

INEQUALITIES FOR TRACE ANOMALIES, LENGTH OF THE RG FLOW, DISTANCE BETWEEN THE FIXED POINTS AND IRREVERSIBILITY

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Abstract

I discuss several issues about the irreversibility of the RG flow and the trace anomalies c , a and a' . First I argue that in quantum field theory: *i*) the scheme-invariant area $\Delta a'$ of the graph of the effective beta function between the fixed points defines the length of the RG flow; *ii*) the minimum of $\Delta a'$ in the space of flows connecting the same UV and IR fixed points defines the (oriented) distance between the fixed points; *iii*) in even dimensions, the distance between the fixed points is equal to $\Delta a = a_{UV} - a_{IR}$. In even dimensions, these statements imply the inequalities $0 \leq \Delta a \leq \Delta a'$ and therefore the irreversibility of the RG flow. Another consequence is the inequality $a \leq c$ for free scalars and fermions (but not vectors), which can be checked explicitly. Secondly, I elaborate a more general axiomatic set-up where irreversibility is defined as the statement that there exist no pairs of non-trivial flows connecting interchanged UV and IR fixed points. The axioms, based on the notions of length of the flow, oriented distance between the fixed points and certain “oriented-triangle inequalities”, imply the irreversibility of the RG flow without a global a function. I conjecture that the RG flow is irreversible also in odd dimensions (without a global a function). In support of this, I check the axioms of irreversibility in a class of $d = 3$ theories where the RG flow is integrable at each order of the large N expansion.

1 Introduction

At the critical points of the RG flow, the trace anomaly in external gravity in even dimensions $d = 2n$ contains three types of terms, constructed with the curvature tensors and their covariant derivatives: *i*) terms \mathcal{W}_i , $i = 0, 1, \dots, I$, such that $\sqrt{g}\mathcal{W}_i$ are conformally invariant; *ii*) the Euler density

$$G_d = (-1)^n \varepsilon_{\mu_1 \nu_1 \dots \mu_n \nu_n} \varepsilon^{\alpha_1 \beta_1 \dots \alpha_n \beta_n} \prod_{i=1}^n R_{\alpha_i \beta_i}^{\mu_i \nu_i};$$

iii) covariant total derivatives \mathcal{D}_j , $j = 0, 1, \dots, J$, having the form $\nabla_\alpha J^\alpha$, J^α denoting a covariant current.

The coefficients multiplying these terms are called ‘‘central charges’’ and can be conveniently normalized by the formula

$$\Theta_{d=2n}^* = \frac{n!}{(4\pi)^n (d+1)!} \left[\frac{c_d (d-2)}{4(d-3)} \mathcal{W}_0 - \frac{2^{1-n}}{d} a_d G_d + \frac{a'_d}{(d-1)} \square^{n-1} R + \sum_{i=1}^I c_d^i \mathcal{W}_i + \sum_{j=1}^J a_d^j \mathcal{D}_j \right]. \quad (1.1)$$

Here \mathcal{W}_0 is the unique term of the form $W_{\mu\nu\rho\sigma} \square^{n-2} W^{\mu\nu\rho\sigma} + \dots$ such that $\sqrt{g}\mathcal{W}_0$ is conformally invariant, where $W^{\mu\nu\rho\sigma}$ is the Weyl tensor and the dots denote cubic terms in the curvature tensors. Its coefficient c_d is normalized so that for free fields (n_s real scalars, n_f Dirac fermions and, in even dimensions, n_v $(n-1)$ -forms) it reads

$$c_d = n_s + 2^{[n]-1} (d-1) n_f + \frac{d!}{2[(n-1)!]^2} n_v. \quad (1.2)$$

The \mathcal{D}_j s, with $j > 0$, are at least quadratic in the curvature tensors.

The three central charges c_d , a_d , a'_d are special among the others. They provide useful tools to investigate the properties of conformal field theories in even dimensions and the space of renormalization-group flows. In four dimensions, the trace anomaly in external gravity at criticality contains only these three quantities:

$$\Theta_* = \frac{1}{120} \frac{1}{(4\pi)^2} \left[c W^2 - \frac{a}{4} G + \frac{2}{3} a' \square R \right].$$

The central charges satisfy positivity conditions ($c \geq 0$, $a \geq 0$), encode the irreversibility of the RG flow ($\Delta a = a_{UV} - a_{IR} \geq 0$) and are sometimes computable exactly in the strongly coupled IR limits of UV-free theories [2]. In this paper I discuss several issues about the irreversibility of the RG flow in even and odd dimensions and its relation with the central charges and the trace anomaly in external gravity.

The formula (1.1) encodes a universal way to fix the relative normalizations of c , a and a' [1], based on the ‘‘pondered’’ Euler density of ref. [3] and the specialness of the $c = a$ theories

of ref. [4]. The existence of a canonical relative normalization for c , a and a' makes it natural to inquire whether the central charges are constrained by universal inequalities such as $c \geq a$, $c \geq a'$, $\Delta c \geq \Delta a'$, etc.

Consider for example free field theories. The values of c and a in four dimensions are

$$c = n_s + 6n_f + 12n_v, \quad a = \frac{1}{3}(n_s + 11n_f + 62n_v),$$

where n_s , n_f and n_v are the numbers of real scalars, Dirac fermions and vectors, respectively. Note that scalars and fermions have $c > a$, while vectors have $c < a$. Analogous inequalities hold in higher even dimensions. The equality $c = a$ holds in two dimensions. Is there a reason why scalars and fermions always have $c \geq a$? Why vectors (or differential forms, in higher dimensions) behave differently from scalars and fermions? The answers to these questions are unexpectedly related to the irreversibility of the RG flow.

The example just discussed is sufficient to exclude universal inequalities between c and a . For our purposes, it is more interesting to investigate possible inequalities between Δc , Δa and $\Delta a'$. In an RG flow, the quantities Δc , Δa and $\Delta a'$ have different properties: $\Delta a'$ is non-negative in unitary flows, Δa is non-negative in unitary renormalizable flows, while Δc can be either positive or negative [2]. Secondly, Δc and Δa are flow invariants of the second type [5], which means that they depend only on the critical limits of a flow and not on the particular flow connecting them. The reason is that the central charges c and a are unambiguously defined at the critical points by the embedding in external gravity. On the other hand, a' is ill-defined at criticality and depends on the renormalization scheme. Consequently, $\Delta a'$ does depend on the particular flow connecting the fixed points, although it is scheme-independent (the scheme dependence cancels out in the difference $a'_{\text{UV}} - a'_{\text{IR}}$). The flow non-invariance of $\Delta a'$ is crucial for the ideas proposed below. Finally, $\Delta a'$ has a nice geometrical interpretation: it is the scheme-invariant area of the graph of the (effective) beta function β_{eff} between the fixed points [7], where $\beta_{\text{eff}}(|x|\mu) = |x|^d \sqrt{|\Theta(x)\Theta(0)|}$, Θ being the trace of the stress tensor and d the space-time dimension. In classically conformal theories $\Delta a'$ is precisely the area of the beta function.

Some inequalities between Δc , Δa and $\Delta a'$ can be excluded immediately, using known results. In particular, the exact formulas for Δc and Δa in supersymmetric theories derived in refs. [2, 6] allow us to compare Δc and Δa in a variety of models and prove that no general inequality between Δc to Δa can hold. Combining the results of [2, 6] with those of [7] it is possible to exclude also inequalities between Δc and $\Delta a'$. The calculations of [2, 6], however, do not provide results for $\Delta a'$. The present knowledge does not allow us to exclude that a universal inequality might relate Δa to $\Delta a'$. This fact inspires some stimulating ideas.

It was recalled above that $\Delta a'$ is non-negative in unitary flows and does depend on the flow connecting the fixed points. This suggests that $\Delta a'$ measures the “length” of the flow. We have also said that Δa is non-negative in unitary renormalizable flows and does not depend on the particular flow connecting the fixed points. This suggests that Δa measures the “distance” between the fixed points. If this is correct, Δa should be the minimum of $\Delta a'$ in the space of

flows connecting the same fixed points. It follows, in particular, that unitary renormalizable flows satisfy the universal inequalities $0 \leq \Delta a \leq \Delta a'$ in even dimensions. This implies the irreversibility of the RG flow. The classically conformal theories and the flows with $\Delta c = \Delta a$ are the “geodesic flows”, because they have $\Delta a = \Delta a'$ [7, 4].

The understanding I offer in this paper starts from these observations. It includes and generalizes previous ideas about the irreversibility of the RG flow in classically conformal theories and in the flows with $\Delta a = \Delta c$, recently reviewed in ref. [1]. Among the other things, I explain why scalars and fermions have necessarily $c \geq a$ and why vectors can have $c < a$.

Several ideas apply also to odd-dimensional quantum field theory. The area $\Delta a'$ of the graph of the effective beta function between the fixed points can be taken as definition of length of the flow also in odd dimensions. Its minimum over the flows F connecting the same UV and IR fixed points defines the distance between them, and the distance is oriented. Moreover, it is still reasonable to expect that the flows with minimal length are classically conformal. However, in odd dimensions the minimum of $\Delta a'$ cannot be expressed as the difference Δa between the values of a central charge a unambiguously and globally defined at criticality [5]. However, it is possible to give a set of simple axioms that imply irreversibility also in odd dimensions.

The flows can be conventionally oriented from the UV to the IR. The distance between the fixed points is therefore “oriented” and can satisfy certain “oriented-triangle inequalities” that are more restrictive than the usual triangle inequalities. The notion of oriented distance and the oriented-triangle inequalities are the basic axioms of irreversibility in odd (and even) dimensions. First, they imply the irreversibility of the RG flow without a “height” (c or a) function. Irreversibility without an a function is defined as the property that, in the realm of unitary theories, there exist no pairs of non-trivial flows connecting interchanged UV and IR fixed points. Second, the oriented-triangle inequalities imply the existence of a local a function, in the smooth regions of the space of flows. Only in even dimensions there exists a global a function.

In support of irreversibility without a global a function in odd dimensions, I check the oriented-triangle inequalities explicitly in a class of three-dimensional flows introduced in ref. [8]. In these classically conformal models, the RG flow is exactly integrable at each order of the large N expansion [9].

The ideas of this paper might have applications also outside quantum field theory. Issues concerning the central charges c and a and the irreversibility of the RG flow have recently attracted an amount of interest in the realm of the so-called AdS/CFT correspondence [10], a conjecture that relates the strongly coupled large N limits (where N is typically a number of colors) of certain conformal field theories and RG flows to supergravity and string duals. In particular, the conformal theories considered in the AdS/CFT correspondence belong to the special class of theories that have $c = a$ [11], at least to the leading order in the large N expansion. These theories are mathematically elegant, relatively simple, and have various properties in common with two-dimensional conformal field theory [4]. In the context of the AdS/CFT conjecture it is possible to derive the irreversibility of the RG flow [12] for holographic

flows interpolating between fixed points belonging to the class $c = a$. This is the maybe best generalization of Zamolodchikov’s c theorem [13]. I hope that the notions of length of the RG flow, oriented distance between the fixed points and the oriented-triangle inequalities might inspire new investigations in the program of holographic renormalization (see [14] for an account of recent developments and references). In particular, it would be interesting to have definitions of length of the flow and distance between the endpoints of the flow in the context of string theory. The AdS/CFT conjecture is certainly the best tool to start a research in this direction.

The paper is organized as follows. In section 2 I collect the main no-go statements which exclude certain inequalities for trace anomalies. In section 3 I inspect the sum rules for a and a' to look for inequalities between Δa and $\Delta a'$. I show that the sum rules are compatible with the statements of this paper, but do not provide an easy way to prove them. In section 4 I collect some observations on irreversibility, which inspired the ideas of this paper. In section 5 I formulate the definitions of length of the RG flow and distance between the fixed points in even dimensions, relate the distance to Δa and use this relation to derive irreversibility in even dimensions. In section 6 I analyse the main topological and metric properties of the spaces of fixed points and flows. In section 7 I check some predictions in the case of free fields. I explain why free scalars and fermions have $a \leq c$ in arbitrary even dimensions, while free vectors can have $a > c$. I analyse also Gaussian non-unitary fields, whose central charges a and c do not obey any general inequality. In sections 8 and 9 I generalize the ideas to odd dimensions. In section 8 I formulate the irreversibility of the RG flow without an a function and show that a good axiomatic set-up for irreversibility is provided by the notion of oriented distance and the oriented-triangle inequalities. I prove that the oriented-triangle inequalities imply irreversibility without a global a function. In section 9 I test the oriented-triangle inequalities in a class of three-dimensional classically conformal flows. Section 10 contains the conclusions.

2 No-go statements

In this section I use known results to exclude some inequalities between the trace anomalies. I recall that there exist classes of flows with $\Delta c = \Delta a$ [4], with $\Delta c = \Delta a'$ [15] and with $\Delta a = \Delta a'$ [7]. Here I prove that no general inequalities of the form $\Delta c \geq f_1 \Delta a$, $\Delta c \geq f_2 \Delta a'$ hold, where $f_{1,2}$ are numerical factors. There survives only the possibility of a general inequality relating Δa and $\Delta a'$.

No $\Delta c - \Delta a$ inequality. In a class of supersymmetric theories in the conformal window the values of Δa and Δc have been computed exactly [2, 6]. For our considerations, it is sufficient to consider the examples treated explicitly in ref. [2], in particular an “electric” theory and its dual “magnetic” theory. Their IR conformal fixed points are related by Seiberg’s “electric-magnetic” duality [16].

The electric theory is N=1 supersymmetric QCD with N_c colors and N_f flavors. In the

conformal window $3N_c/2 \leq N_f \leq 3N_c$ the results are

$$\Delta a = \frac{5}{2} N_c N_f \left(1 - 3 \frac{N_c}{N_f}\right)^2 \left(2 + 3 \frac{N_c}{N_f}\right), \quad \Delta c = \frac{5}{2} N_c N_f \left(1 - 3 \frac{N_c}{N_f}\right) \left(4 - 3 \frac{N_c}{N_f} - 9 \frac{N_c^2}{N_f^2}\right),$$

$$\frac{\Delta c}{\Delta a} = \frac{4 - 3r - 9r^2}{(1 - 3r)(2 + 3r)},$$

where $r = N_c/N_f$. We see that in the conformal window $\Delta c/\Delta a$ ranges from $-\infty$ to $1/2$.

The magnetic theory is $N=1$ supersymmetric QCD with N_c colors, N_f quark flavors and a $N_f \times N_f$ meson superfield. The IR fixed point of the magnetic theory is related to the IR fixed point of the electric theory by means of the duality map $N_c^{\text{mag}} = N_f^{\text{el}} - N_c^{\text{el}}$ and $N_f^{\text{mag}} = N_f^{\text{el}}$ [16]. In particular, duality relates the IR values of c and a in the electric and magnetic models. Instead, the UV values of c and a (and therefore the differences Δc , Δa) are not related to each other. The magnetic theory provides an independent source of information about possible inequalities for Δc and Δa .

In the conformal window the results are

$$\Delta a = \frac{5}{2} \left(1 - 3 \frac{N_c}{N_f}\right)^2 (3N_c^2 - 10N_c N_f + 10N_f^2),$$

$$\Delta c = \frac{5}{2} \left(1 - 3 \frac{N_c}{N_f}\right) \left(8N_f^2 - 38N_c N_f + 33N_c^2 - 9 \frac{N_c^3}{N_f}\right),$$

$$\frac{\Delta c}{\Delta a} = \frac{10 - 10r + 3r^2}{(1 - 3r)(8 - 38r + 33r^2 - 9r^3)}. \quad (2.1)$$

Here $\Delta c/\Delta a$ ranges from $7/8$ to $+\infty$, which has no overlap with the range of $\Delta c/\Delta a$ in the electric theory. From these two examples, it is immediate to conclude that no general inequality can relate Δa and Δc .

No $\Delta c - \Delta a'$ inequality. In classically conformal theories, we have $\Delta a = \Delta a'$ to the fourth loop order in perturbation theory and an argument saying that the relation $\Delta a = \Delta a'$ is actually exact [7]. Here we just need to know that $\Delta a' \sim (1 - 3r)^2$ when r tends to $1/3$. I recall that in the limit $r \gtrsim 1/3$, the perturbative expansion can be used, since the IR fixed point is weakly coupled. Moreover, $\Delta a'$ is always non-negative, because of the sum rule (3.1) (see below). Now, in the electric theory, when r gets close to $1/3$ the ratio $\Delta c/\Delta a'$ takes arbitrarily large negative values. Instead, in the magnetic theory when r gets close to $1/3$ the ratio $\Delta c/\Delta a'$ takes arbitrarily large positive values. This proves that there exists no universal inequality relating Δc to $\Delta a'$.

After this analysis, we remain only with the possibility of an inequality for Δa and $\Delta a'$. The methods of refs. [2, 6] do not allow us to calculate $\Delta a'$ in the supersymmetric conformal windows and compare it to Δa .

3 Sum rules in even dimensions

In even dimension greater than two the embedding in external gravity allows us to derive sum rules for Δa and $\Delta a'$ [17]. It is instructive to inspect these sum rules to see whether we can prove a general inequality for Δa and $\Delta a'$.

In arbitrary even $d = 2n$ dimensions the sum rule for $\Delta a'$ is [3]

$$\Delta a' = \frac{\pi^n (d+1)}{n!} \int d^d x |x|^d \langle \Theta(x) \Theta(0) \rangle. \quad (3.1)$$

By definition, the contact term is excluded from the integral. The integral (3.1) is over all points of spacetime and depends, in general, on the parameters of the theory (ratios between masses, values of the coupling constants at a reference energy, etc.).

When the operator Θ is finite, which happens for example in four-dimensional theories containing no scalars (when scalar fields are present Θ mixes with an improvement term for the stress tensor and the treatment is more complicated [15]), it is possible to parametrize the correlation function $\langle \Theta(x) \Theta(0) \rangle$, using the Callan-Symanzik equation, in terms of a function G of the running coupling constants g at the scale $1/|x|$:

$$\langle \Theta(x) \Theta(0) \rangle = \frac{G(g(1/|x|))}{|x|^{2d}} \equiv \frac{\tilde{G}(g(\mu), \ln |x|\mu)}{|x|^{2d}}.$$

Then the integral (3.1) becomes an integral over the RG flow:

$$\Delta a' = \frac{d(d+1)}{(n!)^2} \pi^d \int_{-\infty}^{+\infty} dt \tilde{G}(g(\mu), t).$$

The behavior of \tilde{G} for large $|t|$ can be studied using the RG equations. The integral is ensured to be convergent, if the flow interpolates between well-defined UV and IR limits.

In the realm of unitary theories we have $\Delta a' \geq 0$, because reflection positivity ensures $\langle \Theta(x) \Theta(0) \rangle \geq 0$ for $|x| \neq 0$. The equality $\Delta a' = 0$ takes place only if the RG flow is trivial ($\Theta = 0$), i.e. the UV and IR fixed points coincide or belong to a family of continuously connected conformal field theories.

The formula for Δa is the sum of the integral (3.1) plus integrals of correlation functions containing more Θ -insertions. For example, in four dimensions two equivalent sum rules for Δa read [17]

$$\begin{aligned} \Delta a &= -5 \frac{\pi^2}{2} \int d^4 x |x|^4 \Gamma'_{x0} - 5 \frac{\pi^2}{2} \int d^4 x d^4 y x^2 y^2 \Gamma'_{xy0} \\ &= -\frac{5\pi^2}{2} \int d^4 x |x|^4 \Gamma'_{x0} - \frac{5\pi^2}{2} \int d^4 x d^4 y d^4 z (x \cdot y) (x \cdot z) \Gamma'_{xyz0}, \end{aligned} \quad (3.2)$$

while in six dimensions the “minimal” sum rule reads [5]

$$\begin{aligned} \Delta a &= \frac{7\pi^3}{36} \left(-6 \int x^6 \Gamma'_{x,0} d^6 x + \int [8(x \cdot y)^3 - 9x^4 y^2] \Gamma'_{x,y,0} d^6 x d^6 y \right. \\ &\quad \left. + 6 \int x^2 (y \cdot z)^2 \Gamma'_{x,y,z,0} d^6 x d^6 y d^6 z \right). \end{aligned} \quad (3.3)$$

Here Γ is the quantum action in external gravity, $\Gamma' = \Gamma - \Gamma_{UV}$ and $\Gamma_{x_1 \dots x_k}$ is the k th functional derivative of Γ with respect to the conformal factor ϕ at the points $x_1 \dots x_k$. For example, $\Gamma'_{x_0} = -\langle \Theta(x) \Theta(0) \rangle$, etc. In arbitrary even dimension $2n$ we have integrals of correlation functions containing up to $n + 1$ insertions of Θ .

It does not seem straightforward to derive general inequalities between Δa and $\Delta a'$ using the sum rules just written, because there is no simple way [17] to apply Osterwalder-Schrader (OS) positivity [18]. However, we can make some observations. Using the vanishing relations of [17] an equivalent form for the four-dimensional Δa sum rules (3.2) can be written, namely

$$\Delta a = \Delta a' + \frac{\pi^2}{96} \lim_{V \rightarrow \infty} \frac{1}{V} \int_V d^4x d^4y d^4z d^4w (x - y)^2 (z - w)^2 \Gamma'_{xyzw}. \quad (3.4)$$

Here the integrand is a positive function times Γ'_{xyzw} . Basically, Γ'_{xyzw} is minus the Θ four-point function. Naively, it is tempting to think that the four-point function is “positive”, in some sense. This suggests that the integral in (3.4) is negative or zero and the inequality

$$\Delta a' \geq \Delta a$$

holds.

The argument is however naive, for the reasons that I now explain. OS positivity states that the integral

$$\int d^4x d^4y d^4z d^4w g(x, y) g^*(\theta z, \theta w) \langle O(x) O(y) O(z) O(w) \rangle \quad (3.5)$$

is non-negative, for every Hermitean operator O and function $g(x, y)$, vanishing together with its derivatives unless $x^0 > y^0 > 0$. Here $\theta(x^0, x^1, x^2, x^3) = (-x^0, x^1, x^2, x^3)$. The positivity condition holds for every choice of the “time” axis x^0 .

The application of OS positivity to our integral (3.4) is problematic, however. The function $g(x, y)$ is $(x - y)^2 / \sqrt{V}$ inside the finite volume V and zero elsewhere, so it does not vanish together with its derivatives if $x^0 > y^0 > 0$ is not true.

Due to the symmetry of the correlation function under the exchange of x and y , the requirement that $g(x, y)$ should vanish together with its derivatives unless $x^0 > y^0$ can be replaced with the requirement that $g(x, y)$ should vanish together with its derivatives at $x^0 = y^0$. As long as $x \neq y$, there is no reason to expect surprises at $x^0 = y^0$, but we do have to pay attention to the coincident points (see below).

I now show that if the points x, y, z, w do not lie on a plane, we can also relax the restriction that the product $g(x, y)g^*(\theta z, \theta w)$ should vanish together with its derivatives unless $x^0, y^0 > 0 > z^0, w^0$. It is easy to prove, using the invariance under translations and rotations, a simple theorem, stating that for every set of points x, y, z and w that do not lie on a plane, there exist an origin and a time axis such that $x^0, y^0 > 0 > z^0, w^0$ with respect to that origin and that axis. When x, y, z and w do lie on a plane (in particular, when two points coincide), there might exist no origin and time axis such that $x^0, y^0 > 0 > z^0, w^0$. Now, the function $g(x, y)$ is invariant under translations and rotations, if we neglect that the integrals of (3.4)

are performed in a finite volume V . Since we have to take the limit $V \rightarrow \infty$ in the end, it is probably legitimate to ignore this nuisance. Then, whenever the points x, y, z and w do not lie on a plane we can apply our simple theorem to (3.5) and choose an origin and a time axis such that OS positivity holds. So, the contributions to the integral (3.5) coming from distinct points appear to be under control.

Ultimately, the true difficulty to apply OS positivity to the integral (3.4) comes from the coincident points. The set of coincident points is of vanishing measure only if the correlation functions have no contact terms. If there are no contact terms, we can surround the points x, y, z and w with infinitesimal spheres, perform the integral outside the spheres and let the radii of the spheres tend to zero in the very end. If there are contact terms, however, this procedure does not return the correct result and we cannot conclude that the integral (3.5) satisfies positivity.

There is no reason to expect that contact terms are absent. Actually, simple perturbative calculations show that contact terms are expected to be there. In momentum space, for example, contact terms are the product of a local function of some momenta times an arbitrary function of the other momenta. Finally, the contact terms of a four-point function are associated with three- or two-point functions. We have no control on the positivity of the flow integrals of the three-point functions.

Moreover, in (3.4) we do not just have the Θ four-point function, but $-\Gamma'_{xyzw}$, which is a combination of four-, three- and two-point functions [17]. The difference between $-\Gamma'_{xyzw}$ and the Θ four-point function is made of other contact terms.

Having shown that it is illegitimate to ignore the contact terms, we cannot rigorously prove the inequality $\Delta a' \geq \Delta a$. On the other hand, this difficulty is more than welcome. If the contact terms were not there, the above arguments would imply the strict inequality $\Delta a' > \Delta a$ any time the flow is nontrivial ($\Theta \neq 0$). This would contradict the claim of ref. [7] that in classically conformal theories the equality $\Delta a' = \Delta a$ holds exactly. Moreover, I would not be allowed to argue, as I do below, that the minimum of $\Delta a'$ over the flows F connecting the same fixed points is precisely Δa .

In conclusion, the inequality $\Delta a' \geq \Delta a$ is possible, and there is room for nontrivial flows satisfying the equality $\Delta a' = \Delta a$. Actually, if we enlarge the class of flows to include nonrenormalizable (e.g. asymptotically safe) theories, then there is also room for flows with $\Delta a < 0$ (see below).

4 Irreversibility of the RG flow in even dimensions

In even dimension greater than two and in odd dimensions the embedding in external gravity is unable to explain many known properties of trace anomalies [5]. Some properties, verified empirically in a number of cases, remain unexplained. To achieve a better understanding, it is helpful to investigate the space of conformal field theories and RG flows connecting them, and study the topological and metric properties of this space.

In even dimensions, the irreversibility of the RG flow [13, 1] is expressed by the existence

of a positive quantity a whose values are always larger in the ultraviolet than in the infrared. The quantity a is interpreted as a counter of the massless degrees of freedom of the theory. In two dimensions, this quantity is Zamolodchikov's c function. In higher even dimensions it is the central charge a , namely the coefficient of the Euler density in the trace anomaly of the theory embedded in external gravity.

In refs. [7, 3, 4] an approach to the irreversibility of the RG flow in even dimensions has been developed. A synthetic review can be found in [1]. The approach of these references does not apply to the most general flow, but only the subclass of classically conformal flows and the subclass of flows that have $\Delta a = \Delta c$. Relevant and irrelevant deformations have to be included in a more complete understanding that possibly applies also to odd dimensions.

In this section I collect some considerations about the irreversibility of the RG flow in even dimensions, which hopefully make the inequality $\Delta a' \geq \Delta a$ more plausible. I stress the different roles played by classically conformal and classically non-conformal theories.

In quantum field theory, there are basically two sources of violations of scale invariance: the dimensionful parameters of the classical lagrangian and the dynamical scale μ introduced by renormalization. Power counting groups the dimensionful parameters into relevant and irrelevant. The relevant parameters are expected to enhance irreversibility, because of the Appelquist-Carazzone decoupling theorem [19]. For example, the theories of massive free scalars and fermions certainly have $a_{UV} > a_{IR}$. Symmetrically, the irrelevant parameters are expected to depress irreversibility, because a non-renormalizable coupling, which does not affect the IR limit, can kill degrees of freedom in the UV limit. Observe that for these considerations, which mostly concern the signs of Δa , $\Delta a'$ and $\Delta a - \Delta a'$, it is necessary to assume only that the theory is unitary and interpolates between well-defined UV and IR fixed points. In particular, it is not necessary to assume that the theory is renormalizable in a conventional sense, nor that it is predictive, i.e. quantizable with finitely many parameters. In the realm of non-renormalizable theories, consistent flows can be defined, for example, in the scenario of Weinberg's asymptotic safety [20].

In view of the observations just made, the irrelevant parameters, such as the Newton constant, are expected to violate the irreversibility of the RG flow ($\Delta a < 0$), unless they are dynamically generated by the renormalization scale μ (from a renormalizable theory). Note that the existence of flows with $\Delta a < 0$ does not contradict irreversibility, as long as $\Delta a < 0$ is the product of an *explicit* violation of scale invariance, that is to say a violation due to a classical dimensionful parameter. It is not surprising that explicit violations produce $\Delta a < 0$, because they can be arbitrarily strong and cover all opposite effects. The theories that contain no explicit violation of scale invariance are precisely the classically conformal theories. There, the dynamical breaking of scale invariance does not mix with the effects of explicit violations. This is the reason why the classically conformal theories occupy a special role in the investigation of irreversibility.

The properties of relevant, marginal and irrelevant parameters at the classical and quantum

levels are summarized in the table

	classic	quantum
relevant	$\Delta a' > 0, \Delta a > 0,$	$\Delta a' > 0, \Delta a > 0,$
marginal	$\Delta a' = 0, \Delta a = 0,$	$\Delta a' \geq 0, \Delta a \geq 0,$
irrelevant	$\Delta a' > 0, \Delta a < 0,$	$\Delta a' > 0, \Delta a > < 0.$

At the classical level, the marginal plane $\Delta a = 0$ separates the space of relevant flows ($\Delta a > 0$) from the space of irrelevant flows ($\Delta a < 0$). At the quantum level, the plane $\Delta a = 0$ moves inside the space of irrelevant flows. Now, $\Delta a'$ is strictly positive in all non-trivial unitary theories, but asymptotically safe flows with $\Delta a \leq 0$ (and $\Delta a' > 0$) are in principle allowed to exist. These considerations rule out the inequality $\Delta a' \leq \Delta a$.

In conclusion, only one universal inequality is not ruled out, namely

$$\Delta a' \geq \Delta a. \quad (4.1)$$

This possibility opens the door to some stimulating ideas.

5 Length of the RG flow, distance between the fixed points and Δa

In this section I elaborate the definitions of length of the RG flow and oriented distance between the fixed points. The RG flow is conventionally oriented from the ultraviolet to the infrared. I conjecture that the distance coincides with Δa . The inequalities $\Delta a' \geq \Delta a \geq 0$ and the irreversibility of the RG flow (in even dimensions) are implied straightforwardly by this conjecture. The generalization of these ideas to odd dimensions is presented in section 8.

Length of the RG flow and distance between the fixed points. The quantity $\Delta a'$ is always positive and does depend on the flow connecting the fixed points. It is therefore a natural candidate to define the length L of the flow F :

$$L(F) = \frac{\pi^n (d+1)}{n!} \int_F d^d x |x|^d \langle \Theta(x) \Theta(0) \rangle = \Delta a'(F). \quad (5.2)$$

A (unitary) flow of zero length is trivial, since $L(F) = 0$ and reflection positivity imply $\Theta \equiv 0$.

The minimum of $L(F)$ in the space $\mathcal{F}_{C_{UV}, C_{IR}}$ of (unitary, renormalizable) flows F connecting the same fixed points C_{UV} and C_{IR} is the distance d between them:

$$d(C_{UV}, C_{IR}) = \min_{F \in \mathcal{F}_{C_{UV}, C_{IR}}} L(F). \quad (5.3)$$

The minimum has to be taken in the space of continuously deformable flows and sequences of flows with concordant orientations. If the space $\mathcal{F}_{C_{UV}, C_{IR}}$ has disconnected components, there might be a minimum in each subspace $\mathcal{F}_1, \mathcal{F}_2, \dots$ of continuously connected flows. In some

situations, $L(F)$ might not admit a minimum, but only an inferior limit. Then the distance is defined as the inferior limit of $L(F)$.

The distance between the conformal field theories from C_1 and C_2 is defined only if there exists a flow, or a sequence of flows with concordant orientations, interpolating between C_1 and C_2 .

The distance between two continuously connected conformal field theories (e.g. two $N=4$ $d=4$ supersymmetric Yang-Mills theories with different values of the gauge coupling g) is zero, because an exactly marginal deformation of a conformal field theory is a trivial RG flow.

The minimization of $\Delta a'$ in the space of flows was first realized to have remarkable properties in ref. [15].

With an abuse of language, I use the words “distance between the fixed points”, even if, strictly speaking, $d(C_{UV}, C_{IR})$ is the distance between suitable projections of the fixed points. The space of conformal theories \mathcal{C} is projected to a space $\pi\mathcal{C}$ and the projection π is such that conformal field theories with zero distance are projected onto the same point of $\pi\mathcal{C}$. Similarly, the space \mathcal{F} is projected onto a space $\pi\mathcal{F}$.

Oriented distance between the fixed points and Δa . I now restrict the attention to renormalizable theories, on which we have a better control. Inspired by the considerations made so far, in particular *i*) the possibility of a universal inequality (4.1), *ii*) the existence of flows with $\Delta a' = \Delta a$ (the classically conformal theories) and *iii*) the independence of Δa from the flow connecting the fixed points, I conjecture that the distance between the fixed points is precisely Δa , i.e.

$$d(C_{UV}, C_{IR}) = \Delta a = a_{UV} - a_{IR} = \min_{F \in \mathcal{F}_{C_{UV}, C_{IR}}} \Delta a'(F). \quad (5.4)$$

The relation (5.4) encodes also the fact that the distance is oriented. This means, in particular, that the distance is not symmetric: strictly speaking $d(C_{UV}, C_{IR})$ is the distance *from* C_{UV} to C_{IR} , not the distance *between* C_{UV} and C_{IR} . The axioms of the oriented distance are elaborated in section 8.

Furthermore, (5.4) allows us to conclude that $\pi\mathcal{C}$ and $\pi\mathcal{F}$ are one-dimensional, a subset of the a -axis.

Formula (5.4) implies the inequalities

$$0 \leq \Delta a \leq \Delta a' \quad (5.5)$$

and therefore the irreversibility of the RG flow as it is commonly stated in even dimensions ($\Delta a \geq 0$).

The observations made in the previous section suggest also that if we extend our considerations to non-renormalizable flows, e.g. asymptotically safe theories, the minimum (5.4) has probably no relation with Δa .

The ideas of the present paper generalize the understanding of ref.s [1, 7, 3, 4], which mostly concerned classically conformal flows and the flows with $\Delta c = \Delta a$. Several pieces of evidence

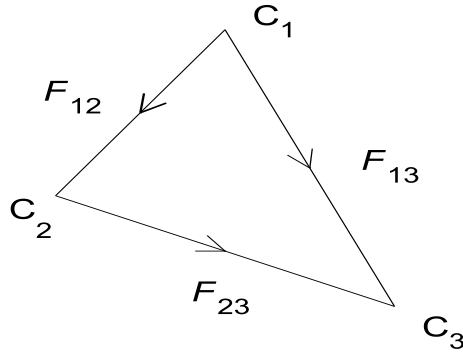


Figure 1: An oriented triangle in the space of flows

suggested that these flows have $\Delta a = \Delta a'$, namely that they saturate the minimum (5.4) (“geodesic” flows). The extended picture covers also non-geodesic flows, for which the strict inequality $\Delta a' > \Delta a$ can hold.

Observe that in two dimensions the three central charges c , a and a' are indistinguishable at the fixed points, since the trace anomaly in external gravity contains only one term, namely $cR/(24\pi)$. With the relative normalization adopted in the introduction we are allowed to write the identifications $c = a = a'$ in two dimensions. This means that all of the flows have $\Delta c = \Delta a = \Delta a'$ and therefore equal and minimal length.

In section 7 I derive some predictions from the statements of this section and test them.

6 Geometry of the spaces of fixed points and flows

In this section I make some observations about the topological and metric properties of the spaces \mathcal{C} and \mathcal{F} of fixed points and flows in even-dimensional quantum field theory. This kind of analysis is extended to odd-dimensional theories in section 8.

All triangles are degenerate. First I show that all triangles are degenerate in the space $\pi\mathcal{C}$. Consider a “triangle” T in the space of flows (Fig. 1), made of the fixed points C_1 , C_2 and C_3 , and the RG flows F_{12} , F_{23} , and F_{13} , connecting C_1 to C_2 , C_2 to C_3 and C_1 to C_3 , respectively. Due to the existence of the central charge a and the relation (5.4), the distance d_{ij} between C_i and C_j is oriented from C_i to C_j and equal to $a_i - a_j$ with $i < j$. We have the “triangle equality”

$$d_{13} = a_1 - a_3 = (a_1 - a_2) + (a_2 - a_3) = d_{12} + d_{23}. \quad (6.6)$$

The geometric meaning of the triangle equality is that the space $\pi\mathcal{C}$ in which the distances are measured is one-dimensional.

The triangle equality is a property of even-dimensional quantum field theory. Odd-dimensional quantum field theory does not admit a global a function and $\pi\mathcal{C}$ can have dimension greater than one.

In higher even dimensions, the central charges c and a do not identify a conformal field theory uniquely. There exist families of continuously connected inequivalent conformal field theories having the same c and the same a [2]. If we embed the space \mathcal{C} of conformal field theories in \mathbf{R}^k for k sufficiently large, we probably find sets of discrete points, lines (one-parameter families of continuously connected conformal field theories), two-dimensional surfaces and three- or higher-dimensional regions. Probably, most theories are isolated points or belong to one-parameter families, and the higher-dimensional regions are exceptional. The space \mathcal{F} is even more complicated, since it is the space of flows connecting the points of \mathcal{C} . We do not know if there exists a flow, or a family of flows, connecting every pair of points of \mathcal{C} ; several pairs of points might not admit a flow connecting them. Some families of flows are continuously connected, others are not. Probably, \mathcal{F} looks like a neural network. Observe that the notion of oriented distance works quite well for a neural network, or in general an environment where the paths connecting the nodes are one-way and some pairs of nodes are connected by no path nor sequence of paths with concordant orientations.

The topology of $\pi\mathcal{C}$ is considerably simpler than the topology of \mathcal{C} : $\pi\mathcal{C}$ is just a set of points and maybe intervals on the a axis. We can expect that also the topology of $\pi\mathcal{F}$ is simpler than the topology of \mathcal{F} .

Finally, regions of \mathcal{C} with different values of a or c are disconnected from one another. This follows from a property of the central charges c and a known as “marginality of the central charge” [2], stating that continuously connected conformal field theories have the same c and a .

7 Inequalities for trace anomalies in free-field theories

I have anticipated, in the introduction, that free scalars and fermions have

$$a_{\text{free}} \leq c_{\text{free}} \tag{7.1}$$

in arbitrary even dimensions $d = 2n$, while vectors, or, more generally, the $(n - 1)$ -differential forms, have $a \geq c$. The values of c are [22]

$$c_{\text{scal}} = 1, \quad c_{\text{ferm}} = 2^{n-1}(d - 1), \quad c_{\text{forms}} = \frac{1}{2} \frac{d!}{[(n - 1)!]^2},$$

while the values of a have been calculated in [21]. Using the procedure of section 2.3 of [5] we can write

$$a = \frac{1}{2} (-1)^{n-1} (-1)^{2S} \frac{(2n + 1)!}{n!(n - 1)!} \Upsilon(0)$$

where S is the spin and

$$\Upsilon(0) = \lim_{s \rightarrow 0} \sum_{k=0}^{\infty} \frac{\delta_k}{\varpi_k^s}.$$

Here ϖ_k are the eigenvalues of an appropriate second-order differential operator and δ_k are their multiplicities on the sphere ($R_{\mu\nu} = \Lambda g_{\mu\nu}$). The differential operator is $-\square + \Lambda d(d-2)/(4(d-1))$ for scalar fields and $-\square + d\Lambda/4$ for fermions. In the case of the differential forms, the sum $\Upsilon(0)$ is defined as the $\zeta_{AT}(0)$ of [21].

Using the values of $\Upsilon(0)$ given in the tables of [21], we have, for scalars and fermions,

d	2	4	6	8	10	12	14
$a_{\text{scal}}/c_{\text{scal}}$	1	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{23}{90}$	$\frac{263}{1080}$	$\frac{133787}{567000}$	$\frac{157009}{680400}$
$a_{\text{ferm}}/c_{\text{ferm}}$	1	$\frac{11}{18}$	$\frac{191}{360}$	$\frac{2497}{5040}$	$\frac{14797}{31104}$	$\frac{92427157}{199584000}$	$\frac{36740617}{80870400}$

etc. We see that (7.1) is satisfied. The ratios a/c decrease when the dimension increases.

Now consider the differential forms. The ratio a/c is equal to

$$\frac{a_{\text{forms}}}{c_{\text{forms}}} = (-1)^{n-1} \frac{d+1}{n} \Upsilon(0).$$

Taking $\Upsilon(0)$ from the tables of [21], we have

d	4	6	8	10	12	14
$a_{\text{forms}}/c_{\text{forms}}$	$\frac{31}{18}$	$\frac{221}{90}$	$\frac{8051}{2520}$	$\frac{1339661}{340200}$	$\frac{525793111}{112266000}$	$\frac{3698905481}{681080400}$

etc. The free $(n-1)$ -differential forms do not satisfy (7.1). Actually, they always have $a > c$.

I now prove that the inequality (7.1) for scalars and fermions follows from the inequality (5.5) of section 5. Consider the RG flows of free massive scalars and fermions. In the ultraviolet the fields are massless: $c_{UV} = c_{\text{free}}$, $a_{UV} = a_{\text{free}}$. Instead, the IR fixed points of these flows are trivial: $c_{IR} = a_{IR} = 0$. The quantities Δa and $\Delta a'$ are calculable exactly and $\Delta a'$ happens to coincide with Δc . (A study of this coincidence can be found in ref. [15].) Then (5.5) implies $c_{\text{free}} = \Delta c = \Delta a' \geq \Delta a = a_{\text{free}}$ and therefore (7.1).

This argument does not generalize to vector fields and differential forms, because there exists no RG flow connecting the free vector with the empty theory and having $\Delta a' = \Delta c$. The Proca theory of massive vectors is singular in the ultraviolet and has $\Delta a' = \infty$, while Δc is finite. This allows a_{vector} to be greater than c_{vector} , avoiding any contradiction with our predictions.

An argument can be given to explain, to some extent, why vectors should better have $a > c$. Let us recall that there exists a remarkable subclass of conformal field theories having $c = a$ [4]. The equality $c = a$ is all but difficult to fulfil, even at the free-field level. However, if both scalars, fermions and vectors (or $(n-1)$ -differential forms) had $a < c$, there would exist no free-field theory with $c = a$. So, having proved that scalars and fermions behave one way, it follows that vectors should behave the opposite way, if we want that the $c = a$ theories are not so rare.

The considerations of the previous sections apply to unitary flows. It is easy to check that, indeed, the inequality $a \leq c$ is violated in non-unitary Gaussian fields with a \square^2 kinetic term. In [15] it was shown that the equality $\Delta a' = \Delta c$ is satisfied in the massive higher-derivative

models, if the masses are chosen appropriately. Using the results of [5] and [15] we can see that a higher-derivative scalar field with a \square^2 -kinetic term has

d	4	6	8	10	12	14
c	-8	-5	-4	$-\frac{7}{2}$	$-\frac{16}{5}$	-3
a	$-\frac{28}{3}$	$-\frac{16}{9}$	$-\frac{52}{45}$	$-\frac{124}{135}$	$-\frac{56302}{70875}$	$-\frac{30544}{42525}$

etc. The equality $a \leq c$ holds only in $d = 4$.

In summary, the understanding of section 5 explains some facts that otherwise would appear to have no reason, namely why free scalars and fermions have $a \leq c$, while the Proca theory of vector fields is singular and vectors can have $a > c$.

8 Irreversibility of the RG flow in odd (and even) dimensions

In this section I formulate a more general notion of irreversibility of the RG flow, which applies also to odd dimensions and does not require the existence of an a function. I give a set of axioms (“oriented-triangle inequalities”) that imply irreversibility without a global a function and the existence of a local a function. I emphasize the conceptual differences between irreversibility in even-dimensional and odd-dimensional theories and study other topological and metric properties of the spaces \mathcal{C} and \mathcal{F} , $\pi\mathcal{C}$ and $\pi\mathcal{F}$.

Length and distance in odd dimensions. The quantity $\Delta a'$ is meaningful and non-negative in the most general quantum field theory [3] and so the definition (5.2) of length $L(F)$ of the flow F applies also in odd dimensions. The factor in front of the integral of formula (5.2) has no meaning in odd dimensions. Using $\Delta a'$, the distance between the fixed points can still be defined as the minimum (or the inferior limit, if the minimum does not exist) of $L(F)$ in the space of flows F connecting the same fixed points. Therefore, (5.3) holds also in odd dimensions. Furthermore, it is still reasonable to conjecture that the geodesic flows are classically conformal, but not all of the classically conformal flows are geodesic in odd dimensions (see next section).

Instead, in odd dimensions it is not possible to associate a central charge a to the fixed points [5] and express the minimum of $\Delta a'$ as the difference between the values of a at the fixed points. So, (5.4) is not ensured to hold. A global a function might not exist. The positivity of the minimum of $\Delta a'$, i.e. the positivity of a distance, is an obvious statement and does not imply that the RG flow is irreversible. As a consequence, the irreversibility of the RG flow in odd dimensions cannot be formulated as in even dimensions.

Oriented distance and oriented triangles. I have shown in section 6 that in even dimensions the triangle equality (6.6) holds, because there exists a global a function. In odd dimensions, instead, the distance satisfies genuine triangle inequalities. With reference to Fig. 1, we have

$$d_{13} \leq d_{12} + d_{23}, \quad d_{12} \leq d_{13} + d_{23}, \quad d_{23} \leq d_{13} + d_{12}. \quad (8.1)$$

The inequalities of the form $d_{ij} \geq |d_{ik} - d_{jk}|$ do not add information, because they are implied by (8.1).

The inequalities (8.1) follow from the definition of distance. Since non-trivial flows have an orientation, conventionally taken as $UV \rightarrow IR$, the distance is oriented. This allows us to postulate more restrictive inequalities. With “oriented triangle” I mean a triangle whose sides are oriented, unless they are associated with trivial flows (zero length), in which case their orientation is unspecified. If the flows of the triangle T are oriented as in Fig. 1, the “oriented-triangle inequalities” are

$$d_{13} \leq d_{12} + d_{23}, \quad d_{13} \geq d_{23}, \quad d_{13} \geq d_{12}. \quad (8.2)$$

I conjecture that the unitary, renormalizable RG flows in arbitrary dimensions satisfy the oriented-triangle inequalities. Observe that (8.2) imply (8.1), but (8.1) do not imply (8.2).

More generally: consider a sequence of flows $F_i : C_{i-1} \rightarrow C_i$, $i = 1, \dots, n$, with concordant orientations, connecting two fixed points C_0 and C_n , with $n-1$ intermediate fixed points C_1, \dots, C_{n-1} ; then the distance $d(C_i, C_{i-1})$ between two consecutive intermediate fixed points is always smaller than (or equal to) the distance between C_0 and C_n . It is easy to see that this more general statement is implied by the oriented-triangle inequalities. For the proof, it is sufficient to apply the oriented-triangle inequalities to the set of triangles $C_{i-1}-C_i-C_n$, $i = 1, \dots, n-1$.

Irreversibility without a height function. The irreversibility of the RG flow is the statement that given a unitary flow F connecting the UV conformal field theory C_1 to the IR conformal field theory C_2 there exists no unitary flow having C_2 as UV fixed point and C_1 as IR fixed point. In particular, this means that the reversed flow \overline{F} of F , obtained by scale inversion, which is the transformation

$$\mu \rightarrow \frac{1}{\mu}, \quad m \rightarrow \frac{1}{m},$$

where μ denotes the dynamical scale and m a generic dimensionful parameter, does not exist or violates basic principles of quantum field theory (typically unitarity or locality).

Strictly speaking, there is no need of a “height” (c or a) function to have irreversibility. If there exists a height function, irreversibility follows *a fortiori*, but if there does not exist a height function, the RG flow can still be irreversible in the more general sense just explained. Using the more general definition, it is easier to disprove irreversibility, eventually: it is sufficient to find two RG flows with interchanged the UV and IR limits interpolating between the same pair of fixed points. At present, no example of this kind is known in the literature, to my knowledge.

If a global a function does not exist (generically, there might still exist a local a function: see below) then it is not possible to define a counter of the (massless) degrees of freedom of the theory. Moreover, the projected spaces $\pi\mathcal{C}$ and $\pi\mathcal{F}$ are not necessarily one-dimensional: there might exist non-degenerate triangles.

Irreversibility from oriented-triangle inequalities. I now prove that the oriented-triangle inequalities imply the irreversibility of the RG flow without a height function. Suppose

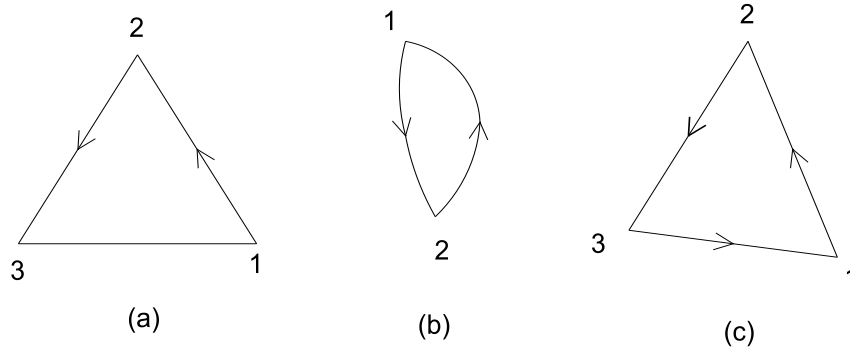


Figure 2: Inadmissible configurations

that the flow F_{13} of the triangle T is trivial ($d_{13} = 0$, see Fig. 2a). Then (8.2) imply that the triangle is trivial. Applying this property to the case $C_1 = C_3$, we conclude there cannot exist two RG flows connecting interchanged fixed points, namely the situation depicted in Fig. 2b is inadmissible. This means precisely that the RG flow is irreversible without a height function. Similarly, a triangle with the orientation shown in Fig. 2c is incompatible with (8.2), unless the triangle is trivial.

Implication of a local a function from irreversibility without an a function. Here I prove that the oriented-triangle inequalities imply also the existence of an a function in the smooth regions of the space \mathcal{F} .

I have explained in section 6 that the topology of the space \mathcal{F} is expected to be involved. Let us consider, however, a two- (or higher-) dimensional subset \mathcal{R} of \mathcal{C} , and assume that for every pair of points of \mathcal{C} there exists a flow in \mathcal{F} (eventually trivial, i.e. of zero length), or a sequence of concordant flows, connecting them. I want to show that under these smoothness assumptions, the oriented-triangle inequalities (8.1) imply the existence of an a function in \mathcal{R} .

Consider a point C in \mathcal{R} . Viewed from C , the set \mathcal{R} can be written as the union of three subsets:

$$\mathcal{R} = \mathcal{R}_+^C \cup \mathcal{R}_0^C \cup \mathcal{R}_-^C. \quad (8.3)$$

\mathcal{R}_+^C is the set of points P of \mathcal{R} such that the flows connecting P to C are oriented from P to C . \mathcal{R}_-^C is the set of points P of \mathcal{R} such that the flows connecting P to C are oriented from C to P . The set \mathcal{R}_0^C is a surface of points having zero distance from C .

The decomposition (8.3) of \mathcal{R} is well-defined because the oriented-triangle inequalities imply irreversibility without an a function, which means that given two points C_1 and C_2 in \mathcal{R} all of the flows connecting them are either oriented from C_1 to C_2 or from C_2 to C_1 , or the distance between C_1 and C_2 is zero.

Moreover, the decomposition (8.3) is made with respect to a reference point C . Similar decompositions with respect to every reference point C of \mathcal{R} are given (obviously, C belongs to \mathcal{R}_0^C). Now, I have proved above that the situation of Fig. 2a is inadmissible. This ensures

that the decomposition (8.3) does not depend on the reference point C in \mathcal{R}_0^C , i.e. $\mathcal{R}_+^C = \mathcal{R}_+^{C'}$, $\mathcal{R}_0^C = \mathcal{R}_0^{C'}$, $\mathcal{R}_-^C \cup \mathcal{R}_-^{C'}$ for every C and C' in \mathcal{R}_0^C .

The decomposition (8.3) can be used to define a “height” function in \mathcal{R} . The \mathcal{R}_0 surfaces are the surfaces with equal height. The space \mathcal{R}_+ is higher than \mathcal{R}_0 , while \mathcal{R}_- is lower than \mathcal{R}_0 , which means that the a function has larger values in the points of \mathcal{R}_+^C than in \mathcal{R}_0^C and a larger value in \mathcal{R}_0^C than in \mathcal{R}_-^C , for every C in \mathcal{R} . Moreover, the values of a should be positive. Apart from this, the values assigned to the function a are arbitrary.

The construction of the a function, however, might not extend to the entire spaces \mathcal{C} and \mathcal{F} . Some pairs of points $C_1, C_2 \in \mathcal{C}$ might not admit a flow or sequence of concordant flows connecting them, and in this case there is no way to determine which point is higher and which is lower. It might still be possible to assign some a -values to such points consistently with the a -values assigned in the smooth subsets of \mathcal{C} , but it is doubtful that this is more than an academic exercise and it would be quite arbitrary to interpret the resulting a function as the “counter of the massless degrees of freedom of the theory”.

In conclusion, the interpretations of $\Delta a'$ as length of the RG flow and its minimum as distance between the fixed points give the triangle equalities (8.1). Secondly, the idea that the distance is oriented suggests to conjecture the more restrictive oriented-triangle inequalities (8.2). Finally, the irreversibility of the RG flow does not require the existence of an a function and can be defined as the statement that two RG flows connecting interchanged UV and IR fixed points are trivial. The oriented-triangle inequalities imply irreversibility without a global a function in \mathcal{F} . In the next section I perform a check of the oriented-triangle inequalities in three dimensions and prove the existence of non-degenerate triangles in odd dimensions. The results support the idea that the RG flow is irreversible in odd dimensions without a global a function.

Marginality. I point out that the triangle inequalities (8.1), and *a fortiori* the oriented-triangle inequalities (8.2), imply that the distance $d(C_{UV}, C_{IR})$ is invariant under exactly marginal deformations of the fixed points C_{UV} and C_{IR} . In other words, $d(C_{UV}, C_{IR}) = d(C'_{UV}, C'_{IR})$ if C_{UV} and C_{IR} are continuously connected to C'_{UV} and C'_{IR} , respectively. More generally, if we consider a triangle T with $d_{23} = 0$, the inequalities (8.2) give $d_{13} \leq d_{12}$, $d_{13} \geq d_{12}$ and therefore $d_{12} = d_{13}$. In this more general formulation of the statement, the conformal theories C_2 and C_3 do not even need to be continuously connected. Observe that only the distance, i.e. $\min_{F \in \mathcal{F}} \Delta a'(F)$, is expected to be marginal, but $\Delta a'$ is not (for other details, see sect. 3.2 of [5]).

Proceeding as in the end of section 6, marginality implies that the “surfaces of equal height” are disconnected from one another.

9 A calculation in three dimensions

In ref.s [8, 9] some classes of three-dimensional classically conformal theories have been constructed. They provide a valid laboratory to test ideas about strongly-coupled quantum field

theory and irreversibility. Those RG flows interpolate between the UV fixed points of four-fermion models and are exactly integrable in the running couplings at each order of the large N expansion. In this section I test the conjecture (8.2) in these three-dimensional flows.

The four-fermion model is a power-counting non-renormalizable theory in three dimensions. It can be renormalized using a construction due to Parisi [23] (see also [24]) in the large N expansion, where N is the number of fermions. Only one diagram, precisely the fermion bubble, contributes to the leading order. The fermion bubble gives a finite contribution that improves the large-momentum behaviors of propagators in such a way that, after resumming the fermion bubbles, the subleading corrections are power-counting renormalizable. The theories of refs [8, 9] are renormalized in a similar way.

The models. I focus on the fermion models with lagrangian

$$\mathcal{L} = \sum_{i=1}^N \bar{\psi}_i \not{\partial} \psi_i + \sum_{j=1}^{rN} \bar{\chi}_j \not{\partial} \chi_j + \lambda \sigma \left(\sum_{i=1}^N \bar{\psi}_i \psi_i + g \sum_{j=1}^{rN} \bar{\chi}_j \chi_j \right). \quad (9.1)$$

The effective σ -propagator is generated by the fermion bubble in the large N limit. The coupling λ is inert and $\lambda^2 N$ is kept fixed in the $1/N$ expansion, while the coupling g runs. The RG flow is integrable in g at each order of the $1/N$ expansion [9]. The model is chiral invariant, namely invariant under

$$\psi \rightarrow \gamma_5 \psi, \quad \chi \rightarrow \gamma_5 \chi, \quad \sigma \rightarrow -\sigma,$$

and has a strong-weak coupling duality

$$r \rightarrow \frac{1}{r}, \quad g \rightarrow 1/g, \quad \lambda \rightarrow \frac{\lambda}{\sqrt{r}}, \quad \psi \leftrightarrow \chi, \quad \sigma \rightarrow \sigma g \sqrt{r},$$

which is exact at each order of the $1/N$ expansion. With four-component (Dirac) fermions we can take $0 \leq g \leq 1$. The beta function reads

$$\beta_g = \frac{8}{3\pi^2 N} \frac{g(g^2 - 1)}{1 + rg^2},$$

while $\beta_\lambda = -\varepsilon \lambda / 2$ ($\lambda_B = \lambda \mu^{\varepsilon/2}$). The fixed points are

$$\text{UV } (g = 0) : \quad \Sigma_N \otimes \Psi_{rN} ; \quad \text{IR } (g = 1) : \quad \Sigma_{N(1+r)} ;$$

where Ψ_N denotes N free fermions and Σ_N is the conformal field theory defined by the lagrangian

$$\mathcal{L} = \sum_{i=1}^N \bar{\psi}_i (\not{\partial} + \lambda \sigma) \psi_i.$$

The flow is classically conformal, so we expect that the distance between the UV and IR fixed points is equal to the length of the flow, defined as

$$L(r) = \frac{16\pi}{3} \int d^3x |x|^3 \langle \Theta(x) \Theta(0) \rangle.$$

First I derive the trace-anomaly formula, then compute $\Delta a'(r) = L(r)$ and use this result to check (8.2).

Regularization and renormalization. The dimensional regularization has to be modified adding an evanescent, RG invariant non-local term to the lagrangian (9.1) [8], to avoid the appearance of $\Gamma[0]$ s. This complicates the study of Θ , since the embedding of a non-local term in external gravity is quite involved. Here I use a more practical regularization convention. The theory is still extended to $3 - \varepsilon$ dimensions, but the cut-off term

$$\frac{1 + rg^2}{2\Lambda} (\partial_\mu \sigma)^2 \quad (9.2)$$

is added to the lagrangian. The limit $\Lambda \rightarrow \infty$ is performed after the $\varepsilon \rightarrow 0$ limit. The $\varepsilon \rightarrow 0$ limit renormalizes the fermion loops, while the $\Lambda \rightarrow \infty$ limit gives sense to the loops containing σ -propagators. The factor in (9.2) is chosen to have manifest duality invariance.

I use a classically conformal minimal subtraction scheme. The poles and the terms proportional to powers of $\ln \Lambda/\mu$ are subtracted away with no finite part, as well as the linear, quadratic and cubic divergences (Λ^k , Λ^k/ε^m , $\Lambda^k(\ln \Lambda/\mu)^m$ for $k > 0$, $m \geq 0$). The renormalized lagrangian contains also a counterterm of the form

$$\Lambda \frac{1 + rg^2}{2} \delta Z_m \sigma^2 \quad (9.3)$$

that cures the linear divergences. Chiral invariance forbids an analogous term for the fermions. To avoid a heavy notation, I go through the derivation as if (9.3) were not there. It is easy to include this term and check that the result (9.6) is unmodified.

The trace-anomaly operator equation. I start from the general integrated formula [17]

$$\left\langle \int d^{3-\varepsilon} x \widehat{\Theta}(x) \right\rangle = -\mu \frac{\partial \Gamma}{\partial \mu}. \quad (9.4)$$

Here Γ is the quantum action and $\widehat{\Theta}$ is the trace of the stress tensor up to terms proportional to the field equations (which are irrelevant for the computation of $\Delta a'$). Formula (9.4) says that the insertion of an integrated trace is equal to an insertion of $-\mu S \partial/\partial \mu$, where S is the action. Since $\mu d\Gamma/d\mu = 0$, we can use the Callan-Symanzik equations and rewrite $-\mu \partial/\partial \mu$ as

$$\beta_\lambda \frac{\partial}{\partial \lambda} + \beta_g \frac{\partial}{\partial g}. \quad (9.5)$$

Now, β_λ is evanescent and $\partial/\partial \lambda$ is a renormalized operator, because the derivative $\partial/\partial \lambda$ of a renormalized correlation function is obviously finite. Therefore the piece $\beta_\lambda \partial/\partial \lambda$ can be omitted. In summary, the $\ln \mu$ -derivative of a correlator can be re-expressed as minus its g -derivative times β_g .

The differentiation of a correlation function with respect to g is equivalent to the insertion of the integrated operator $-\partial S/\partial g$. The g -derivative of the action is done keeping the renormalized couplings and fields constant. Alternatively, we can keep the bare fields constant, since

the difference amounts to terms proportional to the field equations. However, we do have to differentiate the bare parameters. Since λ_B does not depend on g , we just have to differentiate g_B . The result is

$$\int d^3x \hat{\Theta}(x) = \beta_g \frac{\partial g_B}{\partial g} \frac{\partial S}{\partial g_B} = \beta_g \frac{\partial g_B}{\partial g} \int d^3x \left[\lambda_B \sigma_B \sum_{j=1}^{rN} \bar{\chi}_{Bj} \chi_{Bj} + \frac{r g_B}{\Lambda} (\partial_\mu \sigma_B)^2 \right].$$

Using the σ field equation, we obtain also

$$\frac{1}{\beta_g} \int d^3x \hat{\Theta}(x) = \frac{\lambda_B}{1 + r g_B^2} \frac{\partial g_B}{\partial g} \int d^3x \sigma_B \left(\sum_{j=1}^{rN} \bar{\chi}_{Bj} \chi_{Bj} - r g_B \sum_{i=1}^N \bar{\psi}_{Bi} \psi_{Bi} \right).$$

Now, since the left-hand side is a renormalized operator, the insertion of the right-hand side in a correlation function is finite. Therefore, the right-hand side is the renormalized version of the operator obtained suppressing the σ_B everywhere.

Finally, assuming that the integral can be taken away, the trace-anomaly operator formula

$$\hat{\Theta} = \frac{\lambda \beta_g}{1 + r g^2} \sigma \left(\sum_{j=1}^{rN} \bar{\chi}_j \chi_j - r g \sum_{i=1}^N \bar{\psi}_i \psi_i \right) \quad (9.6)$$

is obtained, where the right-hand side is understood to be the renormalized composite operator. It is immediate to check that this expression is duality invariant. (In [8] a non-manifestly duality invariant expression was given. The formula of [8] differs from (9.6) by a term proportional to the σ field equation and an evanescent term coming from (9.2), but gives exactly the same $\Delta a'$.)

We have to justify that the integral can be freely taken away. Since we are using a local regularization, $\hat{\Theta}$ is local. $\hat{\Theta}$ might differ from (9.6) by total derivatives. These can only be $\partial_\mu (\bar{\psi} \gamma_\mu \psi)$, $\partial_\mu (\bar{\chi} \gamma_\mu \chi)$ and $\square \sigma^2 / \Lambda$. The first two terms are absent. This can be seen observing that Θ is invariant under charge conjugation, but $\partial_\mu (\bar{\psi} \gamma_\mu \psi)$ and $\partial_\mu (\bar{\chi} \gamma_\mu \chi)$ are not. The term $\square \sigma^2 / \Lambda$ is a renormalized evanescent total derivative, so it gives no contribution when the cut-off is removed.

Θ two-point function and length of the flow. Using the techniques of [9], we know that in the leading-log approximation it is sufficient to compute the two-point function of the operator $\sigma (\bar{\chi} \chi - r g \bar{\psi} \psi)$ to the leading order. The relevant diagrams are depicted in Fig. 3. The sum of diagrams (b) and (c) vanishes. It remains to compute the diagram (a), which is straightforward in the x space. We obtain

$$\langle \Theta(x) \Theta(0) \rangle = \frac{32 r g^2 (1/|x|) (1 - g^2 (1/|x|))^2}{9 \pi^8 N^2 x^6 (1 + r g^2 (1/|x|))^4},$$

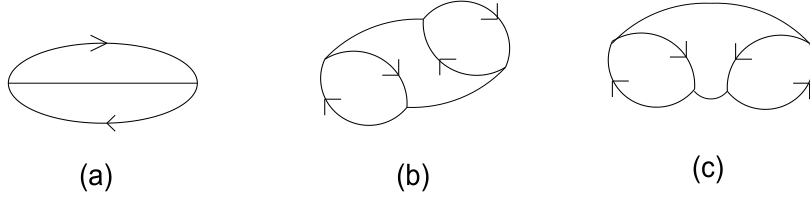


Figure 3: Diagrams for the calculation of $\Delta a'$

where $g(1/|x|)$ is the running coupling. From this expression it is immediate to derive $\Delta a'$. The result is

$$\Delta a'(r) = \frac{16\pi}{3} \int d^3x |x|^3 \langle \Theta(x) \Theta(0) \rangle = \frac{64}{9\pi^4 N} \left(1 - \frac{1}{r+1} \right) \equiv \frac{64}{9\pi^4 N} d(r).$$

The distance $d(r)$ between the fixed points has been defined eliminating an irrelevant factor. In the limit $r \rightarrow 0$ the order $\mathcal{O}(1/N)$ of $\Delta a'(r)$ tends to zero. However, this does not mean that the flow is trivial. Indeed, it was proved in [9] that the limit $r \rightarrow 0$ exists and is a non-trivial RG flow. Therefore we expect that the subleading orders give $\Delta a' > 0$ also in this case.

Test of the prediction. Consider the theory

$$\mathcal{L} = \sum_{i=1}^N \bar{\psi}_i \not{\partial} \psi_i + \sum_{j=1}^{r_1 N} \bar{\chi}_j \not{\partial} \chi_j + \sum_{k=1}^{r_2 N} \bar{\zeta}_k \not{\partial} \zeta_k + \lambda \sigma \left(\sum_{i=1}^N \bar{\psi}_i \psi_i + g_1 \sum_{j=1}^{r_1 N} \bar{\chi}_j \chi_j + g_2 \sum_{k=1}^{r_2 N} \bar{\zeta}_k \zeta_k \right). \quad (9.7)$$

Three sets of fermions are coupled together by means of two couplings g_1 and g_2 , with $0 \leq g_{1,2} \leq 1$. The fixed points are $(g_1, g_2) = (0, 0), (0, 1), (1, 0), (1, 1)$. We study the triangle of flows

$$\begin{aligned} F_{12} : \quad g_2 &\equiv 0, \quad 0 \leq g_1 \leq 1; & F_{13} : \quad g_1 &\equiv g_2, \quad 0 \leq g_1 \leq 1; \\ F_{23} : \quad g_1 &\equiv 1, \quad 0 \leq g_2 \leq 1. \end{aligned} \quad (9.8)$$

with conformal fixed points

$$C_1 = \Sigma_N \otimes \Psi_{N(r_1+r_2)}; \quad C_2 = \Sigma_{N(1+r_1)} \otimes \Psi_{Nr_2}; \quad C_3 = \Sigma_{N(1+r_1+r_2)}.$$

The distances

$$d_{12} = 1 - \frac{1}{1+r_1}, \quad d_{13} = 1 - \frac{1}{1+r_1+r_2}, \quad d_{23} = 1 - \frac{1+r_1}{1+r_1+r_2},$$

do satisfy the oriented-triangle inequalities (8.2). This is an indication in favor of irreversibility in odd dimensions, in non-trivial agreement with the understanding offered in this paper.

The results imply also that there exist non-degenerate triangles in odd dimensions, and therefore it is impossible to define a global a function such that $d(C_{UV}, C_{IR}) = a(C_{UV}) - a(C_{IR})$.

Another consequence is that not all classically conformal flows are geodesic. Indeed, the theory (9.7) depends on two parameters. Choosing more generic values of the couplings than the particular cases (9.8) it is possible to fill the triangle with flows that interpolate continuously between F_{13} and $F_{12}+F_{23}$. All of the flows are classically conformal, but their lengths vary continuously from d_{13} to $d_{12} + d_{23} > d_{13}$.

Conversely, it is still plausible that all geodesic flows are classically conformal.

10 Conclusions

In this paper I have studied several aspects of the irreversibility of the RG flow, and elaborated a conceptual picture that is consistent with the present knowledge and explains some facts that otherwise would appear to be somewhat mysterious. The investigation has direct connections with different research domains, such as the study of the topological and metric properties of the spaces of RG flows and conformal field theories. Little is known today about these difficult subjects. It is often compulsory to proceed empirically, or by means of conjectures and cross-checks, or axioms and logical implications. On the other hand, it is obvious that before spending a lot of effort to obtain rigorous proofs, it is better to have a clear idea of what might be worth trying to prove. I am convinced that the results of this paper are a good starting point to address the future research in this area.

In the most general terms, irreversibility is the statement that there exist no pairs of non-trivial flows of unitary theories connecting interchanged UV and IR fixed points. A primary goal of the paper was to elaborate a simple and clear set of axioms that imply irreversibility. It is worth to recapitulate the guidelines of the arguments. The scheme-invariant area $\Delta a'$ of the graph of the effective beta function between the fixed points is taken as definition of the length of the RG flow. Then the minimum of $\Delta a'$ in the space of flows connecting the same UV and IR fixed points defines the distance between the fixed points. Since the flows are oriented, the distance is oriented and it is possible to postulate “oriented-triangle” inequalities.

These notions form the “axioms of irreversibility”, because they imply the irreversibility of the RG flow, in even and odd dimensions. At the moment, I do not have a proof that (unitary) quantum field theory does satisfy the axioms of irreversibility, but I can test some non-trivial consequences of those axioms. In section 9 I have studied certain triangles made of classically-conformal flows in three dimensions and showed that the oriented-triangle inequalities are fulfilled. At the same time, those results show that there exist non-degenerate triangles in odd dimensions.

In even dimensions, more powerful tools are available, in particular there exists a candidate global a function for irreversibility. This and other arguments lead to a further conjecture, namely that in even dimensions the oriented distance between the fixed points coincides precisely with Δa . Again, a definitive proof of this conjecture is not available, but checks of its consequences are possible. In particular, the chain of inequalities $\Delta a' \geq \Delta a \geq 0$ and a few other facts imply that free massive scalars and fermions (but not vectors) always have $c \geq a$, which is true.

I recall that the existence of a global a function with the mentioned properties implies that all triangles are degenerate in even dimensions.

The no-go statements of section 2 are of course rigorously proved, but there it was easy to give proofs, since it was sufficient to exhibit counter-examples.

As said, one of the purposes of the investigation of this paper is to make a first attempt to characterize the space of conformal field theories and flows in higher dimensions. Several subspaces, i.e. classes of flows and conformal theories, need to be classified and characterized. In the long range, the final goal of this kind of investigation is to establish with sufficient precision to which classes QCD, the Standard Model and Gravity belong, and explain their phenomenological properties, maybe also quantitatively, using this information.

The difficulties of this kind of research smear out when theories in various dimensions are compared with one another. In particular, it is useful to compare even and odd dimensions and, in even dimensions, dimension two and dimension greater than two. Important tools for the classification of fixed points and flows are the definitions of length of the RG flow, distance between the fixed points, oriented distance and irreversibility, with and without a global a function. In odd dimensions a global a function does not exist, but a global a function is not necessary to have irreversibility. I believe that irreversibility holds also in odd dimensions, in the more general sense elaborated here.

The irreversibility of the RG flow has a variety of implications. For example, in even dimensions, where the counter a of degrees of freedom is globally defined, irreversibility might explain why quantizing the theories from the IR is often problematic: it is reasonable to expect that climbing against the stream of irreversibility, the missing degrees of freedoms should be added manually. This might be the reason why *i*) QED has the Landau pole; *ii*) the φ_4^4 theory is probably trivial; *iii*) gravity – seen from the IR – is non-renormalizable; *iv*) examples of IR-free, UV-interacting conformal windows in even dimensions are not known. Instead, in odd dimensions there exist IR-free, UV-interacting RG flows and their quantization does not exhibit particular difficulties, maybe because it is possible to interpolate between the fixed points exactly in the running couplings at each order of the large N expansion [9].

The results of this paper stress once again that to fully understand the properties of quantum field theory we need more powerful tools than the ones we are accustomed to, a new framework and maybe a new language. Hopefully, quantum field theory is going to please us with some interesting surprises in the future.

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