IFUP-TH 2006-11

Renormalization And Causality Violations In Classical Gravity Coupled With Quantum Matter

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Abstract

I prove that classical gravity coupled with quantized matter can be renormalized with a finite number of independent couplings, plus field redefinitions, without introducing higher-derivative kinetic terms in the gravitational sector, but adding vertices that couple the matter stress-tensor with the Ricci tensor. The theory is called "acausal gravity", because it predicts the violation of causality at high energies. Renormalizability is proved by means of a map $\mathcal{M}$ that relates acausal gravity with higher-derivative gravity. The causality violations are governed by two parameters, a and b, that are mapped by $\mathcal{M}$ into higher-derivative couplings. At the tree level causal prescriptions exist, but they are spoiled by the one-loop corrections. Some ideas are inspired by the usual treatments of the Abraham-Lorentz force in classical electrodynamics.

1 Introduction

The necessity of quantizing gravity is a debated issue. Bohr and Rosenfeld [1] showed that a theory in which some fields are quantized and others are not can violate some basic principles of quantum mechanics, for example the indeterminacy principle. Rosenfeld [2] observed that there is no direct evidence for the validity of such principles in situations where the gravitational field is important. Feynman questioned whether gravity must be quantized in his lectures on gravitation [3]. Møller [4] and Rosenfeld [2] gave a specific suggestion to couple a one-half quantum and one-half classical world, in the realm of quantum mechanics. They stated that the spacetime geometry couples to the expectation value of the energy-momentum tensor, calculated on the quantum state of the matter fields. Eppley and Hannah [5] showed that if matter is quantized, but gravity is classical, then, assuming the "Copenhagen" interpretation of quantum mechanics, two scenarios are given: if the gravitational interactions do not collapse the wave-function, gravity can be used to propagate information at superluminal velocity; if, on the other hand, gravity collapses the wave-function, then either the uncertainty principle or energy-momentum conservation can be violated. They also suggested an experiment to establish whether gravity is quantum mechanical. Recently, Mattingly [6] questioned the feasibility of any such experiment. Other arguments advocated to assert that gravity needs to be quantized are weaker, because they are just based on the analogy with the other interactions of nature. None of these observations settle the debate, actually, since experiments are unable, at present, to ensure that the gravitational interactions obey the indeterminacy principle and causality at arbitrarily high energies.

A remarkable fact is that the Standard Model is "ready" for the coupling with gravity, in the sense that the anomaly cancellations survive the embedding in a curved background [7]. Thus it is natural to consider a partially quantized theory where the Standard Model is embedded in external gravity, which is treated classically, and the pure-gravity sector is described just by the Einstein action with a cosmological term. For consistency, no higher-derivative gravitational kinetic terms should be turned on by renormalization.

The investigation of classical gravity coupled with quantum field theory in a curved background is an alternative way to search for new physics beyond the Standard Model. A variety of problems can be treated exactly and physical predictions can be derived. The results can also suggest new experimental observations to determine whether gravity must be quantized or not. Some predictions might hold also for quantum gravity, at least qualitatively.

The main purposes of this paper are to:

1) extend the Møller-Rosenfeld approach [4, 2] to quantum field theory, formulating a minimum principle that generates the field equations of a partially quantized theory,

2) prove that classical gravity coupled with quantum matter is renormalizable with a finite number of independent parameters, without introducing higher-derivative kinetic terms in the

gravitational sector;
3) analyze the physical effects of renormalization in the gravitational sector, such as the violation of causality at short distances.

The quantization of fields in curved space (see for example [8]) has motivated an enormous amount of work. An extension of the Møller-Rosenfeld approach has been proposed by Schwinger and Keldysh [9], in terms of the "in-in" expectation value of the stress tensor, which is both real and causal. The approach formulated here uses out-in expectation values, to make a more direct connection with the standard formulation of quantum field theory. Nevertheless, the other results of this paper do not depend in a crucial way on how the stress-tensor expectation value is interpreted. In particular, with some obvious modifications, properties 2) and 3) hold also in the Schwinger-Keldysh framework.

The renormalizability of the partially quantized theory is proved applying a theorem stating that a term quadratically proportional to the field equations can be reabsorbed by a field redefinition to all orders. For the investigation of this paper, such a theorem is rephrased by a map

$$\mathcal{M}: S_{\mathrm{HD}} \rightarrow S_{\mathrm{AC}} \tag{1.1}$$

that relates a causal theory S_{HD} with instabilities, typically due to higher-derivative (HD) kinetic terms, with an acausal theory S_{AC} without instabilities. Precisely, S_{HD} is higher-derivative classical gravity coupled with quantum matter, whose renormalization is straightforward. The matter fields circulating in the loops generate the higher-derivative counterterms $R_{\mu\nu}R^{\mu\nu}$ and R^2, which are subtracted adding these same terms to the lagrangian, multiplied by independent parameters a and b. Instead, S_{AC} denotes classical Einstein gravity coupled with quantum matter. Its renormalization is less trivial. The counterterms $R_{\mu\nu}R^{\mu\nu}$ and R^2 are subtracted by means of a field redefinition of the metric tensor. The existence of such a field redefinition is obvious to the lowest order. The map $\mathcal{M}$ ensures its existence to all orders. In practice, the map $\mathcal{M}$ replaces the higher-derivative terms $R_{\mu\nu}R^{\mu\nu}$ and R^2 by new vertices belonging to the matter sector, that couple the matter stress tensor to the Ricci tensor, with coupling constants a and b.

A typical feature of higher derivative theories is that the field equations admit unstable solutions. For a discussion in classical higher-derivative gravity, see for example [10]. The field redefinition provided by the map $\mathcal{M}$ eliminates the unstable solutions. On the other hand, the map $\mathcal{M}$ contains power series in momenta that can be resummed exactly. The main outcome of the resummation is the violation of causality at high energies. The causality violation is independent of the interpretation of the stress-tensor expectation value. In particular, it is present also in the Schwinger-Keldysh approach.

The correspondence between instabilities and causality violations is inspired by an analogous correspondence that is usually learnt in connection with the Abraham-Lorentz force of classical electrodynamics [11] and that has been applied also to higher-derivative gravity [12, 13]. The approach (1.1) is not equivalent to the ones existing in the literature and is specifically designed to work efficiently in combination with renormalization.

The map $\mathcal{M}$ is useful to relate the renormalization properties of S_{HD} and S_{AC}, but it is not just a change of variables. The theories S_{HD} and S_{AC} are physically inequivalent, because the unstable solutions of S_{HD} are not solutions of S_{AC}. The map $\mathcal{M}$ is used to show that classical gravity coupled with quantum matter is predictive, because it can be renormalized with a finite set of independent couplings, plus field redefinitions, without introducing higher-derivative kinetic terms in the gravitational sector.

Commonly [14] the Planck scale is considered as the physical cut-off which defines the extreme limit of validity of semi-classical gravity and the attention is confined to predictions that involve energy scales much greater than the Planck length. However, as long as there is no definitive experimental evidence that gravity should be quantized, nor that causality should hold at arbitrarily high energies, there is no compelling reason to consider the model of this paper as an effective one. In such a situation, it belongs to the duties of a theorist to investigate also the consequences that follow from the assumption that the model is a fundamental theory, valid at arbitrarily high energies. This attitude is also the most efficient one to eventually uncover reasons to reject the assumption. As mentioned above, in the acausal theory constructed here, certain power series in the momenta can be resummed exactly, so it is compulsory to take these resummations seriously and inquire about their physical meaning, if any. What happens is that the stress tensor gets averaged in an acausal way, because the average receives contributions also from the future light cone and from spacelike separated points. The causality violations are parametrized by a and $b' = -2(a + 3b)$. At the tree level, there exist causal prescriptions, if a and b' are negative. However, the radiative corrections spoil the causal prescriptions and produce causality violations in any case.

The physical effects of the couplings a and b' can be detected also in causal situations. Experimental bounds on the values of a and b' can be derived from the tests about the validity of Newton's law at short distances.

The map $\mathcal{M}$ cannot be applied to quantum gravity, at least in a straightforward way. This is a weakness of the model if gravity ultimately needs to be quantized. It is a good feature of the model, instead, if gravity does not need to be quantized. Still, the results of this paper might inspire the search for appropriate generalizations of the map $\mathcal{M}$ to quantum gravity.

The paper is organized as follows. The minimization principle for fully and partially quantized field theories is treated in section 2. The map $\mathcal{M}$ is studied in section 3 and worked out explicitly for gravity in the quadratic approximation. A source term is then added to study the physical effects. In section 4 the map $\mathcal{M}$ is used to prove the renormalizability of the theory. Section 5 is devoted to the investigation of causality violations and their relation with instabilities. Section 6 contains the conclusions.

2 Minimum principles for fully and partially quantized field theories

According to the Møller-Rosenfeld approach [4, 2], in quantum mechanics classical gravity couples to the expectation value of the energy-momentum tensor, calculated on the quantum state ψ of the matter fields. The Einstein equations read

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\kappa^2\left\langle \psi\left|T_{\mu\nu}\right|\psi\right\rangle . \tag{2.1}$$

The generalization of this equation to classical gravity coupled with quantized fields has been discussed by various authors in the literature. In the Schwinger-Keldysh [9] approach, the right-hand side of (2.1) is replaced with the "in-in" expectation value of the stress tensor, so it is both real and causal. Functional methods for the calculation of in-in expectation values have been developed [15, 16]. It is important to observe that the renormalization structure does not depend on the interpretation of the right-hand side of (2.1). In particular, the counterterms $R_{\mu\nu}R^{\mu\nu}$ and R^2 calculated in the Schwinger-Kleydish approach are identical to those calculated in the usual approach [16]. The causality violations discussed here, which are due to the renormalization of $R_{\mu\nu}R^{\mu\nu}$ and R^2 *via* metric-tensor field redefinitions, are independent of the generalization of (2.1) to quantum field theory, so they exist also in the Schwinger-Keldysh approach.

Since causality is anyway violated in the end, it is meaningful to study a generalization of (2.1) that is closer to the standard formulation of quantum field theory, where correlation functions are out-in expectation values. The prescrition adopted in this paper is to replace the right-hand side of (2.1) with the real part of the out-in expectation value of the stress tensor.

Specifically, in a fully quantized theory the quantum action $S_q[\varphi_q]$, depending on the quantum fields φ_q, is defined as the real part of the generating functional $\Gamma[\Phi]$ of one-particle irreducible Green functions, under the assumption that *i)* $\varphi_q\equiv\Phi$ is real, if the fields φ are real bosonic, or *ii)* $\varphi_q=\Phi$ is the conjugate of $\overline{\Phi}$, if the fields φ are complex bosonic or fermionic. The variation of S_q with respect to φ_q gives the quantum field equations. This minimum principle applies both to fully quantized theories and to partially classical, partially quantum theories.

Consider a quantum field theory of fields φ. I first assume that the φ's are real bosonic and later generalize the argument to the other types of fields. Define, as usual, the generating functionals

$$Z[J]=\int\mathcal{D}\varphi\ \exp\left[i\int\mathrm{d}^4x\ \left(\mathcal{L}[\varphi(x)]+J(x)\varphi(x)\right)\right]$$

and

$$W[J] = i \ln Z[J], \qquad \Gamma[\Phi] = -W[J[\Phi]] - \int \mathrm{d}^4 x\ J[\Phi](x)\ \Phi(x),$$

of disconnected, connected and one-particle irreducible correlation functions, respectively, where

$$\Phi[J](x) = \langle \varphi(x) \rangle_J = -\frac{\delta W[J]}{\delta J(x)}, \qquad J[\Phi](x) = -\frac{\delta \Gamma}{\delta \Phi(x)}.$$

For the moment it is convenient to work with real fields φ. Then it is natural to take real sources J. Nevertheless, $Z[J]$, $W[J]$ and $\Phi[J]$ are complex functionals of J. The imaginary parts of the T-ordered correlation functions are originated by the $i\varepsilon$-prescription in the propagators. Consequently, if J is real, Φ cannot be a good quantum field and Γ, W, Z cannot be good quantum actions.

In a more general framework, assume that the sources J are complex. Observe that now J are complex sources for real fields φ. The T-anti-ordered Green functions are encoded in the conjugate functionals

$$W^*[J^*], \quad \Phi^*[J^*] = -\frac{\delta W^*[J^*]}{\delta J^*}, \quad \Gamma^*[\Phi^*] = -W^*[J^*] - \ J^* \cdot \Phi^*. \tag{2.2}$$

For convenience, integrals such as $\int \mathrm{d}^4 x\ J(x)\Phi(x)$ are often shortened as $J \cdot \Phi$.

Write $J = J_q + iJ_q'$, where J_q, J_q' are real. Now I prove that there exists a unique functional $J_q'[J_q]$, in perturbation theory, such that $\Phi[J]$ is real.

The reality of $\Phi[J]$ is expressed by the condition

$$\Phi[J] = \Phi^*[J^*], \qquad \text{i.e. } \frac{\delta W}{\delta J}[J_q + iJ_q'] = \frac{\delta W^*}{\delta J^*}[J_q - iJ_q']. \tag{2.3}$$

Formula (2.3) is an equation for $J_q'[J_q]$. Since at the tree level $\Gamma[\Phi]$ is real and coincides with the classical action, J_q' is at least one loop. In the perturbative expansion (2.3) reads

$$J_q' \left\{ \frac{\delta^2 W}{\delta J^2}[J_q] + \frac{\delta^2 W^*}{\delta J^{*2}}[J_q] \right\} = -i\Phi[J_q] + i\Phi^*[J_q] + \mathcal{O}(J_q'^2). \tag{2.4}$$

This equation admits one solution, since J_q', on the left-hand side, is multiplied by the real part of the two-point function, which is certainly invertible. For example, in momentum space for scalar fields

$$\frac{\delta^2 W}{\delta \widetilde{J}(-p)\delta \widetilde{J}(p)} = \frac{1}{p^2 - m^2 + i\varepsilon} + \mathcal{O}(\lambda),$$

where λ collectively denotes the coupling constants that parametrize the interactions of the theory. The left-hand side of (2.4) is just

$$2\ \mathrm{P}\left(\frac{1}{p^2 - m^2}\right) \widetilde{J}_q'(p) + \mathcal{O}(\lambda J_q'),$$

where P denotes the principal part. Returning to coordinate space, the solution reads

$$J'_q[J_q] = -(\partial^2 + m^2)\,\mathrm{Im}\,\Phi[J_q] + \mathcal{O}(\lambda J'_q, J'^2_q).$$

The higher orders can be worked out recursively in powers of λ and in the loop expansion.

Because of its reality, the functional $\Phi[J_q + iJ'_q[J_q]]$ can be taken as the quantum field $\varphi_q[J_q]$, with source J_q. Then the quantum action is

$$S_q[\varphi_q] \equiv \mathrm{Re}\,\Gamma[\varphi_q]$$

and coincides with the Legendre transform of $\mathrm{Re}\,W$, written as a functional of J_q. Indeed, consider

$$W_q[J_q] \equiv \mathrm{Re}\,W[J_q + iJ'_q[J_q]].$$

It is immediate to show, using (2.3), that

$$-\frac{\delta W_q[J_q]}{\delta J_q} = \Phi[J_q + iJ'_q[J_q]] = \Phi^*[J_q - iJ'_q[J_q]] = \varphi_q[J_q].$$

Then, if $J_q[\varphi_q]$ denotes the inverse of $\varphi_q[J_q]$, the Legendre transform gives

$$-W_q[J_q[\varphi_q]] - J_q[\varphi_q] \cdot \varphi_q = \mathrm{Re}\,\Gamma[\Phi] = S_q[\varphi_q],$$

as desired.

Summarizing, there exists a unique complex source J such that the functional $\Phi[J]$ is real. The quantum field φ_q coincides with Φ and the quantum action $S_q[\varphi_q]$ is just the real part of $\Gamma[\Phi]$.

The generating functional $\Gamma[\Phi]$ can be reconstructed from the quantum action $S_q[\varphi_q]$. Indeed, $S_q[\varphi_q]$ contains the reals parts of the T-ordered Green functions. The imaginary parts of the Green functions can be perturbatively calculated from the real parts.

For example, if the theory is unitary, the unitarity equation reads

$$\mathrm{Im}\,T = \frac{1}{2}TT^\dagger, \tag{2.5}$$

where $S = 1 + iT$ is the S-matrix, $SS^\dagger = 1$. Since T is at least of order one in the interactions, (2.5) implies that $\mathrm{Im}\,T$ is at least of order two. So, the equation (2.5) recursively determines the imaginary parts of the correlation functions from the lower-order real parts.

If the theory is not unitary, a more general version of the identity (2.5), with the same structure as (2.5), follows from the largest time equation [17]. It cannot be interpreted as a unitarity equation (the summation over intermediate states is affected by minus signs, due to propagating ghosts), but it can be used to calculate the imaginary parts of correlation functions from the lower-order real parts.

Thus, in complete generality the quantum action $S_q[\varphi_q]$ contains the full information about the theory.

The arguments of this section can be applied also to a partially classical, partially quantum field theory. In that case, let φ_c denote the classical fields, with action $S_c[\varphi_c]$, and φ the quantized fields, with classical action $S[\varphi, \varphi_c]$, embedded in the external φ_c-background. The procedure described above defines the quantum action $S_q[\varphi_q, \varphi_c] = \mathrm{Re}\,\Gamma[\Phi, \varphi_c]$, with $\varphi_q = \Phi =$ real. The total action $S_{\mathrm{tot}}[\varphi_c, \varphi_q]$ of the partially classical, partially quantum theory is obtained adding the classical action S_c of the fields φ_c to S_q, namely

$$S_{\mathrm{tot}}[\varphi_c, \varphi_q] = S_c[\varphi_c] + S_q[\varphi_q, \varphi_c].$$

For example, for classical gravity coupled with quantum matter, φ_c is the metric tensor $g_{\mu\nu}$, S_c is the Einstein action, and S_q is the real part of the Γ functional in external gravity, so

$$S_{\mathrm{tot}}[g, \varphi_q] = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g}\, [R(g) - 2\Lambda] + \mathrm{Re}\,\Gamma[\varphi_q, g]. \tag{2.6}$$

The field equations of gravity are $\delta S_{\mathrm{tot}}[g, \varphi_q]/\delta g^{\mu\nu} = 0$, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = -\kappa^2\, \mathrm{Re}\, \langle T_{\mu\nu} \rangle, \qquad \langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta\Gamma[\varphi_q, g]}{\delta g^{\mu\nu}}. \tag{2.7}$$

The matter field equations are $\delta S_{\mathrm{tot}}[g, \varphi_q]/\delta\varphi_q = 0$ and have to be solved consistently with (2.7). The simplest solution is $\varphi_q = 0$ or $\varphi_q =$constant (if there is a vacuum expectation value). Then the Einstein equations (2.7) describe how the spacetime geometry is affected by quantized matter fields circulating in the loops. Together with (2.7), they generalize the Møller-Rosenfeld approach (2.1) to quantum field theory.

Working with complex bosonic fields and/or fermionic fields φ, $\overline{\varphi}$, denote the associated sources with J, $\overline{J}$. The functionals are $Z[J, \overline{J}]$, $W[J, \overline{J}]$ and $\Gamma[\Phi, \overline{\Phi}]$, which is the Legendre transform of $W[J, \overline{J}]$. Repeating the argument outlined above, if the source $\overline{J}$ is the conjugate of J, then the functional $\overline{\Phi}[J, \overline{J}]$ is not the conjugate of $\Phi[J, \overline{J}]$. Instead, if the sources J, $\overline{J}$ are not the conjugates of each other, the relation between J and $\overline{J}$ can be determined imposing that the functional $\overline{\Phi}[J, \overline{J}]$ is the conjugate of $\Phi[J, \overline{J}]$. In that case, the quantum action is

$$S_q[\varphi_q, \overline{\varphi}_q] = \frac{1}{2} \left(\Gamma[\Phi, \overline{\Phi}] + \Gamma^\dagger[\overline{\Phi}, \Phi] \right),$$

where $\varphi_q = \Phi$, $\overline{\varphi}_q = \overline{\Phi}$. The other arguments extend straightforwardly.

In the presence of non-Abelian gauge fields, the gauge transformation of Φ can be a complex, non-local functional $\langle s\Phi\rangle$, where s denotes the BRST operator. Then the definition of a gauge invariant real quantum functional $S_q[\varphi_q]$ is not evident, at least in the most general framework. It is preferable to define the functional $\Gamma[\Phi]$ using the background field method [18], where now Φ denotes the background field, and the quantum field is set to zero. The background field method ensures manifest gauge invariance and, most of all, the gauge transformation of Φ preserves the reality of Φ. Then it is straightforward to identify Φ with the quantum field φ_q and define the quantum action $S_q[\varphi_q]$ as the real part of Γ.

Even using the background field method, however, the functional Γ depends on the gauge-fixing parameters. Call G the unbroken non-Abelian gauge group of the theory and φ_G the fields that transform non-trivially under G. The φ_G-quantum field equations can be solved setting all φ_G's to zero. The physical justification is that the G-interactions are short-range (and even confining in QCD), so the boundary conditions for the φ_G's are that they tend to zero with an appropriate velocity at infinity, which implies $\varphi_G \equiv 0$ by the unicity of the solution.

Setting all φ_G's to zero removes also the gauge-fixing dependence of Γ. Indeed, at $\varphi_G = 0$, the functional Γ depends only on $g_{\mu\nu}$ and the other G-invariant fields, namely it is a collection of correlation functions containing only insertions of G-invariant operators, so, by the usual BRST arguments, it cannot depend on the gauge-fixing parameters. Abelian gauge fields A need not be set to zero, since Γ is both gauge invariant and gauge-fixing independent at $A \neq 0$.

In Euclidean theories, which are employed, for example, in the study of critical phenomena, the average field $\Phi = \langle\varphi\rangle_{J,\overline{J}}$ is the conjugate of $\overline{\Phi} = \langle\overline{\varphi}\rangle_{J,\overline{J}}$ and the generating functionals $W[J,\overline{J}]$ and $\Gamma[\Phi,\overline{\Phi}]$ are hermitian, if the sources $\overline{J}$ are the conjugates of J. Then the functional $\Gamma[\Phi,\overline{\Phi}]$ is the good quantum action, Φ and $\overline{\Phi}$ being the quantum fields.

3 Field redefinitions that reabsorb terms quadratically proportional to the field equations

In this section I prove that a term quadratically proportional to the field equations can be reabsorbed with a field redefinition. This theorem is used to construct the map $\mathcal{M}$ that relates the higher-derivative theory with the acausal theory.

Consider an action S depending on the fields ϕ_i, where the index i labels both the field type, the component and the spacetime point. Add a term quadratically proportional to the field equations $S_i \equiv \delta S/\delta\phi_i$ and define the modified action

$$S'[\phi_i] = S[\phi_i] + S_i F_{ij} S_j,$$

where F_{ij} is symmetric and can contain derivative operators. Summation over repeated indices (including the integration over spacetime points) is understood. The theorem states that there exists a field redefinition

$$\phi_i' = \phi_i + \Delta_{ij} S_j, \tag{3.1}$$

with Δ_{ij} symmetric, such that, perturbatively in F and to all orders in powers of F,

$$S'[\phi_i] = S[\phi_i']. \tag{3.2}$$

Here is the proof. The condition (3.2) can be written as

$$S[\phi_i] + S_i F_{ij} S_j = S[\phi_i + S_j \Delta_{ij}] = S[\phi_i] + \sum_{n=1}^{\infty} \frac{1}{n!} S_{k_1 \cdots k_n} \prod_{l=1}^{n} (\Delta_{k_l m_l} S_{m_l}),$$

after a Taylor expansion, where $S_{k_1 \cdots k_n} \equiv \delta^n S/(\delta\phi_{k_1} \cdots \delta\phi_{k_n})$. This equality is verified if

$$\Delta_{ij} = F_{ij} - \Delta_{k_1 i} \Delta_{k_2 j} \sum_{n=2}^{\infty} \frac{1}{n!} S_{k_1 k_2 k_3 \cdots k_n} \prod_{l=3}^{n} (\Delta_{k_l m_l} S_{m_l}), \tag{3.3}$$

where the product is meant to be equal to unity when $n = 2$. Equation (3.3) can be solved recursively for Δ in powers of F. The first terms of the solution are

$$\Delta_{ij} = F_{ij} - \frac{1}{2} F_{k_1 i} F_{k_2 j} S_{k_1 k_2} + \cdots \tag{3.4}$$

The theorem just proved is very general. It works both for local and non-local theories. Assume that the spacetime dimension d is greater than two, so that the fields φ have positive dimensionalities d_φ in units of mass. Call "perturbatively local" a functional that can be expanded in powers of the fields and their derivatives. That means, for example, that it does not contain low-energy singularities, such as $1/\partial_\mu$, $1/\Box$, etc. Call "perturbatively local expansion" the expansion in powers of the fields and their derivatives. If $S'[\phi_i]$ and $S[\phi_i]$ are perturbatively local, then F_{xy} has the form

$$F_{xy} = (f_x + f_x^\mu \partial_\mu + f_x^{\mu\nu} \partial_\mu \partial_\nu + \cdots)\, \delta(x - y), \tag{3.5}$$

where $f_x^{\mu_1 \cdots \mu_n}$ are perturbatively local tensorial functionals of the fields ϕ and their derivatives in x. Now I prove that the field redefinition (3.1) is perturbatively local, and the solution of (3.3) can be worked out recursively and has the same form as (3.5), namely

$$\Delta_{xy} = (g_x + g_x^\mu \partial_\mu + g_x^{\mu\nu} \partial_\mu \partial_\nu + \cdots)\, \delta(x - y), \qquad g_x^{\mu_1 \cdots \mu_k} = f_x^{\mu_1 \cdots \mu_k} + \mathcal{O}(f^2). \tag{3.6}$$

The functionals $g_x^{\mu_1 \cdots \mu_m}$, $f_x^{\mu_1 \cdots \mu_m}$ have dimensionalities $2d_\varphi - d - m < 0$. Equation (3.3) splits into separate equations for $g_x^{\mu_1 \cdots \mu_k}$, that can be solved recursively in powers of $f_x^{\mu_1 \cdots \mu_m}$. Each functional $f_x^{\mu_1 \cdots \mu_m}$ can be considered of the same order. At each order in f the solution is worked out term-by-term in the perturbatively local expansion.

Write the perturbatively local expansions of $f_x^{\mu_1\cdots\mu_m}$ and $g_x^{\mu_1\cdots\mu_m}$ as

$$f_x^{\mu_1\cdots\mu_m} = \sum c_f^{(m,p,q)} \mathcal{O}_{p,q}^{\mu_1\cdots\mu_m}[\varphi(x)], \qquad g_x^{\mu_1\cdots\mu_m} = \sum c_g^{(m,p,q)} \mathcal{O}_{p,q}^{\mu_1\cdots\mu_m}[\varphi(x)],$$

where $\mathcal{O}_{p,q}^{\mu_1\cdots\mu_m}[\varphi]$ denotes a basis of local operators constructed with p derivatives and q fields and $c_f^{(m,p,q)}$, $c_g^{(m,p,q)}$ are numerical coefficients, with dimensionalities $2d_\varphi - d - m - p - qd_\varphi < 0$. Finitely many parameters M with positive dimensionalities are contained in the action S. The dimensionalities of M are obviously bounded by d. Each term in the sum of (3.3) is polynomial in M, so (3.3) can be translated into equations for the c_g's that have schematically the form

$$c_g = c_f + \sum_{n=2}^{\infty} P_{n-1}(M) c_g^n, \tag{3.7}$$

where $P_{n-1}(M)$ is a polynomial of degree $n-1$ in M. Thus each c_g receives $\mathcal{O}(f^n)$ contributions from a finite number of coefficients c_f's, which proves that the equations (3.7) can be solved recursively.

If both $S'[\phi_i]$ and $S[\phi_i]$ are local, F_{xy} is local. Even then, in general, Δ_{xy} is only perturbatively local. Actually, the resummation of derivatives in (3.6) can produce a non-local field redefinition. Take an ordinary free field theory $S[\phi_i]$. Then $S_{k_1\cdots k_n} = 0$ for every $n > 2$, while $S_{k_1k_2}$ is field-independent and quadratic in the derivatives. The modified action $S'[\phi_i]$ describes a higher-derivative theory. Equation (3.3) reads

$$\Delta_{ij} = F_{ij} - \frac{1}{2}\Delta_{k_1 i}\Delta_{k_2 j} S_{k_1 k_2}$$

and is solved in matrix form by

$$\Delta = \left(\sqrt{1+2FS} - 1\right) S^{-1}.$$

Clearly, the solution Δ_{ij} is non-local, but perturbatively local. In the next subsection these facts are illustrated explicitly for gravity in the quadratic approximation.

A known situation where the theorem applies is the three-dimensional $U(1)$ gauge theory. The field equations of the Chern-Simons action

$$S[A] = \frac{1}{2\alpha_{\rm CS}} \int \varepsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho$$

are $F^{\mu\nu} = 0$, so there exists a field redefinition $A'_\mu(A, \alpha/\alpha_{\rm CS})$ such that

$$S'[A] = S[A'], \tag{3.8}$$

where S' is the sum of the Chern-Simons action plus the square of the field strength,

$$S'[A] = \frac{1}{\alpha_{\rm CS}} \int \varepsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho - \frac{1}{4\alpha} \int F_{\mu\nu} F^{\mu\nu}.$$

3.1 The map $\mathcal{M}$ for gravity

In pure gravity, the theorem just proved ensures that there exists a field redefinition that maps a class of higher-derivative theories into the Einstein theory. For example, there exists a field redefinition $g \rightarrow g'(g, a, b)$ such that

$$S_{\mathrm{HD}}[g] = S_{\mathrm{E}}[g'], \tag{3.9}$$

where

$$S_{\mathrm{HD}}[g] = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[R(g) + aR_{\mu\nu}R^{\mu\nu}(g) + bR^2(g) \right], \tag{3.10}$$

$$S_{\mathrm{E}}[g] = \frac{1}{2\kappa^2} \int \sqrt{-g} R(g) \tag{3.11}$$

Indeed, the terms $R_{\mu\nu}R^{\mu\nu}$ and R^2 are quadratically proportional to the field equations of the action $S_{\mathrm{E}}[g]$. The lowest-order contributions to the map $\mathcal{M}$ are, from (3.1), (3.4),

$$g'_{\mu\nu}(g, a, b) = g_{\mu\nu} - aR_{\mu\nu} + \frac{1}{2}(a + 2b)g_{\mu\nu}R + \mathcal{O}(a^2, b^2, ab). \tag{3.12}$$

I stress once again that the identity (3.9) does not imply that higher-derivative gravity is physically equivalent to Einstein gravity. Indeed, it is evident, already in the free-field limit, that the degrees of freedom of S_{HD} and S_{E} are different. Nevertheless, formula (3.9) and the more general identity (3.2) are useful to relate the renormalization properties of the two theories. In the next section the identity (3.9) is used to prove that classical gravity coupled with quantum matter is predictive, namely all divergences are renormalized redefining the fields and a finite number of independent couplings.

The field redefinition $g'(g, a, b)$ is the map $\mathcal{M}$ for gravity. It is clearly nonlocal. When a source term is added, the map is in general acausal (see the subsection 3.3 and section 5). Thus, in general the map $g'(g)$ relates higher-derivative gravity with acausal gravity.

In the presence of a cosmological constant, the theorem ensures that there exists a field redefinition $g'(g)$ such that

$$S_{\mathrm{HD}}^{(\Lambda)}[g] = S_{\mathrm{E}}^{(\Lambda)}[g'], \tag{3.13}$$

where

$$S_{\mathrm{HD}}^{(\Lambda)}[g] = S_{\mathrm{HD}}[g] - \frac{\Lambda}{\kappa^2} \int \sqrt{-g} = \frac{1}{2\widetilde{\kappa}^2} \int \sqrt{-g} \left[R(g) - 2\Lambda + \widetilde{a}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu}(g) + \widetilde{b}\widehat{R}^2(g) \right],$$

$$S_{\mathrm{E}}^{(\Lambda)}[g] = \frac{1}{2\widetilde{\kappa}^2} \int \sqrt{-g} \left[R(g) - 2\Lambda \right],$$

and $\kappa^2 = \widetilde{\kappa}^2(1 + 2a\Lambda + 8b\Lambda)$, $\widetilde{a} = a\widetilde{\kappa}^2/\kappa^2$, $\widetilde{b} = b\widetilde{\kappa}^2/\kappa^2$. Indeed, the hatted tensors

$$\widehat{R}_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}\Lambda, \qquad \widehat{R} = R - 4\Lambda, \tag{3.14}$$

vanish on the solutions to the field equations of $S_{\mathrm{E}}^{(\Lambda)}[g]$.

In the next subsection the map $g'(g)$ is worked out explicitly in the quadratic approximation in the absence of a cosmological constant.

3.2 The map $\mathcal{M}$ for gravity in the quadratic approximation

It is instructive to work out the field redefinition explicitly for gravity in the quadratic approximation. The expansion around flat space is defined as

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu},$$

where $\eta_{\mu\nu}$ =diag$(1,-1,-1,-1)$. The trace of $\phi_{\mu\nu}$ is denoted with ϕ. Below, I use the convention $t^2_{\mu_1\cdots\mu_n} \equiv t_{\mu_1\cdots\mu_n}t^{\mu_1\cdots\mu_n}$, where $t_{\mu_1\cdots\mu_n}$ is any tensor.

The identity (3.9) reads, in the quadratic approximation,

$$S'[\phi] = S[\phi'], \tag{3.15}$$

where

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^4x \left\{(\partial_\mu\phi_{\rho\sigma})^2 - (\partial_\mu\phi)^2 + 2(\partial^\mu\phi)(\partial^\nu\phi_{\mu\nu}) - 2(\partial^\mu\phi_{\mu\nu})^2\right\},$$
$$S'[\phi] = S[\phi] + \frac{1}{2}\int \mathrm{d}^4x \left\{a\left(\Box\phi_{\mu\nu} + \partial_\mu\partial_\nu\phi - \partial_\mu\partial^\alpha\phi_{\nu\alpha} - \partial_\nu\partial^\alpha\phi_{\mu\alpha}\right)^2 + 4b\left(\Box\phi - \partial^\mu\partial^\nu\phi_{\mu\nu}\right)^2\right\},$$

and the field transformation is

$$\phi_{\mu\nu} = \frac{1}{\sqrt{1-a\Box}}\left(\phi'_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\phi' + \eta_{\mu\nu}\frac{1}{3\Box}\partial^\rho\partial^\sigma\phi'_{\rho\sigma}\right) + \frac{\eta_{\mu\nu}}{3\sqrt{1-b'\Box}}\left(\phi' - \frac{1}{\Box}\partial^\rho\partial^\sigma\phi'_{\rho\sigma}\right), \tag{3.16}$$

where $b' \equiv -2(a+3b)$.

It is immediate to check that

$$\widetilde{\phi}_{\mu\nu} = \frac{1}{\sqrt{1-a\Box}}\widetilde{\phi}'_{\mu\nu}, \tag{3.17}$$

where $\widetilde{\phi}_{\mu\nu}$ and $\widetilde{\phi}'_{\mu\nu}$ are the traceless parts of $\phi_{\mu\nu}$ and $\phi'_{\mu\nu}$, respectively. If $b' = a$ the transformation (3.16) becomes simply

$$\phi_{\mu\nu} = \frac{1}{\sqrt{1-a\Box}}\phi'_{\mu\nu}. \tag{3.18}$$

Due to (3.17), the gauge-fixing

$$\partial^\mu\widetilde{\phi}'_{\mu\nu} = 0 \tag{3.19}$$

implies also

$$\partial^\mu\widetilde{\phi}_{\mu\nu} = 0. \tag{3.20}$$

Using (3.19) and (3.20), the identity (3.15) simplifies to

$$\frac{1}{2}\int \mathrm{d}^4x \left\{(\partial_\mu\widetilde{\phi}_{\rho\sigma})^2 + a(\Box\widetilde{\phi}_{\mu\nu})^2 - \frac{3}{8}\left[(\partial_\mu\phi)^2 + b'(\Box\phi)^2\right]\right\} = \frac{1}{2}\int \mathrm{d}^4x \left\{(\partial_\mu\widetilde{\phi}'_{\rho\sigma})^2 - \frac{3}{8}(\partial_\mu\phi')^2\right\}$$

and the field redefinition (3.16) becomes

$$\widetilde{\phi}_{\mu\nu} = \frac{1}{\sqrt{1-a\Box}}\widetilde{\phi}'_{\mu\nu}, \qquad \phi = \frac{1}{\sqrt{1-b'\Box}}\phi'. \tag{3.21}$$

3.3 Physical effects

A non-renormalizable theory contains infinitely many vertices, with an arbitrarily high number of derivatives. The usual low-energy expansion is obtained expanding the action in powers of the fields and their momenta and considering the (bosonic) fields and momenta of the same order. However, sometimes it is useful to study different expansions. For example, there are situations where it is possible to resum the expansion in powers of the fields exactly, but it not straightforward to resum the expansion in powers of the momenta [19]. Here, instead, the expansion in powers of the fields is difficult to resum, but it is straightforward to resum certain expansions in powers of the momenta, which lead for example to the square roots of formulas (3.16) and (3.24). The resummation of momenta is meaningful in a regime in which the fields are weak, but not necessarily slowly varying, where it is sufficient to keep only the linear and quadratic terms in ϕ'.

Thus, to illustrate the effects on interactions in the weak-field approximation, add a source term

$$S_{\text{source}}[\phi, T] = -\kappa \int \mathrm{d}^4x \ \phi_{\mu\nu}T^{\mu\nu}, \tag{3.22}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. Then (3.15) extends to

$$S_{\text{HD}}[\phi, T] = S_{\text{AC}}[\phi', T], \tag{3.23}$$

where

$$S_{\text{HD}}[\phi, T] = S'[\phi] + S_{\text{source}}[\phi, T], \qquad S_{\text{AC}}[\phi', T] = S[\phi'] + S_{\text{source}}[\phi', T'(T)]$$

and

$$T'_{\mu\nu}(T) = \frac{1}{\sqrt{1-a\Box}}\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T + \frac{1}{3\Box}\partial_\mu\partial_\nu T\right) + \frac{1}{3\sqrt{1-b'\Box}}\left(\eta_{\mu\nu}T - \frac{1}{\Box}\partial_\mu\partial_\nu T\right), \tag{3.24}$$

T being the trace of $T_{\mu\nu}$. The expansions of (3.16) and (3.24) in powers of a and b' are perturbatively local, in agreement with the conclusions derived previously. At the non-perturbative level in a and b', the operators

$$\frac{1}{\sqrt{1-a\Box}}, \qquad \frac{1}{\sqrt{1-b'\Box}} \tag{3.25}$$

stand for convolutions with the generalized functions

$$\mathcal{C}_4^{(f)}(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{e}^{-ikx}}{\sqrt{1+fk^2}}, \tag{3.26}$$

where $f = a, b'$. The operator $1/\Box$ in (3.24) stands for the convolution with

$$\mathcal{G}_4(t, \mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}\delta(t - |\mathbf{x}|). \tag{3.27}$$

The Fourier transforms (3.26) need prescriptions for the contour integrations. The prescriptions must ensure that the $f \to 0$ limits of $\mathcal{C}_4^{(f)}(x)$ are regular, for the reasons explained below.

The action $S_{\mathrm{AC}}[\phi', T]$ couples the field $\phi'_{\mu\nu}$ with $T'_{\mu\nu}(T)$, which is a sort of spacetime average of the matter stress tensor $T_{\mu\nu}$. In section 5 it is shown that if f is negative $\mathcal{C}_4^{(f)}$ admits a real causal prescription. That prescription, however, does not survive the radiative corrections and ultimately the value of $T'_{\mu\nu}(t, \mathbf{x})$ at time t depend also on the spacetime points that are located in the future light cone of x or are spacelike separated from x. Then, causality is violated.

If a complex prescription is used for (3.26), the conclusions of the previous section apply, and the tree-level quantum action $S_{q\mathrm{AC}}[\phi', T]$ is the real part of $S_{\mathrm{AC}}[\phi', T]$, with the convention that the quantum field $\varphi_q \equiv \phi'$ is real. Neither the choice of the prescription, nor the suppression of the imaginary part of $S_{\mathrm{AC}}[\phi', T]$, affect the perturbative expansion in powers of a and b' and the renormalizability of the theory, discussed in the next section.

Note that resummations similar to the ones that lead to (3.25) are familiar in high-energy physics, where they are produced by the renormalization group. Specifically, the renormalization group is able to resum certain expansions in powers of the couplings and the logarithms of momenta. In gravity the coupling is, in some sense, itself a momentum. Then the gravitational analogue is the resummation of an expansion in powers of momenta and the logarithms of momenta. In section 5 the radiative corrections are included, and produce the expected dependence on the logarithms of momenta, see formulas (5.28) and (5.29).

The identity (3.23) is the map $\mathcal{M}$ for classical gravity in the weak-field approximation. The action S_{HD} contains higher-derivative kinetic terms, while the action S_{AC} does not. Now, assume that the physical theory is $S_{q\mathrm{AC}}[\phi', T]$. That means that the spacetime geometry is described by $\phi'_{\mu\nu}$ and the source of the physical interaction is $T_{\mu\nu}$. However, the spacetime geometry is not affected directly by $T_{\mu\nu}$. Instead, it is sensitive to the "effective stress-tensor" $\mathrm{Re}\, T'_{\mu\nu}$, which is a spacetime average of $T_{\mu\nu}$. Observe that $\mathrm{Re}\, T'_{\mu\nu}$ need not obey the positivity constraints obeyed by $T_{\mu\nu}$. Using the gauge-fixing

$$\partial^\nu \phi'_{\mu\nu} = \frac{1}{2}\partial_\mu \phi', \tag{3.28}$$

the gravitational field equations read

$$\Box \phi'_{\mu\nu} = -\kappa \,\mathrm{Re}\, T'_{\mu\nu}(T) + \frac{\kappa}{2}\eta_{\mu\nu}\,\mathrm{Re}\, T'(T). \tag{3.29}$$

Equation (3.29) is a second-order partial differential equation and must be supplemented with the usual boundary conditions, e.g. $\phi'_{\mu\nu}(t_0, \mathbf{x})$ and $\partial_0\phi'_{\mu\nu}(t_0, \mathbf{x})$ at the initial time t_0.

It is instructive to compare equation (3.29) with the equation generated by the higher-derivative theory. Assume that the physical theory is $S_{\rm HD}[\phi, T]$. Then, with the gauge-fixing analogous to (3.28), the field equation for $\phi_{\mu\nu}$ reads

$$\Box\phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box\phi - a\left(\Box^2\phi_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box^2\phi - \Box\partial_\mu\partial_\nu\phi\right) - 2b\left(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\right)\Box\phi = -\kappa T_{\mu\nu}. \quad (3.30)$$

This equation is a fourth-order partial differential equation and must be supplemented with unusual boundary conditions, e.g. $\partial_0^n\phi'_{\mu\nu}(t_0, \mathbf{x})$, $n = 0, 1, 2, 3$ at the initial time t_0. It has extra solutions that (3.29) does not have. In particular, $\Box\phi_{\mu\nu} \neq 0$ even at $T_{\mu\nu} = 0$. Thus, equations (3.29) and (3.30) are physically inequivalent.

When the higher-derivative local equation (3.30) is converted into the second-order non-local equation (3.29) by the map $\mathcal{M}$, the extra solutions of (3.30) disappear. They are killed by the requirement that the generalized functions (3.26) be regular in the limit $f \to 0$. In practice, the map $\mathcal{M}$ consists of a universal choice of the extra boundary conditions, which suppresses the unwanted degrees of freedom, but in general produces causality violations. These ideas are inspired by known treatments of the Abraham-Lorentz force in classical electrodynamics [11], which are reviewed in section 5. To be precise, a certain ambiguity survives also in (3.29), due to the freedom to choose different prescriptions for $\mathcal{C}_4^{(f)}(x)$.

The causality violations can be physically tested studying, for example, the gravitational force predicted by $S_{q\rm AC}[\phi', T]$. Consider a set of small rigid spheres of masses m_i and radii R_i, moving along trajectories $\mathbf{r}_i(t)$. The mass distributions are $\rho_i(\mathbf{x} - \mathbf{r}_i)$, where $\rho_i(\mathbf{r}) = 3m_i/(4\pi R_i^3)$ for $|\mathbf{r}| \leq R_i$ and $\rho_i(\mathbf{r}) = 0$ for $|\mathbf{r}| > R_i$. The stress-energy tensor reads

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_i \frac{\rho_i(\mathbf{x} - \mathbf{r}_i(t))}{\sqrt{1 - \dot{\mathbf{r}}_i^2(t)}} v_i^\mu(t) v_i^\nu(t), \quad (3.31)$$

where $v_i^\mu(t) = (1, \dot{\mathbf{r}}_i(t))$. The total action, including the kinetic terms of the spheres, is

$$S_{\rm tot}[\phi', \mathbf{r}_i] = S_{q\rm AC}[\phi', T] - \sum_i m_i \int {\rm d}t \sqrt{1 - \dot{\mathbf{r}}_i^2(t)}. \quad (3.32)$$

The equations of motions of $S_{\rm tot}[\phi', \mathbf{r}_i]$ are involved, but some qualitative aspects of their solutions can be studied in the non-relativistic limit, where the time derivatives of (3.29) and (3.30) are negligible and the causality violations disappear. The stress tensor (3.31) simplifies to

$$T_{00}(\mathbf{x}) = \sum_i m_i \rho_i(\mathbf{x} - \mathbf{r}_i),$$

any other component being negligible. From (5.23), the generalized functions (3.26) become

$$\mathcal{C}_4^{(f)}(x) \rightarrow -\frac{\delta(t) K_1(r/\sqrt{-f})}{2\pi^2 f r}. \tag{3.33}$$

For concreteness, assume that the spheres are pointlike, $\rho_i(\mathbf{r}) = m_i \delta^{(3)}(\mathbf{r})$. Using (3.33), $T'_{\mu\nu}$ has components

$$T'_{00}(\mathbf{x}) = \sum_i \frac{2}{3}\overline{\rho}_i^{(a)}(\mathbf{x}) + \frac{1}{3}\overline{\rho}_i^{(b')}(\mathbf{x}), \qquad \overline{\rho}_i^{(f)}(\mathbf{x}) = \frac{m_i K_1(|\mathbf{x}-\mathbf{r}_i|/\sqrt{-f})}{2\pi^2(-f)|\mathbf{x}-\mathbf{r}_i|},$$

and

$$T'_{ij}(\mathbf{x}) = \frac{1}{3}\left(\delta_{ij} - \frac{\partial_i \partial_j}{\triangle}\right)\left(\overline{\rho}_i^{(a)}(\mathbf{x}) - \overline{\rho}_i^{(b')}(\mathbf{x})\right),$$

while $T'_{i0} = 0$. In practice, a pointlike sphere effectively smears out into distributions of mass $\overline{\rho}_i^{(f)}(\mathbf{x})$, which are sensibly different from zero in regions of radii $\sqrt{|f|}$.

The force is $\mathbf{F}_i = -\boldsymbol{\nabla}_i U$. The potential energy U can be read from

$$\frac{1}{2}\,\mathrm{Re}\, S_{\mathrm{source}}[\phi', T'] = -\int \mathrm{d}t\, U, \tag{3.34}$$

and $\phi'_{\mu\nu}$ can be calculated from (3.29), using (3.27). The factor one half in (3.34) is because $\phi'_{\mu\nu}$ is proportional to $T'_{\mu\nu}$. The result is

$$U = -\frac{\kappa^2}{8\pi}\int \frac{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left(\mathrm{Re}\, T'_{\mu\nu}(\mathbf{x})\,\mathrm{Re}\, T'^{\mu\nu}(\mathbf{x}') - \frac{1}{2}\,\mathrm{Re}\, T'(\mathbf{x})\,\mathrm{Re}\, T'(\mathbf{x}')\right). \tag{3.35}$$

For $a, b' < 0$ the integral gives

$$U = -\frac{\kappa^2}{8\pi}\sum_{i<j}\frac{m_i m_j}{r_{ij}}\left(1 - \frac{4}{3}\mathrm{e}^{-r_{ij}/\sqrt{-a}} + \frac{1}{3}\mathrm{e}^{-r_{ij}/\sqrt{-b'}}\right), \qquad r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|, \tag{3.36}$$

where the self-energies have been subtracted away.

The generalization of U when either a or b' is positive is simple, but left to the reader.

So far, the Newton law has been verified down to about 0.1 millimeters [20] without observing any deviations, so the experimental bound on the values of $|a|$ and $|b'|$ is

$$|a|, |b'| < 2.5 \cdot 10^5 (\mathrm{eV})^{-2}. \tag{3.37}$$

4 Renormalization of classical gravity coupled with quantum matter

In this section I show that the divergences of acausal gravity coupled with quantum matter can be removed with a finite number of independent couplings without introducing higher-derivative terms in the gravitational sector. The map $\mathcal{M}$ is used to relate the renormalization of acausal gravity coupled with quantum matter to the renormalization of higher-derivative gravity coupled with quantum matter.

For simplicity, I first consider a theory that does not contain parameters with positive dimensionality in units of mass. The generalization to theories with cosmological constant, masses and super-renormalizable couplings is described later on. Moreover, I use the dimensional regularization technique, which is BRST invariant and does not produce power-like divergences.

The classical action is written as

$$S_{\mathrm{AC}}[g,\varphi,\lambda,\lambda',\kappa]=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R+S_m[\varphi,g,\lambda]+\Delta S_m[\varphi,g,\lambda,\lambda']. \tag{4.1}$$

Here S_m collects the power-counting renormalizable terms of the matter action embedded in external gravity and λ denotes the dimensionless couplings of S_m. For example, in the case of (massless) QED S_m is equal to

$$S_m=\int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\overline{\psi}i\not{D}\psi\right), \tag{4.2}$$

where $D_\mu=\mathcal{D}_\mu+ieA_\mu$ is the covariant derivative. In the case of scalar fields, the action S_m includes also the non-minimal term $R\varphi^2$:

$$S_m=\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)-\frac{1+2\eta}{12}R\varphi^2-\frac{\lambda}{4!}\varphi^4\right). \tag{4.3}$$

In (4.1) ΔS_m collects the terms of dimensionality greater than four, parametrized by couplings λ' with negative dimensionalities in units of mass.

In four dimensions, neither S_m nor ΔS_m include pure-gravity terms, namely $S_m[0,g,\lambda]=\Delta S_m[0,g,\lambda,\lambda']=0$. In higher dimensions this requirement has to be appropriately modified (see below).

The theory is renormalizable if the correction ΔS_m to the matter action is such that the divergences of (4.1) are subtracted away renormalizing the couplings of (4.1) and redefining the fields. The field redefinition of $g_{\mu\nu}$ cannot depend on the matter fields, because the matter fields are quantized (they are integrated in the functional integral), while the metric tensors $g_{\mu\nu}$ is just an external source.

4.1 Renormalizability of the higher-derivative theory

Before proving the renormalizability of the acausal theory, I recall the properties of the higher-derivative theory. The action of the higher-derivative classical gravity coupled with quantum matter is

$$S_{\rm HD}[\overline{g},\varphi,\lambda,a,b,\kappa]=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-\overline{g}}\left(\overline{R}+a\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}+b\overline{R}^2\right)+S_m[\varphi,\overline{g},\lambda], \qquad (4.4)$$

where $\overline{R}_{\mu\nu}=R_{\mu\nu}(\overline{g})$. The metric tensor is denoted with $\overline{g}$ to distinguish it from the metric tensor g of the theory (4.1).

As in the theory (4.1), only the matter fields φ are quantized. By power-counting, the renormalization of S_m generates counterterms of dimensionality four, which can be grouped in four classes: counterterms proportional to the terms of S_m, counterterms proportional to the field equations, BRST-exact counterterms and pure-gravity counterterms. The pure-gravity counterterms are

$$\int \mathrm{d}^4x\sqrt{-\overline{g}}\left(\alpha\overline{R}_{\mu\nu\rho\sigma}\overline{R}^{\mu\nu\rho\sigma}+\beta\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}+\gamma\overline{R}^2\right), \qquad (4.5)$$

but, as usual, the first term of this list is converted into a combination of the other two, up to a total derivative, using the Gauss-Bonnet identity. Thus, the higher-derivative theory (4.4) can be renormalized redefining the matter fields φ and the parameters λ, a and b. The φ-redefinition is just multiplicative ($\varphi_{\rm B}=Z_\varphi^{1/2}\varphi$), so it does not depend on the gravitational background.

The bare action reads

$$S_{\rm HD\ B}=S_{\rm HD}[\overline{g},\varphi_{\rm B},\lambda_{\rm B},a_{\rm B},b_{\rm B},\kappa]=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-\overline{g}}\left(\overline{R}+a_{\rm B}\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}+b_{\rm B}\overline{R}^2\right)+S_m[\varphi_{\rm B},\overline{g},\lambda_{\rm B}]. \qquad (4.6)$$

There is no need to redefine the metric tensor and the Newton constant, so $\overline{g}_{\mu\nu\rm B}=\overline{g}_{\mu\nu}$ and $\kappa_{\rm B}=\kappa$.

Finally, the generating functional $\Gamma_{\rm HD}[\overline{g},\Phi,\lambda,a,b,\kappa]$ of one-particle irreducible Green functions is defined by

$$\int \mathcal{D}\varphi\,\exp\left(iS_{\rm HD\ B}+i\int\sqrt{-\overline{g}}J\varphi\right)=\exp\left(i\Gamma_{\rm HD}[\overline{g},\Phi,\lambda,a,b,\kappa]+i\int\sqrt{-\overline{g}}J\Phi\right),$$

where $J=-(1/\sqrt{-\overline{g}})(\delta\Gamma_{\rm HD}/\delta\Phi)$.

4.2 Usage of the map $\mathcal{M}$

The next step is to use the map $\mathcal{M}$ (3.9) to convert the higher-derivative theory (4.4) into a theory of the form (4.1). Call $\overline{G}(g,a,b)$ the function such that for $\overline{g}=\overline{G}(g,a,b)$

$$\int \mathrm{d}^4x\sqrt{-\overline{g}}\left[\overline{R}(\overline{g})+a\overline{R}_{\mu\nu}\overline{R}^{\mu\nu}(\overline{g})+b\overline{R}^2(\overline{g})\right]=\int \mathrm{d}^4x\sqrt{-g}R(g), \qquad (4.7)$$

where the bar on the curvature tensors means that they are those of the metric $\overline{g}$, and define the correction ΔS_m as

$$\Delta S_m[\varphi,g,\lambda,a,b]=S_m[\varphi,\overline{G}(g,a,b),\lambda]-S_m[\varphi,g,\lambda]. \qquad (4.8)$$

Then, (4.7) and (4.8) ensure that

$$S_{\mathrm{HD}}[\overline{G}(g,a,b),\varphi,\lambda,a,b,\kappa]=S_{\mathrm{AC}}[g,\varphi,\lambda,a,b,\kappa], \tag{4.9}$$

where the parameters λ' of (4.1) are just a and b, whose dimensionalities are -2.

The acausal theory $S_{\mathrm{AC}}[g,\varphi,\lambda,a,b,\kappa]$ does not contain higher-derivatives terms in the pure-gravity sector. However, due to ΔS_m, the matter sector contains vertices that depend non-polynomially on the gravitational field and its derivatives.

The main properties of ΔS_m can be read directly from (4.9). First, the vertices of ΔS_m have dimensionality greater than four. They are constructed with the matter fields, the Ricci tensor and their covariant derivatives. Moreover, they are proportional to the Ricci tensor and polynomial in the matter fields (and their covariant derivatives), of the same degree as S_m. Clearly, $\Delta S_m[0,g,\lambda,a,b]=0$.

Using (3.12) the lowest-order contributions to the correction ΔS_m are

$$\begin{aligned}
\Delta S_m &= \Delta S_m^{(\mathrm{HEAD})}+\Delta S_m^{(\mathrm{QUEUE})},\\
\Delta S_m^{(\mathrm{HEAD})} &= \int \mathrm{d}^4x\sqrt{-g}\,\left[-\frac{a}{2}T_m^{\mu\nu}R_{\mu\nu}+\frac{1}{4}(a+2b)RT_m\right],\\
\Delta S_m^{(\mathrm{QUEUE})} &= \mathcal{O}(a^2,b^2,ab),
\end{aligned} \tag{4.10}$$

where $T_m^{\mu\nu}=-(2/\sqrt{-g})(\delta S_m/\delta g_{\mu\nu})$ is the stress-tensor of the uncorrected matter sector and T_m denotes its trace. Formula (4.10) clarifies the meaning of the couplings a and b in the acausal theory: they multiply the vertices that couple $T_m^{\mu\nu}$ to the Ricci tensor. The other contributions to ΔS_m are either proportional to $T_m^{\mu\nu}$ times derivatives of the Ricci tensor or quadratically proportional to the Ricci tensor.

The correction ΔS_m falls in the class of non-renormalizable perturbations constructed in ref.s [19, 21]. In (4.10), the terms $\Delta S_m^{(\mathrm{HEAD})}$ have dimensionality 6. They are multiplied by independent couplings, a and b, and form the *head* of the perturbation. The terms $\Delta S_m^{(\mathrm{HEAD})}$ have dimensionalities greater than 6 and form the *queue* of the perturbation. Although the queue contains infinitely many vertices, it contains only a finite number of independent matter operators, generated by the functional derivatives of $T_m^{\mu\nu}$ with respect to the metric. The queue does not contain new independent couplings. Its vertices are multiplied by functions of the other couplings (a, b, λ and κ), determined by certain RG consistency conditions, called *reduction equations*, ensuring that the divergences of the theory are removed renormalizing the couplings a, b, λ and κ, together with field redefinitions.

4.3 Renormalizability

The renormalizability of (4.1) is proved using the renormalizability of the higher-derivative theory (4.4) and the map $\mathcal{M}$. Briefly, the divergences of $S_{\mathrm{HD}}[\overline{g},\varphi,\lambda,a,b,\kappa]$ are renormalized redefining φ, λ, a and b at fixed $\overline{g}_{\mu\nu}$ and κ: since g is a function of $\overline{g}$, a and b, the divergences of $S_{\text{AC}}[g, \varphi, \lambda, a, b, \kappa]$ are removed redefining g, φ, λ, a and b at fixed κ.

The acausal theory is renormalizable if there exists a bare metric tensor g_{B}, depending only on g and the couplings, such that the bare action

$$S_{\text{AC B}} \equiv S_{\text{AC}}[g_{\text{B}}, \varphi_{\text{B}}, \lambda_{\text{B}}, a_{\text{B}}, b_{\text{B}}, \kappa] \tag{4.11}$$

produces finite Green functions. The g-redefinition g_{B} that does this job is obtained solving the condition

$$\overline{G}(g_{\text{B}}, a_{\text{B}}, b_{\text{B}}) = \overline{G}(g, a, b). \tag{4.12}$$

More explicitly, calling $g = G(\overline{g}, a, b)$ the inverse of $\overline{g} = \overline{G}(g, a, b)$,

$$g_{\text{B}} = G(\overline{G}(g, a, b), a_{\text{B}}, b_{\text{B}}) = g + \mathcal{O}(\hbar), \tag{4.13}$$

and, to the lowest order, using (3.12),

$$g_{\mu\nu\text{B}} = g_{\mu\nu} + (a - a_{\text{B}})R_{\mu\nu} + \frac{1}{2}(a_{\text{B}} - a + 2b_{\text{B}} - 2b)g_{\mu\nu}R + \hbar\mathcal{O}(\hbar, a, b).$$

Observe that the couplings a and b cancel out in the lowest-order expression, which confirms that $g_{\mu\nu\text{B}}$ is truly a field redefinition, not a redefinition of the couplings.

Using (4.11), (4.9), (4.12) and (4.6), $S_{\text{AC B}}$ is equal to

$$S_{\text{AC B}} = S_{\text{HD}}[\overline{G}(g_{\text{B}}, a_{\text{B}}, b_{\text{B}}), \varphi_{\text{B}}, \lambda_{\text{B}}, a_{\text{B}}, b_{\text{B}}, \kappa] = S_{\text{HD}}[\overline{G}(g, a, b), \varphi_{\text{B}}, \lambda_{\text{B}}, a_{\text{B}}, b_{\text{B}}, \kappa] = S_{\text{HD B}}. \tag{4.14}$$

This equality ensures that the set of Feynman diagrams of the acausal theory is obtained from the set of diagrams of the higher-derivative theory, once $\overline{g}$ on the external legs is replaced with the finite function $\overline{G}(g, a, b)$. Thus, the Green functions of the acausal theory are finite and collected in the generating functional

$$\Gamma_{\text{AC}}[g, \Phi, \lambda, a, b, \kappa] = \Gamma_{\text{HD}}[\overline{G}(g, a, b), \Phi, \lambda, a, b, \kappa]. \tag{4.15}$$

According to the arguments of section 2, the quantum action $S_{q\text{AC}}[g, \varphi_q, \lambda, a, b, \kappa]$ is the real part of $\Gamma_{\text{AC}}[g, \Phi, \lambda, a, b, \kappa]$, with the convention that *i*) g is real and *ii*) $\Phi = \varphi_q$ is real if the fields φ are real bosonic, while Φ is the conjugate of $\overline{\Phi}$ if the fields φ, $\overline{\varphi}$ are complex bosonic or fermionic. The gravitational field equations are given by the variation of $S_{q\text{AC}}[g, \varphi_q, \lambda, a, b, \kappa]$ with respect to the metric tensor $g_{\mu\nu}$. The variation of $S_{q\text{AC}}[g, \varphi_q, \lambda, a, b, \kappa]$ with respect to φ_q generates the quantum field equations of matter.

Thus the theory $S_{q\text{AC}}[g, \varphi_q, \lambda, a, b, \kappa]$ is a predictive formulation of classical gravity coupled with quantum matter. No higher-derivative kinetic terms have been added to the pure-gravity sector. The number of independent couplings is finite: the gravitational couplings are just three, namely the Newton constant κ, which does not renormalize, plus a and b; on the other hand, the number of couplings λ belonging to the matter sector is constrained by power counting.

In the presence of a cosmological constant, the identity (3.13) has to be used. If the matter sector does not contain parameters with positive dimensionalities in units of mass, the parameters a, b renormalize exactly as above, and κ, Λ do not renormalize. In the acausal theory, the true Newton constant is $\widetilde{\kappa}$, which gets renormalized because it is a function of a, b, κ and Λ. Observe that the cosmological constant Λ is the same on the two sides of the map $\mathcal{M}$. If the matter sector contains parameters with positive dimensionalities in units of mass, then there are independent renormalizations of the Newton constant and the cosmological constant.

4.4 Classical gravity coupled with quantum matter in higher dimensions

The construction of this section can be generalized to higher dimensions. Assume first that the cosmological constant is zero and the matter sector does not contain parameters of positive dimensionalities in units of mass.

On the higher-derivative side of the map $\mathcal{M}$, the counterterms can be classified in two subsets, in connection with the expansion of the metric tensor around flat space, $\overline{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$: *i*) "kinetic counterterms", namely counterterms that contain contributions quadratic in h; *ii*) "vertex counterterms", namely counterterms that do not contain contributions quadratic in h.

The kinetic counterterms can always be converted into counterterms quadratically proportional to the Einstein field equations [22], which can be reabsorbed by the map $\mathcal{M}$, plus vertex counterterms. The vertex counterterms can be quadratically proportional to the Einstein field equations or not. Those that are can be reabsorbed by the map $\mathcal{M}$, those that are not must be included in S_m, multiplied by independent couplings. With these arrangements the theorem of section 2 applies.

For example, in six dimensions [23], kinetic counterterms of dimensionality 6 are

$$\int \sqrt{-\overline{g}}\,\overline{R}_{\mu\nu\rho\sigma}\overline{\nabla}_{\alpha}\overline{\nabla}^{\alpha}\overline{R}^{\mu\nu\rho\sigma}, \qquad \int \sqrt{-\overline{g}}\,\overline{\nabla}_{\alpha}\overline{R}_{\mu\nu\rho\sigma}\overline{\nabla}^{\mu}\overline{R}^{\alpha\nu\rho\sigma}, \tag{4.16}$$

etc. Using partial integrations and Bianchi identities, and commuting derivatives, these counterterms can be converted into terms quadratically proportional to the Ricci tensor, which can be reabsorbed by the map $\mathcal{M}$, plus vertex counterterms. Vertex counterterms are

$$\int \sqrt{-\overline{g}}\,\overline{R}_{\mu\nu}{}^{\rho\sigma}\overline{R}_{\alpha\beta}{}^{\mu\nu}\overline{R}_{\rho\sigma}{}^{\alpha\beta}, \qquad \int \sqrt{-\overline{g}}\,\overline{R}_{\mu}^{\nu}\overline{R}_{\nu\alpha}{}^{\rho\sigma}\overline{R}_{\rho\sigma}{}^{\mu\alpha}, \qquad \int \sqrt{-\overline{g}}\,\overline{R}_{\mu}^{\nu}\overline{R}_{\rho}^{\mu}\overline{R}_{\nu}^{\rho}, \tag{4.17}$$

etc. The third of (4.17) can be reabsorbed by the map $\mathcal{M}$, while the other two must be included in S_m, multiplied by independent couplings.

The vertex counterterms can be ignored in the quadratic analysis of causality violations (see the next section).

In the presence of a cosmological constant, or if the matter sector contains parameters with positive dimensionalities in units of mass (which generate the cosmological constant by renormalization), it is convenient to expand the metric $\overline{g}_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ around a maximally symmetric metric $g_{\mu\nu}^{(0)}$, such that

$$R_{\mu\nu\rho\sigma}^{(0)} = \frac{2\Lambda}{(d-1)(d-2)} \left(g_{\mu\rho}^{(0)} g_{\nu\sigma}^{(0)} - g_{\mu\sigma}^{(0)} g_{\nu\rho}^{(0)} \right),$$

where d is the spacetime dimension. The gravitational counterterms are more conveniently rearranged as functions of the hatted Riemann tensor

$$\widehat{\overline{R}}_{\mu\nu\rho\sigma} = \overline{R}_{\mu\nu\rho\sigma} - \frac{2\Lambda}{(d-1)(d-2)} (\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} - \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho})$$

and its covariant derivatives, because $\widehat{\overline{R}}_{\mu\nu\rho\sigma}$ vanishes on the metric $g_{\mu\nu}^{(0)}$. Again, the counterterms can be distinguished into kinetic counterterms (those that contain contributions quadratic in h) and vertex counterterms (those that do not contain contributions quadratic in h). It was shown in [22] that the kinetic counterterms can be converted into terms quadratically proportional to $\widehat{\overline{R}}_{\mu\nu}$ or its covariant derivatives, which are reabsorbed by the map $\mathcal{M}$, plus vertex counterterms, plus a linear combination of $\overline{R}$ and 1, that renormalize the Newton constant and the cosmological constant. There is only one case where this fact is not obvious, namely $\widehat{\overline{R}}_{\mu\nu\rho\sigma}\widehat{\overline{R}}^{\mu\nu\rho\sigma}$. However, the combination

$$\begin{aligned} \widehat{\overline{\mathrm{G}}} &= \widehat{\overline{R}}_{\mu\nu\rho\sigma}\widehat{\overline{R}}^{\mu\nu\rho\sigma} - 4\widehat{\overline{R}}_{\mu\nu}\widehat{\overline{R}}^{\mu\nu} + \widehat{\overline{R}}^2 + \frac{8(d-3)}{(d-1)(d-2)}\Lambda\left(\overline{R} - 2\Lambda\right) \\ &= \overline{R}_{\mu\nu\rho\sigma}\overline{R}^{\mu\nu\rho\sigma} - 4\overline{R}_{\mu\nu}\overline{R}^{\mu\nu} + \overline{R}^2 - 4\frac{(d-3)(d-4)}{(d-1)(d-2)}\Lambda(\overline{R} - \Lambda), \end{aligned}$$

does not contain h-quadratic contributions, thanks a peculiar identity [22],

$$\int \sqrt{-g}\, \widehat{\overline{\mathrm{G}}} = \frac{32(d-3)}{(d-1)(d-2)^2}\Lambda^2 \int \sqrt{-g^{(0)}} + \mathcal{O}\left(h^3\right),$$

which proves that $\widehat{\overline{\mathrm{G}}}$ is a vertex counterterm. The counterterms $\widehat{\overline{R}}_{\mu\nu\rho\sigma}\overline{\nabla}^{\lambda_1}\cdots\overline{\nabla}^{\lambda_n}\widehat{\overline{R}}_{\alpha\beta\gamma\delta}$, $n > 0$, with indices contracted in all possible ways, can be reduced by means of repeated partial integrations, commutations of covariant derivatives and applications of the Bianchi identities.

The counterterm $\int \sqrt{-g}\, \widehat{\overline{\mathrm{G}}}$ and the other vertex counterterms cannot, in general, be reabsorbed by the map $\mathcal{M}$. They have to be included in S_m, multiplied by independent couplings. If the matter sector is a power-counting renormalizable theory (which, in $d > 4$, means just a free theory, or the φ^3 theory in five and six dimensions), embedded in curved space, then the action

$S_{\rm HD}^{(\Lambda)}$ contains a finite number of terms, therefore the acausal theory $S_{\rm AC}^{(\Lambda)}$ has a finite number of independent couplings.

Finally, in three spacetime dimensions the action (4.1) is renormalizable with $\Delta S_m = 0$. Indeed, a three-dimensional power-counting renormalizable theory in curved space generates no higher-derivative pure-gravity counterterm: the Lorentz Chern-Simons term is protected [24]; all other higher-derivative terms constructed with the Riemann and Ricci tensor have at least dimensionality four. So, there is no causality violation in three dimensions. In higher dimensions higher-derivative terms can be generated, but they must be multiplied by parameters with positive odd dimensionalities. If such parameters are not contained in S_m, then causality is not violated. If such parameters are contained in S_m, then the procedure of even-dimensional theories has to be applied and there are causality violations.

5 Higher time derivatives, instabilities and causality violations

In general, the map $\mathcal{M}$ converts a causal classical theory with instabilities, originated by higher derivatives in the kinetic term, into an acausal classical theory without instabilities. This section is devoted to study these properties in more detail, including the effects of the radiative corrections.

To illustrate the logic of the discussion, it is convenient to recall the analysis of the Abraham-Lorentz force (see for example [11]). In classical electrodynamics an effective description of the Larmor formula

$$P = m\tau a^2, \qquad \tau = \frac{2e^2}{3mc^3}, \tag{5.1}$$

for the radiation power emitted by an accelerated particle in the adiabatic approximation is provided by the higher-derivative equation

$$ma(t) = m\tau\dot{a}(t) + F(t), \tag{5.2}$$

where a is the acceleration and $F(t)$ is an external force. The term $m\tau\dot{a}$ is the Abraham-Lorentz force. Equation (5.2) can be integrated one time, to give

$$ma(t) = -\frac{1}{\tau}\int_{-\infty}^{t} \mathrm{d}t' \ \mathrm{e}^{(t-t')/\tau}F(t') + ma_0\mathrm{e}^{t/\tau}, \tag{5.3}$$

where a_0 is the arbitrary constant. The solution (5.3) is causal, since it depends only on the force $F(t')$ at earlier times $t' < t$. The second term is a runaway solution, which is the sign of instability. It is present even when there are no external forces.

Observe that in (5.3) the contribution of the force at earlier times $t' < t$ is exponentially amplified. The reason is that the limits $\tau \to 0$ of equation (5.2) and its solution (5.3) are singular. However, physics suggests that such a limit should exist, since τ in (5.1) is proportional to the square of the charge.

The $\tau \to 0$ limit becomes regular only if the constant a_0 is set equal to

$$a_0 = \frac{1}{m\tau} \int_{-\infty}^{\infty} \mathrm{d}t' \ \mathrm{e}^{-t'/\tau} F(t').$$

Then (5.3) becomes

$$ma(t) = \frac{1}{\tau} \int_{t}^{\infty} \mathrm{d}t' \ \mathrm{e}^{(t-t')/\tau} F(t'). \tag{5.4}$$

The $\tau \to 0^+$ limit of this equation is $F = ma$, as desired. Equation (5.4) is a physically reasonable replacement of the Abraham-Lorentz force. However, (5.4) is not equivalent to (5.2). Every solution of (5.4) solves (5.2), but not vice versa. The runaway solution is eliminated and at $F = 0$ the acceleration vanishes. The effective force felt by the particle is a time average of the true force F. The acceleration of the particle at a time t depends on the force F at future times $t' > t$, so causality is violated. Summarizing, the physics described by equation (5.2) is causal but unstable, while the physics of equation (5.4) is stable but acausal.

The causality violations are short-range, the range being of the order $\Delta t \sim \tau$. Numerically, $\Delta t \sim 10^{-22}$sec. Since quantum effects become important already at time intervals of the order of 137τ, the causality violations predicted by equation (5.4) are unobservable.

Writing

$$ma = \frac{1}{1 - \tau \frac{\mathrm{d}}{\mathrm{d}t}} F, \tag{5.5}$$

it becomes evident that the runaway solution, which is the zero mode of $1 - \tau \mathrm{d}/\mathrm{d}t$, is lost in the inversion of this operator, demanding the regularity of the $\tau \to 0$ limit. The inversion of $1 - \tau \mathrm{d}/\mathrm{d}t$ is the map $\mathcal{M}$ that relates the equations (5.2) and (5.4).

Observe that the presence of instabilities, or causality violations, is related to the sign of τ. No instability nor causality violation occurs for $\tau < 0$.

The rearrangement of equation (5.2) into formula (5.5), interpreted in the usual low-energy expansion, which throws away the unstable solutions, is known in the literature as the *regular reduction* of the order of the differential equation and can be done also for gravity [12, 13].

Although inspired by the arguments just recalled, the map $\mathcal{M}$ differs from the regular reduction in a crucial way. The regular reduction is not a field redefinition, but a manipulation of the field equations. In the case of gravity, the analogue of this operation [12] is a manipulation of the field equations of higher-derivative gravity, coupled with classical or quantum matter, which leaves the metric tensor unchanged. It is not known how to implement the regular reduction for gravity at the level of the action. The construction of this paper, instead, is performed at the level of the action and implemented by iterative field redefinitions of the metric tensor that renormalize the counterterms $R_{\mu\nu}R^{\mu\nu}$ and R^2. Typical signs of the difference between the two approaches are the square roots of (3.25), which appear naturally in the map $\mathcal{M}$, but do not appear in (5.5) and in the approach of [12]. Observe that it is not possible to derive the Abraham-Lorentz force from an action, which is why now I abandon this analogy and proceed with the description of the approach of this paper in lagrangian models.

Consider the higher-derivative theory

$$\mathcal{L}'(q) = \frac{m}{2}\dot{q}^2 + \frac{m\alpha^2}{2}\ddot{q}^2 \equiv \mathcal{L}(q) + \Delta\mathcal{L}(q), \qquad \mathcal{L}(q) = \frac{m}{2}\dot{q}^2.$$

The term $\Delta\mathcal{L}$ is quadratically proportional to the field equations of $\mathcal{L}$. The map $\mathcal{M}$ is

$$q(q') = \frac{1}{\sqrt{1-\alpha^2\frac{d^2}{dt^2}}}\, q', \tag{5.6}$$

so that

$$\int \mathrm{d}t\ \mathcal{L}'(q) = \int \mathrm{d}t\ \mathcal{L}(q'). \tag{5.7}$$

More explicitly,

$$q(t) = \int_{-\infty}^{+\infty} \mathrm{d}t'\ \mathcal{C}(t-t')q'(t'), \qquad \mathcal{C}(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\frac{e^{-ikt}}{\sqrt{1+\alpha^2k^2}}. \tag{5.8}$$

According to the discussion about the Abraham-Lorentz force, the map $\mathcal{M}$ should tend to the identity in the limit $\alpha \to 0$, which is implicit in (5.8). Nevertheless, there might exist different prescriptions to define $\mathcal{C}(t)$. Every prescription has the same perturbative expansion in powers of α.

When $\alpha^2 > 0$ (assume $\alpha > 0$ without loss of generality) the solutions of the field equations of $\mathcal{L}'(q)$ and $\mathcal{L}(q')$ read

$$q(t) = at + b + c\mathrm{e}^{t/\alpha} + d\mathrm{e}^{-t/\alpha}, \qquad q'(t) = a't + b', \tag{5.9}$$

respectively. At finite non-vanishing α, $q(t)$ contains two solutions (one of which is runaway) that are absent in $q'(t)$. When $\alpha^2 < 0$ the exponentials are replaced by sine and cosine functions and there is no runaway solution. Finally, $q(t)$ is singular in the limit $\alpha \to 0$.

If the system is subject to an external time-dependent force $F(t)$, the lagrangian

$$\mathcal{L}'(q, F) = \frac{m}{2}\dot{q}^2 + \frac{m\alpha^2}{2}\ddot{q}^2 + qF(t), \tag{5.10}$$

is mapped by (5.6) into

$$\mathcal{L}(q', F') = \frac{m}{2}\dot{q}'^2 + q'F'(t), \qquad F'(t) = \frac{1}{\sqrt{1-\alpha^2\frac{d^2}{dt^2}}}F(t) = \int_{-\infty}^{+\infty} \mathrm{d}t'\ \mathcal{C}(t-t')F(t'). \tag{5.11}$$

so

$$\int \mathrm{d}t\ \mathcal{L}'(q(q'),F) = \int \mathrm{d}t\ \mathcal{L}(q',F'(F)). \tag{5.12}$$

Consider the function $\mathcal{C}(t)$ in (5.8) and (5.11). For $\alpha^2 > 0$ the integral (5.8) is convergent and gives

$$\mathcal{C}(t) = \frac{1}{\pi|\alpha|}K_0\left(\frac{|t|}{|\alpha|}\right), \qquad \text{if } \alpha^2 > 0. \tag{5.13}$$

Causality is violated, since $F'(t)$ depends on the force $F(t')$ at future times t'. The range of the causality violations is $\Delta t = |\alpha|$.

When $\alpha^2 < 0$ it is necessary to specify a prescription for the contour integration in the complex plane. There is a real causal prescription, which gives the retarded function $\mathcal{C}_{\mathrm{ret}}(t)$,

$$\mathcal{C}_{\mathrm{ret}}(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \frac{e^{-ikt}}{\sqrt{1+\alpha^2(k+i\varepsilon)^2}} = \frac{\theta(t)}{|\alpha|}J_0\left(\frac{|t|}{|\alpha|}\right), \qquad \text{if } \alpha^2 < 0. \tag{5.14}$$

The advanced function is $\mathcal{C}_{\mathrm{adv}}(t) = \mathcal{C}_{\mathrm{ret}}(-t)$. A complex acausal prescription is studied below in arbitrary spacetime dimensions: see formula (5.19).

Summarizing, the theory $\mathcal{L}(q',F')$ has no unstable solution. It violates causality for $\alpha^2 > 0$ and admits a causal prescription for $\alpha^2 < 0$.

Again, the redefinition (5.8) maps two physically inequivalent theories. Once it is known whether q or q' are the physical fields, and whether F or F' are the physical forces, the physics follows from the appropriate lagrangian, (5.10) or (5.12).

5.1 Fields

Consider the scalar theory

$$\mathcal{L}'(\varphi,J) = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) + \frac{1}{2}\alpha^2(\Box\varphi)^2 + \varphi J \tag{5.15}$$

in n spacetime dimensions. The map

$$\varphi' = \sqrt{1-\alpha^2\Box}\varphi \tag{5.16}$$

relates (5.15) with the theory

$$\mathcal{L}(\varphi',J'(J)) = \frac{1}{2}(\partial_\mu\varphi')(\partial^\mu\varphi') + \varphi'J', \qquad J'(x) = \int \mathrm{d}^n x'\ \mathcal{C}_n(x-x')J(x'), \tag{5.17}$$

where

$$\mathcal{C}_n(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{e}^{-ik\cdot x}}{\sqrt{1+\alpha^2 k^2}}. \tag{5.18}$$

Again, the regularity of the $\alpha \to 0$ limit is understood in (5.18).

Fields of higher spins can be treated similarly. In every case, the function $\mathcal{C}_n(x)$ is the essential ingredient of the map $\mathcal{M}$. For gravity in the quadratic approximation the map $\mathcal{M}$ is collected in formulas (3.16) and (3.24) and involves the functions $\mathcal{C}_n(x)$ with α^2 equal to a or b'.

The Fourier transform (5.18) has to be defined with an appropriate prescription. It is convenient to begin with the prescription [25]

$$\mathcal{C}_n^{\mathrm{F}}(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{e}^{-ik\cdot x}}{\sqrt{1+\alpha^2 k^2 + i\varepsilon}} = \frac{\mathrm{e}^{-i\frac{\pi}{4}\left[2(n-1)+(n-2)(\mathrm{sign}(\alpha^2)-1)\right]}}{2^{(n-1)/2}\pi^{(n+1)/2}|\alpha|^n} \frac{K_{(n-1)/2}\left(\sqrt{\frac{x^2}{\alpha^2} - i\varepsilon'}\right)}{\left(\frac{x^2}{\alpha^2} - i\varepsilon'\right)^{(n-1)/4}}, \quad (5.19)$$

which illustrates the main features of the function $\mathcal{C}_n(x)$. Observe that $\mathcal{C}_n^{\mathrm{F}}(x)$ is complex, but recall that the quantum action $S_{q\mathrm{AC}}$ is the real part of the functional Γ_{AC} of (4.15).

In even dimensions the function $\mathcal{C}_n^{\mathrm{F}}(x)$ is quite simple. For example,

$$\mathcal{C}_4^{\mathrm{F}}(x) = \frac{\mathrm{sign}(\alpha^2)}{4\pi^2|\alpha|^4} \frac{i\exp\left(-\sqrt{\frac{x^2}{\alpha^2} - i\varepsilon}\right)}{\left(\frac{x^2}{\alpha^2} - i\varepsilon\right)^{3/2}} \left(1 + \sqrt{\frac{x^2}{\alpha^2} - i\varepsilon}\right). \quad (5.20)$$

The exponential tends to zero or rapidly oscillates for $|x^2| \gg |\alpha^2|$, so the causality violations can be experimentally tested only at distances of the order of

$$\Delta x \sim 2\pi|\alpha| \quad (5.21)$$

and become physically unobservable at distances much larger than this bound.

For $\alpha^2 = -\overline{\alpha}^2 < 0$ there is a real causal prescription, namely the retarded function

$$\mathcal{C}_n^{\mathrm{ret}}(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{e}^{-ik\cdot x}}{\sqrt{1 - \overline{\alpha}^2\,(k_0 + i\varepsilon)^2 + \overline{\alpha}^2 \mathbf{k}^2}}, \quad (5.22)$$

which vanishes for $t < 0$. Indeed, the branch cuts are located in the lower half k_0-plane and if $t < 0$ it is possible to close the contour of integration in the upper half plane ($\mathrm{Im}\, k^0 > 0$). By Lorentz invariance, every point outside the light-cone admits a reference frame in which $t < 0$, so $\mathcal{C}_n^{\mathrm{ret}}(x)$ vanishes identically outside the light-cone. The advanced function $\mathcal{C}_n^{\mathrm{adv}}(x)$ is defined as in (5.22) with $k_0 + i\varepsilon \to k_0 - i\varepsilon$. In the non-relativistic limit,

$$\mathcal{C}_n(x) \to \frac{2\delta(t) K_{n/2-1}(r/\overline{\alpha})}{(2\pi\overline{\alpha})^{n/2} r^{n/2-1}}, \quad (5.23)$$

independently of the prescription, where $x = (t, \mathbf{x})$ and $r = |\mathbf{x}|$. For $\alpha^2 > 0$ no causal prescription exists. When $\alpha \to 0$ the functions $\mathcal{C}_n(x)$ tend to $(2\pi)^n \delta^n(x)$, independently of the prescription. When $\alpha^2 \to \infty$ they tend to zero.

The map $\mathcal{M}: S_{\rm HD} \rightarrow S_{\rm AC}$ is essentially classical, because it applies to a classical theory, or to the classical sector of a partially classical, partially quantum theory. The generalization of the map $\mathcal{M}$ to quantum gravity should convert higher-derivative quantum gravity into acausal quantum gravity, preserving the renormalizability. Higher-derivative quantum gravity is renormalizable, but not unitary [26]. The violation of unitarity is exhibited by the propagation of ghosts, which are the quantum counterparts of the classical instabilities. However, the renormalization of higher-derivative quantum gravity is singular in the limit where a, b' tend to zero. It has been remarked above that the smoothness of these limits is an essential ingredient for the map $\mathcal{M}$, to trade the instabilities for causality violations (check the discussion about the $\tau \rightarrow 0$ limit of the Abraham-Lorentz force).

The quantum map $\mathcal{M}$ should be able convert unitarity violations into causality violations, preserving the renormalization structure. Once again, the map $\mathcal{M}$ cannot be a field redefinition, because a field redefinition preserves the renormalization structure, but does not change the poles of the S-matrix elements (see [27]). A naive application of the map $\mathcal{M}$ in the functional integral restores the ghosts by means of the Jacobian determinant. In conclusion, the construction of a good map $\mathcal{M}$ for quantum gravity has to be left to future investigations.

5.2 Effects of the radiative corrections

At the tree level, the presence of causality violations depends on the sign of α^2. When $\alpha^2 < 0$, causal prescriptions exist for $\mathcal{C}_n(x)$, when $\alpha^2 < 0$ there is no causal prescription. Beyond the tree-level, the logarithmic corrections spoil the causal prescriptions. Nevertheless, the causality violations affect only high energies.

Consider classical gravity coupled with a renormalizable quantum field theory. At one loop the a-running is governed by the trace anomaly of the matter sector in curved space. To the lowest order the beta functions are [28, 29]

$$\frac{1}{\kappa^2}\beta_a = -4c + \mathcal{O}(\lambda), \qquad \frac{1}{\kappa^2}\beta_{b'} = \kappa^2\mathcal{O}(\lambda), \tag{5.24}$$

where λ denotes the matter couplings, including the parameter η of formula (4.3), and

$$c = \frac{12n_v + 6n_f + n_s}{120(4\pi)^2}, \tag{5.25}$$

where n_s is the number real scalars, n_f is the number of Dirac fermions and n_v is the number of vector fields. For example, in QED

$$\beta_a = -\frac{3\kappa^2}{5(4\pi)^2} - \frac{7}{9}\frac{\kappa^2 e^2}{(4\pi)^4} + \kappa^2\mathcal{O}(e^4), \qquad \beta_{b'} = -\frac{16}{27}\frac{e^6\kappa^2}{(4\pi)^8} + \kappa^2\mathcal{O}(e^8).$$

To illustrate the effects of radiative corrections it is sufficient to concentrate on the first contributions to the beta functions. Assume that the interactions of the matter sector are switched off ($\lambda = 0$), namely that the matter sector is a free-field theory in curved space. Then the renormalization of a and b' is exact,

$$\beta_a = -4c\kappa^2, \qquad \beta_{b'} = 0. \tag{5.26}$$

The exactness of formulas (5.26) holds in a larger class of models, those whose matter sector is a conformal field theory $\mathcal{C}$ embedded in external gravity. Then c is not (5.25), but a characteristic quantity of $\mathcal{C}$, called "central charge" c (see for example ref.s [30] for definitions and properties). According to (5.26), the parameter b' does not run, but a does. The a-running is

$$a(-p^2) = \overline{a} - 2c\kappa^2 \ln \frac{-p^2}{\mu^2} = -2c\kappa^2 \ln \frac{-p^2}{\Lambda^2}, \qquad \Lambda \equiv \mu \exp\left(\frac{\overline{a}}{4c\kappa^2}\right), \tag{5.27}$$

where $\overline{a} = a(\mu^2)$ and Λ is the energy scale at which the running coupling switchs its sign.

I stress again that (5.27) is an exact formula for an important class of models. Thus, it is mandatory to investigate the physical effects of the a-running at the non-perturbative level in a and κ^2.

Write the higher-derivative action $S_{\mathrm{HD}}(g)$ (3.10) as

$$S_{\mathrm{HD}}[g] = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[R + \frac{a}{2}W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma} - \frac{b'}{6}R^2\right],$$

where $W^{\mu\nu\rho\sigma}$ is the Weyl tensor. The one-loop quantum functional Γ reads, in the gravity sector,

$$\Gamma_{\mathrm{HD}} = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[R - c\kappa^2 W^{\mu\nu\rho\sigma}\ln\left(\frac{\Box}{\Lambda^2}\right)W_{\mu\nu\rho\sigma} - \frac{b'}{6}R^2\right],$$

up to cubic terms in the curvature tensors.

In the quadratic approximation, with the gauge fixing (3.19)-(3.20), the map $\mathcal{M}$ relating the higher-derivative quantum functional Γ_{HD} with the acausal functional Γ_{AC},

$$\begin{aligned}\Gamma_{\mathrm{HD}} &= \frac{1}{2}\int \mathrm{d}^4x \left\{(\partial_\mu\widetilde{\phi}_{\rho\sigma})^2 - 2c\kappa^2(\Box\widetilde{\phi}_{\mu\nu})\ln\left(\frac{\Box}{\Lambda^2}\right)(\Box\widetilde{\phi}^{\mu\nu}) - \frac{3}{8}\left[(\partial_\mu\phi)^2 + b'(\Box\phi)^2\right]\right\}\\ &= \frac{1}{2}\int \mathrm{d}^4x\left\{(\partial_\mu\widetilde{\phi}'_{\rho\sigma})^2 - \frac{3}{8}(\partial_\mu\phi')^2\right\} = \Gamma_{\mathrm{AC}},\end{aligned}$$

is promoted, by dimensional transmutation, to the renormalization-group invariant form

$$\widetilde{\phi}_{\mu\nu} = \frac{1}{\sqrt{1 + 2c\kappa^2\Box\ln\left(\frac{\Box}{\Lambda^2}\right)}}\widetilde{\phi}'_{\mu\nu}, \qquad \phi = \frac{1}{\sqrt{1 - b'\Box}}\phi'. \tag{5.28}$$

According to the arguments of section 2, the quantum action S_{AC} is the real part of Γ_{AC}, with the convention that ϕ' is real.

The function $\mathcal{C}_4(x)$ mapping the traceless part $\widetilde{\phi}_{\mu\nu}$ is

$$\mathcal{C}_4(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-ip\cdot x}}{\sqrt{1 + 2\xi \frac{-p^2}{\Lambda^2} \ln \frac{-p^2}{\Lambda^2}}}, \tag{5.29}$$

where $\xi = c\kappa^2\Lambda^2$. The prescription for $\ln\left(\Box/\Lambda^2\right)$ is determined by the Feynman prescription for the propagators of the matter fields that circulate in the loops, so in momentum space

$$\ln\left(\frac{-p^2}{\Lambda^2}\right) \to \ln\left(\frac{-p^2 - i\varepsilon}{\Lambda^2}\right).$$

Closing the contour of the p_0-integration in the upper half p_0-plane at infinity, the phase of p_0 ranges from 0 to π and the phase of p^2 crosses the branch cut of the logarithm. The function $\mathcal{C}_4(x)$ receives a contribution from the integral along the cut and does not vanish for $t < 0$. Moreover, since the integral along the cut is not purely imaginary, even $\mathrm{Re}\,\mathcal{C}_4(x)$ is non-vanishing for $t < 0$. Other non-vanishing contributions can come from the cuts of the square root. In conclusion, causality is violated due to the radiative corrections.

A prescription for the other factor of p^2 in (5.29) can be obtained generalizing the prescription (5.19). Then $\mathcal{C}_4(x)$ is

$$\mathcal{C}_4(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-ip\cdot x}}{\sqrt{1 - 2\xi \frac{p^2 + i\varepsilon}{\Lambda^2} \ln\left(-\frac{p^2 + i\varepsilon}{\Lambda^2}\right)}}. \tag{5.30}$$

At sufficiently low energies, the function $\mathcal{C}_4(x)$ is not sensibly different from the identity $(2\pi)^4\delta(x)$. The acausal behavior can be observed starting from energies E such that

$$E^2 a(E^2) \sim 1.$$

So far, the gravitational force has been tested down to distances of the order of 0.1 millimeters, which means energies about $2 \cdot 10^{-3}$eV, without observing acausal behaviors. Thus the value $a(E^2)$ of the coupling a at that energy is bounded by

$$|a(E)| < 2.5 \cdot 10^5 (\mathrm{eV})^{-2}.$$

6 Conclusions

I have proved that classical gravity coupled with quantized matter can be renormalized with a finite number of independent couplings, without adding higher-derivative terms to the gravitational sector. Instead, the theory contains vertices that couple the matter stress-tensor with the Ricci tensor and predicts the violation of causality at small distances.

The proof of renormalizability uses a map $\mathcal{M}$ that relates acausal gravity with higher-derivative gravity. The map $\mathcal{M}$, inspired by known treatments of the Abraham-Lorentz force in classical electrodynamics, trades the instabilities due to higher-derivatives for causality violations. The field equations of a partially classical, partially quantum field theory follow from a suitable minimization principle.

The matter sector is an ordinary power-counting renormalizable theory in curved space, with couplings λ, plus a non-renormalizable perturbation, made by a head and a queue. The head contains two vertices that couple the matter stress-tensor with the Ricci tensor, multiplied by independent couplings a and b'. The queue contains an infinity of higher-dimensioned vertices, polynomial in the matter fields, but non-polynomial in the gravitational field and its derivatives. The queue does not contain new independent couplings, rather its vertices are multiplied by appropriate functions of the other couplings, such that the divergences of the theory are subtracted away renormalizing λ, a, b', the Newton constant and the cosmological constant, together with field redefinitions. The causality violations are due to the resummation of derivatives in the vertices that couple matter with gravity.

The analysis of causality violations has been performed in a regime in which the gravitational field is weak, which means much smaller than the Planck mass, but rapidly varying. For a gravitational field of the order of the Planck mass or higher it is necessary to treat the Einstein equations coupled with matter exactly or with more powerful approximation methods. In principle there might exist causal strong-field configurations. Here it was important to show that there do exist configurations that violate causality at small distances.

The causality violations are governed by the parameters a and b'. Their values need to be experimentally measured. Bounds can be derived from the tests on the validity of Newton's law at short distances. At the tree level, causal prescriptions exist, if a and b' are negative, but the radiative corrections make a and b' run and switch their signs. Thus there always exist configurations that violate causality at sufficiently short distances.

On the higher-derivative side of the map $\mathcal{M}$, a and b' multiply combinations of the terms $R_{\mu\nu}R^{\mu\nu}$ and R^2. The map $\mathcal{M}$ provides a new interpretation of the physical meaning of such terms.

Strictly speaking, the investigation of this paper makes sense only if gravity is ultimately classical in nature. More generally, the knowledge provided by this research may be interesting to suggest experiments to decide whether gravity must be quantized or not.

Although the map $\mathcal{M}$ does not generalize straightforwardly to quantum gravity, some conclusions of this paper could. Quantum gravity, being non-renormalizable, is necessarily non-polynomial in the fields and their derivatives. Quite generally, the resummation of derivatives can produce causality violations, with a mechanism similar to the one illustrated here. Therefore, it is reasonable to expect that high-energy causality violations take place also in quantum gravity.

Acknowledgments

I am grateful to M. Halat for discussions and especially M. Mintchev and G. Morchio for help with the generalized functions. I would also like to thank the referee for pointing out relevant literature on approaches different from the ones pursued here.

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