Quantum Gravity And Renormalization

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Abstract

The properties of quantum gravity are reviewed from the point of view of renormalization. Various attempts to overcome the problem of nonrenormalizability are presented, and the reasons why most of them fail for quantum gravity are discussed. Interesting possibilities come from relaxing the locality assumption, which can inspire the investigation of a largely unexplored sector of quantum field theory. Another possibility is to work with infinitely many independent couplings, and search for physical quantities that only depend on a finite subset of them. In this spirit, it is useful to organize the classical action of quantum gravity, determined by renormalization, in a convenient way. Taking advantage of perturbative local field redefinitions, we write the action as the sum of the Hilbert term, the cosmological term, a peculiar scalar that is important only in higher dimensions, plus invariants constructed with at least three Weyl tensors. We show that the FRLW configurations, and many other locally conformally flat metrics, are exact solutions of the field equations in arbitrary dimensions $d > 3$. If the metric is expanded around such configurations the quadratic part of the action is free of higher-time derivatives. Other well-known metrics, such as those of black holes, are instead affected in nontrivial ways by the classical corrections of quantum origin.

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1 Nonrenormalizability and its possible remedies

The Einstein-Hilbert action

$$S_{\mathrm{EH}}(g) = -\frac{1}{2\kappa^2}\int \sqrt{|g|}R \tag{1.1}$$

is nonrenormalizable in four dimensions, because when the metric tensor $g_{\mu\nu}$ is expanded around the flat-space metric $\eta_{\mu\nu}$, by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}, \tag{1.2}$$

the quadratic terms are properly normalized and the vertices are multiplied by powers of κ, which has dimension -1. The counterterms generated by renormalization have dimensions $2L+2$, and increase with the number of loops L.

In the absence of matter, the theory is finite at one loop [1]. Indeed, using the dimensional regularization the one-loop divergent terms have the form

$$\Delta S(g)=\frac{\hbar\mu^{-\varepsilon}}{(4\pi)^2\varepsilon}\int\sqrt{|g|}\left(aR_{\mu\nu}R^{\mu\nu}+bR^2\right) \tag{1.3}$$

where $\varepsilon=4-D$, D is the continued complex dimension, μ is the subtraction scale and a, b are calculable numbers. Formula (1.3) follows from locality, general covariance, power counting and the identity

$$\sqrt{|g|}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2\right)=\text{total derivative}, \tag{1.4}$$

which can be used to eliminate the third scalar of dimension 4, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, by converting it into a linear combination of $R_{\mu\nu}R^{\mu\nu}$ and R^2. The counterterms (1.3) are proportional to the $S_{\rm EH}$ vacuum field equations and can be eliminated with a redefinition of the metric tensor. Precisely,

$$g'_{\mu\nu}=g_{\mu\nu}-\frac{\kappa^2\hbar\mu^{-\varepsilon}}{(4\pi)^2\varepsilon}\left(2aR_{\mu\nu}-(a+2b)g_{\mu\nu}R\right) \tag{1.5}$$

gives

$$S_{\rm EH}(g')=S_{\rm EH}(g)-\Delta S(g)+\mathcal{O}(\hbar^2).$$

Finiteness is indeed the property that divergences can be subtracted just by means of field redefinitions, with no redefinitions of parameters. The one-loop finiteness of four dimensional pure quantum gravity is however a lucky coincidence. It is spoiled by the presence of matter [1], because in that case the terms (1.3) are no longer proportional to the classical field equations. In the absence of matter, it is spoiled at two loops [2, 3], where there appears a counterterm proportional to

$$\frac{\mu^{-2\varepsilon}\kappa^2\hbar^2}{(4\pi)^4\varepsilon}\int\sqrt{|g|}R_{\mu\nu}{}^{\rho\sigma}R_{\alpha\beta}{}^{\mu\nu}R_{\rho\sigma}{}^{\alpha\beta}, \tag{1.6}$$

which cannot be absorbed into field redefinitions. In higher even dimensions finiteness does not even hold at one loop and in the absence of matter. For example, in six-dimensional pure quantum gravity the counterterm (1.6) appears already at one loop [4].

Supergravity enhances the finiteness properties of quantum gravity, but not enough to solve the problem of renormalizability. N=1 pure supergravity is finite through two loops [5]. Extended supergravities and supergravities in higher dimensions are believed to be finite up to higher orders. However, it is always possible to build candidate counterterms that may appear at even higher orders, and only a miracle would prevent them from being generated by renormalization.

Barring miracles, if we want to remove the divergences of quantum gravity, or supergravity, we need to extend the classical action by including the invariants generated by renormalization, such as (1.6), multiplied by independent parameters. However, those invariants generate counterterms of new types, which must also be included. In the end, the extension contains infinitely many independent terms and parameters, which raises the issue of predictivity.

Renormalization and higher derivatives

Alternative theories of quantum gravity have been investigated for many years. We discuss the most important results of this research. First observe that the divergent terms (1.3) can also be subtracted by extending the classical action (1.1), instead of making field redefinitions. Consider the theory

$$S_{\mathrm{HD}}(g,\alpha,\beta)=-\frac{1}{2\kappa^2}\int\sqrt{|g|}\left(R+\alpha\kappa^2R_{\mu\nu}R^{\mu\nu}+\beta\kappa^2R^2\right), \qquad (1.7)$$

where α and β are two new independent parameters. If we define

$$\alpha'=\alpha+\frac{2a\hbar\mu^{-\varepsilon}}{(4\pi)^2\varepsilon}, \qquad \beta'=\beta+\frac{2b\hbar\mu^{-\varepsilon}}{(4\pi)^2\varepsilon}.$$

we get

$$S_{\mathrm{HD}}(g,\alpha',\beta')=S_{\mathrm{HD}}(g,\alpha,\beta)-\Delta S(g).$$

The problem of this approach is that once the couplings α and β are introduced, there are energy domains where the values of α and β cannot be considered small. The modified theory (1.7) violates unitarity, because its propagators in flat space have a spin-2 unphysical pole of squared mass $1/(\kappa^2\alpha)$, which is a ghost, besides a new scalar physical pole of squares mass $1/(\kappa^2\bar{\alpha})$, where $\bar{\alpha}=-2(\alpha+3\beta)$. Differently from the Fadeev-Popov ghosts commonly introduced by the gauge-fixing, the spin-2 unphysical particle cannot be eliminated by choosing a different gauge-fixing, and does contribute to the S matrix. In the end, it is responsible for the violation of the identity $SS^\dagger=1$.

If we include the cosmological term, we obtain higher-derivative quantum gravity, whose action

$$S_{\mathrm{HD}\Lambda}(g,\alpha,\beta)=-\frac{1}{2\kappa^2}\int\sqrt{|g|}\left(R+2\Lambda+\alpha\kappa^2R_{\mu\nu}R^{\mu\nu}+\beta\kappa^2R^2\right). \qquad (1.8)$$

is renormalizable to all orders by simple power counting [6], because its propagators (upon choosing a suitable higher-derivative gauge-fixing) fall off like $1/(p^2)^2$ for large momenta p. This improvement of the ultraviolet behavior, however, requires α and $\bar{\alpha}$ to be of order one, which makes it impossible to eliminate the unphysical pole from the propagators.

Let us see what happens if we try and suppress the higher-derivative terms nonetheless. The limit $\beta\rightarrow 0$ is costless, but when we further take $\alpha\rightarrow 0$ (at which point we can also take $\Lambda\rightarrow 0$), we find divergences $\sim\alpha^{-r}\ln^s\alpha$, where $r,s\geqslant 0$, $r+s>0$, which replace the factors $\mu^{-\varepsilon}/\varepsilon$ and $\mu^{-2\varepsilon}/\varepsilon$ in (1.3) and (1.6), and multiply all sorts of similar terms. The removal of those divergences takes us back to the problems of the action (1.1), and its nonrenormalizability. In the end, higher-derivative extensions may solve the problem of renormalizability, but the price to pay, that is to say the violation of unitarity, is unacceptable.

Renormalization and local perturbative changes of field variables

We may try and get rid of the unphysical pole by taking a somewhat intermediate attitude, that is to say leaving both α and β at their places, but treat them perturbatively, and include all orders in α and β. In this case, the couplings α and β are called "inessential" [7] and the theory (1.7) is no different from (1.1). Precisely, there exists a map [8]

$$g'_{\mu\nu} = g_{\mu\nu} - \frac{\kappa^2}{2}\left(2\alpha R_{\mu\nu} - (\alpha + 2\beta) g_{\mu\nu} R\right) + \mathcal{O}(R^2), \tag{1.9}$$

such that

$$S_{\mathrm{EH}}(g') = S_{\mathrm{HD}}(g, \alpha, \beta)$$

exactly to all orders in α and β. Said differently, if the unperturbed theory is the same, in this case the quadratic part of the action (1.1) in the expansion (1.2), the actions (1.1) and (1.7) are completely equivalent from the point of view of perturbative quantum field theory. At the level of Feynman diagrams, it means that when we use (1.7) we expand the integrands in powers of α and β before evaluating the integrals. Doing so, we find a huge number of nontrivial divergent terms, which are in one-to-one correspondence with those of (1.1). To emphasize the perturbative equivalence of (1.1) and (1.7) even more, observe that if we use the same procedure (namely, first expand in powers of α and β, then calculate) to evaluate the Jacobian determinant associated with the field redefinition (1.9), the result is identically one in dimensional regularization.

A map similar to (1.9) exists also in the presence of the cosmological constant, which makes (1.8) perturbatively equivalent to the action

$$S_{\mathrm{EH}\Lambda}(g) = -\frac{1}{2\kappa^2}\int \sqrt{|g|}\,(R + 2\Lambda) \tag{1.10}$$

of Einstein gravity with a cosmological constant. More generally, a similar map exists any time a correction is quadratically proportional to the field equations of the uncorrected action [8].

Higher-derivative quantum gravity is defined by the action (1.8) when α is *not* treated perturbatively. Then it is not equivalent to (1.10), because it is defined by expanding around a different unperturbed theory, that is to say the quadratic part of (1.8), instead of the quadratic part of (1.10). Since counterterms are polynomial in the cosmological constant, for the sole purposes of renormalization it is like expanding around the quadratic part of (1.7) versus the quadratic part of (1.1). The integrals of Feynman diagrams must be evaluated at finite α, which kills all nonrenormalizable divergences, but gives birth to the spin-2 ghost.

More higher derivatives

Any attempts to remove the ghost from the actions (1.7) and (1.8) have been unsuccessful so far, so we may try and modify those actions. The ultraviolet behaviors of propagators can be further improved by adding terms with more higher-derivatives and considering the coefficient of $R_{\mu\nu}\Box^n R^{\mu\nu}$ of order one when n has the maximum value, $\Box$ being the covariant D'Alembertian. For example, instead of (1.8) we can consider

$$S_{\mathrm{HD}\Lambda}(g,\alpha,\beta)=-\frac{1}{2\kappa^2}\int\sqrt{|g|}\left(R+2\Lambda+\alpha\kappa^2R_{\mu\nu}R^{\mu\nu}+\beta\kappa^2R^2\right.$$
$$\left.+\gamma\kappa^4R_{\mu\nu}\Box R^{\mu\nu}+\delta\kappa^4R\Box R\right) \qquad (1.11)$$

and take γ of order one, while α,β and δ are treated perturbatively. We can choose a suitable gauge-fixing so that all propagators fall off as $1/(p^2)^3$ for large momenta p, which makes the theory (1.11) super-renormalizable. Then all the diagrams are convergent, except for a few ones.

It is not possible to improve the renormalization of the theory any further by adding higher and higher derivatives. In particular, we cannot make the theory finite, since certain one-loop diagrams remain divergent. This is a simple consequence of power counting: since the higher-derivative corrections are gauge invariant, they not only improve the behaviors of propagators, but also generate vertices with large numbers of derivatives. The two effects sort of compensate each other inside the one-loop diagrams.

The problem of unitarity remains in all these cases: if we switch Λ off and expand the metric tensor around flat space, the propagators do contain unphysical poles.

Renormalization and nonlocality

A possible wayout, put forward by Tomboulis in 1997 [9] is to relax the assumption of locality and add infinitely many higher-derivative terms. Consider the action

$$S_{\mathrm{nl}}(g,\alpha,\beta)=-\frac{1}{2\kappa^2}\int\sqrt{|g|}\left(R+2\Lambda+\kappa^2R_{\mu\nu}h(\Box)R^{\mu\nu}+\kappa^2Rh'(\Box)R\right), \qquad (1.12)$$

where $h(\Box)$ and $h'(\Box)$ are so-far unspecified functions. If we choose them so that

$$\lim_{|z|\rightarrow\infty}\frac{h(z)}{\alpha}=\lim_{|z|\rightarrow\infty}\frac{h'(z)}{\beta}=1,$$

and such limits are reached sufficiently smoothly, we may expect that the ultraviolet behavior of the nonlocal theory (1.12) is the same as the one of Stelle's theory (1.8), which would make (1.12) power counting renormalizable. If we choose the functions so that

$$\lim_{|z|\rightarrow\infty}\frac{h(z)}{z\kappa^2\gamma}=\lim_{|z|\rightarrow\infty}\frac{h'(z)}{z\kappa^2\delta}=1,$$

we expect that the nonlocal theory (1.12) is super-renormalizable, like the theory (1.11). More generally, we can require that there exist polynomials $P_n(z)$ and $P_n'(z)$ of degrees $n \geqslant 0$, such that

$$\lim_{|z|\rightarrow\infty}\frac{h(z)}{P_n(z)}=\lim_{|z|\rightarrow\infty}\frac{h'(z)}{P_n'(z)}=1. \tag{1.13}$$

The limits must be sufficiently smooth to ensure that the surviving divergences are local.

Not everything works as expected, but the basic idea is ultimately right. The main point is that the functions $h(z)$ and $h'(z)$ are unconstrained by renormalization for finite values of z and can be chosen (together with suitable gauge-fixing functions) so that the propagators do not have unphysical poles (aside from those introduced by the gauge-fixing, which do not contribute to the S matrix). This goal can be achieved by demanding that $h(z)$ and $h'(z)$ be entire trascendental functions that have real and positive values on the real axis, no zeros on the complex plane and the same asymptotic behaviors at $\pm\infty$ on the real axis. Moreover, the condition (1.13) must be restricted to $n \geqslant 3$ in a cone $\mathcal{C}$ containing the real axis, or replaced by the milder condition that there exists an integer $n \geqslant 3$ such that $|h(z)|, |h'(z)| \rightarrow |z|^n$ for $|z| \rightarrow \infty$ in $\mathcal{C}$. Then, it can be proved that the theory is super-renormalizable.

The investigation of nonlocal theories is expected to shed light on a mainly unexplored sector of quantum field theory, and recently has been the focus of renewed interest [10, 11]. One of the main objections to this approach to quantum field theory is that "by relaxing nonlocality everything becomes possible". As far as we know now, this is not true, because the conditions (1.13) pose severe restrictions on the functions $h(z)$ and $h'(z)$, and the solutions are not particularly easy to work with. To overcome this difficulty, it is necessary to explore more general approaches to nonlocal theories, and understand whether simpler functions, such as exponentials [11], may be viable or not from the point of view of renormalization. This investigation is in progress [12], and may lead to a better understanding of the relation between nonlocality and renormalization.

It is possible to make the nonlocal theory completely finite by working in odd dimensions, where one-loop divergences are trivial in dimensional regularization, or adding suitable nonminimal terms [13].

Other remedies for nonrenormalizability

Other approaches to try and make sense of nonrenormalizable theories have been explored in the literature. We mention four such methods. Each of them has its own weaknesses. Some have nonperturbative aspects, others can be applied to very few models, others can be applied to a large class of theories, but not quantum gravity.

First, we mention the idea of asymptotic safety, which is a generalization of asymptotic freedom [7]. If the ultraviolet limit of quantum gravity is an interacting conformal field theory and its critical surface is finite dimensional, then it is possible to reduce the free parameters of quantum gravity to a finite number by demanding that the theory lie on the critical surface at high energies. The difficulty of this idea is that it is not known how to implement it in the realm of perturbation theory. However, truncations and consistency checks have been used to provide evidence that ultraviolet fixed points with good critical surfaces may exist [22].

Large N expansions can be used to treat quantum field theories beyond the weakly coupled limit. In some cases, they can also be used to give sense to nonrenormalizable theories [14]. For example, the four fermion model in three spacetime dimensions with Lagrangian

$$L=\sum_{I=1}^{N}\overline{\psi}_{I}i\partial\!\!\!/\psi_{I}+\frac{1}{2MN}\left(\sum_{I=1}^{N}\overline{\psi}_{I}\psi_{I}\right)^{2} \tag{1.14}$$

is nonrenormalizable by power counting, but becomes renormalizable in the large N expansion, where N is the number of fermion copies and M is a mass scale. Unfortunately, this approach works only in a very sparse set of special models. The model (1.14) is also a good example of asymptotically safe theory[15], its UV interacting fixed point being the conformal field theory described by the Lagrangian[16]

$$L_{\mathrm{UV}}=\sum_{I=1}^{N}\overline{\psi}_{I}\left(i\partial\!\!\!/-\frac{\sigma}{\sqrt{N}}\right)\psi_{I},$$

where σ is a scalar field acquiring a nontrivial two-point function from a one-loop "bubble" diagram at the leading order of the large N limit.

Another approach to the problem of nonrenormalizability is the infinite reduction of refs. [17, 18], which amounts to express the infinitely many independent parameters λ_i of a nonrenormalizable theory as functions $f_i(\alpha)$ of a finite subset α_I of them, in a way that is consistent with the renormalization group. This procedure works better when the power counting renormalizable sector is an interacting conformal field theory $\mathcal{C}$ [18]. Then the solution exists and is unique under relatively mild assumptions on $\mathcal{C}$ and the critical exponents of its irrelevant operators. "Queues" made of infinitely many higher-dimensional operators O^i can be consistently turned on, and deform $\mathcal{C}$ into a "quasi finite" theory. Each operator O^i is multiplied by an appropriate power of a scale κ, which can run due to radiative corrections, and a dimensionless coefficient r_i, which depends on the marginal couplings of $\mathcal{C}$ and is determined by requiring that its beta function vanishes. If the power counting renormalizable sector is not conformal, but the marginal sector is interacting, it is still possible to define an infinite reduction, by imposing analyticity requirements on the functions $f_i(\alpha,\kappa)$, where α are the marginal couplings. However, in general the solution is not unique and new independent parameters sporadically appear at high orders [17]. When the marginal sector is not interacting, which is the case of quantum gravity, the infinite reduction is difficult to control. Attempts to apply the infinite reduction to quantum gravity have been made [19], but they only work in higher-derivative theories, because the presence of terms such as $R_{\mu\nu}R^{\mu\nu}$, multiplied by finite coefficients, is crucial.

Another mechanism that can be used to turn vertices that are usually nonrenormalizable into renormalizable ones is the explicit violation of Lorentz symmetry at high energies [20, 21]. It allows us to add Lagrangian terms containing higher space derivatives without being forced to accompany them by terms containing higher time derivatives. In this way, perturbative unitarity can be preserved, while improving the behaviors of propagators in the limit where the space components of momenta become large. These improved behaviors are enough to make several theories renormalizable according to a "weighted power counting"[20]. The divergences of Feynman diagrams are controlled by appropriate weights, which replace the dimensions in units of mass. Energy and space components of momenta obviously have different weights. Weights are also assigned to parameters and fields, and a renormalizable theory is a theory that contains no parameters of negative weights (when the kinetic terms are appropriately normalized and the propagators behave correctly). Weighted power counting gives us full control on the renormalizability of the theory and the locality of counterterms. Examples of vertices that become renormalizable are the dimension-5 vertex $(LH)^2$, which gives Majorana masses to left-handed neutrinos, and the dimension-6 four fermion vertices $\sim \left(\overline{\psi}\psi\right)^2$. Interesting renormalizable Standard Model extensions can be built incorporating these interactions [21]. Unfortunately, the breaking of Lorentz symmetry must be explicit, because a spontaneous breaking is unable to modify ultraviolet behaviors. Quantum gravity turns Lorentz symmetry into a local symmetry. Like the explicit breaking of any gauge symmetry, the explicit breaking of local Lorentz symmetry causes the propagation of ghosts in the physical sector and so violates unitarity.

Other spacetime dimensions

In general dimension d, the constant κ has dimension $(2-d)/2$, which is negative for $d \geqslant 3$. Thus, quantum gravity in higher dimensions has the same problems as in four dimensions. Three dimensions, on the other hand, are exceptional. The constant κ has dimension $-1/2$, nevertheless pure Einstein gravity is finite to all orders in $d=3$. The reason is that the Weyl tensor $C_{\mu\nu\rho\sigma}$ identically vanishes and so the Riemann tensor $R_{\mu\nu\rho\sigma}$ can be expressed by means of the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R. This means that every counterterm of pure gravity without a cosmological constant is proportional to the Ricci tensor, so it vanishes by using the Einstein field equations. All the counterterms can be subtracted by means of field redefinitions, as we did using (1.5) to remove (1.3), which proves that the theory is finite.

The theory remains finite in $d=3$ if the cosmological term is switched on, because it can be reformulated as a Chern-Simons theory [23]. Generically, it is not finite when it is coupled to matter, which can be proved by explicit computation [24]. Nevertheless, under certain conditions, three-dimensional quantum gravity coupled to matter can be made finite by infinite reduction [25].

2 The classical action of quantum gravity

An enormous effort has been spent in the past decades to explore approaches "beyond quantum field theory", but a convincing proposal has yet to emerge from those investigations. In such a situation, it is worth pursuing quantum gravity within its original framework, the one made of quantum field theory and renormalization, and be open to change our attitudes towards a number of fundamental issues, such as locality and predictivity. Even if the ultimate theory of quantum gravity is still missing, we can work with its nonrenormalizable, perturbatively local version. Like any nonrenormalizable theory, it is predictive in the low-energy regime, where only a finite number of terms really contribute to the physical amplitudes, all others being suppressed by powers of the energy divided by the Planck mass. However, in principle it is also possible to use quantum gravity to make physical predictions beyond the low-energy regime, if we identify physical amplitudes that just depend on a finite subset of parameters.

The first thing to do is to organize the classical action of quantum gravity, determined by renormalization, in a convenient way. Among other things, we know that a term proportional to (1.6) must necessarily be there, multiplied by an independent parameter, because renormalization turns it on at two loops [2, 3]. This is the first departure from Einstein gravity. Even if each correction to Einstein gravity is small, the presence of infinitely many such corrections may originate unforeseen effects, probably belonging to energy domains not tested so far, but detectable in the forthcoming future. Some of those effects might even be insensitive to radiative corrections, in which case the classical action of quantum gravity is sufficient to single them out. In extreme situations, such as inside black holes, or close to the event horizon, or in the primordial phases of the universe, classical corrections of quantum origin may play a relevant role.

Using field redefinitions, the classical action of quantum gravity S_{QG} can be written in different, perturbatively equivalent ways as expansions around the Einstein action. For example, we can rearrange the terms proportional to the Einstein vacuum field equations in the way we like. Equivalent actions can be useful to uncover different classes of exact solutions of the field equations, or reduce the effort to study approximate solutions.

The form of S_{QG} we want to study here contains the Hilbert term, the cosmological term, a peculiar scalar $\hat{\text{G}}$ that is nontrivial in dimensions d larger than four, and then invariants constructed with the Weyl tensor $C_{\mu\nu\rho\sigma}$, rather than the Riemann tensor $R_{\mu\nu\rho\sigma}$. Precisely, we take[26], in arbitrary dimensions $d>2$,

$$S_{\mathrm{QG}}=-\frac{1}{2\kappa^{d-2}}\int\sqrt{|g|}\left(R+2\Lambda+\lambda_0\kappa^2\hat{\mathrm{G}}+\lambda_1\kappa^4C_3+\lambda_1'\kappa^4C_3'\right.$$
$$\left.+\sum_{n=2}^{\infty}\lambda_n\kappa^{2n+2}\mathcal{I}_n(\nabla,C)\right)+S_m \tag{2.15}$$

where κ has dimension -1 in units of mass, λ_n are dimensionless constants, S_m are the contributions of matter fields and other gauge fields, $\hat{\mathrm{G}}$ is the special combination [27]

$$\hat{\mathrm{G}}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2+\frac{4(d-3)(d-4)}{(d-1)(d-2)}\Lambda\left(R+\Lambda\right),$$

and $\mathcal{I}_n(\nabla,C)$ collectively denotes the local scalars of dimension $2n+4$ that can be constructed with three or more Weyl tensors $C_{\mu\nu\rho\sigma}$ and covariant derivatives ∇_μ, up to covariant divergences of vectors. Each such scalar must be multiplied by an independent parameter λ_n. The terms $\mathcal{I}_1(\nabla,C)$ have been written explicitly as contractions of three Weyl tensors:

$$C_3=C_{\mu\nu\rho\sigma}C^{\rho\sigma\alpha\beta}C_{\alpha\beta}^{\ \ \ \mu\nu},\qquad C_3'=C_{\mu\rho\nu\sigma}C^{\alpha\mu\beta\nu}C^{\rho\ \ \sigma}_{\ \alpha\ \beta}.$$

For simplicity we assume parity invariance. Parity violating terms may be treated along the same guidelines.

In four dimensions $\sqrt{-g}\hat{\mathrm{G}}$ is the Gauss-Bonnet integrand, which does not contribute to the field equations. Moreover, C_3 and C_3' are proportional to each other, so we can set $\lambda_1'=0$ in $d=4$. Because of the result of Goroff and Sagnotti [2] the term C_3 is switched on at two loops in pure gravity. That result can be interpreted as the running of the coupling constant λ_1, and allows us to infer that quantum gravity predicts $\lambda_1\neq 0$. In principle, the presence of matter can modify this conclusion, but only if the matter fields exactly cancel the C_3-counterterm generated by pure gravity, and the cancellation is consistent with renormalization-group invariance. As far as we know today, this happens only in supergravity.

It is useful to compare S_{QG} with the most general local perturbative extension S_{loc} of the Einstein action [7]

$$S_{\mathrm{loc}}=-\frac{1}{2\kappa^{d-2}}\int\sqrt{-g}\left(R+2\Lambda+\sum_{n=0}^{\infty}\bar{\lambda}_n\kappa^{2n+2}\bar{\mathcal{I}}_n^{(\Lambda)}(\nabla,\hat{R})\right)+S_m, \tag{2.16}$$

where $\bar{\mathcal{I}}_n^{(\Lambda)}(\nabla,\hat{R})$ denotes the scalars of dimensions $2n+4$ that can be constructed with two or more tensors

$$\hat{R}_{\mu\nu\rho\sigma}=R_{\mu\nu\rho\sigma}+\frac{2\Lambda}{(d-1)\,(d-2)}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right), \tag{2.17}$$

as well as their contractions $\hat{R}_{\mu\nu}=\hat{R}^{\rho}{}_{\mu\rho\nu}$ and $\hat{R}=\hat{R}_{\mu}^{\mu}$, and covariant derivatives ∇_μ, up to covariant divergences of vectors.

The action $S_{\rm QG}$ looks like a restriction on $S_{\rm loc}$, but it is actually perturbatively equivalent to $S_{\rm loc}$. Precisely, the two actions can be mapped into each other by means of local field redefinitions and parameter redefinitions, the parameters λ_n and $\bar{\lambda}_n$ being treated perturbatively. The action $S_{\rm loc}$ is preserved by renormalization, since it is the most general local covariant action. Its perturbative equivalence with $S_{\rm QG}$ proves that $S_{\rm QG}$ is also preserved by renormalization, namely all the divergences generated by the Feynman diagrams can be subtracted away by redefining the metric tensor and the parameters λ_n, as well as the matter fields and the parameters contained inside S_m. Using Bianchi identities, commuting covariant derivatives and integrating by parts every (counter)term that does not explicitly appear inside $S_{\rm QG}$ can be canceled in this way [27]. In particular, the scalar $\hat{\rm G}$ is used to write $\hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma}$ as a linear combination of terms that are present in $S_{\rm QG}$, plus terms quadratically proportional to $\hat{R}_{\mu\nu}$. In turn, these can be converted into terms of $S_{\rm QG}$ by redefining the metric tensor.

Spaces of constant curvature play a peculiar role, since in the absence of matter they are exact solutions of the field equations of the most general covariant action. Indeed, once $R_{\mu\nu\rho\sigma} = K(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ is used, with K =constant, any covariant gravitational field equations must reduce to a simple condition

$$f(\kappa^2 K, \kappa^2\Lambda, \lambda)g_{\mu\nu} = 0, \tag{2.18}$$

where f is some function of the parameters of the theory, which can be solved to obtain K. The parametrizations of (2.15) and (2.16), which use hat tensors or Weyl tensors, are such that the solution of (2.18) simply reads

$$K = -\frac{2\Lambda}{(d-1)(d-2)}. \tag{2.19}$$

The scalar $\hat{\rm G}$ generalizes the Gauss-Bonnet integrand in a convenient way. Its main property is that by expanding the metric tensor around a background $\bar{g}_{\mu\nu}$ of constant curvature K equal to (2.19), the integral $\int\sqrt{-g}\hat{\rm G}$ does not contain terms that are linear or quadratic in the quantum fluctuations. The invariants $\int\sqrt{-g}\mathcal{I}_n$ and $\int\sqrt{-g}\mathcal{I}_n^{(\Lambda)}$, $n \geqslant 1$, clearly have the same property.

In every even dimensions $d = 2k$ we can drop one term $\sim \int\sqrt{-g}C^k$ containing k Weyl tensors and no derivatives and add the topological invariant

$$\int\sqrt{-g}{\rm G}_k \equiv \int\sqrt{-g}\,\delta^{\alpha_1\beta_1\cdots\alpha_k\beta_k}_{\mu_1\nu_1\cdots\mu_k\nu_k}R^{\mu_1\nu_1}{}_{\alpha_1\beta_1}\cdots R^{\mu_k\nu_k}{}_{\alpha_k\beta_k} \tag{2.20}$$

instead, which does not contribute to the field equations. The difference between two such actions is a linear combination of other terms $\sim \int\sqrt{-g}C^k$ plus terms containing the Ricci tensor [28]. Writing the Ricci tensor as a linear combination of $\hat{R}_{\mu\nu}$ and $\Lambda g_{\mu\nu}$, we can reabsorb the difference into a perturbative local field redefinition and parameter redefinitions. For example, in six dimensions we can set $\lambda_1' = 0$ and add $\int\sqrt{-g}{\rm G}_3$.

The invariants (2.20) with $k < d/2$ are not topological. Nevertheless, their variations with respect to the metric tensor are free of higher derivatives [29]. The action of Lovelock gravity [29] in d dimensions contains only the invariants (2.20) with $k \leqslant d/2$. Therefore, its field equations are completely free of higher derivatives. Nevertheless, that kind of action is not preserved by renormalization. For example, in four dimensions Lovelock gravity is just Einstein gravity with a cosmological constant and the Gauss-Bonnet term.

The form (2.15) is convenient for various purposes. For example, it allows us to find interesting classes of exact solutions of the field equations, besides the spaces of constant curvature. Examples are all the locally conformally flat metrics (which we just call "conformally flat" from now on) that solve the Einstein equations. In particular, the Friedmann-Lemaître-Robertson-Walker (FLRW) metrics are exact solutions of the S_{QG} field equations in arbitrary dimensions $d > 2$ with a homogeneous and isotropic matter distribution. In four dimensions such solutions coincide with the usual ones, while in higher dimensions they coincide with the usual solutions once the energy density ρ and the pressure p are replaced by suitable functions of ρ and p. Metric independent maps also relate conformally flat solutions of the S_{QG} field equations to conformally flat solutions of the Einstein equations. On the other hand, solutions that are not conformally flat, such as the Schwarzschild and Kerr metrics, are deformed in nontrivial ways by the couplings λ_n.

Another property of S_{QG} is that by expanding the metric tensor around conformally flat backgrounds the quadratic part of the action is free of higher derivatives. Vertices, instead, as well as quadratic terms obtained by expanding around more general backgrounds, do not have this feature. Working perturbatively in the couplings λ_n, every term of the field equations that contains higher derivatives can be unambiguously resolved. Then the solutions of the S_{QG} field equations are uniquely determined by their $\lambda_n \rightarrow 0$ limits. The absence of higher derivatives in the quadratic part of the action is important to prevent the propagation of unphysical degrees of freedom, such as those of higher-derivative quantum gravity (1.7).

The matter action S_m is the most general local one, as long as it has correct unitary propagators. If the classical action has correct propagators, the renormalized one also has. Indeed, in a quantum field theory of matter fields of spins $\leqslant 1/2$ and gauge fields of spins $\leqslant 2$, higher-derivative quadratic terms are not turned on by renormalization if they are absent at the tree level [27].

Field equations

Writing

$$S_{\text{QG}} = -\frac{1}{2\kappa^{d-2}} \int \sqrt{-g}\,(R + 2\Lambda) + S_m + S^{(g)},$$

the S_{QG}-field equations read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^{d-2} T_{\mu\nu} + \kappa^{d-2} T^{(g)}_{\mu\nu}, \tag{2.21}$$

where

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}, \qquad T^{(g)}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S^{(g)}}{\delta g^{\mu\nu}},$$

are the matter energy-momentum tensor and the gravitational self-energy-momentum tensor, respectively. Varying $\int\sqrt{-g}\hat{\mathrm{G}}$ explicitly, we find

$$\begin{aligned}\kappa^{d-2}T^{(g)}_{\mu\nu} = &-\lambda_0\kappa^2\left[2C_{\mu\rho\sigma\alpha}C_\nu{}^{\rho\sigma\alpha} - \frac{1}{2}g_{\mu\nu}C_{\rho\sigma\alpha\beta}C^{\rho\sigma\alpha\beta} - \frac{4(d-4)}{d-2}C_{\mu\rho\nu\sigma}\hat{R}^{\rho\sigma}\right.\\ &\left.-\frac{(d-3)(d-4)}{(d-2)^2}\left(4\hat{R}_{\mu\rho}\hat{R}^\rho_\nu - 2g_{\mu\nu}\hat{R}_{\rho\sigma}\hat{R}^{\rho\sigma} - \frac{2d}{d-1}\hat{R}_{\mu\nu}\hat{R} + \frac{d+2}{2(d-1)}g_{\mu\nu}\hat{R}^2\right)\right]\\ &+\mathcal{O}(\nabla^2C^2) + \mathcal{O}(RC^2) + \mathcal{O}(C^3).\end{aligned} \tag{2.22}$$

Observe that the variation of $\int\sqrt{-g}\hat{\mathrm{G}}$ with respect to the metric is $\mathcal{O}(\hat{R}^2)$, where the notation $\mathcal{O}(\hat{R}^n)$ means terms containing at least n powers of $\hat{R}_{\mu\nu\rho\sigma}$ and its contractions. Clearly, the tensor $T^{(g)}_{\mu\nu}$ of (2.22) is identically zero in three dimensions. In four dimensions, instead, it reduces to the last line of (2.22). For future use we explicitly work out the first non-trivial contributions to $T^{(g)}_{\mu\nu}$ in $d=4$, which are the ones proportional to the *Goroff-Sagnotti (GS) constant* $\Lambda_{\mathrm{GS}} \equiv 3\lambda_1\kappa^4$. Setting $\lambda_1' = 0$ and dropping the Gauss-Bonnet term, we write the four dimensional action as

$$S^{(d=4)}_{\mathrm{QG}} = -\frac{1}{2\kappa^2}\int\sqrt{-g}\left(R + 2\Lambda + \frac{\Lambda_{\mathrm{GS}}}{3}C_3 + \sum_{n=2}^{\infty}\lambda_n\kappa^{2n+2}\mathcal{I}_n(\nabla, C)\right) + S_m.$$

Then we find

$$\begin{aligned}\kappa^2T^{(g)}_{\mu\nu} = \Lambda_{\mathrm{GS}}&\left(\nabla^\rho\nabla^\sigma C^{(2)}_{\mu\rho\sigma\nu} + \nabla^\rho\nabla^\sigma C^{(2)}_{\nu\rho\sigma\mu} - \frac{1}{2}C^{(2)}_{\mu\alpha\rho\sigma}R_\nu{}^{\alpha\rho\sigma} - \frac{1}{2}C^{(2)}_{\nu\alpha\rho\sigma}R_\mu{}^{\alpha\rho\sigma}\right.\\ &\left.+\frac{1}{6}g_{\mu\nu}C_3 - \frac{1}{6}\nabla_\mu\nabla_\nu C_2 + \frac{1}{6}g_{\mu\nu}\nabla^2C_2 + \frac{1}{6}R_{\mu\nu}C_2\right)\\ &+\mathcal{O}(\nabla^4C^2) + \mathcal{O}(\nabla^2C^3) + \mathcal{O}(C^4),\end{aligned} \tag{2.23}$$

where

$$C^{(2)}_{\mu\nu\rho\sigma} = C_{\mu\nu\alpha\beta}C^{\alpha\beta}{}_{\rho\sigma}, \qquad C_2 = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}.$$

In the list of higher orders that appears in the third line of (2.23) it is understood that pairs of covariant derivatives can be replaced by curvature tensors, so $\mathcal{O}(\nabla^4C^2) = \mathcal{O}(\nabla^2RC^2)$, etc.

As promised, when the metric tensor is expanded around the metric $\overline{g}_{\mu\nu}$ of a space of constant curvature, an FLRW metric, or more generally a conformally flat metric, the quadratic part of the expanded action S_{QG} does not contain higher derivatives. We can prove this fact by considering the variation of $T^{(g)}_{\mu\nu}$ with respect to the metric. The first two lines of (2.22) give contributions that contain at most two derivatives of the fluctuation. The third line of (2.22) gives contributions that are proportional to the Weyl tensor, and vanish on conformally flat metrics. If $\overline{g}_{\mu\nu}$ does not belong to these classes of backgrounds then the quadratic part of the action may contain higher derivatives. In general, vertices do contain higher derivatives of $g_{\mu\nu}$, multiplied by the couplings λ_n.

To understand how to deal with such higher derivatives, recall that renormalization, which is responsible for turning on the couplings λ_n, is purely perturbative. To be consistent, the action $S_{\rm QG}$ must be treated perturbatively in the λ_ns. In particular, we must search for solutions of the field equations that are analytic in the λ_ns, at least away from singularities. Such solutions exist and are uniquely determined by their limits $\lambda_n \rightarrow 0$. Indeed, the field equations contain at most two time derivatives at $\lambda_n = 0$. Therefore, working perturbatively in λ_n we can express the terms that contain higher derivatives as nonlocal averages of the external sources. In this way we obtain new field equations that are perturbatively equivalent to (2.21).

Similar methods are commonly used to eliminate runaway solutions caused by higher-time derivatives, as in the case of the Abraham-Lorentz force in classical electrodynamics [30]. For applications to gravity see refs. [31, 32, 8]. The elimination of unphysical solutions has a price, because it generates violations of microcausality [30].

These facts, together with the presence of infinitely many independent couplings, are there to remind us that $S_{\rm QG}$ is not the action of a fundamental theory, but must be viewed as an effective action that can be obtained from a more complete theory in a particular limit or integrating out some massive fields. In the same way as the Fermi theory of weak interactions helped building the Standard Model, studying the properties of $S_{\rm QG}$ can be useful to identify the missing ultimate theory of quantum gravity, which should be unitary, causal (but non necessarily microcausal) and renormalizable with a finite number of independent couplings.

3 Exact solutions

In this section we derive exact solutions of the $S_{\rm QG}$ field equations and relate them to known solutions of the Einstein equations. Any solution of $S_{\rm QG}$ can be perturbatively mapped into a solution of the field equations of any action that is perturbatively equivalent to $S_{\rm QG}$, for example $S_{\rm loc}$.

We begin observing that in four dimensions all the conformally flat metrics that solve the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{3.24}$$

also solve the $S_{\rm QG}$ field equations (2.21), and vice versa. The reason is that when $d = 4$ and $C_{\mu\rho\nu\sigma} = 0$ formulas (2.22) and (2.23) ensure that the gravitational self-energy-momentum tensor $T_{\mu\nu}^{(g)}$ identically vanishes. Moreover, the variation of $T_{\mu\nu}^{(g)}$ with respect to the metric is proportional to the Weyl tensor. Therefore, it also vanishes on conformally flat metrics. If we expand the metric tensor around conformally flat backgrounds in four dimensions, the propagator coincides with the one of Einstein gravity (if the same gauge-fixing is used).

Now, if $d\Omega_{d-2}^2$ denotes the standard metric of the $(d-2)$-dimensional sphere of unit radius, the metrics

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_{d-2}^2\right) \tag{3.25}$$

of homogeneous and isotropic spaces are conformally flat in arbitrary dimensions $\geqslant 4$. Indeed, it is easy to prove that the Weyl tensor vanishes everywhere. The FLRW metrics have the form (3.25) and solve (3.24) with a homogeneous and isotropic distribution of matter, described by an energy-momentum tensor T_μ^ν equal to

$$T_\mu^\nu(\rho, p) = \rho\delta_\mu^0\delta_0^\nu - p\sum_{i=1}^{d-1}\delta_\mu^i\delta_i^\nu, \tag{3.26}$$

where the energy density ρ and the pressure p can be time-dependent.

Thus, the FLRW metrics are exact solutions of the $S_{\rm QG}$ field equations (2.21) in four dimensions.

FLRW solutions in arbitrary dimensions

In higher dimensions we have to take the term $\int\sqrt{-g}\hat{\rm G}$ into account. Nevertheless, in the classes of FLRW metrics and conformally flat metrics we can find metric-independent maps that convert solutions of the Einstein equations into solutions of the $S_{\rm QG}$ field equations, and vice versa. Consider the $S_{\rm QG}$ field equations (2.21) with matter energy-momentum tensor given by (3.26). We want to show that the FLRW metrics (3.25) that solve

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^{d-2}T_{\mu\nu}(\tilde{\rho}, \tilde{p}) \tag{3.27}$$

also solve (2.21), and vive versa, where $\tilde{\rho}$ and $\tilde{p}$ are suitable functions of ρ and p. Inserting (3.27) into (2.21) we find that this statement is true if and only if

$$T_{\mu\nu}(\tilde{\rho}, \tilde{p}) = T_{\mu\nu}(\rho, p) + T_{\mu\nu}^{(g)}. \tag{3.28}$$

Using (3.27) inside (2.22) (and recalling that $C_{\mu\nu\rho\sigma} = 0$) we easily get

$$T^{(g)}{}_\mu^\nu = \Lambda_0\tilde{\rho}\left(\tilde{\rho}\delta_\mu^0\delta_0^\nu - (\tilde{\rho} + 2\tilde{p})\sum_{i=1}^{d-1}\delta_\mu^i\delta_i^\nu\right),$$

where

$$\Lambda_0 = 2\lambda_0\kappa^d\frac{(d-3)(d-4)}{(d-2)(d-1)},$$

so equation (3.28) is equivalent to the pair of metric-independent quadratic equations

$$\rho = \tilde{\rho} - \Lambda_0 \tilde{\rho}^2, \qquad p = \tilde{p} - \Lambda_0 \tilde{\rho}(\tilde{\rho} + 2\tilde{p}), \tag{3.29}$$

for $\tilde{\rho}$ and $\tilde{p}$.

Given ρ and p, we determine $\tilde{\rho}$ and $\tilde{p}$ by solving the equations (3.29). Then the usual FLRW solution with energy density $\tilde{\rho}$ and pressure $\tilde{p}$ solves the S_{QG} field equations with energy density ρ and pressure p. Assuming $\rho\Lambda_0, p\Lambda_0 \ll 1$ the solution can be worked out perturbatively. The cases $d = 3, 4$ can be seen as particular cases of the more general solution.

Observe that in higher dimensions when we expand the metric around FLRW backgrounds the propagator does not coincide with the one obtained in Einstein gravity (even if we use the same gauge-fixing). Nevertheless, formula (2.22) shows that the quadratic part of the expanded action S_{QG} does not contain higher derivatives. Indeed, it is just affected by terms $\sim \tilde{\rho}\bigtriangledown^2$ and $\sim \tilde{p}\bigtriangledown^2$, and terms with fewer derivatives.

Conformally flat solutions in arbitrary dimensions

More generally, if $\tilde{T}_\mu^\nu$ and T_μ^ν are related by the metric independent polynomial equation

$$T_\mu^\nu = \tilde{T}_\mu^\nu - \Lambda_0 \frac{d-1}{d-2}\left(2\tilde{T}_\mu^\rho \tilde{T}_\rho^\nu - \delta_\mu^\nu \tilde{T}_2 - \frac{2}{d-1}\tilde{T}_\mu^\nu \tilde{T} + \frac{1}{d-1}\delta_\mu^\nu \tilde{T}^2\right), \tag{3.30}$$

where $\tilde{T} = \tilde{T}_\rho^\rho$ and $\tilde{T}_2 = \tilde{T}_\rho^\sigma \tilde{T}_\sigma^\rho$, then the conformally flat metrics that solve

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^{d-2} \tilde{T}_{\mu\nu} \tag{3.31}$$

also solve the S_{QG} field equations, and vice versa. The condition (3.30) is obtained inserting (3.31) into (2.22) and (2.21), and using $C_{\mu\nu\rho\sigma} = 0$. Expanding the metric tensor around a conformally flat solution the quadratic part of the action S_{QG} is free of higher derivatives.

4 Approximate solutions

From the observational point of view, deformed black-hole solutions can offer interesting possibilities to test modifications of general relativity. Deviations from the Kerr metric, in particular, are the easiest to detect. Since black-hole solutions are not conformally flat, they are affected in a non-trivial way by the corrections to Einstein gravity contained in S_{QG}.

To illustrate these effects, we work in four dimensions and in the absence of matter, and keep only the GS constant Λ_{GS}, besides the Newton constant $G = \kappa^2/8\pi$ and the cosmological constant Λ. The action reads

$$S'_{\mathrm{QG}} = -\frac{1}{2\kappa^2}\int \sqrt{-g}\left(R + 2\Lambda + \frac{\Lambda_{\mathrm{GS}}}{3} C_3\right). \tag{4.32}$$

We begin looking for spherically symmetric solutions of the form

$$ds^2 = \mathrm{e}^{\nu(r)+\omega(r)}dt^2 - \mathrm{e}^{-\nu(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.33}$$

Inserting the ansatz (4.33) into the field equations (2.21) and using (2.23) we find differential equations for $\nu(r)$ and $\omega(r)$. The Λ_{GS} dependent contributions involve up to four derivatives of these functions. We must search for solutions that are analytic in Λ_{GS}, at least away from singularities. Thus we can work iteratively in Λ_{GS}, which allows us to convert the higher-derivative terms into terms that have at most two derivatives. After this conversion we find two (involved) equations of the form

$$\nu' = F_1(\nu, \omega, r), \qquad \omega' = F_2(\nu, \omega, r), \tag{4.34}$$

for certain functions F_1 and F_2 that are analytic in Λ_{GS}, and two other equations that are automatically satisfied when (4.34) hold. We see that the solutions certainly exist and are uniquely determined by their limits $\Lambda_{\mathrm{GS}} \to 0$. To the lowest order of approximation, the solutions are

$$\begin{aligned} \mathrm{e}^{\nu(r)} &= 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2 + \frac{6\Lambda_{\mathrm{GS}}r_s^2}{r^6}\left(1 - \frac{8r_s}{9r} - \frac{4}{9}\Lambda r^2\right) + \mathcal{O}(\Lambda_{\mathrm{GS}}^2), \\ \omega(r) &= -\frac{4\Lambda_{\mathrm{GS}}r_s^2}{r^6} + \mathcal{O}(\Lambda_{\mathrm{GS}}^2), \end{aligned} \tag{4.35}$$

$r_s = 2Gm$ being the usual Schwarzschild radius. Higher-order corrections show that the solution has the form

$$-g_{rr}^{-1} = \mathrm{e}^{\nu(r)} = 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2 + \frac{r_s}{r}\sum_{n=1}^{\infty}\xi^n P_n, \qquad \omega(r) = \frac{r_s}{r}\sum_{n=1}^{\infty}\xi^n Q_{n-1}, \tag{4.36}$$

where

$$\xi(r) = \frac{\Lambda_{\mathrm{GS}}r_s}{r^5}$$

and P_n, Q_n are polynomials of degree n in r_s/r and Λr^2. It is easy to verify that the expansion of g_{tt} has the same form as the one of $-g_{rr}^{-1}$. Thus the approximation obtained by expanding in powers of Λ_{GS} is valid for $\xi \ll 1$, with r_s/r and Λr^2 bounded.

The violations of microcausality induced by the presence of higher time derivatives can be studied by considering a fluctuation δg around the metric given by (4.33) and (4.35). Higher-time derivative terms provided by $\delta T_{\mu\nu}^{(g)}$ are multiplied by the Weyl tensor $C \sim r_s/r^3$ or by $\nabla C \sim r_s/r^4$:

$$\kappa^2 \delta T_{\mu\nu}^{(g)} \sim \Lambda_{\mathrm{GS}} \nabla C \nabla^3 \delta g + \Lambda_{\mathrm{GS}} C \nabla^4 \delta g$$

Comparing these terms with the ones contained in the Einstein equations and assuming that the derivatives of δg are time ones, for $\xi(r) < 1$ causality violations last for a typical time equal to

$$\tau(r) = r\sqrt{\xi(r)}.$$

In the case of gravitational lensing by a light black hole, taking r around a few times r_s and assuming $\xi(r) \sim 1$ it is necessary to resolve time intervals of about 10^{-4} seconds.

Now we switch to the modified Kerr metric. We study the large distance expansion of the deformed Kerr metric at $\Lambda = 0$. Precisely, we take $r_s, a \sim \varepsilon$ and $\Lambda_{\text{GS}} \sim \varepsilon^4$, $\varepsilon \ll 1$ (i.e. we assume that the constants r_s, a and Λ_{GS} are of orders equal to their dimensions in units of length), and calculate the metric to the order ε^8. Doing so, we automatically exclude orders of Λ_{GS} larger than the first. In Boyer-Lindquist coordinates we write

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi$$

and obtain

$$\begin{aligned} g_{tt} &= 1 - \frac{rr_s}{\rho^2} + \frac{2\Lambda_{\text{GS}}r_s^2}{3\rho^8}(3\rho^2 - 2rr_s - 54a^2\cos^2\theta), \qquad g_{\theta\theta} = -\rho^2 + \frac{6a^2r_s^2}{\rho^6}\Lambda_{\text{GS}}\sin^2\theta, \\ g_{rr} &= -\frac{\rho^2}{\Delta} + \frac{2\Lambda_{\text{GS}}r_s^2}{3\rho^6\Delta}\left(9a^2 + 9\rho^2 + rr_s + r_s^2 - 297a^2\cos^2\theta\right), \\ g_{t\phi} &= \frac{arr_s}{\rho^2}\sin^2\theta\left(1 + \frac{4\Lambda_{\text{GS}}r_s^2}{3\rho^6}\right), \qquad g_{\phi\phi} = -\sin^2\theta\left(a^2 + r^2 + \frac{a^2rr_s}{\rho^2}\sin^2\theta\right), \end{aligned} \tag{4.37}$$

where, as usual,

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - rr_s + a^2.$$

We stress again that renormalization predicts $\Lambda_{\text{GS}} \neq 0$. Therefore, the deviations just reported can be viewed as predictions of quantum gravity. Their practical detectability depends on the actual value of the constant Λ_{GS}. Theoretically, we cannot predict the exact value of Λ_{GS}, but only the Λ_{GS} running, which gives us an estimate of the minimum value of $|\Lambda_{\text{GS}}|$. Using the two-loop result of ref. [2] we find

$$\Delta\Lambda_{\text{GS}}(\ell, \ell') = \Lambda_{\text{GS}}(\ell) - \Lambda_{\text{GS}}(\ell') = f\frac{l_P^4}{\pi^2}\ln\frac{\ell}{\ell'},$$

where $\Lambda_{\text{GS}}(x)$ is the running coupling at the scale x, $l_P = \sqrt{G}$ is the Planck length and f is a numerical factor of order 1. If we take ℓ equal to the diameter of the observable universe and ℓ' equal to the Planck length itself, we obtain

$$|\Delta\Lambda_{\text{GS}}| \sim l_P^4.$$

If the value of $|\Lambda_{\text{GS}}|$ were around l_P^4 there would be no chance to detect the deviations we have worked out so far. We can only hope that $|\Lambda_{\text{GS}}|$ has a much larger value in nature. Light black holes are the ones that are affected more sensibly. Taking a mass equal to 5 solar masses, we need at least

$$|\Lambda_{\text{GS}}| \sim 10^{156}l_P^4 = 10^{44}(\text{eV})^{-4} \tag{4.38}$$

to get $\xi_s \sim 1$. In this case the deviations would be appreciable right outside the black hole. The Schwarzschild radius would be modified in a sensible way and effects on the deflection of light, for example, could be detected. Depending on the precision of our instruments, smaller values of ξ_s could suffice. In case no deviations are observed it is possible to put experimental bounds on $\Lambda_{\rm GS}$. Observe that as long as $|\Lambda_{\rm GS}|$ is much larger than l_P^4, for all practical purposes $\Lambda_{\rm GS}$ does not run throughout the universe.

5 Conclusions

The nonrenormalizability of quantum gravity is a challenging problem that forces us to investigate unexplored sectors of quantum field theory, and maybe change our attitudes towards a number of fundamental issues. Most attempts to make it renormalizable generate violations of unitarity. A possibility to have both renormalizability and unitarity at the same time is to relax the locality assumption. A more conservative standpoint is to work with quantum gravity as a nonrenormalizable theory, which means get used to deal with infinitely many parameters and search for physical predictions that just depend on a finite subset of them. In this perspective, the first thing to do is to organize the classical action of quantum gravity, as determined by renormalization, in the most convenient way.

Using perturbative field redefinitions and parameter redefinitions, we can single out a form $S_{\rm QG}$ that allows us to show that some well known metrics, such as the FLRW metrics, are exact solutions of the field equations, or can be mapped into exact solutions. More generally, all the conformally flat solutions of Einstein gravity can be mapped in a metric independent way into conformally flat solutions of $S_{\rm QG}$, and vice versa. The quadratic terms of the action, generated by expanding the metric around these solutions, are free of higher derivatives. Solutions that are not conformally flat are instead modified in a nontrivial way. We have studied the first corrections to the metrics of the Schwarzschild and Kerr types, by expanding in powers of the GS constant. Renormalization only tells us that this constant is nonvanishing, but is unable to predict its actual value. It would be desirable to put constraints on its magnitude by comparing predictions and observational data.

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