \approx \frac{p_s}{M^2}.$$

Both  $\epsilon$  and  $\eta(p_s)$  are positive quantities, but  $\epsilon$  is infinitesimal, while  $\eta(p_s)$  is small and finite.

Since  $\eta(p_s) \rightarrow 0$  for  $p_s \rightarrow +\infty$ , there is a pinch singularity of the pole located in the first quadrant with the two poles located in the fourth quadrant, and a similar pinch singularity of the pole located in the third quadrant with the two poles located in the second quadrant. The violations of power counting that we find in the next sections can be traced back to this pinching and ultimately to the presence of both the  $+i\eta$  and  $-i\eta$  terms at  $p_s \gg M$ .

### 3 Bubble diagrams

In this section we compute the nonlocal divergent parts of higher-derivative, one-loop scalar bubble diagrams in six and four dimensions, with trivial and nontrivial numerators.

As usual, the ultraviolet divergent part is a sum of powerlike divergences and logarithmic divergences. The powerlike divergences are less interesting than the logarithmic ones for the purpose of singling out inconsistencies, because they depend on the subtraction scheme and can be removed in a renormalization-group invariant way. The one-loop logarithmic divergences, on the contrary, do not depend on the regularization scheme, and provide meaningful tests of the locality of counterterms. For these reasons, we focus our attention mostly on them. We either use the dimensional regularization technique or a sharp cutoff  $\Lambda_{UV}$  on the space momenta of the loops, according to convenience. We always convert the outcome to the cutoff notation.

#### 3.1 Bubble diagram in six dimensions

The bubble diagram with different masses in  $D$  spacetime dimensions gives the loop integral

$$\Sigma(p) = \int_{k_s \leq \Lambda_{UV}} \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} S(k, m_1) S(k-p, m_2), \quad (3.1)$$

where  $S(p, m)$  is given in formula (2.2),  $\Lambda_{UV}$  is an ultraviolet cutoff on the space momenta  $\mathbf{k}$  and  $k_s = |\mathbf{k}|$ .

Since the higher-derivative theory is relativistically invariant,  $\Sigma(p)$  is expected to be a function of  $p^2$  only<sup>1</sup>. It is then convenient to consider a timelike external momentum,  $p^2 > 0$ , and select a Lorentz frame in which  $p^\mu$  has only the time component,  $p^\mu = (p_0, \mathbf{0})$ . Once the loop integral is evaluated, we can retrieve the Lorentz invariant result by means of the replacement  $p_0^2 \rightarrow p^2$ . The values of the bubble for a spacelike external momentum,  $p^2 < 0$ , are obtained by means of the analytic continuation in  $p^2$ .

The first step is to integrate  $k_0$  over the real line, which we can make in two equivalent ways. The first method is by applying the residue theorem, after closing the integration path with a semicircle at infinity, say in the upper half  $k_0$  plane. The second method involves the partial fractioning in  $k_0$ . Since there are only simple poles for  $p_0 \neq 0$ , one ends up with a  $k_0$  integrand of the form:

$$\sum_{i=1}^{12} \frac{c_i}{k_0 - s_i},$$

where  $s_i$  are the poles of the propagators and  $c_i$  are coefficients that depend on the nonintegrated variables  $k_s, p_0, m_1, m_2, M$ . Since each propagator contains six poles, there are twelve terms in total. Then, one integrates over  $k_0$  term by term. In practice,  $1/(k_0 - s)$  gives  $\pm i\pi$ , depending on whether  $\text{Im } s > 0$  or  $\text{Im } s < 0$ .

The next step is to integrate over the space momentum  $\mathbf{k}$ . The angular integration is trivial, because of our choice of the Lorentz frame, and gives the volume  $\Omega_{D-2}$  of the unit sphere in  $D - 2$  dimensions.

Since we are only interested in the ultraviolet divergences for  $k_s \rightarrow \infty$ , we expand the integrand for large  $k_s$ , in order to avoid special functions due to the  $k_s$  integral. Consider the residues calculated at the first step. In each of them, either  $k^2$  or  $(k - p)^2$  is equal to a constant. The two cases are symmetrical, so we just assume  $k^2 = \text{constant}$ . Then, the propagator  $S(k, m_1)$  gives a contribution  $\sim 1/k_s$  for large  $k_s$ , by formula (2.4). Instead, the propagator  $S(k - p, m_2)$  behaves as  $1/((k - p)^2)^3 \sim 1/(p \cdot k)^3 \sim 1/k_s^3$  (having used  $p \cdot k = p_0 k_0$ ). The product of the two behaves as  $1/k_s^4$ , so the  $k_s$  integral diverges like

$$\int^{\Lambda_{UV}} \frac{k_s^{D-2} dk_s}{k_s^4}.$$

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<sup>1</sup>To be rigorous, one should use an ultraviolet regularization that preserves Lorentz invariance, such as the dimensional regularization, instead of a cutoff on the space momenta. However, we are only interested in the logarithmic divergences, which, as already noted, are independent of this choice.

However, it is easy to check that the contributions of such type coming from the poles with  $k^2 = \text{constant}$  compensate analogous contributions coming from the poles with  $(k - p)^2 = \text{constant}$ . In the end, the integrand of (3.1) behaves as  $1/k_s^5$ , so we find that the leading  $\Sigma$  divergence is proportional to

$$\int^{\Lambda_{UV}} \frac{dk_s}{k_s^{7-D}}.$$

We conclude that for  $D < 6$  the bubble diagram is ultraviolet finite, while it is divergent for  $D \geq 6$ . In particular, at  $D = 6$  there is a logarithmic divergence,

$$\int_M^{\Lambda_{UV}} \frac{dk_s}{k_s} = \frac{1}{2} \ln \left( \frac{\Lambda_{UV}^2}{M^2} \right),$$

which leads to the final result

$$\begin{aligned} \Sigma(p) = & \frac{1}{12(4\pi)^3} \frac{M^6}{(M^2 + im_1^2)(M^2 + im_2^2)} \left\{ \frac{1}{(p^2)^2} [m_1^4 + m_2^4 - 4iM^2(m_1^2 + m_2^2) - 6M^4] + \right. \\ & \left. + \frac{3i}{p^2} [2M^2 + i(m_1^2 + m_2^2)] \right\} \log \left( \frac{\Lambda_{UV}^2}{M^2} \right) + (\text{finite}), \end{aligned} \quad (3.2)$$

where by “finite” we mean terms that are finite or infinitesimal for  $\Lambda_{UV} \rightarrow +\infty$ .

The divergent part is nonlocal, equal to the sum of a term proportional to  $1/(p^2)^2$  plus a term proportional to  $1/p^2$ . Differently from the usual divergences of local theories, which are anti-Hermitian, the ones of (3.2) are not, since the coefficients have nontrivial real and imaginary parts. For these reasons, we cannot absorb the divergent part in the usual way, by shifting the bare masses, rescaling the bare fields and adding new local, Hermitian terms to the Lagrangian. We cannot even add nonlocal Hermitian terms. We conclude that the locality and Hermiticity of counterterms are both violated.

Since we are exploring an uncharted territory, we wish to make an explicit check of Lorentz invariance and analyticity. We consider the usual bubble in the case  $p^2 < 0$ , by taking  $p^\mu = (0, \mathbf{p}_s)$ . As in the previous computation, we integrate over  $k_0$  by means of the residue theorem or the partial fractioning in  $k_0$ . Then we have to integrate over the angles, which is a nontrivial operation now. Writing  $k^\mu = (k_0, \mathbf{k}_s)$ , we switch to spherical coordinates, letting  $\theta$  denote the angle between  $\mathbf{p}_s$  and  $\mathbf{k}_s$ . The integral over the remaining three angles is trivial, which leads to the replacement

$$d^5k_s \rightarrow 2\pi^2 dk_s k_s^4 (1 - u^2) du,$$

where  $u \equiv \cos \theta$ . At this point, we should expand the integrand for large  $k_s$ . This cannot be done naively without generating  $u$  poles. For example, consider a typical denominator that is met in the calculation, such as

$$\frac{1}{2uk_s p_s - p_s^2 - 2iM^2}. \quad (3.3)$$

If we expand it for large  $k_s$ , we obtain a divergent  $u$  integral. However, according to (3.3), the  $u$  pole has a positive imaginary part. If we first replace  $u$  by  $u - i\epsilon$ , with  $\epsilon$  arbitrarily small, then the expansion for large  $k_s$  is safe. So doing, no fictitious singularity is generated.

The procedure works as long as the integrand can be arranged so that the powers  $1/(u - i\epsilon)^n$  do not mix with the powers  $1/(u + i\epsilon)^n$ . It can be shown that the bubble diagram has this property. Carrying on the computation to the end, we find (3.2) again.

### 3.2 Bubble diagram in four dimensions

As shown in the previous section, the bubble diagram with unit numerator has nonlocal divergences only in dimensions  $D \geq 6$ . On the other hand, bubble diagrams with nontrivial numerators may have nonlocal divergences also in four dimensions. In this section we study typical one-loop integrals of this type. Their applications to higher-derivative gravity will be considered in subsection 6.

We assume that the propagator has again the form (2.2) and the vertices contain an arbitrary number of derivatives. We study the scalar integrals

$$I_{r,n}(p) \equiv \int \frac{d^D k}{(2\pi)^D} (k \cdot p)^r (k^2)^n S(k, m) S(p - k, m). \quad (3.4)$$

The nonlocal divergent part can be calculated with the method explained in the previous subsection. We set  $p^2 > 0$  and choose  $p^\mu = (p_0, \mathbf{0})$ . First, we integrate on the energy by means of the residue theorem, closing the integration path on the upper half complex plane. In appendix A we show that, if we use the dimensional regularization, the energy integral can always be evaluated by summing the residues, even when it is divergent, because the contribution of the integration path at infinity is always zero.

Then, we remain with the integral on the space momentum  $\mathbf{k}$ . The logarithmic nonlocal divergences  $I_{r,n}^{\text{nl d}}$  of  $I_{r,n}$  are obtained by expanding the integrand in powers of the absolute value  $k_s = |\mathbf{k}|$  and isolating the contributions proportional to  $dk_s/k_s$ .

We report results in various cases, starting from the massless limit. There, we find

$$I_{r,n}^{\text{nl d}} = c_{r,n} \left( \frac{-iM^2}{2} \right)^{r+n} \frac{M^2}{(4\pi)^2 p^2} \log \left( \frac{\Lambda_{UV}^2}{M^2} \right), \quad (3.5)$$



where  $c_{r,n}$  are positive integer numbers. As anticipated, the pole  $1/(4-D)$  of the dimensional regularization has been converted into the logarithm  $\log(\Lambda_{UV}/M)$  of a generic ultraviolet cutoff  $\Lambda_{UV}$  divided by  $M$ . The lowest-order coefficients are

$$\begin{aligned} c_{0,0} = 0, \quad c_{0,1} = 1, \quad c_{1,0} = 0, \quad c_{1,1} = 3, \quad c_{1,2} = 2, \quad c_{2,1} = 1, \quad c_{2,2} = 14, \\ c_{3,1} = 15, \quad c_{1,3} = 12, \quad c_{2,3} = 4, \quad c_{3,2} = 2, \quad c_{3,3} = 60. \end{aligned} \quad (3.6)$$

These results may be justified by evaluating the residues associated with the propagators of  $I_{r,n}$ , as explained in subsection 3.1. The residues with  $k^2 = \text{constant}$  have a superficial degree of divergence  $\omega_1 = D-5+r$ , where  $D-1$  powers come from the  $k_s$  integration measure,  $-1$  and  $-3$  from the propagators and  $r$  from the numerator. Instead, the residues with  $(k-p)^2 = \text{constant}$  have degree of divergence  $\omega_2 = D-5+r+n$ , where the  $n$  additional powers come from the numerator, using  $k^2 \sim 2k \cdot p$ . For  $r=1, n=0$ , we have  $\omega_1 = \omega_2 = D-4$ , so ultraviolet logarithmic divergences are expected from both types of residues in  $D=4$ . However, formula (3.6) shows that the coefficient  $c_{1,0}$  vanishes. This may be interpreted as an eikonal cancellation between the two types of residues. For  $r=0, n=1$ , we have  $\omega_2 > \omega_1 = D-4$ , so the cancellation cannot occur in  $D=4$ . Indeed,  $c_{0,1}$  is different from zero. More generally, cancellations between the ultraviolet divergences of the residues are unlikely to occur for  $n > 0$ . Indeed, all the coefficients (3.6) with  $n > 0$  are nonvanishing.

In the massive case, we find formula (3.5) with coefficients  $c_{r,n}$  that depend on  $m/M$ . The lowest-order ones are

$$c_{0,1} = \frac{M^2}{M^2 + im^2}, \quad c_{1,1} = \frac{3M^2 + im^2}{M^2 + im^2}, \quad c_{1,2} = 2 \left( 1 + \frac{im^2}{M^2} \right).$$

Instead, if we replace the propagators  $S(p, m)$  with the more general ones

$$S(p, m, \mu) = \frac{1}{p^2 - m^2 + i\epsilon} \frac{M^4}{(p^2 - \mu^2)^2 + M^4},$$

we obtain formula (3.5) with

$$\begin{aligned} c_{0,1} &= \frac{M^2}{M^2 + i\Delta m^2}, & c_{1,1} &= \frac{3M^2 + i\Delta m^2}{M^2 + i\Delta m^2}, \\ c_{1,2} &= 2 \left( 1 + \frac{i\Delta m^2}{M^2} + \frac{6i\mu^2}{M^2 + i\Delta m^2} - \frac{2\mu^2 \Delta m^2}{M^2(M^2 + i\Delta m^2)} \right), \end{aligned}$$

where  $\Delta m^2 = m^2 - \mu^2$ .

We see that the locality and Hermiticity of counterterms are violated again. The nonlocal behavior is always of the form  $1/p^2$ , but it must be recalled that the integrals (3.4) contain  $r$  powers of  $p^\mu$  in the numerator, through the term  $(k \cdot p)^r$ . If we divide by those powers, the true nonlocal behavior of the divergent part is  $\sim 1/(p^2)^{(2+r)/2}$ .

## 4 Triangle diagrams

In this section we consider the one-loop three-point function. A peculiar, double-logarithmic structure

$$\ln\left(\frac{Q^2}{\mu^2}\right) \ln\left(\frac{\Lambda_{UV}^2}{M^2}\right)$$

is found, coming from the overlap between the collinear (infrared) region  $\mu^2 \ll k_\perp^2 \ll Q^2$  and the ultraviolet region  $M^2 \ll k_s^2 \ll \Lambda_{UV}^2$ , where  $k_\perp$  is the transverse loop momentum,  $Q$  is the hard scale and  $\mu$  is the virtuality of external legs.

As in the case of the bubble diagram, in order to effectively generate such divergences, we need to introduce nontrivial numerators, which we assume to be scalar for simplicity. Specifically, we consider the following three-index family of amplitudes:

$$I_{r,n,t}(p_1, p_2) = \int \frac{d^D k}{(2\pi)^D} (k \cdot p_1)^r (k^2)^n (k \cdot p_2)^t S(k, m) S(k + p_1, m) S(k + p_2, m). \quad (4.1)$$

The nonlocal divergent part  $I_{r,n,t}^{\text{nl d}}$  can be calculated with a procedure similar to the one of the bubble diagram. First, we use Lorentz invariance and analyticity to impose that the incoming momenta are all timelike, i.e.

$$p_1^2 > 0, \quad p_2^2 > 0, \quad (p_1 - p_2)^2 > 0, \quad (4.2)$$

and choose  $p_1^\mu = (E_1, \mathbf{0})$ , while  $p_2^\mu = (E_2, \mathbf{p}_2)$  remains generic. Then, we integrate on the energy  $k_0$  by means of the residue theorem or a partial fractioning. At that point, we expand the integrand in powers of  $1/k_s$  for  $k_s$  large and integrate term by term over  $u = \cos\theta$ ,  $\theta$  being the angle between the vectors  $\mathbf{p}_2$  and  $\mathbf{k}$ . When the conditions (4.2) hold, this operation is safe. Indeed, the expansion highlights factors

$$\begin{aligned} \frac{1}{(p_1 \cdot k)^3} &= \frac{1}{(E_1 k_0)^3} \sim \frac{1}{(E_1 k_s)^3}, \\ \frac{1}{(p_2 \cdot k)^3} &= \frac{1}{(E_2 k_0 - \mathbf{p}_2 \cdot \mathbf{k})^3} \sim \frac{1}{k_s^3 (\pm E_2 - p_{2s} u)^3}, \\ \frac{1}{[(p_1 - p_2) \cdot k]^3} &\sim \frac{1}{k_s^3 (\pm E_1 \mp E_2 + p_{2s} u)^3}, \end{aligned} \quad (4.3)$$

where  $p_{2s} = |\mathbf{p}_2|$ . The subleading corrections have denominators that are equal to powers of those shown in formula (4.3). We see that every term of the expansion leads to a regular  $u$  integral. At the end, the logarithmic divergences are the coefficients of  $dk_s/k_s$ .

The symmetry relation  $I_{r,n,t}^{\text{nld}}(p_1, p_2) = I_{t,n,r}^{\text{nld}}(p_2, p_1)$  obviously holds, so we can assume, for example,  $r \geq t$ . By explicit calculation of  $I_{r,n,t}$  for different values of its indices, we find that

$$I_{r,n,t}^{\text{nld}}(p_1, p_2) = 0 \quad \text{unless } r + n + t \geq 4. \quad (4.4)$$

This result may be justified by evaluating the residues associated with the propagators of  $I_{r,n,t}$ , as explained in subsection 3.1. The residues with  $k^2 = \text{constant}$  have a superficial degree of divergence  $\omega_0 = D - 8 + r + t$ , while the residues with  $(k + p_1)^2 = \text{constant}$  and  $(k + p_2)^2 = \text{constant}$  have the generally larger degrees  $\omega_{1,2} = D - 8 + r + n + t$ . Barring cancellations, the degree of divergence of  $I_{r,n,t}$  at  $D = 4$  is  $\omega = r + n + t - 4$ , which is nonnegative when  $r + n + t \geq 4$ .

Now we report the results of explicit calculations that confirm that  $I_{r,n,t}$  is indeed nonlocally divergent when the inequality of (4.4) holds. We begin with the massless case  $m = 0$ . The simplest nontrivial integral is

$$I_{1,2,1}^{\text{nld}}(p_1, p_2) = -i \frac{M^8}{128\pi^2} \frac{1}{Q^2} \log \left( \frac{\Lambda_{UV}^2}{M^2} \right), \quad (4.5)$$

where we have defined  $Q^2 \equiv -(p_2 - p_1)^2$ . The term  $1/Q^2$  is reminiscent of the pole-dominance models of form factors and seems to signal — if we insist with some physical interpretation — the propagation of a massless particle in the  $t$ -channel, with an ultraviolet logarithmically divergent coefficient.

A more interesting case is provided by the amplitude

$$I_{2,2,0}^{\text{nld}}(p_1, p_2) = \frac{iM^8}{128\pi^2} \left[ F(Q, p_1, p_2) - \frac{1}{Q^2} \right] \log \left( \frac{\Lambda_{UV}^2}{M^2} \right), \quad (4.6)$$

where

$$F(Q, p_1, p_2) = \frac{2}{\sqrt{(Q^2 + p_1^2 + p_2^2)^2 - 4p_1^2 p_2^2}} \operatorname{arctanh} \frac{\sqrt{(Q^2 + p_1^2 + p_2^2)^2 - 4p_1^2 p_2^2}}{Q^2 + p_1^2 - p_2^2}.$$

In order to understand the dynamic properties of the first term of formula (4.6) (the second term has the same form as the one of the previous amplitude), let us assume the kinematics of the Deep Inelastic Scattering, i.e.

$$Q^2 \gg |p_1^2| \approx |p_2^2| \neq 0.$$

Infrared singularities (soft and/or collinear) are then regulated by nonzero virtualities of the external legs. Using the asymptotic expansion

$$\operatorname{arctanh}(x) = -\frac{1}{2} \ln \left( \frac{1-x}{2} \right) + \mathcal{O}(1-x) \quad \text{for } x \lesssim 1,$$

it is easy to show that

$$F(Q, p_1, p_2) \simeq \frac{1}{Q^2} \ln \left( \frac{Q^2}{\mu_2^2} \right), \quad (4.7)$$

where we have defined

$$\mu_i^2 \equiv -p_i^2 > 0, \quad i = 1, 2.$$

It is convenient to assume  $\mu_2^2 > 0$  in order to have a real logarithm and then avoid absorptive parts related to the “decay” of the  $p_2$  leg. Formula (4.7) exhibits a collinear divergence for  $\mu_2^2 \rightarrow 0$  (or, equivalently,  $Q^2 \rightarrow +\infty$ ), which overlaps the nonlocal divergence already found in the previous cases. The asymmetry of the result, namely the absence of a collinear singularity for  $\mu_1^2 \rightarrow 0$ , is related to the fact that the power  $(k \cdot p_1)^2$  appearing in the numerator screens the singularity related to the emission of a particle collinear to the particle with momentum  $p_1$ . In the case of the previous amplitude  $I_{1,2,1}$ , collinear singularities related to the emission from any leg were screened by the factor  $(k \cdot p_1)(k \cdot p_2)$  in the numerator. By generalizing the example just discussed, we expect that nonlocal divergences overlap the usual, logarithmic infrared singularities found in vertex functions.

It is natural to expect that nonlocal divergences also occur in one-loop box diagrams, pentagon diagrams, etc., and that they overlap the usual logarithmic structures (infrared divergences, small- $x$  logarithms, etc.).

We conclude by briefly reporting results concerning the massive case  $m \neq 0$ . In both  $I_{1,2,1}^{\text{nl,d}}$  and  $I_{2,2,0}^{\text{nl,d}}$ , it is sufficient to replace  $M^8$  with  $M^{10}/(M^2 + im^2)$  in formulas (4.5) and (4.6).

## 5 Power counting for nonlocal divergences

The standard loop integrals in Minkowski spacetime are related to Euclidean integrals by the Wick rotation, so the power counting rules governing their ultraviolet behaviors are the same. This fact is actually far from trivial. A Minkowski integral

$$\int \frac{d^D k}{(2\pi)^D} \prod_i \Delta(k - p_i, m_i)$$

is intuitively expected to be more singular than the corresponding Euclidean integral in the ultraviolet region. Because of the Euclidean metric, the Euclidean integrand falls off when any component of the momentum  $k_\mu$  gets large. At first sight, the Minkowski pseudometric fails to provide an equivalent suppression in several subdomains of integration, such as the regions where the loop momentum is close to cones of the form  $k^2 + a \cdot k + b = 0$  (determined by the poles of the propagators), where  $a$  is a vector and  $b$  is a constant. The reason why this intuitive argument is not correct is that it overlooks the role played by the  $+i\epsilon$  prescription, which allows the Wick rotation.

On the other hand, the higher-derivative propagator  $S(k, m)$  has poles in all quadrants, so the analytic continuation to Euclidean space is not possible. Then, the rules of power counting are no longer guaranteed to coincide in Minkowski and Euclidean spaces and actually turn out to be different.

To illustrate this fact, we work out a formula for the degree of divergence of a generic one-loop diagram in Minkowski higher-derivative theories. Assume that the propagators  $S(k, m)$  behave like  $1/(k^2)^N$  for large  $|k^2|$  and that the vertices contain up to  $N'$  derivatives. Consider a one-particle irreducible diagram with  $V$  vertices, equal to the number of internal lines. We assume that  $V > 1$ , i.e. exclude the tadpoles, because they are independent of the external momenta and cannot originate nonlocal divergences.

Letting  $k$  denote the loop momentum, we integrate over the energy  $k_0$  by means of the residue theorem. In each residue,  $(k - q)^2$  is equal to some constant,  $q$  being a linear combination of external momenta. Making a translation, we can assume that the integrand is evaluated at  $k^2 = \text{constant}$ . Then, by formula (2.4) the propagator  $S(k, m)$  gives a contribution that behaves like  $1/k_s$  for large  $k_s$ , while each one of the other  $V - 1$  propagators behaves like  $1/((p - k)^2)^N \sim 1/(p \cdot k)^N$ , where  $p$  is also a linear combination of the external momenta. If we use analyticity to assume  $p^2 > 0$ , the factors  $1/(p \cdot k)^N$  are regular everywhere.

On the other hand, the vertices provide at most  $N'V$  powers of  $k_s$  and the integration measure is  $k_s^{D-2} dk_s$ . Collecting these pieces of information, the degree of divergence  $\omega_{\text{nl}}$  of the  $k_s$  integral is equal to

$$\omega_{\text{nl}} = D - 1 + N'V - 1 - (V - 1)N = D - 2 + N + V(N' - N). \quad (5.1)$$

An integral with  $\omega_{\text{nl}} < 0$  is ultraviolet convergent, while an integral with  $\omega_{\text{nl}} \geq 0$  may be divergent. The divergent parts are in general nonlocal, because, as shown in formula (4.3), the large  $k_s$  expansion makes the ratios  $1/(p \cdot k)^N$  factorise as  $1/k_s^N$  times nonpolynomial functions of the external momenta.

The condition  $\omega_{\text{nl}} \geq 0$  is necessary to have a divergence, but not sufficient. In many cases, it is possible to enhance it by means of more sophisticated arguments. For example, it is possible to show that the bubble diagram ( $V = 2$ ) benefits from an enhancement of one unit when  $N$  is odd. Then,

$$\omega'_{\text{nl}} = D - 3 + 2N' - N. \quad (5.2)$$

The reason is a simplification between the contributions  $\sim 1/(p \cdot k)^N$  of each propagator, calculated on the poles of the other propagator.

If  $D = 6$ ,  $N = 3$ ,  $N' = 0$ ,  $V = 2$ , which is the case treated in subsection 3.1, we have  $\omega'_{\text{nl}} = 0$ , which confirms that there is a logarithmic divergence. The same diagram in four dimensions has no nonlocal divergences ( $\omega'_{\text{nl}} = -2$ ), unless we equip it with nontrivial numerators. If we take  $N' = 1$ , we raise  $\omega'_{\text{nl}}$  to 0, which is confirmed by the nonvanishing coefficient  $c_{0,1}$  of formula (3.6). On the other hand, it is not enough to have a vertex with one derivative and a vertex with no derivatives (which can be formally obtained by setting  $N' = 1/2$ ), as the vanishing of the coefficient  $c_{1,0}$  confirms.

In the case of the triangle diagram ( $D = 4$ ,  $N = 3$  and  $V = 3$ ), we may distribute the  $r + 2n + t$  derivatives over the three vertices by formally writing  $N' = (r + 2n + t)/3$ . The integrals  $I_{1,2,1}^{\text{nl,d}}$  and  $I_{2,2,0}^{\text{nl,d}}$  have  $N' = 2$  and  $\omega_{\text{nl}} > 0$ , indeed formulas (4.5) and (4.6) shows that they are divergent. Moreover,  $r + 2n + t < 4$  implies  $\omega_{\text{nl}} < 0$ , which agrees with formula (4.4). A better agreement can be obtained by improving the power counting as shown in section 4. Indeed, after a residue is evaluated, a  $k^2$  factor in the numerator does not provide two powers of  $k_s$ , but one at most. This is equivalent to setting  $N' = (r + n + t)/3$ . Then, formula (4.4) follows in all cases, while both  $I_{1,2,1}^{\text{nl,d}}$  and  $I_{2,2,0}^{\text{nl,d}}$  have  $N' = 4/3$ ,  $\omega_{\text{nl}} = 0$ , which implies that the ultraviolet divergence is at most logarithmic, as is actually the case.

Let us inquire which theories have no nonlocal divergences, i.e. when  $\omega_{\text{nl}} < 0$  for every  $V > 1$ . Formula (5.1) shows that this happens when  $D - 2 < N - 2N'$ , which is a very restrictive condition. Higher-derivative theories of gauge fields have  $N' = 2N - 1$ , while those of gravity have  $N' = 2N$ , so neither of the two satisfies the condition. Both are expected to have nonlocal divergences, if their propagators have poles in the first or third quadrants. In the next section we study the case of gravity explicitly.

## 6 Higher derivative gravity

In this section we use the results of the previous ones to work out the nonlocal divergences of the graviton two point function in a relatively simple model of four dimensional higher-

derivative gravity with complex poles. We simplify the calculations as much as possible by choosing a specific Lagrangian and a convenient gauge fixing. The loop integrals are linear combinations of the scalar integrals (3.4).

The simplest model of higher-derivative gravity is the Stelle theory [2], which contains the scalars  $R$ ,  $R^2$  and  $R_{\mu\nu}R^{\mu\nu}$ . However, it is not suitable for our investigation, because its propagators do not have poles in the first or third quadrants. The simplest model with the features we need is the one with Lagrangian

$$\mathcal{L}_{\text{HD}} = -\frac{\sqrt{-g}}{2\kappa^2} \left[ R - \frac{1}{M^4} (D_\rho R_{\mu\nu})(D^\rho R^{\mu\nu}) + \frac{1}{2M^4} (D_\rho R)(D^\rho R) \right]. \quad (6.1)$$

We expand the metric tensor  $g_{\mu\nu}$  around the flat-space metric  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu},$$

where  $\kappa$  is a constant of dimension  $-1$  in units of mass and  $h_{\mu\nu}$  is the quantum fluctuation. After the expansion around flat space, we raise and lower the indices by means of the flat-space metric. We further define  $h \equiv h_\mu^\mu$ .

We choose the De Donder gauge-fixing function

$$\mathcal{G}_\mu(g) = \eta^{\nu\rho} \partial_\rho g_{\mu\nu} - \frac{1}{2} \eta^{\nu\rho} \partial_\mu g_{\nu\rho} = \kappa(2\partial_\nu h_\mu^\nu - \partial_\mu h) \quad (6.2)$$

and perform the gauge fixing as explained in appendix B. The gauge-fixed Lagrangian then reads

$$\mathcal{L}_{\text{grav}} = \mathcal{L}_{\text{HD}} + \frac{1}{4\kappa^2} \mathcal{G}^\mu \left( 1 + \frac{\square^2}{M^4} \right) \mathcal{G}_\mu + \mathcal{L}_{\text{gh}}, \quad (6.3)$$

where  $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$  is the flat-space D'Alembertian, while the ghost Lagrangian is

$$\mathcal{L}_{\text{gh}} = \bar{C}^\mu \left( 1 + \frac{\square^2}{M^4} \right) [\square C_\mu - (2\delta_\mu^\rho \eta^{\nu\sigma} \partial_\nu - \eta^{\rho\sigma} \partial_\mu) \Gamma_{\rho\sigma}^\alpha C_\alpha]. \quad (6.4)$$

The graviton propagator

$$\langle h_{\mu\nu}(p) h_{\rho\sigma}(-p) \rangle_0 = \frac{iM^4}{2(p^2 + i\epsilon)} \frac{\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma}}{(p^2)^2 + M^4}$$

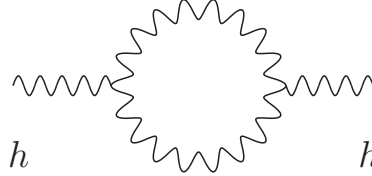
has the same form as that of the propagators of the previous sections, apart from the constant matrices in the numerator.

The ghost sector can be ignored, because it cannot violate the locality of counterterms at one loop. Indeed, after the redefinition  $\bar{C}^{\mu'} = (1 + \square^2/M^4) \bar{C}^\mu$ , the ghost Lagrangian (6.4) turns into the usual one, which is

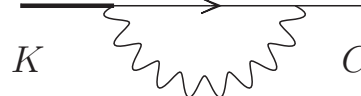
$$\mathcal{L}_{\text{gh}} = \bar{C}^{\mu'} [\square C_\mu - (2\delta_\mu^\rho \eta^{\nu\sigma} \partial_\nu - \eta^{\rho\sigma} \partial_\mu) \Gamma_{\rho\sigma}^\alpha C_\alpha].$$

For this reason, the ghost contribution to the graviton self-energy coincides with the usual one, which has a local divergent part.

It is sufficient to work out the three-graviton vertex, since the one-loop diagrams involving four-leg vertices are tadpoles, which can only have local divergent parts. In the end, we just evaluate two diagrams, which are



(a)



(b)

(6.5)

where the wiggled line represents the graviton  $h_{\mu\nu}$ , the solid line represents the source  $K^{\mu\nu}$  coupled to the  $h_{\mu\nu}$  transformation and the continuous line with the arrow represents the ghosts. Diagram (a) encodes the nonlocal divergences of the graviton self energy. Diagram (b) encodes the nonlocal renormalization of the  $h_{\mu\nu}$  transformation: it is necessary to derive the corrections to the Ward identities satisfied by (a), as explained in appendix B.

## 6.1 Results

Given these ingredients, we are ready to perform the calculation, as well as the consistency checks. We can reduce to the scalar integrals  $I_{r,n}$  of formula (3.4) by means of the Passarino-Veltman decomposition [5], which gives identities such as

$$\mathcal{J}_{\mu_1 \dots \mu_n}(p) \equiv \int \frac{d^D k}{(2\pi)^D} k^{\mu_1} \dots k^{\mu_n} f(k^2, p \cdot k) = \sum_i A_i(p) T_i^{\mu_1 \dots \mu_n}(p), \quad (6.6)$$

where  $A_i(p)$  are scalar integrals and  $T_i^{\mu_1 \dots \mu_n}(p)$  are completely symmetric tensors built with  $\eta^{\mu\nu}$  and  $p^\mu$ . Since the graviton two-point functions has four indices, we just need formula (6.6) for  $n = 1, 2, 3, 4$ . The cases  $n = 1, 2$  give, for example,

$$\begin{aligned} \mathcal{J}_\mu &= \frac{p^\mu}{p^2} \int \frac{d^D k}{(2\pi)^D} (p \cdot k) f(k^2, p \cdot k), \\ \mathcal{J}_{\mu\nu} &= \frac{\eta^{\mu\nu}}{3} \int \frac{d^D k}{(2\pi)^D} \left[ k^2 - \frac{(p \cdot k)^2}{p^2} \right] f(k^2, p \cdot k) - \frac{p^\mu p^\nu}{3p^2} \int \frac{d^D k}{(2\pi)^D} \left[ k^2 - 4 \frac{(p \cdot k)^2}{p^2} \right] f(k^2, p \cdot k). \end{aligned}$$



The one-loop nonlocal divergent part of the graviton two-point function is equal to

$$\begin{aligned} \langle h_{\mu\nu}(p) h_{\rho\sigma}(-p) \rangle_1^{\text{nl d}} &= \frac{\kappa^2 M^8}{240\pi^2 (p^2)^2} [(68r + i)(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}) + (373r - 4i)\eta_{\mu\nu}\eta_{\rho\sigma} \\ &\quad - \frac{1}{8p^2}(125ir^2 + 544r + 8i)(p_\mu p_\rho \eta_{\nu\sigma} + p_\mu p_\sigma \eta_{\nu\rho} + p_\nu p_\rho \eta_{\mu\sigma} + p_\nu p_\sigma \eta_{\mu\rho}) \\ &\quad + \frac{1}{4p^2}(255ir^2 - 1522r + 36i)(p_\mu p_\nu \eta_{\rho\sigma} + p_\rho p_\sigma \eta_{\mu\nu}) \\ &\quad - \frac{1}{2(p^2)^2}(185r^3 + 75ir^2 - 1048r + 24i)p_\mu p_\nu p_\rho p_\sigma] \ln\left(\frac{\Lambda_{UV}^2}{M^2}\right), \quad (6.7) \end{aligned}$$

where  $r \equiv p^2/M^2$ . It is easy to check that it is doubly transverse, i.e.

$$p^\nu p^\sigma \langle h_{\mu\nu}(p) h_{\rho\sigma}(-p) \rangle_1^{\text{nl d}} = 0$$

up to local terms. However, it is not transverse, since  $p^\nu \langle h_{\mu\nu}(p) h_{\rho\sigma}(-p) \rangle_1^{\text{nl d}}$  does not vanish. The reason is that the gauge transformation is itself affected by nonlocal divergences. The correct Ward identity is (B.5), derived from diagram (b) as explained in appendix B. It is easy to show that (6.7) does satisfy (B.5), which provides a good check of the result.

Coherently with what we found in the previous sections, the divergences are nonlocal and truly complex. It is impossible to subtract them away by means of reparametrizations and (local as well as nonlocal) field redefinitions that preserve Hermiticity.

In conclusion, Minkowski higher-derivative theories of gravity violate the locality and Hermiticity of counterterms, when the propagators have poles in the first or third quadrants. Gauge symmetries are unable to protect those properties.

## 7 Conclusions

We have shown that Minkowski higher-derivative quantum field theories whose propagators have complex poles are generically inconsistent, because they generate nonlocal, non-Hermitian ultraviolet divergences. Bubble diagrams, for example, contain logarithmic divergences multiplied by inverse powers of D'Alembertians. Triangle diagrams present more involved nonlocal divergences, where ultraviolet effects mix with standard infrared effects.

Contrary to intuitive expectations, the introduction of higher derivative terms in the Lagrangian does not have a regulating effect, because the constraints coming from power counting are much weaker. Indeed, the contribution of one propagator calculated on the pole of another propagator does not decay fast enough. This unusual behavior can also be

explained by the appearance of pinch singularities, unrelated to the usual absorptive parts of amplitudes, which occur because the extra excitations introduced by the higher derivatives come with effective prescriptions of both signs.

We have extended the calculations to higher-derivative quantum gravity and proved, in particular, that gauge symmetries are unable to protect the locality and Hermiticity of counterterms. The problems we have outlined add up to the well-known problems that higher-derivative Minkowski theories have with perturbative unitarity.

## Acknowledgments

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## A Residue theorem in dimensional regularization

In this appendix we show that the dimensional regularization allows us to evaluate the energy integrals in a straightforward way, even when they are divergent: it is sufficient to sum the residues, while the contribution of the integration path at infinity is always negligible.

The dimensional regularization is defined as follows. The integral on the space momenta is continued to  $D-1$  dimensions and done first. The energy integral is not modified and done second. Strictly speaking, the two can be exchanged when the energy integral is convergent. However, we show that it is always legitimate to integrate on the energy first, if we apply the residue theorem.

Without loss of generality, we can write a one-loop integral as a linear combination of integrals of the form

$$\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{a(\mathbf{k})E^s}{E^{2r} + \sum_{i=1}^{2r} b_i(\mathbf{k})E^{2r-i}}$$

where  $a(\mathbf{k})$ ,  $b_i(\mathbf{k})$  are polynomials of  $\mathbf{k}$  and  $r \geq 1$ ,  $s \geq 0$  are integers. The denominators contain the prescriptions to move the poles away from the real axis.

The energy integral is divergent if  $s+1 \geq 2r$ , so we write  $s = 2r + n - 1$  and take  $n \geq 0$ . When  $|E|$  is much larger than all the other scales, its divergent contributions are

$$\int_{|E| \sim \infty} \frac{dE}{2\pi} \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} a(\mathbf{k}) E^{n-1} \left( 1 - \sum_{i=1}^{2r} \frac{b_i(\mathbf{k})}{E^i} - \sum_{i,j=1}^{2r} \frac{b_i(\mathbf{k})b_j(\mathbf{k})}{E^{i+j}} + \dots \right).$$

All of the  $\mathbf{k}$  integrals are integrals of polynomials of  $\mathbf{k}$  and give zero by the rules of the dimensional regularization. Thus, if we close the integration path by means of a semicircle

at infinity, in the lower or upper half plane, we add a vanishing contribution. This allows us to safely apply the residue theorem without having to worry about the closure of the integration path.

## B Gauge fixing and WTST identities in higher-derivative gravity

To handle the WTST identities [4] of quantum gravity in a compact form, we use the Batalin-Vilkovisky formalism [6]. We collect the fields into the row

$$\Phi^\alpha = \{h_{\mu\nu}, C_\mu, \bar{C}_\mu, B_\mu\},$$

where  $C_\mu$ ,  $\bar{C}_\mu$  and  $B_\mu$  are the ghosts and the antighosts of diffeomorphisms and the Lagrange multipliers for the gauge fixing, respectively. We introduce conjugate sources

$$K_\alpha = \{K^{\mu\nu}, K_C^\mu, K_{\bar{C}}^\mu, K_B^\mu\}$$

and define the antiparentheses of two functionals  $X$  and  $Y$  of  $\Phi$  and  $K$  as

$$(X, Y) \equiv \int \left( \frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K_\alpha} - \frac{\delta_r X}{\delta K_\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} \right),$$

where the integral is over the spacetime points associated with repeated indices and the subscripts  $l$ ,  $r$  in  $\delta_l$ ,  $\delta_r$  denote the left and right functional derivatives, respectively.

The total action is then

$$S(\Phi, K) = S_{\text{HD}} + (S_K, \Psi) + S_K,$$

where  $S_{\text{HD}} = \int \mathcal{L}_{\text{HD}}$  is the classical action,  $\Psi(\Phi)$  is a functional of the fields that performs the gauge fixing, called gauge fermion, and the terms

$$S_K = - \int (\partial_\mu C_\nu + \partial_\nu C_\mu - 2\Gamma_{\mu\nu}^\rho C_\rho) K^{\mu\nu} + \int g^{\nu\rho} C_\rho [(\partial_\mu C_\nu) + g^{\sigma\alpha} C_\alpha (\partial_\sigma g_{\mu\nu})] K_C^\mu - \int B_\mu K_{\bar{C}}^\mu$$

collect the symmetry transformations coupled to the external sources  $K$ . The Lagrangian  $\mathcal{L}_{\text{HD}}$  is given by formula (6.1).

The action  $S$  satisfies the master equation

$$(S, S) = 0, \tag{B.1}$$

which collects the gauge invariance of  $S_{\text{HD}}$  and the closure of the symmetry transformations.

The generating functional  $Z$  of the correlation functions and the generating functional  $W$  of the connected correlation functions are defined by the formulas

$$Z(J, K) = \int [d\Phi] \exp \left( iS(\Phi, K) + i \int \Phi^\alpha J_\alpha \right) = \exp iW(J, K),$$

while the generating functional  $\Gamma(\Phi, K) = W(J, K) - \int \Phi^\alpha J_\alpha$  of the one-particle irreducible diagrams is the Legendre transform of  $W(J, K)$  with respect to  $J$ . Formula (B.1) implies that  $\Gamma$  satisfies an identical master equation

$$(\Gamma, \Gamma) = 0, \tag{B.2}$$

which collects the WTST identities in a compact form.

We choose the gauge fermion

$$\Psi = -\frac{1}{2} \int \bar{C}^\mu \left( 1 + \frac{\square^2}{M^4} \right) \left( \frac{1}{2} B_\mu - \frac{1}{\kappa} \mathcal{G}_\mu \right),$$

where  $\mathcal{G}_\mu$  is given in formula (6.2). Observe that the indices of all the fields  $\Phi^\alpha$  and the sources  $K_\alpha$  are raised and lowered by means of the flat-space metric. We find

$$(S_K, \Psi) = -\frac{1}{4} \int B^\mu \left( 1 + \frac{\square^2}{M^4} \right) B_\mu + \frac{1}{2\kappa} \int B^\mu \left( 1 + \frac{\square^2}{M^4} \right) \mathcal{G}_\mu + S_{\text{gh}},$$

where  $S_{\text{gh}} = \int \mathcal{L}_{\text{gh}}$  and  $\mathcal{L}_{\text{gh}}$  is given in formula (6.4). We can integrate  $B_\mu$  out, which is equivalent to replacing it with the solution of its own field equation:

$$B_\mu = \frac{1}{\kappa} \mathcal{G}_\mu.$$

So doing, we get

$$(S_K, \Psi) \rightarrow \frac{1}{4\kappa^2} \int \mathcal{G}^\mu \left( 1 + \frac{\square^2}{M^4} \right) \mathcal{G}_\mu + S_{\text{gh}}.$$

The gauge-fixed Lagrangian is thus (6.3).

Now we work out the WTST identity satisfied by the graviton two-point function. Expand the functional  $\Gamma$  as

$$\Gamma = S + \Gamma_1 + \Gamma_2 + \dots,$$

where  $\Gamma_i$  collects the contributions of the  $i$ -loop diagrams. Note that  $\Gamma_i$  cannot depend on  $B$ ,  $K_{\bar{C}}$  and  $K_B$ , because no one-particle irreducible diagrams can be built with external legs of this type. The master equation (B.2) gives, at one loop,

$$(S, \Gamma_1) = 0.$$

Expanding the antiparentheses on the left-hand side of this equation and setting  $\bar{C} = B = K = 0$ , we get

$$\int \frac{\delta S_{\text{HD}}}{\delta h_{\mu\nu}} \frac{\delta_l \Gamma_1}{\delta K^{\mu\nu}} \Big|_{\bar{C}=B=K=0} = \int \frac{\delta_r S_K}{\delta K^{\mu\nu}} \frac{\delta_l \Gamma_1}{\delta h_{\mu\nu}} \Big|_{\bar{C}=B=K=0}. \quad (\text{B.3})$$

This identity encodes the modified gauge invariance of the classical action  $S_{\text{HD}}$ . Indeed, it has the form

$$\int \frac{\delta S_{\text{HD}}}{\delta h_{\mu\nu}} \Delta_1 h_{\mu\nu} + \int \frac{\delta \tilde{\Gamma}_1}{\delta h_{\mu\nu}} \Delta_0 h_{\mu\nu} = 0, \quad (\text{B.4})$$

where  $\tilde{\Gamma}_1 = \Gamma_1|_{\bar{C}=B=K=0}$  and

$$\Delta_0 h_{\mu\nu} = -\frac{\delta_r S_K}{\delta K^{\mu\nu}} = \partial_\mu C_\nu + \partial_\nu C_\mu - 2\Gamma_{\mu\nu}^\rho C_\rho, \quad \Delta_1 h_{\mu\nu} = \frac{\delta_l \Gamma_1}{\delta K^{\mu\nu}} \Big|_{\bar{C}=B=K=0}.$$

In turn, the gauge invariance of  $S_{\text{HD}}$ , combined with formula (B.4), gives

$$\int \frac{\delta(S_{\text{HD}} + \tilde{\Gamma}_1)}{\delta h_{\mu\nu}} (\Delta_0 h_{\mu\nu} + \Delta_1 h_{\mu\nu}) = 0,$$

up to two-loop corrections. This identity states that the corrected action  $S_{\text{HD}} + \tilde{\Gamma}_1$  is invariant under the corrected gauge transformations  $\Delta h_{\mu\nu} = \Delta_0 h_{\mu\nu} + \Delta_1 h_{\mu\nu}$ . The derivatives  $\delta \tilde{\Gamma}_1 / \delta h_{\mu\nu}$  and  $\delta_l \Gamma_1 / \delta K^{\mu\nu}|_{\bar{C}=B=K=0}$  are calculated through the diagrams (a) and (b) shown in (6.5), respectively.

Precisely, we can write

$$\begin{aligned} \tilde{\Gamma}_1 &= -i \int h^{\mu\nu}(x) \langle h_{\mu\nu}(x) \rangle_1^{\text{1PI}} d^D x - \frac{i}{2} \int h^{\mu\nu}(x) \langle h_{\mu\nu}(x) h_{\rho\sigma}(y) \rangle_1^{\text{1PI}} h^{\rho\sigma}(y) d^D x d^D y + \mathcal{O}(h^3), \\ \Delta_1 h_{\mu\nu} &= \frac{\delta_l \Gamma_1}{\delta K^{\mu\nu}} \Big|_{\bar{C}=B=K=0} = \frac{\delta_l W_1}{\delta K^{\mu\nu}} \Big|_{\bar{C}=B=K=0} = \langle \partial_\mu C_\nu + \partial_\nu C_\mu \rangle_{1,J}^{\text{1PI}} - 2 \langle \Gamma_{\mu\nu}^\rho C_\rho \rangle_{1,J}^{\text{1PI}} + \mathcal{O}(Ch), \end{aligned}$$

where the one-loop correlation functions  $\langle \dots \rangle_{1,J}$  are evaluated at nonvanishing external sources  $J$ .

Note that  $\langle h_{\mu\nu} \rangle_1^{\text{1PI}}$  has no nonlocal divergences, because it is a tadpole. On the other hand,  $\langle \partial_\mu C_\nu \rangle_{1,J}^{\text{1PI}} = 0$ , because the insertion is linear in the fields. As far as the term  $-2 \langle \Gamma_{\mu\nu}^\rho C_\rho \rangle_{1,J}^{\text{1PI}}$  is concerned, we are just interested into its nonlocal divergent part to the zeroth order in  $h_{\mu\nu}$ , which we denote by  $\Delta_1 h_{\mu\nu}|_{h=0}^{\text{nl d}}$ . We get, in momentum space,

$$\begin{aligned} \Delta_1 h_{\mu\nu}|_{h=0}^{\text{nl d}}(p) &= -\frac{\kappa^2 M^8}{96\pi^2 (p^2)^4} \left[ (3ir^2 + 2r - 2i)p^2 (p_\mu \delta_\nu^\rho + p_\nu \delta_\mu^\rho) \right. \\ &\quad \left. + p^2 (3r - 2i)p^\rho \eta_{\mu\nu} + 4(-4r + 3i)p^\rho p_\mu p_\nu \right] C_\rho(p) \ln \left( \frac{\Lambda_{UV}^2}{M^2} \right), \end{aligned}$$

where  $r = p^2/M^2$ . The quadratic part of  $S_{\text{HD}}$  is, in momentum space,

$$\frac{1}{4} \int h^{\mu\nu}(-p) \left[ 1 + \frac{(p^2)^2}{M^4} \right] p^2 (\Pi_{\mu\rho}\Pi_{\nu\sigma} + \Pi_{\mu\sigma}\Pi_{\nu\rho} - 2\Pi_{\mu\nu}\Pi_{\rho\sigma}) h^{\rho\sigma}(p),$$

where  $\Pi_{\mu\nu} = \eta_{\mu\nu} - p_\mu p_\nu/p^2$ . Inserting these expression in (B.3), we find the Ward identity

$$p^\nu \langle h_{\mu\nu}(p) h_{\rho\sigma}(-p) \rangle_1^{\text{nl d}} = - (\eta_{\rho\sigma} p^2 - p_\rho p_\sigma) p_\mu \frac{\kappa^2 M^8}{96\pi^2 (p^2)^3} (1 + r^2) (3r - 2i) \ln \left( \frac{\Lambda_{UV}^2}{M^2} \right), \quad (\text{B.5})$$

up to local corrections.

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