IFUP-TH 2006/24

Renormalizable Acausal Theories Of Classical Gravity Coupled With Interacting Quantum Fields

Damiano Anselmi and Milenko Halat
Dipartimento di Fisica "Enrico Fermi", Università di Pisa,
Largo Pontecorvo 3, I-56127 Pisa, Italy,
and INFN, Sezione di Pisa, Pisa, Italy
damiano.anselmi@df.unipi.it, milenko.halat@df.unipi.it

Abstract

We prove the renormalizability of various theories of classical gravity coupled with interacting quantum fields. The models contain vertices with dimensionalities greater than four, a finite number of matter operators and a finite or reduced number of independent couplings. An interesting class of models is obtained from ordinary power-counting renormalizable theories, letting the couplings depend on the scalar curvature R of spacetime. The divergences are removed without introducing higher-derivative kinetic terms in the gravitational sector. The metric tensor has a non-trivial running, even if it is not quantized. The results are proved applying a certain map that converts classical instabilities, due to higher derivatives, into classical violations of causality, whose effects become observable at sufficiently high energies. We study acausal Einstein-Yang-Mills theory with an R-dependent gauge coupling in detail. We derive all-order formulas for the beta functions of the dimensionality-six gravitational vertices induced by renormalization, such as $RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Such beta functions are related to the trace-anomaly coefficients of the matter subsector.

1 Introduction

The renormalization of quantum field theory in curved space has been widely studied [1]. Treating the metric tensor as a c-number and neglecting its quantum fluctuations, classical gravity coupled with quantized fields can be used as a low-energy effective field theory, to include the radiative corrections to the Einstein field equations generated by the matter fields circulating in the loops. Moveover, it provides an interesting arena and a laboratory to test ideas about renormalizability beyond power counting. Although there exist persuasive reasons to believe that gravity must be quantized, definitive theoretical arguments and experimental proofs are still missing [2, 3, 4, 5, 6]. Thus it is meaningful to study the physical consequences of the assumption that classical gravity coupled with quantized fields is a fundamental theory, valid at arbitrarily high energies, instead of just an effective one.

When matter is embedded in a curved background, renormalization generates the gravitational counterterms $R_{\mu\nu}R^{\mu\nu}$ and R^2 [7]. Such counterterms can be renormalized in two ways. One possibility is to add the same terms to the lagrangian, if they are not already present, multiplied by independent couplings, and reabsorb the divergences into those couplings. The resulting theory is higher-derivative. Expanding the metric tensor around flat space, the lagrangian contains higher-derivative kinetic terms, which are responsible for instabilities at the classical level and violations of unitarity at the quantum level. In particular, higher-derivative quantum gravity is renormalizable, but not unitary [8]: the propagator falls off sufficiently rapidly at high energies to ensure power-counting renormalizability, but propagates ghosts. An alternative way to remove counterterms, that applies only when they have an appropriate form, is to use field redefinitions. Applied to $R_{\mu\nu}R^{\mu\nu}$ and R^2, field redefinitions can convert the undesirable higher-derivative kinetic terms into new types of vertices that couple gravity to matter.

In a recent paper [9] it was shown that in classical gravity coupled with quantum matter the second method of subtraction can be consistently implemented to all orders in the perturbative expansion, redefining the metric tensor by means of a certain map $\mathcal{M}$. Since the terms $R_{\mu\nu}R^{\mu\nu}$ and R^2 are proportional to the vacuum field equations, a metric redefinition $g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$ can obviously reabsorb them into the Einstein term to the first order in $\delta g_{\mu\nu}$. In the presence of matter the redefinition generates vertices that couple the matter stress-tensor to the Ricci tensor. The map $\mathcal{M}$ promotes such a field redefinition to all orders. This is possible because $R_{\mu\nu}R^{\mu\nu}$ and R^2 are not only proportional, but also *quadratically* proportional to the Einstein vacuum field equations. If gravity is classical, the map $\mathcal{M}$ preserves the renormalizability of the theory. However, while the map $\mathcal{M}$ eliminates the higher-derivative kinetic terms, and therefore the instabilities, in general it produces classical violations of causality, detectable in principle at high energies.

In this paper we prove the renormalizability of new types of acausal theories of classical gravity coupled with quantized fields. In a class of theories the matter sector contains all composite operators that have dimensionalities smaller than or equal to four and the gravitational sector contains arbitrary functions of the metric, therefore an infinite number of independent couplings. The arbitrariness of such theories can be reduced, preserving the renormalizability, by appropriate *reductions of couplings*. Specifically, we prove the renormalizability of the models that are obtained from ordinary power-counting renormalizable theories embedded in curved space, when the couplings are allowed to depend on the scalar curvature R of spacetime. In these models the arbitrariness is reduced to a few functions of R. Every models has a higher-derivative version and an acausal version, and we can switch between the two using the map $\mathcal{M}$.

In particular, we study acausal Einstein-Yang-Mills theory with an R-dependent gauge coupling. Renormalization induces new pure gravitational vertices, such as $RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. We work out exact formulas for the beta functions of the dimensionality-six gravitational vertices, which are related to the trace-anomaly coefficients of the matter subsector.

Our arguments, in particular the map $\mathcal{M}$, do not extend to quantum gravity. Nevertheless, our techniques and results might have some impact also on the task of quantizing gravity.

The paper is organized as follows. In section 2 we recall the results of [9] that are used here. In section 3 we apply them to describe acausal Einstein-Yang-Mills theory explicitly. In section 4 we prove a first generalization of the results of [9], proving the renormalizability of a more general class of theories, but with essentially the same types of vertices as in [9]. In section 5 we prove the renormalizability of more general acausal theories, where the matter sector contains all composite operators that have dimensionalities smaller than or equal to four. We prove the existence of consistent reductions of couplings and the renormalizability of the models obtained giving an R-dependence to the couplings of power-counting renormalizable theories. In particular, we study acausal Einstein-Yang-Mills theory with an R-dependent gauge coupling. Section 6 contains the conclusions. In appendices A, B and C we show how to work out the map $\mathcal{M}$ without using bitensors, write the map $\mathcal{M}$ for gravity to the second order in arbitrary spacetime dimension and perform a detailed analysis of the renormalizability of Einstein-Yang-Mills theory with an R-dependent gauge coupling using the Batalin-Vilkovisky formalism [10, 11].

2 The map $\mathcal{M}$ and its usage

In this section we set the notation and recall the results of ref. [9] that are used or generalized in this paper.

2.1 Action and field equations for partially quantum, partially classical theories

We consider partially quantum, partially classical field theories. Let φ_c denote the classical fields, with action $S_c[\varphi_c]$, and φ the quantized fields, with classical action $S[\varphi, \varphi_c]$, embedded in the external φ_c-background. Call $\Gamma[\Phi, \varphi_c]$ the generating functional of one-particle irreducible diagrams obtained quantizing the fields φ in the φ_c-background, where $\Phi = \langle \varphi \rangle$. Then the total action $S_{\text{tot}}[\varphi_c, \varphi_q]$ of the partially classical, partially quantum theory is defined as

$$S_{\text{tot}}[\varphi_c, \varphi_q] = S_c[\varphi_c] + \operatorname{Re}\Gamma[\Phi, \varphi_c], \tag{2.1}$$

where $\varphi_q = \Phi$ and Φ is real if the fields φ are real bosonic, while Φ is the conjugate of $\overline{\Phi} = \langle \overline{\varphi} \rangle$ if the fields φ are complex or fermionic. For example, for classical gravity coupled with quantum matter, φ_c is the metric tensor $g_{\mu\nu}$ and S_c is the Einstein action, so

$$S_{\text{tot}}[g, \varphi_q] = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g}\, [R(g) - 2\Lambda] + \operatorname{Re}\Gamma[\varphi_q, g]. \tag{2.2}$$

The field equations are obtained functionally variating the action with respect to $g_{\mu\nu}$ and φ_q. For example, a simple way to solve the matter field equations is to set $\varphi_q = 0$ (or φ_q equal to its expectation value, if there is a spontaneous symmetry breaking). Then the gravitational field equations $\delta S_{\text{tot}}[g, 0]/\delta g_{\mu\nu} = 0$, i.e.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = -\kappa^2 \operatorname{Re}\langle T_{\mu\nu} \rangle, \qquad \langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta\Gamma[\varphi_q, g]}{\delta g^{\mu\nu}} \tag{2.3}$$

describe how the spacetime geometry is affected by the quantized matter fields circulating in the loops.

Another approach to the semi-classical theory, due to Schwinger and Keldysh [12], is to replace $\operatorname{Re}\langle T_{\mu\nu} \rangle$ in (2.3) with the "in-in" expectation value of the stress tensor, which is both real and causal. Functional methods for the calculation of in-in expectation values have been developed [13, 14]. It is important to observe that the renormalization structure of the theory does not depend on the interpretation of the right-hand side of the Einstein equations in (2.3). In particular, the counterterms $R_{\mu\nu}R^{\mu\nu}$ and R^2 are identical in the Schwinger-Kleydish approach [14]. The high-energy causality violations discussed here are an effect due to the renormalization of $R_{\mu\nu}R^{\mu\nu}$ and R^2 by means of field redefinitions of the metric tensor, so they are independent of the generalization of (2.3) to quantum field theory.

2.2 The map $\mathcal{M}$

Consider an action S depending on the fields ϕ and modify it into

$$S'[\phi] = S[\phi] + S_i F_{ij} S_j, \tag{2.4}$$

where F_{ij} is symmetric and can contain derivative operators, $S_i \equiv \delta S/\delta\phi_i$ are the S-field equations, the index i stands also for the spacetime point and summation over repeated indices, including the integration over spacetime points, is understood. There exists a field redefinition

$$\phi_i' = \phi_i + \Delta_{ij} S_j, \tag{2.5}$$

with Δ_{ij} symmetric, such that, perturbatively in F and to all orders in powers of F,

$$S'[\phi] = S[\phi']. \tag{2.6}$$

Indeed, after a Taylor expansion, it is immediate to see that this equality is verified if

$$\Delta_{ij} = F_{ij} - \Delta_{k_1 i}\Delta_{k_2 j}\sum_{n=2}^{\infty}\frac{1}{n!}S_{k_1 k_2 k_3\cdots k_n}\prod_{l=3}^{n}(\Delta_{k_l m_l}S_{m_l}). \tag{2.7}$$

where $S_{k_1\cdots k_n} \equiv \delta^n S/(\delta\phi_{k_1}\cdots\delta\phi_{k_n})$ and for $n = 2$ the product is meant to be unity. Equation (2.7) can be solved recursively for Δ in powers of F. The first terms of the solution are

$$\begin{aligned}\Delta_{ij} = F_{ij} - \frac{1}{2}F_{ik_1}S_{k_1k_2}F_{k_2j} + \frac{1}{2}F_{ik_1}S_{k_1k_2}F_{k_2k_3}S_{k_3k_4}F_{k_4j}\\ -\frac{1}{3!}F_{ik_1}S_{k_1k_2k_3}F_{k_3k_4}S_{k_4}F_{k_2j} + \mathcal{O}(F^4).\end{aligned}$$

For example, take an ordinary free field theory

$$S[\phi] = \frac{1}{2}\phi_i S_{ij}\phi_j.$$

Then $S_{k_1\cdots k_n} = 0$ for every $n > 2$, while $S_{k_1k_2}$ is field-independent and quadratic in the derivatives. The modified action

$$S'[\phi] = \frac{1}{2}\phi_i\left(S_{ij} + 2S_{ik}F_{km}S_{mj}\right)\phi_j$$

describes a higher-derivative theory. Equation (2.7) simplifies to

$$\Delta_{ij} = F_{ij} - \frac{1}{2}\Delta_{k_1i}\Delta_{k_2j}S_{k_1k_2}.$$

Its solution reads, in matrix and vector form,

$$\Delta = \left(\sqrt{1+2FS} - 1\right)S^{-1}, \qquad \phi' = \sqrt{1+2FS}\,\phi.$$

The map is not just a change of variables, since it changes the degrees of freedom of the theory.

In the interacting case, we use the free-field limit results to write

$$\Delta = \left(\sqrt{1+2FS} - 1\right)S^{-1} + \mathcal{O}(\phi), \qquad \phi' = \sqrt{1+2FS}\,\phi + \mathcal{O}(\phi^2), \tag{2.8}$$

where F and S are the matrices F_{ij} and S_{ij} calculated at $\phi = 0$. Thus in the acausal theory every ϕ'-leg gets multiplied by $1/\sqrt{1+2FS}$.

With a source term the map gives

$$S_{\mathrm{HD}}[\phi, J] \equiv S'[\phi] + \phi_k J_k = S[\phi'] + \phi'_k J'_k(J) \equiv S_{\mathrm{AC}}[\phi', J],$$

where

$$J'(J) = \frac{1}{\sqrt{1+2FS}} J. \tag{2.9}$$

The action $S_{\mathrm{HD}}[\phi, J]$ describes a higher-derivative theory, while the action $S_{\mathrm{AC}}[\phi', J]$ describes, in general, an acausal theory. To see this more clearly, take for example $F_{ij} = \alpha^2\delta_{ij}/2$, $S_{ij} = -\Box\delta_{ij}$ then

$$J'(x) = \frac{1}{\sqrt{1-\alpha^2\Box}} J = \int \mathrm{d}^n x' \ \mathcal{C}_n(x-x')J(x'), \tag{2.10}$$

where n is the spacetime dimension and

$$\mathcal{C}_n(x) = \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{\mathrm{e}^{-ip\cdot x}}{\sqrt{1+\alpha^2 p^2}}. \tag{2.11}$$

The Fourier transform (2.11) has to be defined with an appropriate prescription. The degrees of freedom that are responsible for the instabilities in the higher-derivative model are suppressed demanding that the prescription be regular in the limit $\alpha \to 0$. For example,

$$\mathcal{C}_n^{\mathrm{F}}(x) = \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{\mathrm{e}^{-ip\cdot x}}{\sqrt{1+\alpha^2 p^2 + i\varepsilon}}. \tag{2.12}$$

Observe that $\mathcal{C}_n^{\mathrm{F}}(x)$ is complex, but the definition (2.1) of the action takes care of this.

Although the map is perturbative in F, formula (2.12) allows us to study some effects of the resummation of derivatives. The function $\mathcal{C}_n^{\mathrm{F}}(x)$ does not vanish outside the past light cone, so causality is violated. In some cases causal prescriptions for $\mathcal{C}_n(x)$ exist, but when the radiative corrections are taken into account, α^2 runs and its logarithmic dependence in general spoils the causal prescriptions. The violation of causality is the price paid for the elimination of instabilities.

The function $\mathcal{C}_n^{\mathrm{F}}(x)$ tends to zero or rapidly oscillates for $|x^2| \gg |\alpha^2|$, so the causality violations can be experimentally tested only at distances of the order of

$$\Delta x \sim |\alpha| \tag{2.13}$$

and become physically unobservable at distances much larger than this bound.

The functional derivatives $S_{ijk\cdots}$ are bi-, tri-tensor densities, etc., and involve several Dirac delta functions. It can be cumbersome to preserve manifest general covariance using these objects. Fortunately, it is not really necessary to work with them, because they appear only in the intermediate formulas. In the appendix we show how to work without them, using only tensors and tensor densities.

Another approach to remove instabilities is known in the literature as the *regular reduction* of the order of the differential equation. It mimics a manipulation usually learnt in connection with the Abraham-Lorentz force in classical electrodynamics [15] and extends it to the case of gravity [16, 17]. The regular reduction is not a field redefinition, but a manipulation of the field equations, which leaves the metric tensor unchanged. The map $\mathcal{M}$, on the other hand, is designed to work efficiently in combination with renormalization.

2.3 Usage of the map $\mathcal{M}$

The map $\mathcal{M}$ can be used to convert a higher-derivative theory of classical gravity coupled with quantum matter into an acausal theory, preserving the renormalizability. Consider the higher-derivative theory

$$S_{\mathrm{HD}}[\overline{g},\varphi,\lambda,a,b,\kappa]=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-\overline{g}}\left(\overline{R}+a\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}+b\overline{R}^2\right)+S_m[\varphi,\overline{g},\lambda], \tag{2.14}$$

where $\overline{R}_{\mu\nu}$, $\overline{R}$ are the Ricci tensor and scalar curvature of the metric $\overline{g}$. Here S_m is the power-counting renormalizable matter action embedded in curved background. For simplicity assume that S_m does not contain masses and super-renormalizable parameters and use the dimensional-regularization technique. Then no cosmological constant is generated by renormalization. The arguments below can be generalized straightforwardly to include these parameters, together with the cosmological constant.

Obviously S_{HD} is renormalizable, but physically unsatisfactory due to the higher-derivative kinetic terms in the gravitational sector. However, the theorem just proved ensures that there exists a map $\overline{g}=\overline{G}(g,a,b)$ such that

$$\int \mathrm{d}^4x\sqrt{-\overline{g}}\left[\overline{R}(\overline{g})+a\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}(\overline{g})+b\overline{R}^2(\overline{g})\right]=\int \mathrm{d}^4x\sqrt{-g}R(g)$$

and reabsorbs the higher-derivative gravitational kinetic terms into the Einstein term. Applied to (2.14), the map generates new vertices that couple matter with gravity, and defines a new action S_{AC} such that

$$S_{\mathrm{HD}}[\overline{G}(g,a,b),\varphi,\lambda,a,b,\kappa]=S_{\mathrm{AC}}[g,\varphi,\lambda,a,b,\kappa]. \tag{2.15}$$

The action S_{AC} has the form

$$S_{\mathrm{AC}}[g,\varphi,\lambda,\lambda',\kappa]=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R+S_m[\varphi,g,\lambda]+\Delta S_m[\varphi,g,\lambda,\lambda'], \tag{2.16}$$

where

$$\Delta S_m=\int \mathrm{d}^4x\sqrt{-g}\,\left[-\frac{a}{2}T_m^{\mu\nu}R_{\mu\nu}+\frac{1}{4}(a+2b)RT_m\right]+\mathcal{O}(a^2,b^2,ab), \tag{2.17}$$

where $T_m^{\mu\nu} = -(2/\sqrt{-g})(\delta S_m/\delta g_{\mu\nu})$ is the stress-tensor of the uncorrected matter sector and T_m denotes its trace. Precisely, ΔS_m does not contain any kinetic contributions and is made of vertices that are either proportional to $T_m^{\mu\nu}$ and (covariant derivatives of) the Ricci tensor, or quadratically proportional to (covariant derivatives of) the Ricci tensor. More details are given in the appendix.

Formula (2.15) is the relation between the classical actions. Analogous relations hold for the bare and renormalized actions, when the matter fields are quantized, and the generating functionals Γ:

$$\Gamma_{\mathrm{AC}}[g,\Phi,\lambda,a,b,\kappa] = \Gamma_{\mathrm{HD}}[\overline{G}(g,a,b),\Phi,\lambda,a,b,\kappa]. \tag{2.18}$$

The total actions $S_{\mathrm{tot\ AC}}$ and $S_{\mathrm{tot\ HD}}$ of (2.1) follow from their definitions. Observe that the resummed map $\overline{g} = \overline{G}(g,a,b)$ is complex, in general, due to the prescription (2.12). The acausal action $S_{\mathrm{tot\ AC}}$ is defined taking the real part of Γ_{AC} with g and Φ real, *after* applying the map $\mathcal{M}$ to Γ_{HD}, which is not the same as applying the map $\mathcal{M}$ to $S_{\mathrm{tot\ HD}}$.

The map $\mathcal{M}$ preserves the renormalizability of the theory. Indeed, the function $\overline{g} = \overline{G}(g,a,b)$ is finite and does not depend on the quantum fields, so Γ_{AC} is convergent because Γ_{HD} is.

Finally, the map is not just a change of variables, but changes the physics, since it eliminates the unwanted degrees of freedom at the price of introducing violations of causality at small distances. In the expansion around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}$, $\eta_{\mu\nu}$ =diag$(1,-1,-,1-1)$, the traceless part $\widetilde{\phi}_{\mu\nu}$ of $\phi_{\mu\nu}$ is mapped as [9]

$$\widetilde{\phi}_{\mu\nu} = \frac{1}{\sqrt{1-a\Box}}\widetilde{\phi}'_{\mu\nu}. \tag{2.19}$$

The other components of $\phi_{\mu\nu}$ are multiplied by $1/\sqrt{1-a\Box}$ or $\sqrt{1-b'\Box}$, where $b' = -2(a+3b)$ (see [9] for other explicit formulas). Thus, the causality violations due to the map $\mathcal{M}$ become detectable at distances of the order of $\sqrt{|a|}$, $\sqrt{|b'|}$ or smaller.

3 Acausal Einstein-Yang-Mills theory

For definiteness, in this section and in other sections of the paper, we consider an explicit model, non-Abelian Yang-Mills theory coupled with classical gravity. However, most of the properties that we uncover are valid, with minor modifications, in every power-counting renormalizable theory coupled with classical gravity. We recall a number of results from the literature [18, 19, 20] that will be useful in section 5, where we let the gauge coupling depend on the scalar curvature R of spacetime. We apply the map $\mathcal{M}$ and describe the acausal Einstein-Yang-Mills theory in detail, in particular its renormalization. The metric tensor has a non-trivial running although it is not quantized. The runnings of a and b' are related with the coefficients of the trace anomaly of Yang-Mills theory in external gravity and can be studied exactly in a large class of models, those that interpolate between UV and IR conformal fixed points.

The lagrangian of the higher-derivative renormalizable theory is

$$\frac{\mathcal{L}_{\mathrm{HD}}}{\sqrt{-\overline{g}}}=\frac{\overline{R}}{2\kappa^2}+\xi\overline{W}^2+\zeta\overline{G}_B+\frac{\eta}{(n-1)^2}\overline{R}^2-\frac{1}{4\alpha}F_{\mu\nu}^aF^{a\mu\nu}\ , \tag{3.1}$$

where $F_{\mu\nu}^a$ is the Yang-Mills field strength, α is the squared gauge coupling, indices are raised and lowered with the metric $\overline{g}_{\mu\nu}$, $\overline{W}^2$ is the square of the Weyl tensor and $\overline{G}_B$ is the Gauss-Bonnet density,

$$\overline{W}^2=\overline{R}_{\mu\nu\rho\sigma}\overline{R}^{\mu\nu\rho\sigma}-\frac{4}{n-2}\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}+\frac{2}{(n-1)(n-2)}\overline{R}^2,\qquad \overline{G}_B=\overline{R}_{\mu\nu\rho\sigma}\overline{R}^{\mu\nu\rho\sigma}-4\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}+\overline{R}^2. \tag{3.2}$$

We use the dimensional-regularization technique, $n=4-\varepsilon$ being the continued spacetime dimension, and the minimal subtraction scheme. Since $\sqrt{-g}\overline{G}_B$ is a total derivative in four dimensions, assuming that the metric tends to flat space at infinity with an appropriate velocity, the integral

$$\int \mathrm{d}^n x\sqrt{-g}\ \overline{G}_B \tag{3.3}$$

is "evanescent" [21]. This means that a counterterm proportional to (3.3), for example $1/\varepsilon$ times (3.3), is not a true divergence, but amounts to a finite local correction to the quantum action, that is to say a scheme redefinition. Thus the parameter ζ does not affect the physical quantities and should not be considered as a new physical coupling. Moreover, keeping n arbitrary and expanding the metric around flat space, the integral (3.3) does not contain kinetic contributions [22]. Writing $\overline{W}^2$ as a linear combination of $\overline{G}_B$, $\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}$ and $\overline{R}^2$, the relations with the couplings a and b used in the previous section read

$$\frac{a}{2\kappa^2}=\frac{4(n-3)}{n-2}\xi,\qquad \frac{b}{2\kappa^2}=\frac{\eta}{(n-1)^2}-\frac{n(n-3)}{(n-1)(n-2)}\xi. \tag{3.4}$$

Renormalization in curved background is achieved by means of the usual renormalization constants, plus additional renormalization constants for ξ, ζ and η. The metric $\overline{g}_{\mu\nu}$ and the Newton constant κ are not renormalized. In the minimal subtraction scheme, α has beta function

$$\mu\frac{\mathrm{d}\alpha}{\mathrm{d}\mu}=\widehat{\beta}(\alpha)=-\varepsilon\alpha+\beta(\alpha),\qquad \beta(\alpha)=-\frac{22}{3}\frac{\alpha^2C(G)}{(4\pi)^2}+\mathcal{O}(\alpha^3).$$

The bare coupling ξ_{B} is related with the renormalized coupling ξ by the formula

$$\xi_{\mathrm{B}}=\mu^{-\varepsilon}\left(\xi+L_\xi\right),\qquad L_\xi=\sum_{i=1}^{\infty}\frac{\xi_i(\alpha)}{\varepsilon^i}. \tag{3.5}$$

Standard RG relations implied with the finiteness of the beta function β_ξ give, in the minimal subtraction scheme,

$$\mu\frac{\mathrm{d}\xi}{\mathrm{d}\mu}=\widehat{\beta}_\xi=\varepsilon\xi+\beta_\xi,\qquad \beta_\xi=\varepsilon L_\xi-\mu\frac{\mathrm{d}L_\xi}{\mathrm{d}\mu}=\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\alpha\xi_1(\alpha)\right) \tag{3.6}$$

and recursively relate the functions $\xi_i(\alpha)$ by

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\alpha\xi_{k+1}(\alpha)\right)=\beta(\alpha)\frac{\mathrm{d}\xi_k(\alpha)}{\mathrm{d}\alpha}. \tag{3.7}$$

Formulas similar to (3.5), (3.7) and (3.6) hold for the couplings ζ and η and define beta functions β_ζ, β_η, which are, as well as β_ξ, just functions of α. The lowest-order results [20] are

$$\begin{aligned}\beta_\xi&=\frac{\dim G}{(4\pi)^2}\left(-\frac{1}{10}+\frac{2}{9}\frac{\alpha C(G)}{(4\pi)^2}\right)+\mathcal{O}(\alpha^2),\\ \beta_\zeta&=-\frac{\dim G}{(4\pi)^2}\left(\frac{31}{180}+\frac{17}{12}\frac{\alpha^2C^2(G)}{(4\pi)^4}\right)+\mathcal{O}(\alpha^3),\\ \beta_\eta&=\frac{\dim G}{(4\pi)^2}\frac{187}{54}\frac{\alpha^3C^3(G)}{(4\pi)^6}+\mathcal{O}(\alpha^4).\end{aligned} \tag{3.8}$$

Formulas for the beta functions of a and b can be written using the relations (3.5) and (3.6), e.g.

$$\beta_a=\frac{8(n-3)}{n-2}\kappa^2\beta_\xi(\alpha),$$

and are valid in every power-counting renormalizable theory coupled with classical gravity.

In flat space there are many examples of quantum field theories whose renormalization-group flow interpolates between ultraviolet and infrared conformal field theories. Conformal field theories are characterized, among the other things, by certain quantities, called "central charges" (for definitions and properties, see for example [23]). Popular models interpolating between UV and IR conformal fixed points are provided by supersymmetric gauge theories in the "conformal window", where the values of the central charges can be calculated exactly also at the interacting fixed points [23]. Massless QCD can be considered a model with analogous features, because it interpolates between a free UV theory of quarks and gluons and a free IR theory of massless pions. When these models are coupled with classical gravity, the running of a can be studied throughout the RG flow. Indeed, the function $\beta_\xi(\alpha)$ is well-behaved and interpolates between (minus) the UV and IR values of the central charge c,

$$\lim_{\mathrm{UV\ (IR)}}\beta_\xi(\alpha)=-c_{\mathrm{UV\ (IR)}}.$$

In massless free field theories the quantity c is equal to

$$c=\frac{12n_v+6n_f+n_s}{120(4\pi)^2},$$

where $n_{v,f,s}$ are the numbers of vectors, fermions and scalars. At a conformal fixed point the a-running is just

$$a(-p^2) = \overline{a} - 2c\kappa^2 \ln \frac{-p^2}{\mu^2},$$

where $\overline{a} = a(\mu^2)$, so (2.11) (with $\alpha^2 \to a$) is renormalization-group improved to

$$\mathcal{C}_n(x) = \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{\mathrm{e}^{-ip\cdot x}}{\sqrt{1 + a(-p^2)p^2}}.$$

Since the radiative corrections to a are just logarithmic, the large-distance behavior of $\mathcal{C}_n(x)$ is still causal. The violations of causality can be appreciated at energies E such that

$$a(E^2)E^2 \sim 1.$$

Now we apply the map $\mathcal{M}$ to the higher-derivative model (3.1) and describe the acausal theory that we obtain. For the reasons explained above, we can keep $n = 4$ without loosing information. From appendix B, we have

$$\begin{aligned}\overline{g}_{\mu\nu} = {} & g_{\mu\nu} + aR_{\mu\nu} - \frac{a+2b}{2} g_{\mu\nu} R + \frac{3a^2}{4} \Box R_{\mu\nu} - \frac{3a(a+2b)}{4} \nabla_\mu \nabla_\nu R - abRR_{\mu\nu} + \frac{1}{2} a^2 R_\mu^\lambda R_{\lambda\nu} \\ & - \frac{3}{2} a^2 R_{\mu\alpha\nu\beta} R^{\alpha\beta} + \frac{1}{8} g_{\mu\nu} \left\{ 3(a+2b)(a+6b) \Box R + 2a(3a+4b) R_{\alpha\beta} R^{\alpha\beta} - a^2 R^2 \right\} \\ & + \mathcal{O}(a^3, a^2 b, ab^2). \end{aligned} \tag{3.9}$$

The acausal Einstein-Yang-Mills lagrangian has the form

$$\mathcal{L}_{\text{EYM-AC}} = \frac{1}{2\kappa^2} \sqrt{-g} R - \frac{1}{4\alpha} \sqrt{-g} \left\{ F_{\mu\nu}^a F^{a\mu\nu} \ H(g) + T_{\mu\nu} K^{\mu\nu}(g) + \Upsilon_{\mu\nu\rho\sigma} L^{\mu\nu\rho\sigma}(g) \right\}, \tag{3.10}$$

where $T_{\mu\nu}$ is the unperturbed stress tensor and $\Upsilon_{\mu\nu\rho\sigma}$ is the traceless operator $F_{\mu\nu}F_{\rho\sigma}$,

$$T_{\mu\nu} = -F_{\mu\alpha}^a F_\nu^{a\ \alpha} + \frac{1}{4} g_{\mu\nu} F^2, \tag{3.11}$$

$$\Upsilon_{\mu\nu\rho\sigma} = F_{\mu\nu}^a F_{\rho\sigma}^a + \frac{1}{2} \left(g_{\mu\rho} T_{\nu\sigma} - g_{\mu\sigma} T_{\nu\rho} - g_{\nu\rho} T_{\mu\sigma} + g_{\nu\sigma} T_{\mu\rho} \right) - \frac{1}{12} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) F^2. \tag{3.12}$$

where $F^2 \equiv F_{\alpha\beta}^a F^{a\alpha\beta}$. The coefficients $H(g)$, $K^{\mu\nu}(g)$ and $L^{\mu\nu\rho\sigma}(g)$ are non-polynomial tensorial functions of the metric tensor. Moreover, $K^{\mu\nu}(g)$ is proportional to the Ricci tensor or its covariant derivatives, while $L^{\mu\nu\rho\sigma}(g)$ and $H(g) - 1$ are squarely proportional to the Ricci tensor or its covariant derivatives.

Thus, the acausal Einstein-Yang-Mills theory is just Yang-Mills theory with two composite operators besides F^2, which are coupled with suitable metric-dependent external sources. To the second order, using the results of appendix B we find

$$H(g) = 1 + \frac{1}{6} a^2 R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{24} a^2 R^2, \qquad L^{\mu\nu\rho\sigma} = a^2 R^{\mu\rho} R^{\nu\sigma}, \tag{3.13}$$

$$K^{\mu\nu}(g) = 2aR^{\mu\nu} + \frac{3}{2} a^2 \Box R^{\mu\nu} + a^2 RR^{\mu\nu} - 3a^2 R^{\mu\alpha\nu\beta} R_{\alpha\beta} - \frac{3a(a+2b)}{2} \nabla^\mu \nabla^\nu R. \tag{3.14}$$

The renormalization of the acausal theory works as follows: the couplings a and b renormalize according to the formulas (3.4) and (3.8). Moreover, the metric tensor $g_{\mu\nu}$, although it is not quantized, has a non-trivial running, induced by the a- and b- runnings, which can be determined using the non-renormalization of $\overline{g}_{\mu\nu}$ in (3.9). Thus,

$$\mu\frac{\mathrm{d}\overline{g}_{\rho\sigma}}{\mathrm{d}\mu}=0 \qquad \Longrightarrow \qquad \mu\frac{\mathrm{d}g_{\rho\sigma}}{\mathrm{d}\mu}=-\beta_a R_{\rho\sigma}+\frac{1}{2}(\beta_a+2\beta_b)g_{\mu\nu}R+\mathcal{O}(a\kappa^2,b\kappa^2).$$

The Gauss-Bonnet term is sent into itself by the map $\mathcal{M}$. The n-dependence of the map $\mathcal{M}$ away from $n=4$ generates a number of new evanescent terms in the acausal theory (see appendix B).

4 First generalization of renormalizable theories

In the rest of the paper we study more general renormalizable theories of classical gravity coupled with quantized fields. The divergences are subtracted with a finite or reduced set of independent couplings. The acausal models can still be obtained from higher-derivative models using the map $\mathcal{M}$. We generalize the constructions of the previous sections in two main ways. In the present section we focus on theories whose non-renormalizable perturbation has a head of the form (2.17), namely proportional to the Ricci tensor and the energy-momentum tensor of the unperturbed matter sector. In section 5 we study non-renormalizable deformations whose heads contain more general matter operators, let the couplings depend on the scalar curvature of spacetime, and so on.

We consider theories of the form

$$S_{\mathrm{AC}}=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R+S_m(\varphi,g,\lambda)+\Delta S_m(\varphi,g,\lambda'). \tag{4.1}$$

Here $S_m(\varphi,g,\lambda)$ denotes the power-counting renormalizable matter action embedded in external gravity, with couplings λ. It includes the non-minimal terms of dimensionality equal to four. For the moment, we set the masses and other super-renormalizable parameters to zero. Instead, $\Delta S_m(\varphi,g,\lambda')$ denotes the vertices of dimensionalities greater than four, multiplied by couplings λ'. Neither S_m nor ΔS_m include purely gravitational terms.

Assume that ΔS_m contains only a finite number of matter operators, all those that have dimensionality four or smaller than four, are gauge invariant, covariant under diffeomorphisms, invariant under the global symmetries of the theory, not necessarily scalar, and can be contracted with tensors constructed with the metric in a non-trivial way. For example, in the case of Yang-Mills theory, the operators are just F^2, plus $T^{\mu\nu}$ (from now on called $T_m^{\mu\nu}$) and $\Upsilon^{\mu\nu\rho\sigma}$, formulas (3.11) and (3.12). They can be contracted with a variety of tensors constructed with the metric, giving e.g.

$$RF^2, \qquad R_{\mu\nu}T_m^{\mu\nu}, \qquad R_{\mu\nu\rho\sigma}\Upsilon^{\mu\nu\rho\sigma}, \qquad R_{\mu\rho}R_{\nu\sigma}\Upsilon^{\mu\nu\rho\sigma}, \qquad \nabla_\nu\nabla_\sigma R_{\mu\rho}\Upsilon^{\mu\nu\rho\sigma}, \tag{4.2}$$

etc. Moreover, assume that ΔS_m is proportional to the Ricci tensor or its covariant derivatives and the vertices that depend linearly on the Ricci tensor are also proportional to the stress tensor $T_m^{\mu\nu}$ of the unperturbed action. Collecting these assumptions into explicit formulas, ΔS_m has the structure

$$\begin{aligned} \Delta S_m &= \Delta_1 S_m + \Delta_2 S_m, \qquad & \Delta_1 S_m &= \int \mathrm{d}^4 x \sqrt{-g} f_{\mu\nu}(g_{\rho\sigma}) T_m^{\mu\nu}, \\ \Delta_2 S_m &= \mathcal{O}(R_{\mu\nu}^2, \varphi), \qquad & T_m^{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}(x)}, \end{aligned} \tag{4.3}$$

where $f_{\mu\nu}(g_{\rho\sigma})$ is a tensor depending on the metric, proportional to the Ricci tensor (or its covariant derivatives), of the form

$$f_{\mu\nu}(g_{\rho\sigma}) = \lambda_1' R_{\mu\nu} + \lambda_2' R g_{\mu\nu} + \mathcal{O}(\nabla^2 R_{\rho\sigma}).$$

and $\Delta_2 S_m$ contains the prescribed set of matter operators. Thus for example, the second term of (4.2) belongs to $\Delta_1 S_m$ while the forth belongs to $\Delta_2 S_m$. Instead, the first, third and fifth term of (4.2) do not belong to ΔS_m, unless they are multiplied by at least 1, 2 and 1 extra powers of the Ricci tensor, respectively.

A simple example is a generalization of acausal Einstein-Yang-Mills theory, described by a lagrangian of the form (3.10), where however $H(g)$, $K^{\mu\nu}(g)$ and $L^{\mu\nu\rho\sigma}(g)$ do not have necessarily the expressions (3.13) and (3.14), inherited applying the map $\mathcal{M}$ to (3.1), but are arbitrary functions of the metric, subject only to the restrictions that $K^{\mu\nu}(g)$ be proportional to the Ricci tensor or its covariant derivatives, and $L^{\mu\nu\rho\sigma}(g)$ and $H(g) - 1$ be squarely proportional to the Ricci tensor or its covariant derivatives.

We wish to prove that the action (4.1) is renormalizable in the form (4.1). We first prove that this result follows from the renormalizability of the theory

$$S_{\mathrm{HD}} = \frac{1}{2\kappa^2} \int \mathrm{d}^4 x \sqrt{-g} R + \int \mathrm{d}^4 x \sqrt{-g} R_{\mu\nu} \mathcal{T}^{\mu\nu\rho\sigma}(\lambda') R_{\rho\sigma} + S_m(\varphi, g, \lambda) + \Delta S_m(\varphi, g, \lambda, \lambda'). \tag{4.4}$$

Here ΔS_m is not the same as in (4.1), but it is subject to the same restrictions as the ΔS_m of (4.1). We use the same notation, since no confusion can arise, because what is important here is just the structure of ΔS_m, not its precise value. The tensor $\mathcal{T}^{\mu\nu\rho\sigma}$ is a (possibly differential) operator that depends only on the metric and can contain covariant derivatives acting on the Ricci tensors to its left and to its right. It can contain other couplings λ'.

Later we prove the renormalizability of (4.4). First we prove that the renormalizability of (4.4) implies the renormalizability of (4.1), applying the map $\mathcal{M}$.

The theorem of section 2 ensures that there exists a map $g'_{\mu\nu} = g'_{\mu\nu}(g) = g_{\mu\nu} + \Delta g_{\mu\nu}(g)$, with $\Delta g_{\mu\nu}$ proportional to the Ricci tensor and its covariant derivatives, that reabsorbs the pure gravitational part of (4.4) inside the Einstein term, namely such that

$$\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R+\int \mathrm{d}^4x\sqrt{-g}R_{\mu\nu}\mathcal{T}^{\mu\nu\rho\sigma}R_{\rho\sigma}=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g'}R(g'). \tag{4.5}$$

On the other hand, acting on $S_m+\Delta S_m$ the redefinition produces terms that fall again in the classes S_m, $\Delta_1 S_m$ and $\Delta_2 S_m$, as we now prove.

First, the matter operators generated by this process can have at most dimensionality four, since S_m and ΔS_m contain only, and all, such operators. Second, to the second and higher orders in $\Delta g_{\mu\nu}$ the terms generated by the field redefinition are certainly quadratic in the Ricci tensor or its covariant derivatives, so they fall in the class $\Delta_2 S_m$. Finally, to the first order in $\Delta g_{\mu\nu}$,
i) the terms generated varying S_m are certainly proportional to the energy-momentum tensor $T_m^{\mu\nu}$ of the unperturbed action S_m and proportional to the Ricci tensor, so they fall in the class $\Delta_1 S_m$;
ii) the terms generated varying $\Delta_1 S_m$ can be of two types:

a) those obtained varying the metric outside $T_m^{\mu\nu}$ are still proportional to $T_m^{\mu\nu}$ and at least linearly proportional to the Ricci tensor, so they fall either in the class $\Delta_1 S_m$, or in the class $\Delta_2 S_m$;

b) those obtained varying the metric inside $T_m^{\mu\nu}$ are not necessarily proportional to $T_m^{\mu\nu}$, but they are at least squarely proportional to the Ricci tensor, so they fall in the class $\Delta_2 S_m$;
iii) the terms generated varying $\Delta_2 S_m$ fall necessarily in the class $\Delta_2 S_m$.

Thus, it is sufficient to prove the renormalizability of (4.4). We first describe how the renormalization of the theory (2.14) works, namely the particular case $\Delta S_m=0$. We write

$$S_{\mathrm{HD}}^{(0)}(\varphi,g,\lambda,a,b,\kappa)=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left(R+aR_{\mu\nu}R^{\mu\nu}+bR^2\right)+S_m(\varphi,g,\lambda) \tag{4.6}$$

and denote the quantities relative to this theory with the subscript 0. The counterterms generated by $S_{\mathrm{HD}}^{(0)}$ are of the following types:
1) counterterms proportional to the vertices of $S_m(\varphi,g,\lambda)$. They are subtracted multiplying the matter fields and the couplings λ contained in S_m by appropriate renormalization constants. The wave-function renormalization constants coincide with the flat-space ones. The renormalization constant of a vertex that survives the flat-space limit coincides with the flat-space one. If the vertex vanishes in the flat-space limit, as the non-minimal term $R\varphi^2$ for scalar fields φ, its renormalization constant has to be calculated anew.
2) BRST-exact terms, not included in (4.6), which do not affect the physical sector of the theory;
3) purely gravitational counterterms, proportional to

$$R_{\mu\nu}R^{\mu\nu}, \qquad R^2. \tag{4.7}$$

As usual, the Gauss-Bonnet identity is used to convert $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ into a linear combination of (4.7). The counterterms (4.7) obviously fall in the pure gravitational sector of (4.4).

Proceeding inductively, denote with $\Gamma_0^{(n)}$ the generating functional of one-particle irreducible diagrams of the theory $S_{\rm HD}^{(0)}$ renormalized up to n loops. Then its $(n+1)$-loop divergent part $\Gamma_{0\,\rm div}^{(n)}$ is local and has the form

$$\Gamma_{0\,\rm div}^{(n)} = \frac{\partial S_m}{\partial\lambda}\Delta_n\lambda + \varphi\frac{\delta S_m}{\delta\varphi}\Delta_n Z_\varphi + \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left(\Delta_n a\, R_{\mu\nu}R^{\mu\nu} + \Delta_n b\, R^2\right), \qquad (4.8)$$

plus BRST-exact terms, where $\Delta_n\lambda$, $\Delta_n Z_\varphi$, $\Delta_n a$ and $\Delta_n b$ are appropriate divergent constants. The divergences (4.8) are subtracted by the redefinitions

$$\varphi \to \varphi - \varphi\Delta_n Z_\varphi, \qquad \lambda \to \lambda - \Delta_n\lambda, \qquad a \to a - \Delta_n a, \qquad b \to b - \Delta_n b, \qquad (4.9)$$

and the counterterms are collected in

$$S_{\rm HD}^{(0)}\left(\varphi - \varphi\Delta_n Z_\varphi, g, \lambda - \Delta_n\lambda, a - \Delta_n a, b - \Delta_n b, \kappa\right). \qquad (4.10)$$

Now consider the theory (4.4). We prove that its divergences are renormalized with the same redefinitions (4.9), plus redefinitions of the parameters contained in ΔS_m. Consider first the diagrams that do not contain insertions of ΔS_m. Their divergent parts have to be reabsorbed with the redefinitions (4.9), but now those redefinitions have to be performed on the full action $S_{\rm HD}$,

$$S_{\rm HD}\left(\varphi - \varphi\Delta_n Z_\varphi, g, \lambda - \Delta_n\lambda, a - \Delta_n a, b - \Delta_n b, \kappa\right),$$

which generates the additional counterterms

$$\Delta S_m\left(\varphi - \varphi\Delta_n Z_\varphi, g, \lambda - \Delta_n\lambda, a - \Delta_n a, b - \Delta_n b, \kappa\right).$$

Writing again $\Delta S_m = \Delta_1 S_m + \Delta_2 S_m$, we study the effects of the redefinitions (4.9) separately inside $\Delta_1 S_m$ and $\Delta_2 S_m$. Inside $\Delta_2 S_m$, the redefinitions (4.9) obviously generate terms of type $\Delta_2 S_m$, which can be removed renormalizing the couplings λ' contained in $\Delta_2 S_m$. Instead, inside $\Delta_1 S_m$ the redefinitions (4.9) generate

$$-\Delta_n\lambda\frac{\partial\Delta_1 S_m}{\partial\lambda} - \varphi\Delta_n Z_\varphi\frac{\delta\Delta_1 S_m}{\delta\varphi} = -\int \mathrm{d}^4x\sqrt{-g}f_{\mu\nu}(g_{\rho\sigma})\left[\Delta_n\lambda\frac{\partial T_m^{\mu\nu}}{\partial\lambda} + \varphi\Delta_n Z_\varphi\frac{\delta T_m^{\mu\nu}}{\delta\varphi}\right], \qquad (4.11)$$

which are not of type $\Delta_1 S_m$, in general. We keep the left-overs (4.11) in mind and proceed.

Now, consider the Feynman diagrams that do contain insertions of ΔS_m. The counterterms that contain at least two insertions of vertices of type $\Delta_1 S_m$ or one insertion of vertices of type $\Delta_2 S_m$ fall in the classes $\Delta_2 S_m$ or $R_{\mu\nu}\mathcal{T}^{\mu\nu\rho\sigma}R_{\rho\sigma}$. Indeed, they are certainly quadratic in the Ricci tensor, since they carry at least two Ricci tensors on the external legs. Moreover, simple power counting shows that they are made of matter operators that have at most dimensionality four, since the matter operators contained in $S_m + \Delta S_m$ have at most dimensionality four.

It remains to consider the Feynman diagrams that contain one vertex of type $\Delta_1 S_m$ and no vertex of type $\Delta_2 S_m$. Such diagrams are collected in the expression

$$\int \mathrm{d}^4 x \sqrt{-g} f_{\mu\nu}(g_{\rho\sigma}) \langle T_m^{\mu\nu} \rangle_0 \,, \tag{4.12}$$

where

$$\langle T_m^{\mu\nu} \rangle_0 = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma_0[\Phi, g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}. \tag{4.13}$$

Proceeding again by induction, assume that the divergences have been renormalized up to the order n included. Then the $(n+1)$-loop divergent part of (4.12) is obtained replacing Γ_0 in (4.13) with $\Gamma_{0\,\mathrm{div}}^{(n)}$, given by (4.8). We thus see that the pure gravitational counterterms of (4.12) are squarely proportional to the Ricci tensor or its covariant derivatives, namely they belong to the purely gravitational sector of (4.4). On the other hand, the matter contributions to the $(n+1)$-loop counterterms of (4.12) are

$$-2 \int \mathrm{d}^4 x f_{\mu\nu}(g_{\rho\sigma}) \left(\Delta_n \lambda \frac{\partial}{\partial \lambda} + \varphi \Delta_n Z_\varphi \frac{\delta}{\delta \varphi} \right) \frac{\delta S_m}{\delta g_{\mu\nu}} = \Delta_n \lambda \frac{\partial \Delta_1 S_m}{\partial \lambda} + \varphi \Delta_n Z_\varphi \frac{\delta \Delta_1 S_m}{\delta \varphi},$$

which are cancelled by the left-overs (4.11). Observe that the couplings λ' contained in $\Delta_1 S_m$ are not renormalized. This is of course a consequence of the finiteness of the energy-momentum tensor.

We have therefore proved that the theory (4.1) of classical gravity coupled with quantum matter is renormalizable in the form (4.1). The matter action $S_m + \Delta S_m$ is non-polynomial and ΔS_m contains infinitely many independent couplings. The set of independent couplings, however, is considerably smaller than the set of independent couplings of quantum gravity, since ΔS_m contains only lagrangian terms of the form (4.3).

4.1 Non-vanishing cosmological constant

When the theory contains masses and super-renormalizable parameters, the cosmological constant is turned on by renormalization. The results just proved generalize as follows. First, the non-renormalizable perturbation $\Delta_1 S_m$ should be linearly proportional to the tensor

$$\widehat{R}_{\mu\nu} = R_{\mu\nu} - \Lambda g_{\mu\nu} \tag{4.14}$$

or its covariant derivatives, instead of just the Ricci tensor. The hatted tensor vanishes on the solutions of the field equations of the Einstein action with a cosmological constant. Similarly, the vertices of type $\Delta_2 S_m$ should be squarely proportional to (4.14) and its covariant derivatives. With these assumptions the perturbations will be written $\widehat{\Delta}_1 S_m$ and $\widehat{\Delta}_2 S_m$, respectively, with $\widehat{\Delta} S_m = \widehat{\Delta}_1 S_m + \widehat{\Delta}_2 S_m$. Then the arguments of the previous subsection can be repeated to prove the renormalizability of the action

$$S_{\mathrm{HD}}^{(\Lambda)} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,(R-2\Lambda) + \int \mathrm{d}^4x\sqrt{-g}\widehat{R}_{\mu\nu}\mathcal{T}^{\mu\nu\rho\sigma}\widehat{R}_{\rho\sigma} + S_m(\varphi, g, \lambda) + \widehat{\Delta} S_m(\varphi, g, \lambda, \lambda').$$

The Newton constant κ and the cosmological constant Λ have non-trivial renormalizations. In replacement of (4.5) the identity

$$\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,(R-2\Lambda) + \int \mathrm{d}^4x\sqrt{-g}\widehat{R}_{\mu\nu}\mathcal{T}^{\mu\nu\rho\sigma}\widehat{R}_{\rho\sigma} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g'}\left(R(g') - 2\Lambda\right). \quad (4.15)$$

can be applied to (4.15) to prove the renormalizability of the acausal theory

$$S_{\mathrm{AC}}^{(\Lambda)} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,(R-2\Lambda) + S_m(\varphi, g, \lambda) + \widehat{\Delta} S'_m(\varphi, g, \lambda').$$

5 More general renormalizable theories of acausal classical gravity coupled with interacting quantum fields

The non-renormalizable perturbation $\Delta_1 S_m$ (4.3) of the theories considered in the previous section has a special form. Precisely, it contains a unique matter operator, the energy-momentum tensor $T_m^{\mu\nu}$ of the unperturbed matter action S_m. In this section we prove the renormalizability of more general theories.

In the higher-derivative models that we consider here, renormalization turns on pure gravitational counterterms that do not fall in the class (2.4), so we need to study the map $\mathcal{M}$ for more general gravitational actions. The most general modified action has the form

$$S'[\phi] = S[\phi] + S_i F_{ij} S_j + G_i S_i + H, \quad (5.1)$$

where G_i and H are functions of the fields ϕ, not necessarily proportional to the field equations. It is not possible, in general, to find a map that implements the equality (2.6). We describe what it is convenient to do in this case.

Without loss of generality, we can assume that all kinetic terms are contained in $S[\phi]$ and $S_i F_{ij} S_j$, while $G_i S_i$ and H are vertices. In the general case, for fields of spin smaller than or equal to 2 in arbitrary dimension, this property follows for example from the results of [22]. Here we refresh the proof for four-dimensional gravity. For the purpose of studying the non-trivial counterterms induced by renormalization, it is sufficient to work strictly in four dimensions.

Assume first that the cosmological constant vanishes and $S[\phi]$ is the Einstein action, so S_j is proportional to the Ricci tensor. We wish to prove that every scalar density that contains a non-trivial quadratic part (when the metric is expanded in around flat space) is quadratic in the Ricci tensor, up to vertex terms and total derivatives.

Every scalar density that contains a non-trivial quadratic part must necessarily be of the form

$$\sqrt{-g}R_{\mu\nu\rho\sigma}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}R_{\alpha\beta\gamma\delta}, \tag{5.2}$$

with indices contracted in some way by means of the metric tensor. We consider all possible contractions.

First consider the case in which some indices of one Riemann tensor are contracted with themselves. Then (5.2) reduces to an expression of the form

$$\sqrt{-g}R_{\mu\nu\rho\sigma}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}R_{\alpha\beta}. \tag{5.3}$$

If no λ is contracted with μ, ν, ρ or σ, then also the left Riemann tensor is actually a Ricci tensor or a curvature scalar. If some λ is contracted with an index of the Riemann tensor, say μ, then adding total derivatives and commuting the covariant derivatives among themselves (which amounts to add vertex terms), it is possible to move the derivative ∇^{μ} till it acts directly on the Riemann tensor, obtaining

$$\sqrt{-g}\nabla^{\mu}R_{\mu\nu\rho\sigma}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n-1}}R_{\alpha\beta}. \tag{5.4}$$

Then the Bianchi identity can be used to replace the Riemann tensor with a second Ricci tensor.

Assume now that the indices of no Riemann tensor are contracted with themselves. We can distinguish two cases: *i*) the case where at least one λ is contracted with an index of a Riemann tensor, and *ii*) the case where no λ is contracted with an index of a Riemann tensor. In case *i*) we can assume, eventually adding total derivatives, that the index λ is contracted with an index of the right Riemann tensor,

$$\sqrt{-g}R_{\mu\nu\rho\sigma}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{j-1}}\nabla^{\lambda}\nabla_{\lambda_j}\cdots\nabla_{\lambda_{2n-1}}R_{\lambda\beta\gamma\delta}. \tag{5.5}$$

Then, adding vertex terms we can commute the covariant derivatives till ∇^{λ} acts directly on the right Riemann tensor and use the Bianchi identity to reduce (5.5) to the form (5.3). In case *ii*) we can write, eventually after adding total derivatives and commuting covariant derivatives,

$$\sqrt{-g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \qquad \sqrt{-g}R_{\mu\nu\rho\sigma}\Box^{2n-2}\nabla_{\alpha}\nabla^{\alpha}R^{\mu\nu\rho\sigma}, \tag{5.6}$$

for $n=0$ and $n>0$, respectively. The $n=0$ case is dealt with using the Gauss-Bonnet identity, as usual. The $n>0$ case is treated using the Bianchi identity to write

$$-\sqrt{-g}R_{\mu\nu\rho\sigma}\Box^{2n-2}\nabla_{\alpha}\left(\nabla^{\sigma}R^{\mu\nu\alpha\rho}+\nabla^{\rho}R^{\mu\nu\sigma\alpha}\right).$$

Both terms have now the form (5.5), so the arguments given above can be applied again.

We have just considered gravity without a cosmological constant. Now we generalize the result to the case in which $S[\phi]$ is the Einstein action with a cosmological constant. As before, it is convenient to use hatted tensors

$$\widehat{R}_{\mu\nu\rho\sigma}=R_{\mu\nu\rho\sigma}-\frac{\Lambda}{3}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})$$

and expand the metric around maximally symmetric solutions [22], which have $\widehat{R}_{\mu\nu\rho\sigma}=0$. The scalar densities contributing to the quadratic part in this expansion are classified studying all possible contractions of

$$\sqrt{-g}\widehat{R}_{\mu\nu\rho\sigma}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}}\widehat{R}_{\alpha\beta\gamma\delta}. \tag{5.7}$$

Each such scalar density is quadratically proportional to the hatted Ricci tensor $\widehat{R}_{\mu\nu}$, plus a linear combination of total derivatives, vertices, $\sqrt{-g}R$ and $\sqrt{-g}$.

The proof follows the same arguments used above, replacing unhatted tensors with hatted tensors everywhere. The only caveat is that the commutation of two covariant derivatives generates the sum of a term containing an extra hatted Riemann tensor, which is a vertex $\mathcal{O}\left(\widehat{R}^3\right)$, plus a term of type (5.7) with $n\rightarrow n-1$, multiplied by an extra power of Λ. For example, on a vector V_μ

$$[\nabla_\mu,\nabla_\nu]V_\rho=-R_{\mu\nu\rho\sigma}V^\sigma=-\widehat{R}_{\mu\nu\rho\sigma}V^\sigma-\frac{\Lambda}{3}(g_{\mu\rho}V_\nu-g_{\nu\rho}V_\mu).$$

The Λ-term lowers the number of curvature tensors, so the procedure just described stops after a finite number of steps. The case $n=0$ is treated using the identity

$$\widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\mu\nu\rho\sigma}=G_B+4\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu}-\widehat{R}^2-\frac{4}{3}\Lambda R+\frac{8}{3}\Lambda^2. \tag{5.8}$$

The integral of $\sqrt{-g}G_B$ is an irrelevant infinite additional constant. The second and third terms on the right-hand side of (5.8) are quadratically proportional to the field equations of $S[\phi]$. The forth and fifth terms on the right-hand side of (5.8) give counterterms that can be reabsorbed renormalizing the Newton constant and the cosmological constant, respectively.

Coming back to (5.1), we can thus assume that G_iS_i and H in (5.1) are vertices and the kinetic terms are contained only in $S[\phi]+S_iF_{ij}S_j$. We also assume that, in a weak field approximation, $S_i=\mathcal{O}(\phi)$. If we apply the same map $\phi'(\phi)$ of section 2.2, which is such that $S[\phi]+S_iF_{ij}S_j=S[\phi']$, then G_iS_i and H are mapped into vertices and we can write

$$S'[\phi]=S[\phi']+G_i'[\phi']S_i[\phi']+H'[\phi'] \tag{5.9}$$

for suitable new functions G_i' and H'. The undesirable higher-derivative kinetic terms have been removed. The discussion about violations of causality is exactly the same as before, since the map $\phi'(\phi)$ is the same.

Applying new maps $\mathcal{M}$ iteratively, we can eliminate every newly-generated term that is quadratically proportional to S_i. To isolate such new terms in the action (5.9) rewrite it as

$$S''[\phi'] \equiv S'[\phi[\phi']] = S[\phi'] + S'_i F'_{ij}[\phi'] S'_j + \widetilde{G}'_i[\phi'] S'_i + \widetilde{H}'[\phi'], \tag{5.10}$$

where $S'_i = S_i[\phi']$. Since the new terms are certainly vertices, we have $\widetilde{F}'_{ij} = \mathcal{O}(\phi')$. Applying a map $\mathcal{M}$ of the form $\phi'' = \phi' + \mathcal{O}(\phi'^2)$ we can reabsorb also $S'_i F'_{ij} S'_j$ inside $S[\phi']$. In this way we end up with an F''_{ij} that is $\mathcal{O}(\phi''^2)$. Repeating the procedure indefinitely, we reach $F^{\infty}_{ij} = 0$, i.e. the complete elimination of the terms that are quadratic in S_i. The final action reads

$$S^{\infty}[\phi^{\infty}] = S'[\phi[\phi^{\infty}]] = S[\phi^{\infty}] + G^{\infty}_i[\phi^{\infty}] S_i[\phi^{\infty}] + H^{\infty}[\phi^{\infty}] \tag{5.11}$$

where G^{∞}_i and H^{∞} do not contain terms proportional to $S_i[\phi^{\infty}]$. Every ϕ^{∞}-leg in $S^{\infty}[\phi^{\infty}]$ is multiplied by $1/\sqrt{1+2FS}$, because

$$\phi^{\infty} = \sqrt{1+2FS}\,\phi + \mathcal{O}(\phi^2),$$

where F and S are the matrices F_{ij} and S_{ij} calculated at $\phi = 0$.

We can now use the results just derived to prove the renormalizability of a more general class of acausal and higher-derivative theories. We assume again, for simplicity, that the theories do not contain masses, super-renormalizable couplings and the cosmological constant, because it is straightforward to include them. Consider the higher-derivative theory

$$S_{\mathrm{HD}} = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + \overline{V}(g) + R_{\mu\nu} \mathcal{T}^{\mu\nu\rho\sigma} R_{\rho\sigma} \right] + S_m(\varphi, g, \lambda) + \int \mathrm{d}^4x \sqrt{-g} \sum_I \mathrm{O}_I(\varphi, g) \overline{K}^I(g), \tag{5.12}$$

where $\overline{V}(g)$ is a vertex function of the metric, namely $\overline{V}(g) = \mathcal{O}(\phi^3)$ when the metric is expanded around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}$. Moreover, S_m is the power-counting renormalizable matter action embedded in curved space and $\mathrm{O}_I(\varphi, g)$ is a basis of covariant, gauge- (or BRST-) invariant local operators, not necessarily scalar, of dimensionality smaller than or equal to four. The sources $\overline{K}^I(g)$ are arbitrary tensorial functions of the metric.

The renormalizability of (5.12) is easily proved. First, according to the arguments of this section, the most general pure gravitational sector without a cosmological constant can be written in the form appearing in (5.12). In the notation (5.1), S is the Einstein action,while $\sqrt{-g} R_{\mu\nu} \mathcal{T}^{\mu\nu\rho\sigma} R_{\rho\sigma} = S_i F_{ij} S_j$ and $\sqrt{-g}\overline{V} = G_i S_i + H$. Second, the most general matter sector with operators of dimensionality smaller than or equal to four, coupled with gravity in an arbitrary way, is precisely the one appearing in (5.12). By power-counting, renormalization cannot generate matter operators with dimensionalities greater than four, therefore (5.12) is renormalizable.

Applying the map $\mathcal{M}$ to (5.12) we obtain the acausal theory

$$S_{\text{AC}}=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,[R+V(g)]+S_m(\varphi,g,\lambda)+\int \mathrm{d}^4x\sqrt{-g}\sum_I \mathrm{O}_I(\varphi,g)K^I(g),\tag{5.13}$$

where $V(g)$ and $K^I(g)$ are other functions of the metric, and $V(g)$ is a vertex. With the map $\mathcal{M}$ leading to (5.11), we can assume that $V(g)$ does not contain terms quadratically proportional to the Ricci tensor, or its covariant derivatives, but can contain terms that are linearly proportional to it.

A simple example that illustrates the more general class of theories just considered is the generalized acausal Einstein-Yang-Mills theory, which has lagrangian

$$\frac{\mathcal{L}_{\text{EYM-AC}}}{\sqrt{-g}}=\frac{1}{2\kappa^2}R+V(g)-\frac{1}{4\alpha}\left\{F_{\mu\nu}^aF^{a\mu\nu}\ H(g)+T_{\mu\nu}K^{\mu\nu}(g)+\Upsilon_{\mu\nu\rho\sigma}L^{\mu\nu\rho\sigma}(g)\right\},\tag{5.14}$$

where now $H(g)$, $K^{\mu\nu}(g)$ and $L^{\mu\nu\rho\sigma}(g)$ are unconstrained functions of the metric and V is a vertex function of the metric.

5.1 Consistent reductions of couplings

The arbitrary functions contained in (5.14) can be restricted in various ways, preserving renormalizability. A special class of theories are those obtained from ordinary power-counting renormalizable theories in curved space, once the couplings are allowed to depend on the spacetime scalar curvature R. Then the subtraction of divergences can be performed renormalizing the R-dependent couplings, and multiplying the fields with R-dependent wave-function renormalization constants, paying some attention to the position of covariant derivatives. The lagrangian is uniquely determined up to a few functions of R. The renormalization group, critical exponents and beta functions are of course R-dependent.

We illustrate these facts in an explicit model. Consider the perturbed higher-derivative Einstein-Yang-Mills theory

$$\begin{aligned}S_{\text{EYM-HD}}^{(f)}=\int \mathrm{d}^nx\sqrt{-g}\,\Bigg[&\frac{1}{2\kappa^2}R-\frac{1}{4\alpha}F_{\mu\nu}^a(L_AA)F^{a\mu\nu}(L_AA)\,(1+\alpha Rf(R))+(\xi+\gamma R)\,W^2\\&+(\zeta+\rho R)\,G_B+\frac{\eta+\sigma R}{(n-1)^2}R^2+\frac{\tau}{n-1}R\Box R+R_{\mu\nu}\mathcal{T}_4^{\mu\nu\rho\sigma}R_{\rho\sigma}\Bigg],\end{aligned}\tag{5.15}$$

where $f(R)$ and L_A are power series in R, while γ, ρ, σ, τ are new coupling constants and $\mathcal{T}_4^{\mu\nu\rho\sigma}$ is a (possibly differential) tensor operator such that $R_{\mu\nu}\mathcal{T}_4^{\mu\nu\rho\sigma}R_{\rho\sigma}$ is a linear combination of the terms listed in appendix C, formula (C.22), where Q_i, $i=1,\ldots 8$, are arbitrary functions of the scalar curvature R with $Q_j(R)=\mathcal{O}(R^2)$ for $j=1,2,3,4$. We prove that (5.15) is renormalizable and describe the acausal theory that is obtained applying the map $\mathcal{M}$ to it. We also derive all order formulas for the beta functions of γ, ρ, σ, τ and $f \equiv f(0)$, which are related to the trace-anomaly coefficients of Yang-Mills theory in curved space. Note that (5.15) contains a single pure-gravitational term that is not quadratic in the Ricci tensor, that is to say $RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, multiplied by the coupling $\gamma + \rho$.

Clearly, the theory (5.15) is nothing but Yang-Mills theory with an R-dependent gauge coupling. It contains a unique matter operator, F^2, and is obtained perturbing the renormalizable higher-derivative Einstein-Yang-Mills theory (3.1) with the vertex RF^2. The R-power series $f(R)$ and $L_A(R)$ and the pure gravitational vertices listed in (C.22) are induced by renormalization.

To prove the renormalizability of (5.15), it is convenient to start from (3.1), deform it with $Rf(R)F^2$ and work perturbatively in $f(R)$. To the first order in $f(R)$ the Feynman diagrams coincide with those of (3.1) with one insertion of the composite operator F^2. The renormalization of F^2 in curved space has been studied by Hathrell in QED [19]. Hathrell's derivation can be extended to non-Abelian Yang-Mills theory in curved space using the Batalin-Vilkovisky formalism [10, 11]. First, the action (5.15) is completed with a gauge-fixing term and sources for the BRST transformations of the fields. Then the BRST cohomology is extended to a σ-cohomology, defined by the antiparenthesis with the complete action. Appendix C contains the details and the complete proof of renormalizability, including the characterization of the pure gravitational terms.

Hathrell's strategy (which we do not review here) can be applied in complete analogy with the QED case. The result is formally identical to the one found by Hathrell (formula (3.26) of [19]), apart from the σ-exact terms. Below we do not specify the σ-exact terms, which can be dealt with using the formalism of appendix C. We obtain the renormalized operator

$$\frac{1}{4\alpha}[F^a_{\mu\nu}F^{a\mu\nu}] = -\frac{\varepsilon\alpha\mu^\varepsilon}{4\widehat{\beta}\alpha_{\mathrm{B}}}F^a_{\mathrm{B}\mu\nu}F^{a\mu\nu}_{\mathrm{B}} + \frac{\alpha}{\widehat{\beta}}\left[(\varepsilon L_\xi - \beta_\xi)\,W^2 + (\varepsilon L_\zeta - \beta_\zeta)\,G_B \right.$$
$$\left. + (\varepsilon L_\eta - \beta_\eta)\,\frac{R^2}{(n-1)^2} + \frac{4}{n-1}\left(L_\eta - \frac{\alpha\beta_\eta}{\beta}\right)\Box R\right] + \sigma X, \qquad (5.16)$$

where the subscript B denotes the bare quantities. We can immediately read the renormalization constant Z_{F^2} of the operator F^2 in flat space,

$$Z_{F^2} = Z_\alpha\left(1 - \frac{\beta(\alpha)}{\varepsilon\alpha}\right)$$

and the f-beta function

$$f_{\mathrm{B}} = \mu^{-\varepsilon}fZ^{-1}_{F^2}, \qquad \frac{\mathrm{d}f}{\mathrm{d}\ln\mu} = \varepsilon f + \beta_f, \qquad \beta_f = f\frac{\mathrm{d}\ln Z_{F^2}}{\mathrm{d}\ln\mu} = f\left(\beta' - \frac{2\beta}{\alpha}\right), \qquad (5.17)$$

where the prime denotes differentiation with respect to the gauge coupling α.

Comparing the terms of dimensionality six in the bare and renormalized actions, we can derive the beta functions of γ, ρ, σ and τ. Factorizing one power of R, we have

$$- f_{\mathrm{B}}\frac{1}{4}F^{a}_{\mathrm{B}\mu\nu}F_{\mathrm{B}}^{a\mu\nu}+\gamma_{\mathrm{B}}W^{2}+\rho_{\mathrm{B}}G_{B}+\frac{\sigma_{\mathrm{B}}}{(n-1)^{2}}R^{2}+\frac{\tau_{\mathrm{B}}}{n-1}\square R=$$
$$-f\frac{\mu^{-\varepsilon}}{4}[F^{a}_{\mu\nu}F^{a\mu\nu}]+\gamma\mu^{-\varepsilon}W^{2}+\rho\mu^{-\varepsilon}G_{B}+\frac{\sigma\mu^{-\varepsilon}}{(n-1)^{2}}R^{2}+\frac{\tau\mu^{-\varepsilon}}{n-1}\square R,$$

up to σ-exact terms. Using (5.16) we find

$$\gamma_{\mathrm{B}}=\gamma\mu^{-\varepsilon}+\mu^{-\varepsilon}f\frac{\alpha^{2}}{\widehat{\beta}}\left(\beta_{\xi}-\varepsilon L_{\xi}\right),\qquad\rho_{\mathrm{B}}=\rho\mu^{-\varepsilon}+\mu^{-\varepsilon}f\frac{\alpha^{2}}{\widehat{\beta}}\left(\beta_{\zeta}-\varepsilon L_{\zeta}\right),$$
$$\sigma_{\mathrm{B}}=\sigma\mu^{-\varepsilon}+\mu^{-\varepsilon}f\frac{\alpha^{2}}{\widehat{\beta}}\left(\beta_{\eta}-\varepsilon L_{\eta}\right),\qquad\tau_{\mathrm{B}}=\tau\mu^{-\varepsilon}+\mu^{-\varepsilon}4f\frac{\alpha^{2}}{\widehat{\beta}}\left(\frac{\alpha\beta_{\eta}}{\beta}-L_{\eta}\right).$$

Differentiating with respect to $\ln\mu$ and simplifying by means of (3.6), we immediately derive the beta functions

$$\widehat{\beta}_{\gamma}=\frac{\mathrm{d}\gamma}{\mathrm{d}\ln\mu}=\varepsilon\gamma-f\alpha^{2}\frac{\mathrm{d}\beta_{\xi}}{\mathrm{d}\alpha},\qquad\widehat{\beta}_{\rho}=\varepsilon\rho-f\alpha^{2}\frac{\mathrm{d}\beta_{\zeta}}{\mathrm{d}\alpha},$$
$$\widehat{\beta}_{\sigma}=\varepsilon\sigma-f\alpha^{2}\frac{\mathrm{d}\beta_{\eta}}{\mathrm{d}\alpha},\qquad\widehat{\beta}_{\tau}=\varepsilon\tau-4f\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\alpha^{2}\beta_{\eta}}{\beta}\right).$$

Next, consider the Feynman diagrams of higher order in f, obtained with multiple insertions of RF^2 and its BRST completion $R\sigma X$, where X is the unspecified functional appearing in (5.16). Diagrams with no gauge fields, ghosts and BRST sources on the external legs contribute to the pure gravitational counterterms. Since every insertion of RF^2 plus its completion carries a factor of R, the pure gravitational counterterms with multiple insertions are certainly squarely proportional to the Ricci tensor and can be renormalized redefining the couplings contained in $\mathcal{T}_4^{\mu\nu\rho\sigma}$ (whose form is so far unspecified). BRST invariance, parity invariance and power counting ensure that the divergent diagrams that carry gauge fields, ghost and BRST sources on the external legs can only give counterterms proportional to F^2, plus σ-exact contributions, and carry a power of R at least equal to the number of insertions. The σ-exact contributions contain also gauge-field redefinitions, that can only have the form $A^a_\mu \rightarrow P(R)A^a_\mu$, for a certain function P. These divergences can be subtracted renormalizing the parameters that are contained in the functions $f(R)$ and L_A of (5.15) and in the BRST-exact sector added to (5.15). The pure gravitational contributions are further restricted by the analysis of appendix C to a linear combination of eight independent terms, multiplied by arbitrary functions of the scalar curvature.

Finally, we can read the renormalizable acausal Einstein-Yang-Mills theory associated with (5.15) applying the map $\mathcal{M}$ in the form (5.11). Such a map reabsorbs every term that is quadratically proportional to the Ricci tensor. There exists a function $\overline{g}(g)$ such that if we replace g with

$\overline{g}(g)$ in (5.15) we obtain an acausal action of the form

$$S_{\text{EYM-AC}}(g) = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + (\gamma+\rho)\overline{RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}}\right] \\ -\frac{1}{4\alpha}\int \mathrm{d}^4x\ \sqrt{-\overline{g}}F^a_{\mu\nu}(L_A(\overline{g})A)F^a_{\rho\sigma}(L_A(\overline{g})A)\overline{g}^{\mu\rho}\overline{g}^{\nu\sigma}\left[\,1+\alpha R(\overline{g})f(R(\overline{g}))\right]. \tag{5.18}$$

Here $\overline{g}$ should be understood as the given function $\overline{g}(g)$ and $\sqrt{-g}\overline{RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}}$ denotes the terms generated by the vertex $\sqrt{-\overline{g}}R(\overline{g})R_{\mu\nu\rho\sigma}(\overline{g})R^{\mu\nu\rho\sigma}(\overline{g})$, once every contribution quadratically proportional to the Ricci tensor is eliminated by the iterative procedure explained in the previous subsection. In the end, the surviving vertex has basically the same form as $RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, but every ϕ-leg gets multiplied by the function $1/\sqrt{1+2FS}$ of (2.9) and (2.12), which is responsible for the classical violations of causality.

The theory (5.18) is more general than the ones considered in section 4, because the list $\Delta S^{(\text{HEAD})}$ of vertices that have dimensionality six does not contain just the stress tensor multiplied by the Ricci tensor, but also the vertex RF^2 and the pure gravitational vertex $RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. Precisely,

$$\Delta S^{(\text{HEAD})} = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{f}{4}F^a_{\mu\nu}F^{a\mu\nu}R - \frac{a}{2}R_{\mu\nu}T^{\mu\nu} + (\gamma+\rho)RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right].$$

On the other hand, the theory is less general than (5.14), because the arbitrariness contained in (5.14) is reduced in (5.18) to a few functions of R.

The arguments of this section can be easily applied to classical Einstein gravity coupled with quantum electrodynamics with an R-dependent electric charge, or any other power-counting renormalizable quantum field theory with R-dependent couplings.

6 Conclusions

In this paper we have studied new renormalizable acausal theories of classical gravity coupled with interacting quantum fields. Performing systematic field redefinitions of the metric tensor, the divergences are removed without introducing higher-derivative kinetic terms in the gravitational sector. Previous results are generalized and new theorems are proved. In particular, it is shown how to treat quantized fields which interact through R-dependent vertices.

We have studied Einstein-Yang-Mills theory with an R-dependent gauge coupling in detail. The perturbation RF^2 induces extra gravitational terms, one of which, $RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, is not squarely proportional to the Ricci tensor. General formulas for the beta functions of the vertices with dimensionality six are derived. They are expressed in terms of the trace-anomaly coefficients of the matter sector embedded in curved background. The renormalization-group flow depends on the scalar curvature of spacetime.

Our results can be extended to all power-counting renormalizable quantum field theories with R-dependent coupling constants coupled with classical gravity. More general renormalizable theories contain all matter operators with dimensionalities smaller than or equal to four, multiplied by arbitrary tensorial functions of the metric. In complete generality, the pure gravitational counterterms that contain higher-derivative kinetic contributions can be removed using the map $\mathcal{M}$, which trades the instabilities due to higher-derivatives for classical violations of causality.

The causality violations introduced by the map $\mathcal{M}$ are governed by the parameters a and b', and can be detected at high energies. The renormalization-group flow of a and b' can be studied exactly in a large class of models, for example those that interpolate between UV and IR conformal fixed points.

Appendices

A Working without bitensors

In the case of gravity the functional derivatives $S_{ijk\cdots}$ are bi-, tri-tensor densities, and so on, and they involve several delta functions. It is unpleasant to work with these objects, but it is not strictly necessary for our purposes. Here we show how to use the map $\mathcal{M}$ working only with tensors and tensor densities. Write

$$S'[g_i] = S[g_i] + \int_{ij} \sqrt{-g} S_i F_{ij} S_j, \tag{A.1}$$

where F_{ij} is symmetric and can contain derivative operators. The index i labels also the spacetime point, as usual. Now the summation-integration over repeated indices is not understood, but written explicitly on the integral sign as shown in (A.1). Define the field-equation tensor

$$S_i = \frac{\widetilde{\delta} S}{\widetilde{\delta} g_i}, \qquad \frac{\widetilde{\delta}}{\widetilde{\delta} g_i} \equiv \frac{1}{\sqrt{-\det g_i}} \frac{\delta}{\delta g_i}.$$

Then, there exists a field redefinition

$$g_i' = g_i + \int_j \Delta_{ij} S_j, \tag{A.2}$$

such that, perturbatively in F and to all orders in powers of F,

$$S'[g_i] = S[g_i']. \tag{A.3}$$

The condition (A.3) can be written as

$$S[g_i] + \int_{ij} \sqrt{-g} S_i F_{ij} S_j = S\left[g_i + \int_j \Delta_{ij} S_j\right] = S[g_i] + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\substack{k_1 \cdots k_n \\ m_1 \cdots m_n}} \widetilde{S}_{k_1 \cdots k_n} \prod_{l=1}^{n} (\Delta_{k_l m_l} S_{m_l}),$$

after a Taylor expansion, where $\widetilde{S}_{k_1 \cdots k_n} \equiv \delta^n S/(\delta g_{k_1} \cdots \delta g_{k_n})$ if an n-tensor density. The equality is verified if

$$\Delta_{ij} = F_{ij} - \frac{1}{\sqrt{-\det g_i}} \int_{\substack{k_1 \cdots k_n \\ m_3 \cdots m_n}} \Delta_{k_1 i} \Delta_{k_2 j} \sum_{n=2}^{\infty} \frac{1}{n!} \widetilde{S}_{k_1 k_2 k_3 \cdots k_n} \prod_{l=3}^{n} (\Delta_{k_l m_l} S_{m_l}), \tag{A.4}$$

where the product is meant to be 1 for $n = 2$.

Let A_i be tensor and

$$F = \frac{1}{2} \int_{ij} \sqrt{-g_i} A_i F_{ij} A_j$$

the scalar quadratic form defined by F_{ij}. Then

$$B_i \equiv \int_j F_{ij} A_j = \frac{\widetilde{\delta} F}{\widetilde{\delta} A_i} \tag{A.5}$$

is a tensor. Moreover, the symmetry property of F_{ji} reads

$$\frac{1}{\sqrt{-\det g_i}} F_{ji} = F_{ij} \frac{1}{\sqrt{-\det g_j}}. \tag{A.6}$$

Observe that (A.4) implies that Δ_{ij} satisfies the same symmetry property:

$$\frac{1}{\sqrt{-\det g_i}} \Delta_{ji} = \Delta_{ij} \frac{1}{\sqrt{-\det g_j}}. \tag{A.7}$$

Although F_{ij}, Δ_{ij} and $\widetilde{S}_{ij}$ are bitensors, we do not really need to use them explicitly. For example,

$$C_{k_1} \equiv \frac{1}{\sqrt{-\det g_{k_1}}} \int_{k_2} \widetilde{S}_{k_1 k_2} B_{k_2} \tag{A.8}$$

is a tensor that can be calculated as

$$C_{k_1} = \frac{1}{\sqrt{-\det g_{k_1}}} \int_{k_2} \frac{\delta \widetilde{S}_{k_1}}{\delta g_{k_2}} B_{k_2} = \frac{1}{\sqrt{-\det g_{k_1}}} \left. \delta \widetilde{S}_{k_1}(\delta g) \right|_{\delta g = B}, \tag{A.9}$$

without working out the bi-tensor density $S_{k_1 k_2}$ explicitly. Since $\widetilde{S}_{k_1}$ is a tensor density, also $\left. \delta \widetilde{S}_{k_1}(\delta g) \right|_{\delta \phi = B}$ is a tensor density, so C_{k_1} is a tensor.

Define the tensors

$$E_i^{(n)} = \int_j F_{ij} D_j^{(n)} \qquad D_i^{(n)} = \frac{1}{\sqrt{-\det g_i}} \int_j \widetilde{S}_{ij} E_j^{(n-1)},$$

recursively calculable from $D_i^{(0)} \equiv S_i$, with a chain of operations (A.9) and (A.5), and the tensor dentities

$$\widetilde{D}_i^{(n)} = \sqrt{-\det g_i} D_i^{(n)}.$$

So

$$g'_i = g_i + \Delta_{ij} S_j = g_i + E_i^{(0)} - \frac{1}{2} E_i^{(1)} + \frac{1}{2} E_i^{(2)} \tag{A.10}$$
$$-\frac{1}{3!} \frac{1}{\sqrt{-\det g_i}} \int_{k_1,k_2,k_3} F_{k_1 i} \widetilde{S}_{k_1 k_2 k_3} E_{k_2}^{(0)} E_{k_3}^{(0)} + \mathcal{O}(F^4).$$

The first line is immediate to calculate, with the operations already given. The second line is less immediate. Write

$$\int_{k_2,k_3} \widetilde{S}_{k_1 k_2 k_3} E_{k_2}^{(n)} E_{k_3}^{(m)} = \int_{k_3} \frac{\delta \widetilde{D}_{k_1}^{(n+1)}}{\delta g_{k_3}} E_{k_3}^{(m)} - \int_{k_2,k_3} \widetilde{S}_{k_1 k_2} \frac{\delta E_{k_2}^{(n)}}{\delta g_{k_3}} E_{k_3}^{(m)} \tag{A.11}$$

In this form, the second line of (A.10) can be calculated straightforwardly, using the same procedure as for (A.9). Clearly (A.11) a tensor density and the second line of (A.10) is a tensor.

At the fourth order, we find objects such as

$$\int_{k_2,k_3} \widetilde{S}_{k_1 k_2 k_3} E_{k_2}^{(1)} E_{k_3}^{(0)}, \qquad \int_{k_2,k_3,k_4} \widetilde{S}_{k_1 k_2 k_3 k_4} E_{k_2}^{(0)} E_{k_3}^{(0)} E_{k_4}^{(0)},$$

etc. The first of these can be commuted using (A.11). The second can be computed writing

$$\int_{k_2,k_3,k_4} \widetilde{S}_{k_1 k_2 k_3 k_4} E_{k_2}^{(0)} E_{k_3}^{(0)} E_{k_4}^{(0)} = \int_{k_2,k_3,k_4} \frac{\delta(\widetilde{S}_{k_1 k_2 k_3} E_{k_2}^{(0)} E_{k_3}^{(0)})}{\delta g_{k_4}} E_{k_4}^{(0)} - 2 \int_{k_2,k_3,k_4} \widetilde{S}_{k_1 k_2 k_3} E_{k_2}^{(0)} \frac{\delta E_{k_3}^{(0)}}{\delta g_{k_4}} E_{k_4}^{(0)},$$

and so on. The procedure can be iterated to all orders.

B The map M for gravity

We work out the map $g'(g,a,b)$ such that

$$S'[g] \equiv \int \mathrm{d}^n x \sqrt{-g} \left[R + a R_{\mu\nu} R^{\mu\nu} + b R^2 \right] = \int \mathrm{d}^n x \sqrt{-g'} R(g') \equiv S[g']$$

to the orders a^2, ab, b^2, in generic spacetime dimension n. We have

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^n x \sqrt{-g} R(g), \qquad D_i^{(0)} = S_i = -\frac{1}{2\kappa^2} \left(R^{\mu_i \nu_i} - \frac{1}{2} g^{\mu_i \nu_i} R \right),$$

where $i = (\mu_i, \nu_i, x_i)$ collects the spacetime indices and the spacetime point and $g_i = g_{\mu_i \nu_i}(x_i)$. Next, we have

$$F_{ij} = 2\kappa^2 \left\{ \frac{a}{2} (g_{\mu_i \mu_j} g_{\nu_i \nu_j} + g_{\mu_i \nu_j} g_{\nu_i \mu_j}) + \overline{b} g_{\mu_i \nu_i} g_{\mu_j \nu_j} \right\} \delta^{(n)}(x_i - x_j), \qquad \overline{b} = \frac{4b + a(4-n)}{(n-2)^2}.$$

Multiplying F_{ij} by S_i we find.

$$E_i^{(0)}(g) = -a R_{\mu_i \nu_i} + \frac{v}{2} g_{\mu_i \nu_i} R, \qquad v = a + (n-2)\overline{b}.$$

Next, applying (A.11) we get

$$E_i^{(1)}(g) = \frac{a^2}{2}\Box R_{\mu_i\nu_i} - \frac{av}{4}(n-2)\nabla_{\mu_i}\nabla_{\nu_i}R - \frac{a}{4}\left((n-4)v-4a\right)RR_{\mu_i\nu_i} - a^2R_{\mu_i}^{\lambda}R_{\lambda\nu_i}$$
$$-a^2R_{\mu_i\alpha\nu_i\beta}R^{\alpha\beta} + g_{\mu_i\nu_i}\left\{\frac{v}{4}\left((n-1)v-2a\right)\Box R + \frac{a(v-2\bar{b})}{2}R_{\alpha\beta}R^{\alpha\beta} + uR^2\right\},$$

where

$$u = \frac{(n-4)}{8}\left(v^2-2a\bar{b}\right) - \frac{a^2}{4}.$$

The map $\mathcal{M}$ reads, to the second order in F,

$$g_i' = g_i + \Delta_{ij}S_j = g_i + E_i^{(0)} - \frac{1}{2}E_i^{(1)} + \mathcal{O}(F^3).$$

The inverse map reads

$$g_i = g_i' - E_i^{(0)}(g') + \delta E_i^{(0)}(g')\Big|_{\delta g'=E^{(0)}(g')} + \frac{1}{2}E_i^{(1)}(g') + \mathcal{O}(F^3),$$

where, to the same order,

$$\delta E_i^{(0)}(g')\Big|_{\delta g'=E^{(0)}(g')} = E_i^{(1)}(g') + 2a^2R_{\mu_i}^{\lambda}R_{\lambda\nu_i} + \frac{a}{4}\left((n-6)v-4a\right)RR_{\mu_i\nu_i}$$
$$+g_{\mu_i\nu_i}\left(a\bar{b}R_{\alpha\beta}R^{\alpha\beta} - uR^2\right).$$

C Batalin-Vilkovisky formalism for perturbed Einstein-Yang-Mills theory

When gauge couplings depend on spacetime, the fields and their BRST transformations are renormalized in a non-trivial way. To keep track of such renormalizations it is necessary to use the Batalin-Vilkovisky formalism [10, 11]. We focus on higher-derivative Einstein-Yang-Mills theory and perturb it with the vertex fRF^2.

Consider first the unperturbed case $f=0$. We introduce Faddeev-Popov ghosts C^a, antighosts $\overline{C}^a$ and a Lagrange multiplier B^a for the gauge-fixing. The fields are collectively denoted with $\Phi^i = (A_\mu^a, \overline{C}^a, C^a, B^a)$. We add BRST sources $K_i = (K_a^\mu, K_{\overline{C}}^a, K_C^a, K_B^a)$ for every field Φ^i and extend the action (3.1) as

$$\mathcal{S}_{\text{EYM-HD}} = \int \mathrm{d}^n x\sqrt{-g}\left[\frac{R}{2\kappa^2} + \xi W^2 + \zeta G_B + \frac{\eta}{(n-1)^2}R^2 - \frac{1}{4\alpha}F_{\mu\nu}^a F^{a\mu\nu}\right] \qquad \text{(C.1)}$$
$$+s\Psi(\Phi,g) - \int \mathrm{d}^n x\sqrt{-g}\left[\left(sA_\mu^a\right)K_a^\mu + \left(s\overline{C}^a\right)K_{\overline{C}}^a + \left(sC^a\right)K_C^a + \left(sB^a\right)K_B^a\right],$$

where the BRST transformations are

$$sA_\mu^a = \partial_\mu C^a + f^{abc}A_\mu^b C^c, \qquad sC^a = -\frac{1}{2}f^{abc}C^b C^c, \qquad s\overline{C}^a = B^a, \qquad sB^a = 0.$$

We choose the gauge fixing $\nabla^\mu A^a_\mu = 0$, which breaks the gauge symmetry but preserves general covariance:

$$\Psi(\Phi, g) \equiv \int \mathrm{d}^n x \sqrt{-g} \left[-\frac{\lambda}{2} \overline{C}^a B^a + \overline{C}^a \nabla^\mu A^a_\mu \right]. \tag{C.2}$$

Define the antiparenthesis

$$(X, Y) = \int \mathrm{d}^n x \sqrt{-g(x)} \left\{ \frac{\widetilde{\delta}_r X}{\widetilde{\delta}\Phi^i(x)} \frac{\widetilde{\delta}_l Y}{\widetilde{\delta} K_i(x)} - \frac{\widetilde{\delta}_r X}{\widetilde{\delta} K_i(x)} \frac{\widetilde{\delta}_l Y}{\widetilde{\delta}\Phi^i(x)} \right\}, \tag{C.3}$$

where the tilded derivatives are normal derivatives divided by $\sqrt{-g}$. A canonical transformation of fields and sources is a transformation that preserves the antiparenthesis. It is generated by a functional $\mathcal{F}(\Phi, K')$ and reads

$$\Phi^{i\,\prime} = \frac{\widetilde{\delta}\mathcal{F}}{\widetilde{\delta} K'_i}, \qquad K_i = \frac{\widetilde{\delta}\mathcal{F}}{\widetilde{\delta}\Phi^i},$$

The generating functional of the identity transformation is

$$I(\Phi, K') = \int \mathrm{d}^n x \sqrt{-g} \sum_i \Phi^i K'_i.$$

The BRST invariance is generalized to the identity

$$(\mathcal{S}_{\text{EYM-HD}}, \mathcal{S}_{\text{EYM-HD}}) = 0. \tag{C.4}$$

Define also the generalized BRST operator

$$\sigma X \equiv (\mathcal{S}_{\text{EYM-HD}}, X), \tag{C.5}$$

which is nilpotent, $\sigma^2 = 0$, because of the identity (C.4).

The identity (C.4) ensures also, in every regularization scheme,

$$(\Gamma_{\text{EYM-HD}}, \Gamma_{\text{EYM-HD}}) = 0, \tag{C.6}$$

where $\Gamma_{\text{EYM-HD}}$ is the generating functional of one-particle irreducible diagrams, including the diagrams that have ghosts, Langrange multipliers and BRST-sources on their external legs.

Proceeding inductively, assume that $\Gamma^{(n)}_{\text{EYM-HD}}$ is the generating functional renormalized up to the n-th loop included. General renormalization theory and (C.6) ensure that the $(n+1)$-loop divergences $\Gamma^{(n+1)}_{\text{div}}$ are local and σ-closed, $\sigma\Gamma^{(n+1)}_{\text{div}} = 0$. The most general solution to this condition has the form

$$\Gamma^{(n+1)}_{\text{div}} = \mathcal{G}_n + \sigma\mathcal{R}_n, \tag{C.7}$$

where $\mathcal{G}_n$ is a gauge-invariant functional depending only on A^a_μ (and the metric tensor). By parity invariance, $\mathcal{G}_n$ can only be the sum of a term proportional to F^2, which is reabsorbed renormalizing the squared gauge coupling α, plus pure gravitational terms with dimensionality four, that are reabsorbed renormalizing ξ, ζ, η and τ. The σ-exact terms can be easily classified. They are equal to σ acting on a functional with dimensionality three and ghost number -1. For future use, we list all of the scalars, vectors and tensors that have dimensionality three or less and ghost number -1. There are 17 such objects:

$$\begin{aligned}
&K_a^\mu A_\nu^a, \quad K_C^a C^a, \quad K_{\overline{C}}^a \overline{C}^a, \quad K_B^a B^a, \quad \overline{C}^a B^a, \quad \overline{C}^a \nabla_\mu A_\nu^a, \quad \left(\nabla_\mu \overline{C}^a\right) A_\nu^a, \quad \overline{C}^a A_\mu^a, \\
&f^{abc}\overline{C}^a A_\mu^b A_\nu^c, \quad f^{abc}\overline{C}^a \overline{C}^b C^c, \quad K_{\overline{C}}^a K_B^a, \quad f^{abc} K_B^a A_\mu^b A_\nu^c, \quad f^{abc} K_B^a \overline{C}^b C^c, \\
&f^{abc} K_B^a K_B^b C^c, \quad K_B^a \nabla_\mu A_\nu^a, \quad \left(\nabla_\mu K_B^a\right) A_\nu^a, \quad K_B^a A_\mu^a.
\end{aligned} \tag{C.8}$$

The counterterms are obtained acting with σ on linear combinations of these objects, but must not contain B^a, K_B^a and $K_{\overline{C}}^a$, because the action (C.1) provides no vertices with B^a, K_B^a or $K_{\overline{C}}^a$ on the external legs. There are only three linear combinations that have this property, namely

$$\overline{C}^a K_{\overline{C}}^a + K_B^a B^a + \lambda \overline{C}^a B^a - \overline{C}^a \nabla^\mu A_\mu^a, \qquad \left(K_a^\mu + \nabla^\mu \overline{C}^a\right) A_\nu^a, \qquad K_C^a C^a. \tag{C.9}$$

The first of these, however, is trivial, since it is itself σ-exact. Precisely, it is equal to $\sigma\left(\overline{C}^a K_B^a\right)$. In the unperturbed theory, the counterterms are necessarily scalar, so we can drop the vectors of (C.8) and trace the second of (C.9). We remain with two scalar combinations. A convenient basis is for example the one obtained taking the third term of (C.9) and the difference between the first two, namely

$$I_1(\Phi, K) \equiv -K_a^\mu A_\mu^a + \overline{C}^a K_{\overline{C}}^a + K_B^a B^a + \lambda \overline{C}^a B^a - \nabla^\mu\left(\overline{C}^a A_\mu^a\right), \qquad I_2(\Phi, K) \equiv K_C^a C^a. \tag{C.10}$$

In this framework the functional $\mathcal{R}_n$ of (C.7) has the form

$$\mathcal{R}_n(\Phi, K) = \int \mathrm{d}^n x \sqrt{-g}\left[-\delta_A^{(n)} I_1 + \delta_C^{(n)} I_2\right],$$

where $\delta_A^{(n)}$ and $\delta_C^{(n)}$ are divergent constants. The total derivative contained in the first term of (C.10) can be dropped.

The basis (C.10) is such that the σ-exact counterterms are reabsorbed by a renormalization $\lambda' = \lambda Z_{n\lambda}$ of the gauge-fixing parameter λ and a canonical transformation

$$\Phi^{i\,\prime} = Z_{ni}^{1/2}\Phi^i, \qquad K_i^{(n)\,\prime} = Z_{ni}^{-1/2} K_i,$$

generated by

$$\mathcal{F}_n(\Phi, K') = \int \mathrm{d}^n x \sqrt{-g} \sum_i Z_{ni}^{1/2} \Phi^i K_i' = I(\Phi, K') - \mathcal{R}_n(\Phi, K') + \text{higher orders}.$$

where

$$Z_{n\overline{C}} = Z_{nB} = Z_{nA}^{-1} = Z_{n\lambda}^{-1}, \qquad Z_{nK_i} = Z_{ni}^{-1}, \qquad Z_{nA}^{1/2} = 1 - \delta_A^{(n)}, \qquad Z_{nC}^{1/2} = 1 - \delta_C^{(n)}. \tag{C.11}$$

The Ward identities (C.11) that relate the wave-function renormalization constants easily extend to the complete renormalized theory:

$$Z_{\overline{C}}=Z_B=Z_A^{-1}=Z_\lambda^{-1},\qquad Z_{K_i}=Z_i^{-1},\qquad Z_i=\prod_{i=1}^{\infty}Z_{ni}.$$

Another parametrization, which is more convenient in the perturbed model, amounts to use the basis

$$J_1(\Phi,K)\equiv\left(K_a^\mu+\nabla^\mu\overline{C}^a\right)A_\mu^a,\qquad J_2(\Phi,K)\equiv K_C^aC^a, \tag{C.12}$$

instead of (C.10). Then

$$\mathcal{F}_n(\Phi,K')=\int\sqrt{-g}\left[Z_{nA}^{1/2}K_a^\mu{}'A_\mu^a+\overline{C}^aK_{\overline{C}}^a{}'+K_B^a{}'B^a+Z_{nC}^{1/2}K_C^a{}'C^a+\left(Z_{nA}^{1/2}-1\right)\left(\nabla^\mu\overline{C}^a\right)A_\mu^a\right]$$

and again $Z_{nA}^{1/2}=1-\delta_A^{(n)}$, $Z_{nC}^{1/2}=1-\delta_C^{(n)}$. Here $\overline{C}^a$, B^a, K_B^a and λ are non-renormalized and the unique non-trivial redefinitions are

$$\begin{aligned}&A_\mu^a\rightarrow Z_{nA}^{1/2}A_\mu^a,\qquad K_a^\mu\rightarrow Z_{nA}^{-1/2}K_a^\mu+\nabla^\mu\overline{C}^a(Z_{nA}^{-1/2}-1),\\&K_{\overline{C}}^a\rightarrow K_{\overline{C}}^a-\nabla^\mu A_\mu^a+\nabla^\mu(A_\mu^aZ_{nA}^{1/2}),\qquad C^a\rightarrow Z_{nC}^{1/2}C^a,\qquad K_C^a\rightarrow Z_{nC}^{-1/2}K_C^a,\end{aligned} \tag{C.13}$$

besides the renormalization of the gauge coupling.

Now define a map $\Sigma_{\mathcal{L}}$, $\mathcal{L}=(L,L_A,L_C)$ made of a redefinition

$$\alpha\rightarrow\alpha(fR)=\alpha L(fR) \tag{C.14}$$

of the gauge coupling, plus the addition of unspecified pure gravitational terms (described below), plus the canonical transformation generated by

$$\mathcal{F}(\Phi,K')=\int\mathrm{d}^nx\sqrt{-g}\left[L_AK_a^\mu{}'A_\mu^a+\overline{C}^aK_{\overline{C}}^a{}'+K_B^a{}'B^a+L_CK_C^a{}'C^a+(L_A-1)\left(\nabla^\mu\overline{C}^a\right)A_\mu^a\right],$$

where the functions $\mathcal{L}$ depend fR. The unique non-trivial redefinitions of fields and BRST sources are

$$\begin{aligned}&A_\mu^a\rightarrow L_AA_\mu^a,\qquad K_a^\mu\rightarrow L_A^{-1}K_a^\mu+\left(\nabla^\mu\overline{C}^a\right)(L_A^{-1}-1),\\&K_{\overline{C}}^a\rightarrow K_{\overline{C}}^a-\nabla^\mu A_\mu^a+\nabla^\mu(A_\mu^aL_A),\qquad C^a\rightarrow L_CC^a,\qquad K_C^a\rightarrow L_C^{-1}K_C^a,\end{aligned} \tag{C.15}$$

The maps $\Sigma_{\mathcal{L}}$ form a group:

$$\Sigma_{\mathcal{L}}\Sigma_{\mathcal{L}'}=\Sigma_{\mathcal{L}\mathcal{L}'},\qquad\mathcal{L}\mathcal{L}'=(LL',L_AL_A',L_CL_C').$$

Moreover, the basis (C.12) is invariant:

$$\Sigma_{\mathcal{L}}J_1=J_1,\qquad\Sigma_{\mathcal{L}}J_2=J_2.$$

We assert that the perturbed theory obtained applying the map $\Sigma_{\mathcal{L}}$ to the unperturbed one is renormalizable and that the subtraction of divergences is again a map $\Sigma_{\mathcal{L}}$, namely a renormalization of the gauge coupling of the form (C.14), plus a canonical transformation of the form (C.15), plus a suitable renormalization of the pure gravitational terms.

The action of the perturbed theory is

$$\mathcal{S}_{\text{EYM-HD}}^{(f)} = \Sigma_{\mathcal{L}}\mathcal{S}_{\text{EYM-HD}} = \int \mathrm{d}^n x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{4\alpha L}F_{\mu\nu}^a(L_A A)F^{a\mu\nu}(L_A A) - \frac{\lambda}{2}B^a B^a + B^a\nabla^\mu A_\mu^a\right.$$
$$\left. + L_A^{-1}\left(K_a^\mu + \nabla^\mu\overline{C}^a\right)\left[\partial_\mu(L_C C^a) + f^{abc}L_A L_C A_\mu^b C^c\right] + \frac{L_C}{2}f^{abc}K_C^a C^b C^c - K_{\overline{C}}^a B^a\right\} + \Delta S_g, \quad \text{(C.16)}$$

where $\Delta_g S$ denote the pure gravitational terms, so far unspecified. The Batalin-Vilkovisky analysis has to be applied with the perturbed σ-operator σ_f, defined by

$$\sigma_f X \equiv \left(\mathcal{S}_{\text{EYM-HD}}^{(f)}, X\right).$$

It is easy to verify that σ_f is nilpotent, namely

$$\left(\mathcal{S}_{\text{EYM-HD}}^{(f)}, \mathcal{S}_{\text{EYM-HD}}^{(f)}\right) = 0.$$

Calling $\Gamma_{\text{EYM-HD}}^{(f)}$ the generating functional of one-particle irreducible diagrams, we have also

$$(\Gamma_{\text{EYM-HD}}^{(f)}, \Gamma_{\text{EYM-HD}}^{(f)}) = 0. \quad \text{(C.17)}$$

Proceeding inductively, call $\Gamma_{\text{EYM-HD}}^{(f,n)}$ the perturbed generating functional renormalized up to the n-th loop included. Renormalization theory and (C.17) ensure that the $(n+1)$-loop divergences $\Gamma_{\text{div}}^{(f,n+1)}$ are local and σ_f-closed: $\sigma_f\Gamma_{\text{div}}^{(f,n+1)} = 0$. The solution is

$$\Gamma_{\text{div}}^{(f,n+1)} = \mathcal{G}_{nf} + \sigma_f\mathcal{R}_{nf}. \quad \text{(C.18)}$$

Let us analyze the vertices of the theory, that can be read from (C.16). We can write

$$\mathcal{S}_{\text{EYM-HD}}^{(f)} = \mathcal{S}_{\text{EYM-HD}} + \int \mathrm{d}^n x\sqrt{-g}\left\{\overline{K}_1\mathrm{O}_1 + \overline{K}_{2\mu}\mathrm{O}_2^\mu + \overline{K}_{3\mu\nu}\mathrm{O}_3^{\mu\nu}\right\} + \Delta S_g,$$

where O_1, O_2^μ and $\mathrm{O}_3^{\mu\nu}$ are f-independent operators, with dimensionalities 4, 3 and 2, respectively, constructed with the fields, the BRST sources and their derivatives, while the gravitational sources read

$$\overline{K}_1 = P_1(fR), \qquad \overline{K}_{2\mu} = fP_2(fR)\nabla_\mu R, \qquad \overline{K}_{3\mu\nu} = f^2P_3(fR)\nabla_\mu R\nabla_\nu R. \quad \text{(C.19)}$$

the P_i's being functions of fR. Every f-dependence is contained in the $\overline{K}$'s.

The counterterms (C.18) are local, covariant, have dimensionality four and are constructed with the $\overline{K}$'s, the matter fields Φ^i, the BRST sources K_i, the curvature tensors and their covariant derivatives. The $\overline{K}$'s have non-negative dimensionalities in units of mass. There is one ingredient (P_1) of dimensionality zero, which is why renormalization turns on arbitrary functions of fR.

Let us study the σ_f-cohomology. We postpone the discussion about the pure gravitational terms, which are trivially σ_f-closed, and first concentrate on the terms that depend on Φ^i and K_i. The σ_f-closed terms of type $\mathcal{G}_{nf}$ can contain F^2, $T_{\mu\nu}$ and $\Upsilon_{\mu\nu\rho\sigma}$, with A_μ^a replaced by $L_A A_\mu^a$. However, power counting excludes both $T_{\mu\nu}$ and $\Upsilon_{\mu\nu\rho\sigma}$, since they have dimensionality four and the only dimensionless $\overline{K}$ is scalar. Therefore, only F^2 remains.

The functional $\mathcal{R}_{nf}$ appearing in the exact σ_f-terms of (C.18) is again a linear combination of (C.8), with coefficients constructed with the sources $\overline{K}$'s, the curvature tensors and their covariant derivatives, such that $\sigma_f \mathcal{R}_{nf}$ does not contain B^a, K_B^a and $K_{\overline{C}}^a$. There are no σ_f-exact terms with dimensionality two or less, so $\overline{K}_{3\mu\nu}$ can be dropped. We can drop also $\overline{K}_{2\mu}$ together with the terms $\overline{C}^a A_\mu^a$ and $K_B^a A_\mu^a$ of (C.8), because the counterterms constructed with these objects are easily converted, by means of a partial integration, into products of a scalar function times a combination of the other terms (C.8).

The σ- and σ_f- cohomologies are in one-to-one correspondence. To relate them, start from

$$\sigma X = (\mathcal{S}_{\text{EYM-HD}}, X) \equiv Y \tag{C.20}$$

and perform the canonical transformation (C.15). Using the invariance of the antiparenthesis and denoting the transformed functionals with a tilde, we obtain

$$(\widetilde{\mathcal{S}}_{\text{EYM-HD}}, \widetilde{X}) = \widetilde{Y}. \tag{C.21}$$

The transformed action $\widetilde{\mathcal{S}}_{\text{EYM-HD}}$ differs from $\mathcal{S}_{\text{EYM-HD}}^{(f)}$ because of the coupling redefinition:

$$\widetilde{\mathcal{S}}_{\text{EYM-HD}} = \mathcal{S}_{\text{EYM-HD}}^{(f)} + \int \mathrm{d}^n x \sqrt{-g} \frac{1-L}{4\alpha L} F_{\mu\nu}^a(L_A A) F^{a\mu\nu}(L_A A) \equiv \mathcal{S}_{\text{EYM-HD}}^{(f)} + \widetilde{\Delta}_L,$$

therefore (C.21) can be written as

$$\sigma_f \widetilde{X} = \widetilde{Y} - (\widetilde{\Delta}_L, \widetilde{X}) = \widetilde{Y} - \widetilde{(\Delta_L, X)}.$$

Assume that X is a linear combination of the terms (C.8). Since Δ_L depends only on A_μ^a, only the term proportional to $K_a^\mu A_\nu^a$ in X contributes to (Δ_L, X). So, (Δ_L, X) does not contain B^a, K_B^a or $K_{\overline{C}}^a$. Moreover, the canonical transformation (C.15) is such that functionals (not) containing B^a, K_B^a and $K_{\overline{C}}^a$ are mapped into functionals (not) containing B^a, K_B^a and $K_{\overline{C}}^a$. Thus, an X such that σX does (not) contain B^a, K_B^a and $K_{\overline{C}}^a$ is mapped into an $\widetilde{X}$ such that $\sigma_f \widetilde{X}$ does (not) contain B^a, K_B^a and $K_{\overline{C}}^a$, and viceversa. Having dropped both $\overline{K}_{2\mu}$ and $\overline{K}_{3\mu\nu}$, we can focus on scalar functionals X, $\widetilde{X}$.

These properties ensure that most general $\widetilde{X}$ can be obtained applying the canonical transformation (C.15) to the most general X. Since the latter is a linear combination of J_1 and J_2, and $\widetilde{J}_1 = J_1$, $\widetilde{J}_2 = J_2$, also the former is a linear combination of J_1 and J_2.

Finally, we have

$$\Gamma_{\text{div}}^{(f,n+1)} = U_n \ F_{\mu\nu}^{a}(L_A A)F^{a\mu\nu}(L_A A) + \sigma_f(V_n J_1 + W_n J_2) + \text{ pure gravitational terms,}$$

where U_n, V_n and W_n are functions of fR. Now it is immediate to see that the divergences are inductively subtracted by a map of the form $\Sigma_{\mathcal{L}}$ with $\mathcal{L} = (1 - 4\alpha L U_n, 1 - V_n, 1 - W_n)$ and we conclude that the theory (C.16) is renormalizable.

The pure gravitational counterterms can be constructed with the $\overline{K}$'s, the curvature tensors and their covariant derivatives. The list of independent terms is

$$\begin{array}{lllll} Q_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, & Q_2 R_{\mu\nu}R^{\mu\nu}, & Q_3 R^2, & Q_4 \Box R, & f^2 Q_5 R^{\mu\nu}\nabla_\mu R \nabla_\nu R, \\ f^3 Q_6 \nabla^\mu R \nabla_\mu R \Box R, & f^4 Q_7 \left(\nabla^\mu R \nabla_\mu R\right)^2, & f^2 Q_8 \left(\Box R\right)^2, & & \end{array} \qquad \text{(C.22)}$$

where Q_i, $i = 1, \ldots 8$ are functions of fR. Thus $\Delta_g S$ is linear combination of such terms. We see that there is only one vertex, $RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, that is not squarely proportional to the Ricci tensor.

References

[1] B. DeWitt, *Dynamical theory of groups and fields*, Gordon And Beach Science Publishers, New York, 1965;

N.D. Birrel and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge, 1982.

[2] N. Bohr and S. Rosenfeld, Kgl. Dansk Vidensk. Selsk, Math.-fys. Medd. 12 (1933) 8.

[3] C. Møller, in *Les theories relativistes de la gravitation*, edited by A. Lichnerowics and M.A. Tonnelat, CNRS, Paris, 1962.

S. Rosenfeld, On quantization of fields, Nucl. Phys. 40 (1963) 353.

[4] R.P. Feynman, F.B. Moringo and W.G. Wagner, *Feynman lectures on gravitation*, Penguin Books ltd, 2000.

[5] K. Eppley and E. Hannah, The necessity of quantizing the gravitational field, Found. Phys. 7 (1977) 51.

[6] J. Mattingly, Why Eppley and Hannah's thought experiment fails, Phys. Rev. D 73 (2006) 064025 and arXiv:gr-qc/0601127.

[7] G. 't Hooft and M. Veltman, One-loop divergences in the theory of gravitation, Ann. Inst. Poincarè, 20 (1974) 69.

[8] K.S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D 16 (1977) 953;

E.S. Fradkin and A.A. Tseytlin, Renormalizable asymptotically free quantum theory of gravity, Nucl. Phys. B 201 (1982) 469.

[9] D. Anselmi, Renormalization and causality violations in classical gravity coupled with quantum matter, JHEP 01 (2007) 062 and arXiv:hep-th/0605205.

[10] I.A. Batalin and G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generators, Phys. Rev. D 28 (1983) 2567; erratum ibid. D 30 (1984) 508.

[11] S. Weinberg, *The quantum theory of fields*, vol. II, Cambridge University Press, New York, 1996.

[12] J. Schwinger, Brownian motion of a quantum oscillator, J. Math. Phys. 2 (1961) 407;

L.V. Keldysh, Diagram technique for nonequilibrium processes, Sov. Phys. JETP 20 (1965) 1018.

[13] R.D. Jordan, Effective field equations for expectation values, Phys. Rev. D. 33 (1986) 444.

[14] L.H. Ford and R.P. Woodard, Stress tensor correlators in the Schwinger-Keldysh formalism, Class. Quant. Grav. 22 (2005) 1637 and arXiv:gr-qc/0411003.

[15] J.D. Jackson, *Classical electrodynamics*, John Wiley and Sons, Inc. (1975), chap. 17.

[16] L. Bel and H. Sirouss Zia, Regular reduction of relativistic theories of gravitation with a quadratic lagrangian, Phys. Rev. D 32 (1985) 3128.

[17] L. Parker and J.Z. Simon, Einstein equation with quantum corrections reduced to second order, Phys. Rev. D 47 (1993) 1339 and arXiv:gr-qc/9211002.

[18] S.J. Hathrell, Trace anomalies and $\lambda\varphi^4$ theory in curved space, Ann. Phys. (NY) 139 (1982) 136.

[19] S.J. Hathrell, Trace anomalies and QED in curved space, Ann. Phys. (NY) 142 (1982) 34.

[20] M.D. Freeman, Renormalization of non-Abelian gauge theories in curved space-time, Ann. Phys. (NY) 153 (1984) 339.

[21] J. Collins, *Renormalization*, Cambridge University Press, Cambridge, UK, 1984, § 13.5.

[22] D. Anselmi, Absence of higher derivatives in the renormalization of propagators in quantum field theories with infinitely many couplings, Class. Quantum Grav. 20 (2003) 2355 and arXiv:hep-th/0212013.

[23] D. Anselmi, D.Z. Freedman, M.T Grisaru and A. Johansen, Nonperturbative formulas for central functions in supersymmetric gauge theories, Nucl. Phys. B 526 (1998) 543 and hep-th/9708042.