

DEFORMED DIMENSIONAL REGULARIZATION FOR ODD (AND EVEN) DIMENSIONAL THEORIES

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Abstract

I formulate a deformation of the dimensional-regularization technique that is useful for theories where the common dimensional regularization does not apply. The Dirac algebra is not dimensionally continued, to avoid inconsistencies with the trace of an odd product of gamma matrices in odd dimensions. The regularization is completed with an evanescent higher-derivative deformation, which proves to be efficient in practical computations. This technique is particularly convenient in three dimensions for Chern-Simons gauge fields, two-component fermions and four-fermion models in the large N limit, eventually coupled with quantum gravity. Differently from even dimensions, in odd dimensions it is not always possible to have propagators with fully Lorentz invariant denominators. The main features of the deformed technique are illustrated in a set of sample calculations. The regularization is universal, local, manifestly gauge-invariant and Lorentz invariant in the physical sector of spacetime. In flat space power-like divergences are set to zero by default. Infinitely many evanescent operators are automatically dropped.

1 Introduction

The dimensional-regularization technique [1, 2] is the most efficient technique for the calculation of Feynman diagrams in quantum field theory. When gauge bosons couple to fermions in a chiral invariant way, gauge invariance is manifest. When gauge bosons couple to chiral currents, the definition of γ_5 in even dimensions ($\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ [2, 3, 4]) breaks the Lorentz symmetry in the dimensionally continued spacetime and generates axial anomalies. Gauge invariance survives if and only if the one-loop gauge anomalies vanish. This is a restriction on the matter content of the theory. The Adler-Bardeen theorem [5] ensures that there exists a subtraction scheme where anomalies vanish to all orders in perturbation theory, once they vanish at one loop.

At the practical level, calculations with the dimensional-regularization technique in parity violating theories are not more difficult than calculations in parity invariant theories. The reason is that the presence of γ_5 does not break the continued Lorentz symmetry in the denominators of propagators, but only in the vertices and numerators of propagators. Therefore, using appropriate projectors, the Feynman integrals can be decomposed into a basis of fully Lorentz invariant integrals. The complication introduced by γ_5 is only algebraic and can be easily treated with calculators.

On the other hand, the ordinary dimensional regularization is not universal, in the sense that there exist models that cannot be dimensionally regularized in the ordinary framework. The dimensionally continued Dirac algebra has the property that the trace of an odd product of gamma matrices is always equal to zero (see for example [4]). However, if D denotes the physical spacetime dimension and $d = D - \varepsilon$ is its continuation, the trace of the product of D gamma matrices should tend to the epsilon tensor in the limit $\varepsilon \rightarrow 0$, when D is odd. The three-dimensional four-fermion model in the large N limit is another example of theory that cannot be regularized with an ordinary dimensional continuation. This model is not power-counting renormalizable, but becomes renormalizable in the large N expansion (where N is the number of fermion copies), after the resummation of fermion bubbles [6]. Nevertheless, the effective propagator obtained after this resummation originates $\Gamma[0]$ s in subleading diagrams. Ways to regulate these $\Gamma[0]$ s have been already presented in ref.s [7, 8]. Here I consider a more general framework.

In this paper I show that although the dimensional-regularization technique does not apply to every model in a naive way, there always exist deformations of the dimensional-regularization technique that regularize a theory consistently at each order of the loop expansion in a manifestly gauge-invariant way (up to the known anomalies) and preserve Lorentz invariance in the physical subsector of spacetime. These deformations are obtained combining variants of the usual dimensional technique with evanescent higher-derivative corrections, multiplied by an extra cut-off.

Both the dimensional and higher-derivative regularizations have virtues and weak points. Fortunately, the weak points of the two techniques have an empty intersection, so an appropriate combination of the two can enhance the virtues of both.

The higher-derivative regularization is gauge invariant and in principle universal, but it regulates only higher-loop divergences. One-loop divergences have to be treated separately [9]. Using appropriate Pauli-Villars fields, Fadeev and Slavnov have shown that it is possible to regulate the one-loop divergences in a gauge-invariant way when the theory contains non-Abelian gauge fields [9]. Presumably, the construction can be extended to gravity. However, calculations with the higher-derivative technique are cumbersome. In quantum gravity and other non-renormalizable theories exponentials are necessary and the large number of additional vertices makes computations hard. Moreover, the higher-derivative technique produces power-like divergences (linear, quadratic, etc.). In the presence of gravity the powers of the cut-off can be arbitrarily high.

Power-like divergences are RG invariant, namely they do not depend on the dynamical scale μ , because only logarithms $\log \Lambda/\mu$ force the introduction of the RG scale. Due to this, there always exists a subtraction scheme where power-like divergences are absent. This scheme is called “classically conformal scheme”, because when the theory is classically conformal (namely it does not contain masses, nor dimensionful couplings at the classical level) no mass nor dimensionful coupling is generated by renormalization. Then, the dynamically generated scale μ is the unique dimensionful constant of the theory at the quantum level. The dimensional regularization automatically selects the classically conformal scheme. In the other regularization frameworks it is possible to reach this scheme manually fine-tuning the local counterterms.

In three dimensions the dimensional regularization is inconsistent if the theory contains, for example, two-component fermions coupled with Chern-Simons gauge fields. In this case it is incorrect to set the trace of an odd product of gamma matrices to zero. Nevertheless, this inconsistency does not show up at one-loop, because one-loop diagrams in odd dimensions have no logarithmic divergence. Since power-like divergences can be ignored, at least in the classically conformal subtraction scheme, this is equivalent to say that the theory is convergent at one-loop.

Now, given that the difficulties of the higher-derivative regularization are confined to one loop, while the difficulties of the dimensional technique show up only beyond one-loop, it is reasonable to argue that a suitable combination of the two techniques can provide a consistent and universal regularization framework. A generic higher-derivative deformation, however, is difficult to handle in practical computations. I show in a number of examples that if the higher-derivative deformation is also evanescent, calculations can still be done efficiently.

When the Standard Model is coupled with quantum gravity the dimensional regularization breaks the continued *local* Lorentz symmetry, because of the presence of γ_5 in the interactions. A similar breaking takes place in three-dimensional quantum gravity coupled with Chern-Simons

gauge fields, two-component fermions and so on. In ref. [10] it is shown that it is possible to dimensionally regularize quantum gravity coupled with parity violating matter in such a way that the propagators of the graviton, its ghosts and auxiliary fields have fully Lorentz invariant denominators. Instead of gauge-fixing the residual local Lorentz symmetry choosing a symmetric vielbein, a derivative Lorentz gauge fixing of the form $\partial^\mu \omega_\mu^{ab}$ and a clever use of the auxiliary fields do the job. Combining the results of [10] with the ones of the present paper it is possible to extend the regularization studied here to odd-dimensional quantum gravity coupled with parity-violating matter.

Recapitulating, the purpose of this paper is to study deformations of the dimensional technique that regularize in a manifestly gauge-invariant way also models where the usual formulation does not apply. I explore several types of deformations and search for the one that makes calculations more efficient. It turns out that the most convenient framework is the one in which the higher-derivative deformation is also evanescent. Another important point is that in odd dimensions, differently from even dimensions, it is not always possible to have propagators with fully Lorentz invariant denominators. This complicates the evaluation of integrals, but not too much. I illustrate the evaluation of some standard diagrams to convince the reader that computations are still reasonably doable. Another advantage of the deformed technique, in even and odd dimensions, is that infinitely many evanescent operators, that are present in the usual, undeformed approach, are automatically dropped.

In the literature various alternative definitions of γ_5 and the ε tensor have been proposed. I use the 't Hooft-Veltman prescription, which is known to be fully consistent. To my knowledge, there exists no definition of γ_5 that commutes with all $\gamma_{\mu s}$, is fully consistent and manifestly gauge invariant modulo the Adler-Bell-Jackiw anomalies. It is out of the purposes of this paper to review the history of alternative proposals.

It is worth to remind that evanescent operators do not affect the S-matrix, but produce at most scheme changes. Some properties of the “theory of evanescent operators” are collected in Collins’ book [4]. More recent references are [11, 12].

There is a variety of reasons for which it is good to have manifestly gauge-invariant regularization techniques. For example, the existence of a regularization with these properties is useful to prove the absence of gauge anomalies. In other approaches it is necessary to resort to lengthy cohomological classifications [13] or deal with explicit and involved cut-off dependencies, as in the exact-renormalization-group approach [14]. Other theoretical applications concern the study of renormalizability and finiteness beyond power counting, for example the construction of consistent irrelevant deformations of renormalizable theories [15] in even and odd dimensions, and three-dimensional quantum gravity coupled with matter [16], which, under certain conditions, can be quantized as a finite theory [17].

The plan of the paper is as follows. In section 2 I describe the technique and its main features. I write the deformed actions that regularize Dirac fermions, Chern-Simons gauge fields and gravity and study the structure of the renormalized actions to all orders in perturbation theory. Then I proceed with sample calculations: the vacuum polarization in section 3 and the axial anomalies in section 4. In section 5 I study Chern-Simons gauge theories coupled with two-component fermions in three dimensions, and work out the vacuum polarization and the one-loop fermion self-energy. In section 6 I study the three-dimensional four-fermion models in the large N expansion and analogous scalar models. In section 7 I give a general recipe to perform the evanescent higher-derivative deformation of $SO(d)$ invariant theories in flat space. In section 8 I prove that power-like divergences are absent in flat space. In section 9 I collect the conclusions. In the appendix I show how to calculate some useful integrals and comment on alternative non-evanescent higher-derivative deformations.

I work in the Euclidean framework, so the integrals are already Wick rotated. No information is lost, since divergences are the same in the Euclidean and Minkowskian frameworks [4]. With an abuse of language, I call the $SO(d)$, $SO(D)$ and $SO(-\varepsilon)$ Euclidean invariances (continued, physical and evanescent, respectively) *Lorentz* symmetries, since no confusion can arise.

2 The technique

If the continued gamma matrices satisfy the d -dimensional Dirac algebra $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ a standard argument [4] proves that the trace of an odd product of gamma matrices is necessarily zero. It is useful to recall here the proof. Using the cyclicity of the trace and the Dirac algebra we have immediately

$$d \operatorname{tr}[\gamma^a] = \operatorname{tr}[\gamma^a \gamma^e \gamma_e] = \operatorname{tr}[\gamma_e \gamma^a \gamma^e] = 2 \operatorname{tr}[\gamma^a] - \operatorname{tr}[\gamma^a \gamma_e \gamma^e] = (2 - d) \operatorname{tr}[\gamma^a], \quad (2.1)$$

whence it follows that $\operatorname{tr}[\gamma^a] = 0$. Next, consider

$$d \operatorname{tr}[\gamma^a \gamma^b \gamma^c] = \operatorname{tr}[\gamma^a \gamma^b \gamma^c \gamma^e \gamma_e] = \operatorname{tr}[\gamma_e \gamma^a \gamma^b \gamma^c \gamma^e]. \quad (2.2)$$

Using $\operatorname{tr}[\gamma^a] = 0$ it follows that the tensor $\operatorname{tr}[\gamma^a \gamma^b \gamma^c]$ is completely antisymmetric. After a few manipulations (2.2) gives

$$d \operatorname{tr}[\gamma^a \gamma^b \gamma^c] = (6 - d) \operatorname{tr}[\gamma^a \gamma^b \gamma^c], \quad (2.3)$$

whence $\operatorname{tr}[\gamma^a \gamma^b \gamma^c] = 0$. Repeating the argument, it is possible to prove that the trace of the product of an arbitrary odd number of gamma matrices is equal to zero.

This fact is incompatible with the existence of $2^{\lfloor D/2 \rfloor}$ -component spinors in odd D , because then the trace of the product of D gamma matrices must give the epsilon tensor in the physical limit $d \rightarrow D$. For example, in $D = 3$

$$\operatorname{tr}[\gamma^a \gamma^b \gamma^c] \rightarrow 2i\varepsilon^{abc}.$$

It is worth to observe that this odd-dimensional problem is essentially different from the problem of γ_5 in even dimensions. The standard definitions of γ_5 and the ε tensor break the continued Lorentz invariance [2, 4, 3], but do not affect the d -dimensional Dirac algebra $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$. In odd dimensions, instead, the continued Lorentz symmetry should be broken at the level of the Dirac algebra ¹.

2.1 Algebra of gamma matrices

We have spacetime indices $\mu, \nu, \rho \dots$ running from 1 to d ; Lorentz indices $a, b, c \dots$ running from 1 to d ; physical Lorentz indices $\bar{a}, \bar{b}, \bar{c} \dots$ running from 1 to D ; evanescent Lorentz indices $\hat{a}, \hat{b}, \hat{c} \dots$ running from D to d (with D excluded). There exists an epsilon tensor $\varepsilon_{\bar{a}_1 \dots \bar{a}_D}$ in the physical portion of spacetime, defined as usual, but there exists no epsilon tensor in the evanescent portion of spacetime.

The algebra of gamma matrices is the tensor product of a physical (D dimensional) Dirac algebra, and an evanescent ($-\varepsilon$ dimensional) Dirac algebra. The two commute:

$$\{\gamma^{\bar{a}}, \gamma^{\bar{b}}\} = 2\delta^{\bar{a}\bar{b}}, \quad \{\gamma^{\hat{a}}, \gamma^{\hat{b}}\} = 2\delta^{\hat{a}\hat{b}}, \quad [\gamma^{\bar{a}}, \gamma^{\hat{b}}] = 0, \quad (2.4)$$

where $\bar{a} = 1, \dots, D$ and $D < \hat{a} < d$ (here ε can be imagined to be real and negative).

Spinors have $2^{[D/2]-\varepsilon/2} = 2^{[D/2]} \cdot 2^{-\varepsilon/2}$ components, where $[n]$ denotes the integral part of n . Write $\psi^{\bar{\alpha}\hat{\alpha}}$, where $\bar{\alpha} = 1, \dots, 2^{[D/2]}$ is the physical spinor index and $\hat{\alpha} = 1, \dots, 2^{-\varepsilon/2}$ is the evanescent spinor index. The gamma matrices $\gamma_{\bar{\alpha}\hat{\alpha}, \bar{\beta}\hat{\beta}}^{\bar{a}}$ act as the usual $2^{[D/2]} \times 2^{[D/2]}$ Hermitean Dirac matrices $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}^{\bar{a}}$ on the physical spinor indices and the identity on the evanescent spinor indices:

$$\gamma_{\bar{\alpha}\hat{\alpha}, \bar{\beta}\hat{\beta}}^{\bar{a}} = \bar{\gamma}_{\bar{\alpha}\bar{\beta}}^{\bar{a}} \delta_{\hat{\alpha}\hat{\beta}}.$$

The trace of an arbitrary product of matrices $\gamma^{\bar{a}}$ follows immediately from the definition. In particular, when $D = 3$ the trace of the product of three such matrices is $2^{1-\varepsilon/2}i$ times the epsilon tensor. The matrix γ_5 is defined as $\gamma_5 = \gamma^1 \dots \gamma^D$ (equal to 1 if D is odd).

The gamma matrices $\gamma_{\bar{\alpha}\hat{\alpha}, \bar{\beta}\hat{\beta}}^{\hat{a}}$ act as formal $2^{-\varepsilon/2} \times 2^{-\varepsilon/2}$ Hermitean Dirac matrices $\hat{\gamma}_{\hat{\alpha}\hat{\beta}}^{\hat{a}}$ ($\{\hat{\gamma}^{\hat{a}}, \hat{\gamma}^{\hat{b}}\} = 2\delta^{\hat{a}\hat{b}}$) on the evanescent spinor indices and the identity on the physical spinor indices:

$$\gamma_{\bar{\alpha}\hat{\alpha}, \bar{\beta}\hat{\beta}}^{\hat{a}} = \delta_{\bar{\alpha}\bar{\beta}} \hat{\gamma}_{\hat{\alpha}\hat{\beta}}^{\hat{a}}.$$

The trace of an odd product of matrices $\gamma^{\hat{a}}$ is zero because of the arguments recalled above. The trace of an even product of these matrices is defined in the usual way.

The so defined physical and evanescent gamma matrices clearly commute. Moreover, the trace of a product of gamma matrices factorizes into the product of a trace in the physical spinor indices and a trace in the evanescent spinor indices.

¹I assume that the trace is cyclic. Non-cyclic trace functionals have been studied in the literature [18].

The contradiction (2.3) is avoided thanks of the commutativity of the physical and evanescent Dirac matrices. Obviously, $\text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}] = 2^{1-\varepsilon/2}i\varepsilon^{\bar{a}\bar{b}\bar{c}}$ and

$$d \text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}] = \text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}(\gamma^{\bar{e}}\gamma_{\bar{e}} + \gamma^{\hat{e}}\gamma_{\hat{e}})] = \text{tr}[\gamma_{\bar{e}}\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}\gamma^{\bar{e}}] + \text{tr}[\gamma_{\hat{e}}\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}\gamma^{\hat{e}}].$$

The first piece is in $D = 3$ and the usual manipulations prove that it is equal to $(6-D) \text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}] = 3 \text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}]$. The second piece, because of the *commutation* rule $[\gamma^{\bar{a}}, \gamma^{\hat{b}}] = 0$, gives back $\text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}\gamma_{\hat{e}}\gamma^{\hat{e}}] = -\varepsilon \text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}]$. In total,

$$d \text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}] = (3 - \varepsilon) \text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\bar{c}}],$$

which is consistent. Finally, it is immediate to show that $\text{tr}[\gamma^{\bar{a}}\gamma^{\bar{b}}\gamma^{\hat{c}}] = \text{tr}[\gamma^{\bar{a}}\gamma^{\hat{b}}\gamma^{\hat{c}}] = \text{tr}[\gamma^{\hat{a}}\gamma^{\hat{b}}\gamma^{\hat{c}}] = 0$.

In the deformed dimensional technique, the Dirac action can be regularized without making use of the hatted Dirac matrices: see formula (2.14) below. Then the hatted Dirac matrices appear nowhere in Feynman rules and diagrams. Fermion traces just get an extra factor $2^{-\varepsilon/2}$, which does not change the physical results. In this case, it is consistent to work directly with $2^{[D/2]}$ -component spinors $\psi^{\bar{\alpha}}$. In practice, this amounts to identify $\gamma^{\bar{a}}$ with $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}^{\bar{a}}$, replace the matrices $\gamma^{\hat{a}}$ with the identity and ignore the evanescent spinor indices $\hat{\alpha}\hat{\beta}\dots$

In this framework, which I adopt in the rest of the paper, the renormalization structure of the theory simplifies considerably. For example, infinitely many evanescent operators are automatically dropped. To be more explicit, observe that the matrices $\gamma^{\hat{a}}$ allow the construction of infinite sets of evanescent operators with the same dimensionality. Examples are the four-fermion operators

$$(\bar{\psi}\gamma_{\mu_1\dots\mu_n}\psi)^2, \quad n = 1, 2, \dots, \quad (2.5)$$

$\gamma_{\mu_1\dots\mu_n}$ being the completely antisymmetric product of n gamma matrices. The operators (2.5) are evanescent for $n > D$, but not zero. Using the deformed dimensional regularization with no evanescent spinor indices, the operators (2.5) are exactly zero when $n > D$. This deformed framework admits only a finite number of evanescent operators with a given dimensionality. They are constructed with the evanescent components $\hat{k}, \hat{A}, \hat{g}$ of momenta, gauge vectors and the metric tensor (if gravity is quantized).

An example of regularized Dirac action that does make use of the hatted Dirac matrices is given in Appendix B, formula (B.2).

2.2 Propagators and integrals

Momenta are split into physical and evanescent components, $p_a = (p_{\bar{a}}, p_{\hat{a}})$, where $a = (\bar{a}, \hat{a})$. The regularized propagators are chosen so that they tend to zero in the usual way when \bar{p}^2 tends to infinity and tend to zero in a power-like way also when \hat{p}^2 tends to infinity. The typical behavior

of a bosonic propagator considered in this paper is

$$\frac{1}{\bar{p}^2 + (m + \hat{p}^2/\Lambda)^2}, \quad (2.6)$$

where Λ is the cut-off of the higher-derivative deformation. The typical behavior of a fermionic propagator is, roughly speaking, the square root of (2.6). Alternative propagators are studied in the next sections.

Integrals are split as

$$\int \frac{d^d p}{(2\pi)^d} f(p) = \int \frac{d^D \bar{p}}{(2\pi)^D} \int \frac{d^{-\varepsilon} \hat{p}}{(2\pi)^{-\varepsilon}} f(\bar{p}, \hat{p}) \quad (2.7)$$

and can be calculated in the following way. First calculate the $-\varepsilon$ dimensional integral using the usual formulas of the dimensional regularization. Then calculate the remaining D dimensional integral, using again the formulas of the dimensional regularization. The final D integral is well-defined (in the sense of the dimensional regularization, with complex ε) even if D is strictly integer.

Sometimes it is convenient to do the D integral before the $-\varepsilon$ integral. This exchange can be rigorously done only after D is temporarily continued to complex values. The procedure is: analytically continue the integral to complex D (without touching the gamma matrices $\gamma^{\bar{a}}$), exchange the D integral with the $-\varepsilon$ integral, calculate the D integral, take the limit $D \rightarrow$ integer, calculate the $-\varepsilon$ integral. (The last two steps can be freely interchanged.) More details are given together with the examples.

The regularization is removed letting ε tend to zero at fixed Λ and then letting $\Lambda \rightarrow \infty$ (see the examples for further details). To show that the regularization is a good regularization, it is necessary to prove that a convergent integral gives back the initial integral when the regularization is removed. The key ingredient to prove this statement is the theorem (see for example [4]) stating that if $\hat{f}(\hat{P})$ is a regular function of $\hat{P} \equiv (\hat{p}_1, \dots, \hat{p}_L)$ tending to zero at least as $1/(\hat{P}^2)^\gamma$ for some $\gamma > 0$ when $\hat{P}^2 \equiv \hat{p}_1^2 + \dots + \hat{p}_L^2 \rightarrow \infty$, then

$$\lim_{\varepsilon \rightarrow 0} \int \frac{d^{-L\varepsilon} \hat{P}}{(2\pi)^{-L\varepsilon}} \hat{f}(\hat{P}) = \hat{f}(0), \quad \text{where } \frac{d^{-L\varepsilon} \hat{P}}{(2\pi)^{-L\varepsilon}} \equiv \prod_{i=1}^L \frac{d^{-\varepsilon} \hat{p}_i}{(2\pi)^{-\varepsilon}}. \quad (2.8)$$

In practice, when ε tends to zero the $-\varepsilon$ integral acts as a delta function projecting the function \hat{f} onto $\hat{P} = 0$.

Now, consider a completely convergent Feynman integral, namely the integral associated with a Feynman diagram G that is superficially convergent and has no subdivergences, with L loops, and assume that the propagators are of the form (2.6), or equivalent. Temporarily continue D to complex values, in the way explained above. The integral can be written as

$$\int \frac{d^{-L\varepsilon} \hat{P}}{(2\pi)^{-L\varepsilon}} \hat{f}(\hat{P}), \quad \text{with } \hat{f}(\hat{P}) = \int \frac{d^{LD} \bar{P}}{(2\pi)^{LD}} f(\bar{P}, \hat{P}), \quad (2.9)$$

with obvious notation. The limit $\widehat{P}^2 \rightarrow \infty$ of $\widehat{f}(\widehat{P})$ is studied letting any subset s_p of momenta $\widehat{p}_1, \dots, \widehat{p}_L$ become large. Each s_p is associated with a subgraph of G . Since, by assumption, every subgraph has a negative degree of divergence, Weinberg's theorem [19] ensures that there exists a $\gamma > 0$ such that the $\widehat{f}(\widehat{P})$ of (2.9) tends to zero at least as $1/(\widehat{P}^2)^\gamma$, when $\widehat{P}^2 \rightarrow \infty$. This is true also for the temporarily continued D (i.e. for values of D slightly different from its physical, integer, value). Then formula (2.8) can be used, so the limit $\varepsilon \rightarrow 0$ trivializes the $-\varepsilon$ integration and returns the initial convergent \overline{P} integral

$$\int \frac{d^{LD}\overline{P}}{(2\pi)^{LD}} f(\overline{P}, 0).$$

Now I prove that Feynman diagrams are indeed regularized. Consider a generic diagram. The integrand is a polynomial Q in momenta, times a certain number n of propagators (2.6), that I denote with $P(k, m)$. Let k denote loop momenta, while external momenta are not written explicitly. First study the \widehat{k} -integrations. For \widehat{k} large at fixed \overline{k} ,

$$f(\overline{k}) \equiv \int \frac{d^{-\varepsilon}\widehat{k}}{(2\pi)^{-\varepsilon}} [P(k, m)]^n Q(k) \sim \int \frac{d^{-\varepsilon}\widehat{k}}{(2\pi)^{-\varepsilon}} \sum_p \frac{1}{\widehat{k}^p} \sim \sum_p \frac{\Gamma(p/2 + \varepsilon/2)}{\Gamma(p/2)}, \quad (2.10)$$

where p are integers (coefficients multiplying the terms of the sum \sum_p are understood). The factor $\Gamma(p/2)$ is common type of factor in dimensional regularization and absolutely harmless, even when $p/2 \leq 0$, because it appears in the denominator. The integral is regularized because the gamma function appearing in the numerator has an argument shifted by $\varepsilon/2$. Multiple integrals produce shifts by $q\varepsilon/2$, with q positive integer.

A similar argument can be repeated for the behavior of the \overline{k} -integration, namely

$$\int \frac{d^D\overline{k}}{(2\pi)^D} f(\overline{k}),$$

after the \widehat{k} -integration. To study the large- \overline{k} behavior of $f(\overline{k})$, rescale \overline{k} by a factor λ in $f(\overline{k})$. At the same time, rescale \widehat{k} by a factor $\sqrt{\lambda}$ in the \widehat{k} -integral that defines $f(\overline{k})$, see (2.10). Then $f(\lambda\overline{k}) \sim \sum_p \lambda^{-p/2-\varepsilon/2}$ and therefore

$$\int \frac{d^D\overline{k}}{(2\pi)^D} f(\overline{k}) \sim \sum_p \frac{\Gamma(p/4 + \varepsilon/4 - D/2)}{\Gamma(p/4 + \varepsilon/4)}. \quad (2.11)$$

The gamma function in the numerator has an argument shifted by $\varepsilon/4$, so the integral is regularized. Multiple integrals produce shifts by $q\varepsilon/4$, with q positive integer.

The counterterms are local both in the physical and evanescent components of the external momenta. To prove this, it is sufficient to observe that if the propagators are of the form (2.6) or equivalent, after a sufficient number of differentiations with respect to the physical or evanescent

components of the external momenta, every integral gets a negative overall degree of divergence. Therefore, once the subdivergences have been inductively subtracted, (2.8) can be used and the result is finite. This implies that the divergent part is polynomial both in the physical and evanescent components of the external momenta.

Summarizing, the \widehat{P} integral, combined with the $\varepsilon \rightarrow 0$ limit, is just a sort of delta-function projecting onto $\widehat{P} = 0$, eventually collecting poles in ε . In practice, the evanescent sector of space-time dresses Feynman diagrams with appropriate (gauge-invariant) regularizing distributions. This emphasizes the mathematical meaning of the regularization used here and its elegance.

2.3 Dirac action

The Dirac action

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} (\overline{\mathcal{D}} + m) \psi \quad (2.12)$$

is trivialized by the regularization. Indeed, write

$$\text{Det} (\overline{\mathcal{D}} + m) = \exp \left(\int d^d x \int \frac{d^d p}{(2\pi)^d} \text{tr} \ln [\overline{\mathcal{D}} + m + i\overline{\mathcal{P}} + i\overline{\mathcal{A}}(x)] \right). \quad (2.13)$$

The integrand does not depend on \widehat{p} and the \widehat{p} -integral of 1 is zero in dimensional regularization. The point is that the lagrangian (2.12) is incomplete in d dimensions. It does not provide a propagator behaving like the “square root” of (2.6) or equivalent.

The Dirac fields can be efficiently regularized with an extra non-chiral evanescent higher-derivative term:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} \left(\overline{\mathcal{D}} + m - \frac{\widehat{D}^2}{\Lambda} \right) \psi, \quad \langle \psi(p) \bar{\psi}(-p) \rangle_{\text{free}} = \frac{-i\overline{\mathcal{P}} + m + \widehat{p}^2/\Lambda}{\overline{\mathcal{P}}^2 + (m + \widehat{p}^2/\Lambda)^2}. \quad (2.14)$$

The renormalization is studied first taking $\varepsilon \rightarrow 0$ at fixed Λ and then letting $\Lambda \rightarrow \infty$. The converse does not work, since the argument (2.13) shows that the naive limit $\Lambda \rightarrow \infty$ at $\varepsilon \neq 0$ produces zero, not the initial theory. For a better behavior of integrals, the sign of the evanescent piece is related to the sign of m . Here m is assumed to be positive.

The regularized Dirac action (2.14) does not contain hatted Dirac matrices, because the hatted kinetic term is higher-derivative. The hatted Dirac matrices, which appear nowhere in Feynman rules and diagrams, can be ignored and it is consistent to work with $2^{\lfloor D/2 \rfloor}$ -component spinors $\psi^{\bar{\alpha}}$.

2.4 Chern-Simons action

Consider the Chern-Simons action,

$$\mathcal{L}_{\text{ChS}} = -\frac{i}{2\alpha} \varepsilon_{\bar{a}\bar{b}\bar{c}} F^{\bar{a}\bar{b}} A^{\bar{c}}. \quad (2.15)$$

For the moment I restrict to Abelian gauge fields. Because of the epsilon tensor, the evanescent components $A^{\hat{a}}$ of the gauge field do not have a kinetic term. It is necessary to introduce a second cut-off Λ and suitable higher-derivative terms. The simplest possibility, reported in formula (B.4) of appendix B, is not very convenient for calculations. It is more convenient to regularize the Chern-Simons gauge field with an evanescent higher-derivative deformation,

$$\mathcal{L}_{\text{ChS}} = -\frac{i}{2\alpha}\varepsilon_{\bar{\mu}\bar{\nu}\bar{\rho}}F^{\bar{\mu}\bar{\nu}}A^{\bar{\rho}} + \frac{1}{\alpha\Lambda}F_{\bar{\mu}\bar{\nu}}^2 - \frac{1}{2\alpha\Lambda^3}F_{\hat{\mu}\hat{\nu}}\widehat{\partial^2}F_{\hat{\mu}\hat{\nu}}. \quad (2.16)$$

The gauge-fixing term can be deformed accordingly:

$$\mathcal{L}_{\text{gf}} = \lambda \left(\overline{\partial A} - \frac{1}{\Lambda^2}\widehat{\partial^2}\widehat{\partial A} \right)^2. \quad (2.17)$$

The consequent ghost action reads

$$\mathcal{L}_{\text{ghost}} = \overline{C} \left(-\overline{\partial^2} + \frac{\widehat{\partial^2}^2}{\Lambda^2} \right) C. \quad (2.18)$$

The propagators are

$$\begin{aligned} \langle A_{\mu}(p) A_{\nu}(-p) \rangle_{\text{free}} &= \frac{\alpha}{2(\bar{p}^2 + \hat{p}^4/\Lambda^2)} \left[\varepsilon_{\bar{\mu}\bar{\nu}\bar{\rho}}p_{\bar{\rho}} + \frac{\hat{p}^2}{\Lambda}\delta_{\bar{\mu}\bar{\nu}} + \Lambda\delta_{\hat{\mu}\hat{\nu}} + \frac{p_{\mu}p_{\nu}(1/(\alpha\lambda) - \hat{p}^2/\Lambda)}{\bar{p}^2 + \hat{p}^4/\Lambda^2} \right], \\ \langle C(p) \overline{C}(-p) \rangle_{\text{free}} &= \frac{1}{\bar{p}^2 + \hat{p}^4/\Lambda^2}. \end{aligned} \quad (2.19)$$

All denominators have the same structure as the Dirac propagator (2.14) at $m = 0$. It is not possible to have $SO(d)$ invariant denominators. Nevertheless, since denominators have a coherent structure (up to masses) the use Feynman parameters in calculations is efficient.

The generalization to non-Abelian gauge fields is straightforward,

$$\mathcal{L}_{\text{ChS}} = -\frac{i}{2\alpha}\varepsilon_{\bar{\mu}\bar{\nu}\bar{\rho}} \left(F_{\bar{\mu}\bar{\nu}}^i A_{\bar{\rho}}^i - \frac{1}{3}f_{ijk}A_{\bar{\mu}}^i A_{\bar{\nu}}^j A_{\bar{\rho}}^k \right) + \frac{1}{\alpha\Lambda}(F_{\bar{\mu}\bar{\nu}}^i)^2 - \frac{1}{2\alpha\Lambda^3}F_{\hat{\mu}\hat{\nu}}^i \widehat{D}^{2ij} F_{\hat{\mu}\hat{\nu}}^j,$$

where i, j, k are indices of the adjoint representation of the gauge group. The propagators are of course the same as before.

2.5 Gravity

For completeness, I comment on the regularization of gravity coupled with parity-violating matter. Here an additional problem appears: the Lorentz symmetry is a local symmetry, so the breaking of $SO(d)$ to $SO(D) \otimes SO(-\varepsilon)$ is more delicate. For example, it is not possible to gauge-fix the local Lorentz symmetry using the symmetric gauge. The quadratic part of the Einstein lagrangian

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{g}R \quad (2.20)$$

does not depend on the antisymmetric part of the quantum fluctuation $\tilde{\phi}_{\mu a} = e_{\mu a} - \delta_{\mu a}$, for which additional terms must be provided. An economical arrangement is worked out in ref. [10]. Start from the ordinary, $SO(d)$ invariant situation and gauge-fix diffeomorphisms and the Lorentz symmetry with the gauge-fixings $\partial_\mu(\sqrt{g}g^{\mu\nu})$ and $\mathcal{D}^\mu\omega_\mu^{ab}$, respectively. The gauge-fixing terms are

$$\mathcal{L}_{\text{gf}} = \frac{1}{2\lambda}(\partial_\mu\sqrt{g}g^{\mu\nu})^2 + \frac{1}{2\xi}\sqrt{g}(\mathcal{D}^\mu\omega_\mu^{ab})^2. \quad (2.21)$$

The first term is Lorentz invariant, the second is diffeomorphism invariant. This diagonalizes the ghost action. In particular, the propagator of the Lorentz ghosts C^{ab} is just $1/p^2$ times the identity.

Now, decompose the second term of (2.21) as

$$\frac{1}{2\xi}\sqrt{g} \left[(\tilde{\mathcal{D}}^\mu\omega_\mu^{\bar{a}\bar{b}})^2 + (\tilde{\mathcal{D}}^\mu\omega_\mu^{\hat{a}\hat{b}})^2 + 2(\tilde{\mathcal{D}}^\mu\omega_\mu^{\bar{a}\hat{b}})^2 \right] \quad (2.22)$$

and interpret this decomposition in the context of the theory with broken Lorentz symmetry. The covariant derivatives \mathcal{D}_μ have been replaced by new covariant derivatives $\tilde{\mathcal{D}}_\mu$, defined with the spin connections $\omega_\mu^{\bar{a}\bar{b}}$, $\omega_\mu^{\hat{a}\hat{b}}$ of the reduced Lorentz group $SO(D) \otimes SO(-\varepsilon)$. The first two terms of (2.22) can be viewed as the gauge-fixings of $SO(D) \otimes SO(-\varepsilon)$. The third term of (2.22) can be viewed as an addition to the lagrangian (2.20), that provides the missing propagator for the antisymmetric part of $\tilde{\phi}_{\mu a}$. This addition is legitimate, because it is a scalar density under diffeomorphisms, a scalar under $SO(D) \otimes SO(-\varepsilon)$ rotations, and a true regularization term, in the sense that it formally disappears in the physical limit $d \rightarrow D$. The rearrangement is

$$\mathcal{L}' = \frac{1}{2\kappa^2}\sqrt{g}R + \frac{1}{\xi}\sqrt{g}(\tilde{\mathcal{D}}^\mu\omega_\mu^{\bar{a}\bar{b}})^2, \quad \mathcal{L}'_{\text{gf}} = \frac{1}{2\lambda}(\partial_\mu\sqrt{g}g^{\mu\nu})^2 + \frac{1}{2\xi}\sqrt{g} \left[(\tilde{\mathcal{D}}^\mu\omega_\mu^{\bar{a}\bar{b}})^2 + (\tilde{\mathcal{D}}^\mu\omega_\mu^{\hat{a}\hat{b}})^2 \right]. \quad (2.23)$$

The quadratic part of the sum $\mathcal{L}' + \mathcal{L}'_{\text{gf}}$ is clearly equal to the quadratic part of $\mathcal{L} + \mathcal{L}_{\text{gf}}$. This ensures that the propagator of $\tilde{\phi}_{\mu a}$ is unmodified and in particular its denominators are $SO(d)$ invariant. The difference $\mathcal{L}' + \mathcal{L}'_{\text{gf}} - \mathcal{L} - \mathcal{L}_{\text{gf}}$ is made of cubic terms, due to the difference between the covariant derivatives \mathcal{D}_μ and $\tilde{\mathcal{D}}_\mu$. Another modification is in the ghost action, where the mixed Lorentz ghosts $C^{\bar{a}\hat{b}}$ are suppressed, as well as the companion antighosts. The propagator of the reduced Lorentz ghosts is still $1/p^2$ times the identity. Complete details (and a way to avoid certain IR nuisances due to the higher-derivative gauge fixing) can be found in ref. [10].

In summary, there exists a way to dimensionally regularize gravity coupled with the Standard Model or, in general, parity violating matter, in such a way that all propagators have $SO(d)$ invariant denominators. This property simplifies perturbative calculations.

2.6 Renormalization structure and stability of the deformed actions

Here I study the structure of the renormalized action in the deformed dimensional-regularization framework at arbitrarily high orders in perturbation theory.

According to renormalization theory, every allowed counterterm should appear in the renormalized action, multiplied by an independent coupling. The coupling has to run appropriately, to ensure RG invariance. Evanescent operators are an exception to this rule. This is important for the following reason.

The deformed actions (2.14), (2.16), (2.23) and the ones discussed in the next sections produce convenient propagators for efficient perturbative calculations. However, those actions are written with *ad hoc* deformations. For example, the coefficients of the two Chern-Simons deformations in (2.16) are related to each other in such a way to produce the nice propagator (2.19). If the deformations in (2.16) were multiplied by independent parameters, then the propagator would be much more complicated.

Now, the evanescent sector of the theory does not mix into the physical sector [4, 11]. This means that evanescent operators do not affect the S matrix and the physical correlation functions, but produce at most scheme changes. So, it is unnecessary to multiply the evanescent operators by new independent couplings: evanescent counterterms, such as

$$\frac{1}{\varepsilon} \widehat{\psi} \mathcal{D} \psi, \quad \frac{1}{\varepsilon} \frac{1}{\Lambda} F_{\mu\nu}^2, \quad \frac{1}{\varepsilon} \frac{1}{\Lambda} F_{\hat{\mu}\hat{\nu}}^2, \quad \frac{1}{\varepsilon} \sqrt{g} (\omega_{\mu}^{\hat{a}\hat{b}})^2,$$

etc., can be subtracted just as they come, at higher orders (starting from one loop). This procedure violates RG invariance in physical correlation functions only by contributions that vanish in the physical limit $\varepsilon \rightarrow 0$ and therefore have no physical significance.

Concluding, the complete renormalized action \mathcal{L}_R has a non-evanescent sector $\mathcal{L}_{\text{non-ev}}$ and an evanescent sector \mathcal{L}_{ev} . The non-evanescent sector has the usual structure: it contains renormalization constants for every field and coupling, and every allowed non-evanescent term is multiplied by independent parameters. The structure of the evanescent sector, instead, is completely free. Symbolically,

$$\mathcal{L}_R = \mathcal{L}_{\text{non-ev}}[Z_{\phi}^{1/2} \phi, \lambda Z_{\lambda} \mu^{p\varepsilon}] + \mathcal{L}_{\text{ev}}[\phi, \varepsilon]. \quad (2.24)$$

In particular, the structures of the evanescent sectors in (2.14), (2.16) and (2.23) are *tree-level* structures and do not need to be preserved at higher orders. It is possible to carry out every calculation with the propagators produced by the tree-level structures (2.14), (2.16) and (2.23) and subtract the evanescent counterterms, at higher orders, just as they come.

3 Sample calculation: vacuum polarization

In this and the next sections I illustrate the calculation of Feynman diagrams with the deformed technique. I start from fermions coupled with external gauge fields in four dimensions.

Consider the lagrangian (2.14). There are two vertices, with one or two photon legs,

$-ie\mu^{\varepsilon/2}\gamma_{\mu} + \frac{e\mu^{\varepsilon/2}}{\Lambda}(2\hat{p} + \hat{k})_{\mu}$

$-2e^2\mu^{\varepsilon}\hat{\delta}_{\mu\nu}/\Lambda$

(3.1)

The one-loop vacuum polarization reads

$$VP = e^2\mu^{\varepsilon} \int \frac{d^D\bar{p}}{(2\pi)^D} \frac{2^{\varepsilon}\pi^{\varepsilon/2}}{\Gamma(-\varepsilon/2)} \int_0^{\infty} t^{-1-\varepsilon/2} dt \frac{\text{tr}[\gamma_{\bar{a}}(-i\bar{p} + \tilde{t}/\Lambda)\gamma_{\bar{b}}(-i\bar{p} + i\bar{k} + \tilde{t}/\Lambda)]}{(\bar{p}^2 + \tilde{t}^2/\Lambda^2)((\bar{p} - k)^2 + \tilde{t}^2/\Lambda^2)}, \quad (3.2)$$

where $\tilde{t} = t + m\Lambda$ and $t = \hat{p}^2$. The external indices and momenta have been projected onto the physical spacetime (e.g. $\hat{k} = 0$), so the evanescent contributions to the vertices can be ignored in the calculation.

In the intermediate steps of a calculation, it is convenient to continue D to complex values, and later replace it with its integer value. The general recipe is as follows. First work out the numerators using the Dirac-algebra conventions of section 2, and keep D equal to its physical, therefore integer, value. The integrand is a tensor depending on internal and external momenta. Using several projectors, constructed with $\delta_{\bar{b}}^{\bar{a}}$, $\delta_{\bar{b}}^{\hat{a}}$, the epsilon tensor and the external momenta, decompose the integral into the sum of a certain number of scalar integrals, multiplied by suitable projectors. The scalar integrals can be calculated separately. In these integrals, continue D to complex values, using the conventions of the dimensional regularization (the continuation is straightforward, at this point). Calculate the D -integral and the t integral (the order of these integrations is not crucial after the D continuation) and then let D tend back to its physical value. The properties of the deformed regularization ensure that after the t integral this limit is smooth. The intermediate continuation to complex D s does not touch the physical Dirac algebra and so avoids the inconsistencies mentioned in section 2.

In the example (3.2), working out the Dirac algebra in $D = 4$ we obtain $VP = Ak_{\bar{a}}k_{\bar{b}} + Bk^2\delta_{\bar{a}\bar{b}}$, where A and B are certain scalar integrals. It is immediate to prove gauge invariance, namely $A + B = 0$, and it remains to calculate

$$(DB + A)k^2 = \frac{2^{[D/2]}e^2 2^{\varepsilon}\pi^{\varepsilon/2}\mu^{\varepsilon}}{\Gamma(-\varepsilon/2)} \int \frac{d^D\bar{p}}{(2\pi)^D} \int_0^{\infty} t^{-1-\varepsilon/2} dt \frac{(D-2)\bar{p} \cdot (\bar{p} - k) + D\tilde{t}^2/\Lambda^2}{(\bar{p}^2 + \tilde{t}^2/\Lambda^2)((\bar{p} - k)^2 + \tilde{t}^2/\Lambda^2)}. \quad (3.3)$$

Now I illustrate some ways to calculate (3.3). It is tempting to do the t integral immediately, using the formula (A.11) of the appendix, but this procedure is not efficient. The result is for

$D = 4$ and $m = 0$,

$$3Bk^2 = \frac{2^{1+\varepsilon} \pi^{1+\varepsilon/2} e^2 \mu^\varepsilon \Lambda^{-\varepsilon/2}}{\Gamma(-\varepsilon/2) \sin(\pi\varepsilon/4)} \int \frac{d^4 \bar{p}}{(2\pi)^4} \frac{(3a - b + k^2)ba^{-\varepsilon/4} - (3b - a + k^2)ab^{-\varepsilon/4}}{ab(b - a)}, \quad (3.4)$$

where $a = \bar{p}^2$ and $b = (\bar{p} - k)^2$. The \bar{p} integration in (3.4) is hard to do, although it is always possible to work out the divergent part of this expression taking two derivatives with respect to the external momentum k and then setting k to zero.

Instead of doing the t integral immediately, it is better to temporarily continue to complex D as explained above, use Feynman parameters, and then integrate over \bar{p} and t in any preferred order. The D integration in (3.3) gives

$$(D - 1)Bk^2 = -\frac{e^2 \mu^\varepsilon \Lambda^{-\varepsilon/2} (D - 1) \Gamma(2 - D/2)}{2^{D - [D/2] - 1 - \varepsilon} \pi^{D/2 - \varepsilon/2} \Gamma(-\varepsilon/2)} \int_0^1 dx k_x^2 \int_0^\infty t^{-1 - \varepsilon/2} dt \left[k_x^2 + (t + m)^2 \right]^{D/2 - 2}, \quad (3.5)$$

after a rescaling of t . It is convenient to take the $D \rightarrow 4$ limit before integrating over t and x . The spurious divergence proportional to $\Gamma(2 - D/2)$ is killed by the t integral, as promised. There does not exist a domain of (complex) values for ε where the t integral is convergent, as it stands. It is necessary to split it into a finite sum of integrals that separately admit convergence domains. This is done in the appendix. The expansion in powers of ε can be studied with the help of formula (A.5):

$$B \frac{\pi^2}{e^2} = -\frac{1}{3} \left(\frac{1}{\varepsilon} - \frac{1}{2} \ln \frac{\Lambda}{4\pi\mu} \right) + \frac{1}{6} \gamma_E + \frac{1}{2} \int_0^1 dx x(1-x) \ln \left[x(1-x)k^2/\mu^2 + m^2/\mu^2 \right].$$

After the identification $1/\varepsilon \sim \ln \Lambda/\mu$, this expression agrees with the known one [20], up to a change of scheme.

4 Axial anomalies

Now I illustrate the calculation of anomalies with the deformed technique, using the regularized lagrangian (2.14). The axial transformation $\delta_5 \psi = i\alpha \gamma_5 \psi$, $\delta_5 \bar{\psi} = i\alpha \bar{\psi} \gamma_5$ is associated with the current $J_5^{\bar{a}} = \bar{\psi} \gamma_5 \gamma^{\bar{a}} \psi$. Using the field equations of (2.14) at $m = 0$, which are $\bar{D}\psi = \widehat{D}^2 \psi / \Lambda$, the divergence of the axial current equals $\partial_{\bar{a}} J_5^{\bar{a}} = 2\bar{\psi} \gamma_5 \widehat{D}^2 \psi / \Lambda$. The axial anomaly is

$$\mathcal{A} = \langle \partial_{\bar{a}} J_5^{\bar{a}} \rangle = \frac{2}{\Lambda} \langle \bar{\psi} \gamma_5 \widehat{D}^2 \psi \rangle = -\frac{2}{\Lambda} \text{Tr} \left[\gamma_5 \widehat{D}^2 \frac{1}{\bar{D} - \widehat{D}^2 / \Lambda} \right].$$

The evanescent part $A_{\hat{\mu}}$ of the gauge vector can be set to zero, since here it appears only as an external leg. This amounts to replace \widehat{D}^2 with minus the squared evanescent momentum

$\hat{p}^2 = t$. To calculate the trace Tr it is convenient to choose a basis of plane waves and use $e^{-ipx}\partial_\mu e^{ipx} = \partial_\mu + ip_\mu$, obtaining

$$\mathcal{A} = \frac{2^\varepsilon \pi^{\varepsilon/2}}{\Lambda \Gamma(-\varepsilon/2)} \int \frac{d^D \bar{p}}{(2\pi)^D} \int_0^\infty t^{-\varepsilon/2} dt \sum_{n=0}^\infty \text{tr} \left[\gamma_5 \frac{1}{i\bar{p} + \bar{\not{D}} + t/\Lambda} \left(\frac{-ie\mu^{\varepsilon/2}}{i\bar{p} + \bar{\not{D}} + t/\Lambda} \not{A} \right)^n \right]. \quad (4.1)$$

The denominator has been expanded in powers of the gauge field. It is understood that the power $(AB)^n$, with A and B non-commuting operators, is a symbolic notation to denote the product $ABABAB\dots$ (n times). Only the terms with $n = 2, 3, 4$ can give non-vanishing contributions. In the limit $\varepsilon \rightarrow 0$ the contributions with $n > 4$ are killed by the factor $1/\Gamma(-\varepsilon/2)$, since the t - \bar{p} integral is convergent in that case. If the gauge fields are Abelian, only $n = 2$ gives a non-vanishing contribution.

The trace of (4.1) is calculated strictly in four dimensions, according to the prescriptions of the deformed regularization technique. For $n = 2$ we have immediately

$$\mathcal{A} = \frac{2^{2+\varepsilon} e^{2\pi\varepsilon/2} \mu^\varepsilon \Lambda^{-\varepsilon/2}}{\Gamma(-\varepsilon/2)} \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} k_1^{\bar{a}} A_1^{\bar{b}} k^{\bar{c}} A_2^{\bar{d}} \int \frac{d^D \bar{p}}{(2\pi)^D} \int_0^\infty \frac{s^{-\varepsilon/4} ds}{((\bar{p} + k_1)^2 + s)(\bar{p}^2 + s)((\bar{p} - k_2)^2 + s)}.$$

Using Feynman parameters, it is convenient to integrate first over \bar{p} , then over s . The \bar{p} integral is already convergent, so here there is no need to keep D different from 4 in the intermediate steps. The s can be done using (A.11). After these two integrations the limit $\varepsilon \rightarrow 0$ gives immediately the known result,

$$\mathcal{A} = \langle \partial_{\bar{a}} J_5^{\bar{a}} \rangle = -\frac{e^2}{4\pi^2} \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} k_1^{\bar{a}} A_1^{\bar{b}} k^{\bar{c}} A_2^{\bar{d}} = \frac{e^2}{16\pi^2} \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} F^{\bar{a}\bar{b}} F^{\bar{c}\bar{d}}.$$

Summarizing, the calculation of anomalies with the deformed technique is not more complicated than with the usual technique. It is simpler at the level of algebraic manipulations of numerators, because the Dirac algebra stays in the physical spacetime. On the other hand, the deformed calculation involves a splitting of integrations. The regularization is due to the integration over a sort of (squared) mass s .

5 Chern-Simons theories in flat space

I consider Abelian Chern-Simons $U(1)$ gauge theory coupled with two-component fermions in three dimensions,

$$\mathcal{L} = -\frac{i}{2\alpha} \varepsilon_{\mu\nu\rho} F^{\mu\nu} A^\rho + \bar{\psi}(\bar{D} + m)\psi. \quad (5.1)$$

This example is instructive, because the known dimensional-regularization techniques do not apply. The regularized gauge-fixed lagrangian is the sum of (2.16) plus (2.17) plus (2.18) plus (2.14).

Vacuum polarization. The vacuum polarization is made of two contributions: one can be derived directly from formula (3.5) setting $D = 3$; the second contribution comes from the trace of the product of three gamma matrices in (3.2).

The first contribution is of the form $Ak_{\bar{a}}k_{\bar{b}} + Bk^2\delta_{\bar{a}\bar{b}}$. Multiplying by $k_{\bar{b}}$ it is immediate to prove that $A + B = 0$. The trace gives, from (3.5)

$$2Bk^2 = -\frac{e^2 \mu^\varepsilon \Lambda^{-\varepsilon/2}}{2^{-\varepsilon} \pi^{1-\varepsilon/2} \Gamma(-\varepsilon/2)} \int_0^1 dx k_x^2 \int_0^\infty t^{-1-\varepsilon/2} dt \left[k_x^2 + (t+m)^2 \right]^{-1/2}. \quad (5.2)$$

The t integral can be evaluated with the technique explained in the appendix, formula (A.6), and gives a certain hypergeometric function. After taking the $\varepsilon \rightarrow 0$ limit, the integration over x is immediate.

The second contribution is, after the \bar{p} integration,

$$-\frac{2^\varepsilon \pi^{\varepsilon/2} e^2 \mu^\varepsilon \Lambda^{-\varepsilon/2}}{4\pi \Gamma(-\varepsilon/2)} \varepsilon_{\bar{a}\bar{b}\bar{c}} k^{\bar{c}} \int_0^1 dx \int_0^\infty t^{-1-\varepsilon/2} (t+m) dt \left[k_x^2 + (t+m)^2 \right]^{-1/2}$$

and can be worked out in the same way as (5.2).

The final result is [21, 22]

$$VP|_{D=3} = -\frac{e^2(k^2\delta_{\bar{a}\bar{b}} - k_{\bar{a}}k_{\bar{b}})}{8\pi k} \left[2\frac{m}{k} + \left(1 - \frac{4m^2}{k^2} \right) \arctan \frac{k}{2m} \right] + \frac{e^2}{4\pi} \varepsilon_{\bar{a}\bar{b}\bar{c}} k^{\bar{c}} \left(1 - 2\frac{m}{k} \arctan \frac{k}{2m} \right). \quad (5.3)$$

The last term of (5.3) contains a local contribution, which survives in the massless limit and is known as parity anomaly [22]. Its sign depends on the sign of the mass, which here was taken to be positive. On the other hand, this local term is trivial in perturbation theory, because it can be reabsorbed with a local counterterm, proportional to the Chern-Simons action of the gauge field. For non-perturbative aspects related to this issue, especially in non-Abelian gauge theories, the reader is referred to the literature [22, 23].

One-loop fermion self-energy. The electron self-energy is an interesting diagram because both the gauge-field and the fermion propagators participate. The diagram constructed with the second vertex of (3.1) does not contribute, because it is a massless tadpole. The other diagram can be computed as follows. First, use Feynman parameters to have one denominator. Secondly, integrate over \bar{p} . This is done analytically continuing in D and then setting $D = 3$. No spurious pole in $D - 3$ appears. The third step is the integral over the evanescent components $t = \bar{p}^2$ of the loop momentum. The t integral has the structure of (A.10) with $g = 3$ and can be evaluated using formula (A.9). Forth, take the limit $\varepsilon \rightarrow 0$ and fifth, integrate the result over x .

The result is

$$SE = \frac{i\alpha k}{24\pi} \left[2 + 3\frac{m^2}{k^2} - \frac{3}{k} (ik + m) \left(1 + \frac{m^2}{k^2} \right) \arctan \frac{k}{m} \right] +$$

$$+\frac{1}{8\pi\lambda}\left[1+\frac{ik}{2k^3}(ik+m)^2\arctan\frac{k}{m}-\frac{imk}{2k^2}\right]$$

and can be checked with an ordinary cut-off method (which however produces also a linear divergence).

6 Large N expansion

In this section I study the deformed regularization of certain three-dimensional fermion and scalar models in the large N expansion.

3D four-fermion models in the large N expansion. The four-fermion theory in three dimensions is described by the lagrangian

$$\mathcal{L} = \bar{\psi}(\not{\partial} + m)\psi + \frac{1}{2}M\sigma^2 + \lambda\sigma\bar{\psi}\psi,$$

where λ and M are parameters and σ is an auxiliary field. This theory, despite its non-renormalizability by power-counting, can be defined in the large N expansion [6], resumming the fermion bubbles (one-loop σ self-energy) into an effective σ propagator. The ordinary dimensional continuation does not regularize completely (even if the spinors are four-component), because the resummation of fermion bubbles gives an effective σ propagator of the form

$$\frac{1}{\mu^\varepsilon(k^2)^{(1-\varepsilon)/2} + M} \quad (6.1)$$

that produces $\Gamma(0)$ s in subleading Feynman diagrams. Two ways to circumvent this difficulty have been used in ref.s [7, 8, 24]: a non-local improvement of the dimensional technique, valid only with four-component spinors, and a higher-derivative regularization. Here I consider a more general framework. Recall that the purpose of this paper is to work out an efficient regularization technique that is also universal, and in particular admits a straightforward extension to curved space and non-Abelian gauge theories. In curved space it is extremely heavy to deal with non-local regularizations, containing evanescent powers of derivative operators. Moreover, when the spinors are two-component the usual dimensional continuation of the Dirac algebra is inconsistent.

It is possible to avoid all this choosing the regularization

$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + m - \frac{\widehat{\partial^2}}{\Lambda}\right)\psi + \frac{1}{2}\sigma\left(M - \frac{\overline{\partial^2}}{\Lambda} + \frac{\widehat{\partial^2}^2}{\Lambda^3}\right)\sigma + \lambda\sigma\bar{\psi}\psi. \quad (6.2)$$

To keep the notation to a minimum, I do not make an explicit distinction between bare and renormalized quantities. The lagrangian (6.2) can be read either as the bare lagrangian or as the renormalized lagrangian (up to evanescent counterterms: see (2.24)). In the latter case, it

is understood that appropriate renormalization constants multiply fields and parameters, and a factor $\mu^{\varepsilon/2}$ multiplies the vertex $\lambda\sigma\bar{\psi}\psi$. Recall that $Z_\lambda = 1$.

The first interesting quantity to consider in this model is the σ self-energy. It is necessary to calculate this diagram for generic values of ε, Λ and the external momentum (\bar{k}, \hat{k}) . I report here only the result, giving details of the calculation in the simpler scalar model studied below:

$$B_f(k, m) = \frac{\lambda^2 N \mu^\varepsilon}{4\pi\bar{k}} 2^{3\varepsilon/2} \pi^{\varepsilon/2} \Lambda^{-\varepsilon/2} \Gamma(-2 + \varepsilon/2) u^{-\varepsilon/4} \left[\Upsilon \sin\left(\frac{\varepsilon}{2}\psi\right) + \Psi \cos\left(\frac{\varepsilon}{2}\psi\right) \right], \quad (6.3)$$

where

$$\begin{aligned} \psi &= \arctan \frac{\bar{k}}{2\tilde{m}}, & u &= \bar{k}^2 + 4\tilde{m}^2, & \tilde{m} &= m + \frac{\hat{k}^2}{4\Lambda}, \\ \Upsilon &= -(2 - \varepsilon)\bar{k}^2 - 8\tilde{m}^2, & \Psi &= 2\varepsilon\bar{k}\tilde{m}. \end{aligned} \quad (6.4)$$

The spinor is assumed to be two-component. Observe that the evanescent components $k_{\hat{\mu}}$ of the momentum appear only inside \tilde{m} , so \hat{k}^2/Λ plays the role of a mass. The $\varepsilon \rightarrow 0, \Lambda \rightarrow \infty$ limit is convergent,

$$B_f(k, m) \rightarrow -\frac{\lambda^2 N}{4\pi\bar{k}} \left[(\bar{k}^2 + 4m^2) \arctan \frac{\bar{k}}{2m} \right] + \frac{\lambda^2 N}{2\pi} m. \quad (6.5)$$

In the usual cut-off approach [21] a linear divergence is generated, which is cancelled by means of a fine-tuning. The result (6.5) agrees with the known one up to a scheme change, because the last term can be reabsorbed with a redefinition of M . Observe that any \hat{k} dependence has disappeared in the limit.

The effective σ propagator $\Sigma(k, m)$ is obtained resumming the geometric series of the one-loop σ self-energies:

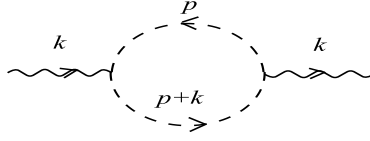
$$\Sigma(k, m) = \frac{1}{M - B_f(k, m) + \bar{k}^2/\Lambda + \hat{k}^4/\Lambda^3}. \quad (6.6)$$

Now I prove that the term $\bar{k}^2/\Lambda + \hat{k}^4/\Lambda^3$ corrects the UV behavior of $B_f(k, m)$ and regularizes the $\Gamma[0]$ s in subleading diagrams, avoiding the problems of the naive propagator (6.1). See also [7, 8, 24] on this issue. The proof is an adaptation of the argument that leads to (2.10) and (2.11).

Consider a generic Feynman diagram. The integrand is a polynomial Q in momenta, due to the vertices and propagators other than (6.6), times a certain number n of propagators (6.6). Consider the \hat{k} -integrations. For \hat{k} large at fixed \bar{k} , $B_f(k, m) \sim \hat{k}^{2-\varepsilon}/\Lambda$, so if $\Re \varepsilon > -2$

$$f(\bar{k}) \equiv \int \frac{d^{-\varepsilon}\hat{k}}{(2\pi)^{-\varepsilon}} [\Sigma(k, m)]^n Q(k) \sim \int \frac{d^{-\varepsilon}\hat{k}}{(2\pi)^{-\varepsilon}} \sum_{p,q} \frac{\hat{k}^{-\varepsilon q}}{\hat{k}^p} \sim \sum_{p,q} \frac{\Gamma(p/2 + (q+1)\varepsilon/2)}{\Gamma(p/2 + q\varepsilon/2)}, \quad (6.7)$$

where q and p are integers, $q \geq 0$. Consequently, no $\Gamma[0]$ is generated in the numerator provided that $\Re \varepsilon > -2$. The requirement $\Re \varepsilon > -2$ is compatible with the usual conditions for



the existence of convergence domains for the integrals. Indeed, these conditions have the form $\delta_{UV} < \Re \varepsilon < \delta_{IR}$, for some integers $\delta_{IR} \geq 0$ and $\delta_{UV} < \delta_{IR}$ (see Appendix A for more details). The inequality $\delta_{IR} \geq 0$ is due to the fact that the integrands are regular for $\hat{k} \rightarrow 0$. The subsets $\Re \varepsilon > -2$ and $\delta_{UV} < \Re \varepsilon < \delta_{IR}$ have always a non-empty intersection.

Now consider the \bar{k} -integration

$$\int \frac{d^D \bar{k}}{(2\pi)^D} f(\bar{k})$$

on the assumption that the \hat{k} -integration has already been done. To study the large- \bar{k} behavior of $f(\bar{k})$, rescale \bar{k} by a factor λ in $f(\bar{k})$. At the same time, rescale \hat{k} by a factor $\sqrt{\lambda}$ in the \hat{k} -integral that defines $f(\bar{k})$, see (6.7). Observing that $B_f(k, m)$ goes into $\lambda^{1-\varepsilon/2} B_f(k, m/\lambda)$ and repeating the argument used for (6.7), we find, for $\Re \varepsilon > -2$,

$$f(\lambda \bar{k}) \sim \sum_{p,q} \lambda^{-p-(q+1)\varepsilon/2}, \quad \text{therefore} \quad \int \frac{d^D \bar{k}}{(2\pi)^D} f(\bar{k}) \sim \sum_{p,q} \frac{\Gamma(p/2 + (q+1)\varepsilon/4 - D/2)}{\Gamma(p/2 + (q+1)\varepsilon/4)},$$

where q and p are integers, $q \geq 0$. Again, no $\Gamma[0]$ is generated.

3D scalar conformal field theories in the large N expansion. The three-dimensional scalar model

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} \sum_{i=1}^N \left[(\partial_\mu \varphi_i)^2 + i\lambda \sigma \varphi_i^2 \right], \quad (6.8)$$

where σ is a dynamical field (see [8]), describes a conformal field theory, the UV (Wilson-Fischer) fixed point of the $O(N)$ sigma model. Although this theory can be regularized also in a standard way, it is instructive to describe how to proceed in the deformed framework. For massive scalars the regularized lagrangian reads

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} \sum_{i=1}^N \left[\left(\widehat{\partial}_\mu \varphi_i \right)^2 + \left(\frac{\widehat{\partial}^2}{\Lambda} \varphi_i - m \varphi_i \right)^2 + i\lambda \sigma \varphi_i^2 \right] + \frac{1}{2\Lambda} \sigma^2.$$

First I describe the calculation of the scalar bubble (one-loop σ self-energy) for $m = 0$, then I add the mass. The integral over the physical components \bar{p} of the loop momentum can be done easily, and gives

$$-\frac{i\lambda^2 N \mu^\varepsilon}{16\pi \bar{k}} \ln \frac{2t + \widehat{k}^2 - i\bar{k}\Lambda + 2\widehat{p} \cdot \widehat{k}}{2t + \widehat{k}^2 + i\bar{k}\Lambda + 2\widehat{p} \cdot \widehat{k}}. \quad (6.9)$$

The next task is the integral over the angle between \widehat{p} and \widehat{k} . This is done expanding (6.9) in powers of $\widehat{p} \cdot \widehat{k}$. By symmetric integration, it is easy to show that the angular integration of a power $(\widehat{p} \cdot \widehat{k})^n$, multiplied by a function of \widehat{p}^2 and \widehat{k}^2 , is equivalent to the substitution

$$(\widehat{p} \cdot \widehat{k})^n \rightarrow \frac{\Gamma(n/2 + 1/2)\Gamma(-\varepsilon/2)}{\sqrt{\pi}\Gamma(n/2 - \varepsilon/2)} (\widehat{p}^2 \widehat{k}^2)^{n/2} \text{ if } n \text{ is even,}$$

while it gives 0 if n is odd. At this point the integral over \widehat{p}^2 is done term-by-term in the expansion. Finally, the series in n is resummed. The result is

$$B_s(k) = -\frac{\lambda^2 N \mu^\varepsilon}{8\pi \bar{k}} 2^{2\varepsilon} \pi^{\varepsilon/2} \Gamma(\varepsilon/2) (\widehat{k}^4 + 4\bar{k}^2 \Lambda^2)^{-\varepsilon/4} \sin\left(\frac{\varepsilon}{2} \arctan \frac{2\bar{k}\Lambda}{\widehat{k}^2}\right). \quad (6.10)$$

The calculation can be extended to massive scalars. The intermediate result (6.9) is modified adding $2m\Lambda$ both to the numerator and denominator of the fraction inside the logarithm. Finally, with a simple replacement, the generalization of (6.10) is

$$B_s(k, m) = -\frac{N\lambda^2 \mu^\varepsilon 2^{3\varepsilon/2} \pi^{\varepsilon/2} \Lambda^{-\varepsilon/2}}{8\pi \bar{k}} \Gamma(\varepsilon/2) u^{-\varepsilon/4} \sin\left(\frac{\varepsilon}{2} \psi\right), \quad (6.11)$$

where u and ψ are defined as in (6.4).

The σ self-energies can be resummed into the effective σ -propagator, which can be used to calculate the $\mathcal{O}(1/N)$ subleading corrections to anomalous dimensions and other quantities [8]. Despite the complicated structure of (6.11), the high-energy behavior of B_s is simpler, and sufficient to calculate the divergent parts of Feynman diagrams. It is also not difficult to calculate the beta functions of the RG flows of [8] that interpolate in a classically conformal way between fixed points of the type (6.8).

With the same procedure and some more algebra it is possible to derive the result (6.3).

Summarizing, it is possible to evaluate complete amplitudes such as $B_f(k, m)$ and $B_s(k, m)$, that contain two arbitrary cut-offs and depend on a mass and generic physical and evanescent momenta. This is another indication that the regularization defined here can be used efficiently.

7 Evanescent higher-derivative deformation in flat space

Several fields admit an $SO(d)$ invariant dimensional regularization. Other fields do not. If a theory contains fields of both types it might be convenient to deform also the regularization of the $SO(d)$ invariant fields, so that all propagators have denominators with the same structure (in the massless limit). This makes the use of Feynman parameters more efficient.

Scalar fields. For scalar fields, the lagrangian

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} \left[(\overline{D}_\mu \varphi)^2 + \left(\frac{\widehat{D}^2}{\Lambda} \varphi - m\varphi \right)^2 \right] \quad (7.1)$$

produces denominators with the same structure as for fermions also at $m \neq 0$.

Gauge vectors. In (Abelian and non-Abelian) Yang-Mills theory, the lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}, \quad \mathcal{L}_{\text{vector}} = \frac{1}{4\alpha} \left(F_{\bar{\mu}\bar{\nu}}^2 - 2F_{\bar{\mu}\hat{\nu}} \frac{\widehat{D}^2}{\Lambda^2} F_{\bar{\mu}\hat{\nu}} + F_{\hat{\mu}\hat{\nu}} \frac{\widehat{D}^2}{\Lambda^4} F_{\hat{\mu}\hat{\nu}} \right), \quad (7.2)$$

$$\mathcal{L}_{\text{gf}} = \lambda \left(\overline{\partial A} - \frac{1}{\Lambda^2} \widehat{\partial}^2 \widehat{\partial A} \right)^2, \quad \mathcal{L}_{\text{ghost}} = \overline{C} \left(-\overline{\partial D} + \widehat{\partial}^2 \frac{\widehat{\partial D}}{\Lambda^2} \right) C, \quad (7.3)$$

where D_μ denotes the covariant derivative, while $D^2 = D_\mu D^\mu$, $\partial D = \partial_\mu D^\mu$ etc., produces the propagators

$$\begin{aligned} \langle A_\mu(p) A_\nu(-p) \rangle_{\text{free}} &= \frac{\alpha}{\bar{p}^2 + \hat{p}^4/\Lambda^2} \left[\bar{\delta}_{\mu\nu} + \frac{\Lambda^2}{\bar{p}^2} \widehat{\delta}_{\mu\nu} + \left(\frac{1}{2\lambda\alpha} - 1 \right) \frac{p_\mu p_\nu}{\bar{p}^2 + \hat{p}^4/\Lambda^2} \right], \\ \langle C(p) \overline{C}(-p) \rangle_{\text{free}} &= \frac{1}{\bar{p}^2 + \hat{p}^4/\Lambda^2}. \end{aligned} \quad (7.4)$$

Again, the structure of denominators simplifies the use of Feynman parameters in the \bar{p} integration. However, there appears a denominator \hat{p}^2 , which can be responsible of IR divergences. To avoid this nuisance it is safer to further deform with an evanescent mass. This is achieved replacing \widehat{D}^2 and $\widehat{\partial}^2$ in (7.2-7.3) with $\widehat{D}^2 - \widehat{m}^2$ and $\widehat{\partial}^2 - \widehat{m}^2$, respectively.

There exists a simple recipe to construct a manifestly gauge invariant higher-derivative deformation of an $SO(d)$ invariant lagrangian in flat space. Start from the $SO(d)$ invariant lagrangian and perform the replacements

$$A_\mu \rightarrow A_\mu, \quad \partial_\mu \rightarrow \partial_\mu, \quad \delta^{\mu\nu} \rightarrow \delta^{\bar{\mu}\bar{\nu}} + \frac{\widehat{m}^2 - \widehat{D}^2}{\Lambda^2} \delta^{\hat{\mu}\hat{\nu}}, \quad \delta_\nu^\mu \rightarrow \delta_\nu^\mu. \quad (7.5)$$

Observe that lower and upper indices have to be kept distinct during the replacement. The rules (7.5) imply also

$$\begin{aligned} F_{\mu\nu} &\rightarrow F_{\mu\nu}, \quad D_\mu = \partial_\mu + iA_\mu \rightarrow D_\mu, \quad \delta_{\mu\nu} \rightarrow \delta_{\bar{\mu}\bar{\nu}} + \frac{\Lambda^2}{\widehat{m}^2 - \widehat{D}^2} \delta_{\hat{\mu}\hat{\nu}}, \\ D^\mu &\rightarrow D^{\bar{\mu}} + \frac{\widehat{m}^2 - \widehat{D}^2}{\Lambda^2} D^{\hat{\mu}}, \quad D^2 \rightarrow \overline{D}^2 + \frac{(\widehat{m}^2 - \widehat{D}^2)}{\Lambda^2} \widehat{D}^2. \end{aligned} \quad (7.6)$$

The transformation rules of A^μ , $F^{\mu\nu}$ etc., follow consequently. Gauge invariance is manifest. The replacement is local, namely the deformation of a local lagrangian is a local lagrangian. Indeed, only the replacement of $\delta_{\mu\nu}$ contains a non-local term, but this affects the propagator, not the lagrangian.

Since $\delta^{\mu\nu}$ is deformed into a derivative operator, it is necessary to specify the position of the tensor $\delta^{\mu\nu}$ before the replacement. The position of $\delta^{\mu\nu}$ is determined observing that the

quadratic part of the lagrangian is correctly deformed only if $\delta^{\mu\nu}$ is placed in between the fields (the deformation would have no effect otherwise). For example, if Φ_μ is a vector field,

$$\Phi^2 \equiv \Phi_\mu \delta^{\mu\nu} \Phi_\nu \rightarrow \overline{\Phi}^2 + \widehat{\Phi} \frac{\widehat{m}^2 - \widehat{D}^2}{\Lambda^2} \widehat{\Phi}.$$

After the replacement there is no need to distinguish between upper and lower indices.

At $\widehat{m} = 0$ the replacement (7.5) produces immediately the quadratic part of (7.2-7.3) and the propagators (7.4). In the gauge-fixing and ghost terms (7.3) the covariant box \widehat{D}^2 can be replaced with the simple box $\widehat{\partial}^2$. As a consistency check, observe that when Λ^2 is (formally) set equal to $-\widehat{D}^2$ or $-\widehat{\partial}^2$ in (7.2-7.3) and (7.4) the standard $SO(d)$ invariant lagrangian as well as the $SO(d)$ invariant propagators are obtained. Moreover, the replacement (7.5) becomes the identity in this formal limit.

At $\widehat{m} \neq 0$ the evanescent mass \widehat{m} takes care of the IR nuisances mentioned above, due to the denominator $1/\widehat{p}^2$ in (7.4). Observe that at $\widehat{m} \neq 0$ the calculations done so far do not become conceptually more difficult than at $\widehat{m} = 0$.

As far as fermions are concerned, the action (2.14) is produced with the additional rule

$$\gamma^\mu \rightarrow \gamma^{\bar{\mu}} - \frac{1}{\Lambda} D^{\bar{\mu}} \quad (7.7)$$

so that

$$\mathcal{D} = \gamma^\mu D_\mu \rightarrow \left(\gamma^{\bar{\mu}} - \frac{1}{\Lambda} D^{\bar{\mu}} \right) D_\mu = \overline{\mathcal{D}} - \frac{\widehat{D}^2}{\Lambda}.$$

With Chern-Simons gauge fields it is possible to proceed as follows. Inspired by the replacement (7.7), write

$$\varepsilon^{\mu\rho\nu} \partial_\rho = -\frac{i}{2} \text{tr}[\gamma^\mu \overline{\mathcal{D}} \gamma^\nu] \rightarrow -\frac{i}{2} \text{tr} \left[\left(\gamma^{\bar{\mu}} - \frac{1}{\Lambda} \partial^{\bar{\mu}} \right) \left(\overline{\mathcal{D}} - \frac{\widehat{\partial}^2}{\Lambda} \right) \left(\gamma^{\bar{\nu}} - \frac{1}{\Lambda} \partial^{\bar{\nu}} \right) \right]. \quad (7.8)$$

This replacement is however incomplete, because it is not gauge invariant, namely it is not annihilated by the contractions with ∂_ν and ∂_μ . It is possible to complete it changing a sign in (7.8) and adding a term:

$$\varepsilon^{\mu\rho\nu} \partial_\rho \rightarrow -\frac{i}{2} \text{tr} \left[\left(\gamma^{\bar{\mu}} - \frac{1}{\Lambda} \partial^{\bar{\mu}} \right) \left(\overline{\mathcal{D}} + \frac{\widehat{\partial}^2}{\Lambda} \right) \left(\gamma^{\bar{\nu}} - \frac{1}{\Lambda} \partial^{\bar{\nu}} \right) \right] - \frac{i}{\Lambda} \delta^{\bar{\mu}\bar{\nu}} \left(\overline{\partial}^2 - \frac{\widehat{\partial}^2}{\Lambda^2} \right). \quad (7.9)$$

Expanding and reorganizing, the expression (2.16) is immediately recovered.

8 Absence of power-like divergences

The second cut-off Λ appears explicitly in the regularized lagrangian, so (gauge invariant) power-like divergences can in principle be generated. This does happen if regularized lagrangians such

as (B.1) and (B.4) are used. However, when the higher-derivative deformation is also evanescent, as in (2.14), (2.16), (7.1) and (7.2-7.3), then the power-like divergences are set to zero by default. In this section I prove this statement, confirmed by the results of the calculations of the previous sections, and derive other properties of the counterterms.

Consider first the regularized Dirac action (2.14). If the evanescent components $A_{\hat{\mu}}$ of the gauge vector and the evanescent spacetime coordinates $x^{\hat{\mu}}$ are rescaled as follows

$$A_{\hat{\mu}} = \tilde{A}_{\hat{\mu}}\sqrt{\Lambda}, \quad x^{\hat{\mu}} = \tilde{x}^{\hat{\mu}}/\sqrt{\Lambda}, \quad (8.1)$$

then the Λ dependence reduces just to a factor $\Lambda^{\varepsilon/2}$ in front of the lagrangian. The same holds for the regularized Chern-Simons lagrangian (2.16-2.18) and the deformed Yang-Mills lagrangian (7.2-7.3). At $\hat{m} \neq 0$ it is necessary to rescale also \hat{m} to $\hat{\tilde{m}}\sqrt{\Lambda}$. Correspondingly, in the Feynman diagrams a rescaling of the integrated momenta, $\hat{p} \rightarrow \hat{\tilde{p}}\sqrt{\Lambda}$, factorizes a certain power of $\Lambda^{\varepsilon/2}$, so the divergent parts have the form

$$\frac{\Lambda^{q\varepsilon/2}}{\varepsilon^n} \quad (8.2)$$

times a function of external momenta (with components $\tilde{k}, \hat{\tilde{k}}, \hat{\tilde{m}}$, etc., where q and n are integers). Then, replacing the tilded-hatted objects with the original hatted ones, only negative powers of Λ are produced, but no positive power. On the other hand, it is evident that the expansion of (8.2) in powers of ε produces only logarithms of Λ . This proves that no power-like divergences are generated.

After the rescaling (8.1), the evanescent components of fields, coordinates and momenta have non-canonical dimensionalities. For example, in $D = 3$ the Chern-Simons gauge field $A_{\mu} = (A_{\bar{\mu}}, A_{\hat{\mu}})$ has, before the rescaling, dimensionality 1. After the rescaling (8.1) the physical components $A_{\bar{\mu}}$ keep their dimensionality 1, but the tilded evanescent components $\tilde{A}_{\hat{\mu}}$ acquire dimensionality $1/2$. Similarly, $p_{\bar{\mu}}$ has dimensionality 1, but $\tilde{p}_{\hat{\mu}}$ has dimensionality $1/2$. It is easy to check that the dimensionalities of the physical and tilded-evanescent components of fields and momenta are always strictly positive. By the theorem of locality of the counterterms, the counterterms are polynomial in these quantities, and therefore polynomial also in the untilded quantities. In particular, the poles in ε cannot multiply arbitrary negative powers of Λ .

Summarizing, at Λ fixed the poles in ε can multiply logarithms of Λ , a finite number of negative powers of Λ , and no positive power of Λ .

The argument just outlined does not work if different fields are regularized in an incoherent way. For example, consider two-component fermions coupled with Yang-Mills theory in three dimensions. The fermions cannot be regularized in a $SO(d)$ invariant way, but can be regularized as in (2.14). If the gauge fields are regularized in a $SO(d)$ invariant way, the rescaling (8.1) does not reduce the Λ dependence of the complete lagrangian to just a factor $\Lambda^{\varepsilon/2}$ in front of it. When the theory contains some fields that do not admit an $SO(d)$ invariant regularization, it is

convenient to deform also the fields that do admit one, using the replacement (7.5-7.6). Then the argument based on the rescaling (8.1) applies as described above.

The evanescent deformation (7.5-7.6) always exists in flat space and it is immediate to generalize it to curved space, as long as gravity is not quantized. Instead, I have no simple generalization of the evanescent deformation to quantum gravity. The difficulty is to find a suitable higher-derivative deformation of the Einstein lagrangian. Combining the regularization of this paper with the one of ref. [10] it is possible to dimensionally regularize also odd-dimensional parity-violating theories coupled with quantum gravity in a manifestly gauge-invariant way, for example the models of [16, 17]. However, mixed ($SO(d)$ invariant and $SO(d)$ non-invariant) denominators appear in Feynman diagrams and it might be necessary to eliminate power-like divergences manually.

9 Conclusions

The results of this paper suggest that there always exists an appropriate deformation of the dimensional-regularization technique that regularizes consistently and in a manifestly gauge-invariant way also the models to which the naive dimensional technique does not apply. In the deformed framework, the spacetime dimension is still analytically continued to complex values, but manifest Lorentz invariance is restricted to the physical subsector of spacetime. Then it is possible to use the ordinary (uncontinued) Dirac algebra, gaining a certain simplification of the renormalization structure. The regularization is completed with an evanescent higher-derivative deformation, which makes use of an extra cut-off. At higher orders, evanescent counterterms can be subtracted just as they come, without spoiling the convenient tree-level structure of the regularized lagrangian.

The virtues of the deformed regularization are that it is universal, local, manifestly gauge invariant (up to the known anomalies) and Lorentz invariant in the physical sector of spacetime. In flat space it kills power-like divergences by default. Infinitely many evanescent operators are automatically dropped. I have paid special attention to the efficiency of practical computations.

The existence of a universal regularization technique with the properties just mentioned is useful to quickly prove the absence of gauge anomalies in the models where the ordinary dimensional technique is inconsistent, in particular when composite operators of high dimensionalities are considered or the theory contains non-renormalizable interactions. Alternative proofs of the absence of gauge anomalies are provided by the algebraic-renormalization approach [13], which does not need an explicit regularization framework and makes use of an involved cohomological classification. Another popular framework uses the exact-renormalization-group techniques [14], but heavy cut-off dependencies are generated and Slavnov-Taylor identities are imposed step-by-step.

The technique of this paper has applications to the study of finiteness and renormalizability beyond power-counting [15, 17], but can be convenient also in four-dimensional renormalizable theories, to reduce the number of evanescent counterterms.

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A Appendix: useful integrals

In this appendix I collect some useful integrals. Let $d = D - \varepsilon$, where D denotes the physical spacetime dimension and d denotes the continued spacetime dimension.

In dimensional regularization, an integral $I(\varepsilon)$ is said to admit a convergence domain if there exists an open set \mathcal{D}_I of the complex plane, such that $I(\varepsilon)$ is convergent for $\varepsilon \in \mathcal{D}_I$. If an integral $I(\varepsilon)$ admits a convergence domain \mathcal{D}_I , then it is first evaluated for $\varepsilon \in \mathcal{D}_I$ and later extended to the complex plane (up to eventual poles) by analytical continuation. An integral that admits no convergence domain can be calculated if it can be split into a finite sum of integrals that separately admit convergence domains. For the types of integrals that appear in perturbative quantum field theory, these operations are consistent and unambiguous.

The situation where the integral does not admit a convergence domain is frequent in dimensional regularization. For example, the integral of 1 does not admit a convergence domain, but can be calculated writing it as the sum of two integrals that separately admit convergence domains:

$$\int \frac{d^d p}{(2\pi)^d} 1 = \int \frac{d^d p}{(2\pi)^d} \frac{m^2}{p^2 + m^2} + \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{p^2 + m^2} = \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} m^d + \frac{d \Gamma(-d/2)}{2 (4\pi)^{d/2}} m^d = 0.$$

The first integral is convergent for $0 < \Re d < 2$, while the second integral is convergent for $-2 < \Re d < 0$.

An integral frequently met in the paper is

$$I_1 = \int_0^\infty t^{-1-\varepsilon/2} dt \ln(k_x^2 + (t+m)^2). \quad (\text{A.1})$$

There is no complex domain of ε that makes this integral convergent. To calculate (A.1), first rewrite it as $2a_1 + a_2$, where

$$a_1 = \int_0^\infty t^{-1-\varepsilon/2} dt \ln(t+m), \quad a_2 = \int_0^\infty t^{-1-\varepsilon/2} dt \ln \frac{k_x^2 + (t+m)^2}{(t+m)^2}.$$

The integral a_1 still does not admit a convergence domain. Multiplying and dividing the integrand by $t+m$, a_1 can be split into the sum of two integrals that separately admit convergence domains:

$$a_1 = \int_0^\infty t^{-\varepsilon/2} dt \frac{\ln(t+m)}{t+m} + m \int_0^\infty t^{-1-\varepsilon/2} dt \frac{\ln(t+m)}{t+m} = \frac{2\pi m^{-\varepsilon/2}}{\varepsilon \sin(\pi\varepsilon/2)}. \quad (\text{A.2})$$

The integral a_2 admits a convergence domain. It can be safely calculated expanding the logarithm in powers series as follows:

$$a_2 = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (k_x^2)^n \int_0^\infty t^{-1-\varepsilon/2} dt (t+m)^{-2n}, \quad (\text{A.3})$$

then integrating term-by-term and resumming. The result is

$$a_2 = \frac{4\pi m^{-\varepsilon/2}}{\varepsilon \sin(\pi\varepsilon/2)} \left[-1 + (1 + k^2 x(1-x)/m^2)^{-\varepsilon/4} \cos\left(\frac{\varepsilon}{2} \arctan \frac{k}{m} \sqrt{x(1-x)}\right) \right]. \quad (\text{A.4})$$

The total is

$$I_1 = 2a_1 + a_2 = \frac{4\pi}{\varepsilon \sin(\pi\varepsilon/2)} \left(k^2 x(1-x) + m^2 \right)^{-\varepsilon/4} \cos\left(\frac{\varepsilon}{2} \arctan \frac{k}{m} \sqrt{x(1-x)}\right).$$

It is now safe to expand in powers of ε , obtaining

$$I_1 = \frac{8}{\varepsilon^2} - \frac{2}{\varepsilon} \ln(k^2 x(1-x) + m^2) + \mathcal{O}(1). \quad (\text{A.5})$$

To evaluate formula (5.2) it is necessary to calculate the integral

$$I_2 = \int_0^\infty t^{-1-\varepsilon/2} dt \left[k_x^2 + (t+m)^2 \right]^{-1/2}. \quad (\text{A.6})$$

This can be done with the same procedure as for (A.1). First expand the integrand in powers of $k_x^2/(t+m)^2$. Then integrate each term of the expansion over t . Finally, resum the power series.

The result is

$$I_2 = -\frac{\pi m^{-1-\varepsilon/2}}{\sin(\pi\varepsilon/2)} {}_2F_1 \left[\frac{1}{2} + \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4}; 1; -\frac{k_x^2}{m^2} \right]. \quad (\text{A.7})$$

When ε tends to zero, the behavior of I_2 is

$$I_2 = -\frac{2}{\varepsilon} \left(k_x^2 + m^2 \right)^{-1/2} + \mathcal{O}(1), \quad (\text{A.8})$$

which, inserted into (5.2), gives (5.3), after a straightforward integration over x .

The result (A.7) can be generalized immediately to give

$$\begin{aligned} F_g &= \int_0^\infty t^{-1-\varepsilon/2} dt \left[k_x^2 + (t+m)^2 \right]^{-g/2} = \\ &= m^{-g-\varepsilon/2} \frac{\Gamma(g+\varepsilon/2)\Gamma(-\varepsilon/2)}{\Gamma(g)} {}_2F_1 \left[\frac{g}{2} + \frac{\varepsilon}{4}, \frac{g+1}{2} + \frac{\varepsilon}{4}; \frac{g+1}{2}; -\frac{k_x^2}{m^2} \right]. \end{aligned} \quad (\text{A.9})$$

It is also straightforward to calculate integrals of the form

$$\int_0^\infty t^{-1-\varepsilon/2} dt P(t) \left[k_x^2 + (t+m)^2 \right]^{-g/2}, \quad (\text{A.10})$$

$P(t)$ being an arbitrary polynomial in t . The result is a sum of terms of the form (A.9), with ε shifted by integer numbers.

Finally, calculating the $-\varepsilon$ integration before the D integration, such as in (3.4), it is frequent to meet integrals such as

$$I[p, n] \equiv \int_0^\infty ds \frac{s^p}{\prod_{i=1}^n (s + a_i)} = \frac{\pi(-1)^n}{\sin p\pi} \sum_{i=1}^n \frac{a_i^p}{\prod_{j \neq i} (a_i - a_j)}. \quad (\text{A.11})$$

I have checked this formula in various cases (up to $n = 4$ included with different a_s , for special values of the a_s with higher n). It satisfies the recursion relation

$$I[p, n] = I[p + 1, n + 1] + a_{n+1} I[p, n + 1].$$

B Appendix: non-evanescent higher-derivative deformations

For completeness, in this appendix I collect some alternative higher-derivative deformations, which are equally consistent, but less efficient in practical computations.

An obvious alternative to the regularized Dirac action (2.14) is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} \left(\overline{\mathcal{D}} - \frac{D^a D_a}{\Lambda} \right) \psi. \quad (\text{B.1})$$

Calculations with the propagator induced by this action are however more involved. A less obvious alternative is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} \left(\overline{\mathcal{D}} + i\widehat{\mathcal{D}} \right) \psi. \quad (\text{B.2})$$

The evanescent correction is imaginary, and Hermiticity (or reflection positivity, in the Euclidean framework) is retrieved in the limit $\varepsilon \rightarrow 0$. Here no additional cut-off Λ is needed. The lagrangian (B.2) is useful when fermions are massless, because the propagator

$$\frac{1}{i\overline{\mathcal{D}} - \widehat{\mathcal{D}}} = \frac{-i\overline{\mathcal{D}} - \widehat{\mathcal{D}}}{p^2} \quad (\text{B.3})$$

has an $SO(d)$ invariant denominator. The violation of chiral invariance is due to $[\gamma_5, \gamma^{\hat{a}}] = 0$. The computation of the axial anomaly with this action resembles the usual computation in

dimensional regularization [4]. However, when the fermions are massive the propagator is not a simple modification of (B.3), but

$$\frac{(-i\widehat{p} - \widehat{p} + m)(p^2 + m^2 + 2m\widehat{p})}{(p^2 + m^2)^2 - 4\widehat{p}^2 m^2},$$

which makes computations rather hard.

The simplest higher-derivative deformation of the Chern-Simons lagrangian (2.15) is

$$\mathcal{L}_{\text{ChS}} = -\frac{i}{2\alpha}\varepsilon_{\bar{a}\bar{b}\bar{c}}F^{\bar{a}\bar{b}}A^{\bar{c}} + \frac{1}{4\Lambda}F_{\mu\nu}^2. \quad (\text{B.4})$$

The gauge-fixing term $(\partial_\mu A^\mu)^2/(2\lambda)$ produces, in the Feynman gauge $\lambda = \Lambda$, the propagator

$$\langle A_\mu(p) A_\nu(-p) \rangle_{\text{free}} = \frac{\Lambda}{p^2}\delta_{\mu\nu} + \frac{\Lambda(p_{\bar{\mu}}p_{\bar{\nu}} - \bar{p}^2\delta_{\bar{\mu}\bar{\nu}})}{p^2(\bar{p}^2 + \alpha^2 p^4/(4\Lambda^2))} + \frac{\alpha}{2}\frac{\varepsilon_{\bar{\mu}\bar{\nu}\bar{\rho}}p_{\bar{\rho}}}{\bar{p}^2 + \alpha^2 p^4/(4\Lambda^2)}. \quad (\text{B.5})$$

where p^4 stands for $(p^2)^2$. It is not easy to use this propagator for explicit computations, because of the structure of its denominators. Away from the Feynman gauge the propagator is even more involved.

References

- [1] C.G. Bollini and J.J Giambiagi, Dimensional renormalization: the number of dimensions as a regularizing parameter, *Nuovo Cim.* B12 (1972) 20.
- [2] G. 't Hooft and M.J.G. Veltman, Regularization and renormalization of gauge theories, *Nucl. Phys.* B44 (1972) 189.
- [3] P. Breitenlohner and D. Maison, Dimensional renormalization and the action principle, *Commun. Math. Phys.* 52 (1977) 11.
- [4] J.C. Collins, *Renormalization*, Cambridge University Press, Cambridge, 1984.
- [5] S.L. Adler and W.A. Bardeen, Absence of higher order corrections in the anomalous axial vector divergence, *Phys. Rev.* 182 (1969) 1517.
- [6] G. Parisi, The theory of nonrenormalizable interactions. I – The large N expansion, *Nucl. Phys.* B 100 (1975) 368.
- [7] D. Anselmi, Large- N expansion, conformal field theory and renormalization-group flows in three dimensions, *JHEP* 0006 (2000) 042 and arXiv:hep-th/0005261.
- [8] D. Anselmi, “Integrability” of RG flows and duality in three dimensions in the $1/N$ expansion, *Nucl. Phys.* B 58 (2003) 440 and arXiv:hep-th/0210123.

- [9] L.D. Fadeev and A.A. Slavnov, *Gauge fields, Introduction to quantum theory*, The Benjamin/Cummings Publishing Company, 1980, § 4.4.
- [10] D. Anselmi, A note on the dimensional regularization of the Standard Model coupled with quantum gravity, *Phys. Lett. B* 596 (2004) 90 and arXiv:hep-th/0404032.
- [11] M.J. Dugan and B. Grinstein, On the vanishing of evanescent operators, *Phys. Lett. B* 256 (1991) 239.
- [12] S. Herrlich and U. Nierste, Evanescent operators, scheme dependencies and double insertions, *Nucl. Phys. B* 455 (1995) 39 and arXiv:hep-ph/9412375;
K. Adel and Y.P. Yao, Renormalization of four-quark operators, effective theory and the role of evanescent operators, *Phys. Rev. D* 53 (1996) 374 and hep-ph/9408341.
- [13] For a review see for example O. Piguet and S.P. Sorella, *Algebraic renormalization: perturbative renormalization, symmetries and anomalies*, Lecture Notes in Physics, Springer-Verlag 1995.
- [14] M. Bonini, M. D’Attanasio and G. Marchesini, BRS symmetry for Yang-Mills theory with exact renormalization group, *Nucl. Phys. B* 437 (1995) 163 and arXiv:hep-th/9410138.
- [15] D. Anselmi, Consistent irrelevant deformations of interacting conformal field theories, *JHEP* 0310 (2003) 045 and arXiv:hep-th/0309251.
- [16] D. Anselmi, Renormalization of quantum gravity coupled with matter in three dimensions, *Nucl. Phys. B* 687 (2004) 143 and arXiv:hep-th/0309249.
- [17] D. Anselmi, Finiteness of quantum gravity coupled with matter in three spacetime dimensions, *Nucl. Phys. B* 687 (2004) 124 and arXiv:hep-th/0309250.
- [18] D. Kreimer, The role of γ_5 in dimensional regularization, arXiv:hep-th/9401354.
- [19] S. Weinberg, High-energy behavior in quantum field theory, *Phys. Rev.* 118 (1960) 838.
- [20] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Westview Press, 1995.
- [21] B. Rosenstein, B. Warr and S.H. Park, Dynamical symmetry breaking in four fermi interaction models, *Phys. Rept.* 205 (1991) 59.
- [22] A.N. Redlich, Gauge noninvariance and parity nonconservation of three-dimensional fermions, *Phys. Rev. Lett.* 52 (1984) 18.

- [23] E. Witten, An $SU(2)$ anomaly, *Phys. Lett.* B117 (1982) 324.
- [24] D. Anselmi, Inequalities for trace anomalies, length of the RG flow, distance between the fixed points and irreversibility, *Class. Quantum Grav.* 21 (2004) 29 and arXiv:hep-th/0210124.