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KINEMATIC SUM RULES FOR TRACE ANOMALIES

D. Anselmi

Dipartimento di Fisica, Università di Pisa, via F. Buonarroti 2, 56126 Pisa, Italia

Abstract

I derive a procedure to generate sum rules for the trace anomalies a and a'. Linear combinations of $\Delta a \equiv a_{\rm UV} - a_{\rm IR}$ and $\Delta a' \equiv a'_{\rm UV} - a'_{\rm IR}$ are expressed as multiple flow integrals of the two-, three- and four-point functions of the trace of the stress tensor. Eliminating $\Delta a'$, universal flow invariants are obtained, in particular sum rules for Δa. The formulas hold in the most general renormalizable quantum field theory (unitary or not), interpolating between UV and IR conformal fixed points. I discuss the relevance of these sum rules for the issue of the irreversibility of the RG flow. The procedure can be generalized to derive sum rules for the trace anomaly c.

1 Introduction

Quantum field theory of particles and fields of spin 0, 1/2 and 1 can be, without loss of generality, embedded in external gravity. The gravitational embedding can be useful to study properties of the ultraviolet and infrared fixed points of the renormalization-group (RG) flow. The correlation functions of the stress-tensor $T_{\mu\nu}$ are encoded in the induced action for the gravitational background and define quantities which characterize conformal and running quantum field theories.

At criticality, the trace anomaly in external gravity defines two central charges, denoted by c and a:

$$\Theta_* = \frac{1}{(4\pi)^2}\left[c\,W^2 - \frac{a}{4}\,\mathrm{G} + \frac{2}{3}\,a'\,\Box R\right], \tag{1.1}$$

where W is the Weyl tensor and $\mathrm{G} = 4R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 16R_{\mu\nu}R^{\mu\nu} + 4R^2$ is the Euler density. In free-field theories, we have $c = (n_s + 6n_f + 12n_v)/120$ and $a = (n_s + 11n_f + 62n_v)/360$, where n_s, n_f and n_v are the numbers of real scalar fields, Dirac fermions and vectors, respectively. The quantity a' is more peculiar, because it does not have a definite value at criticality.

Off-criticality, c and a depend on the energy scale. In a variety of cases, it is possible to compute the exact IR values of c and a in asymptotically-free supersymmetric gauge theories [1, 2].

The exact results of [1, 2] show that the UV values of the central charge a is always larger than its IR value, as conjectured by Cardy in [3]. The property $a_{\mathrm{UV}} \geq a_{\mathrm{IR}}$ is named "irreversibility of the RG flow", or "a-theorem". Sometimes, the name "c-theorem" is borrowed from the two-dimensional theorem of Zamolodchikov [4]. The central charge c does not satisfy an analogous property in four dimensions.

The study of the gravitational embedding in four dimensions is in general a difficult task. The perturbative calculations have been pioneered by Hathrell [5, 6], who worked out the values of c, a, and a' to the second (c) and third (a, a') loop orders. More general methods combine conformal properties and renormalization-group techniques [7]. Recently, Cappelli *et al.* [8] have classified the structure of the stress-tensor three-point function off criticality and obtained sum rules for the anomalies c and a. This classification is rather involved. Conceivably, the classification of the four-point function, which is potentially useful for the investigation of the irreversibility of the RG flow, is even harder. Another approach to the sum rules for trace anomalies is the one of [5, 6] and [15].

Other important results concern the induced action Γ for the gravitational background. Riegert [9] and others [10, 11] integrated the critical expression (1.1) of the trace anomaly with respect to the conformal factor. This procedure gives Γ up to conformally invariant terms. The conformally invariant terms missed by this method have not been written in closed form, so far. The Riegert action is made of some non-local terms, containing c and a, plus a unique, arbitrary, local term, $\int \sqrt{g}R^2$, multiplied by a'. The locality of this term explains why a' has no definite value at criticality and can be shifted by an arbitrary constant. This shift does not depend on the energy and disappears in the difference between the values of a' at two energies, e.g. $\Delta a' \equiv a'_{\rm UV} - a'_{\rm IR}$. Nevertheless, $\Delta a'$ remains dependent on the flow connecting the two fixed points, as shown in [12].

A considerable simplification occurs in conformally flat backgrounds. The correlation functions containing an arbitrary number of insertions of the trace Θ of the stress tensor can be studied. The restricted embedding looses track of the central charge c, but keeps track of a. The Riegert action specialised to conformally flat metrics is local and complete. It contains only two independent terms, multiplied by a and a'.

The so-specialised Riegert action encodes the UV and IR expressions of the Θ-correlators, in terms of a and a'. In this paper, I derive sum rules for the trace anomalies in the most general renormalizable (not necessarily unitary) quantum field theory interpolating between UV and IR conformal fixed points. The formulas are obtained exploiting the fact that the Θ-correlators have to tend to the UV and IR limits encoded in Γ. These sum rules are called "kinematic". The procedure naturally extends to more general background metrics, to derive sum rules for the trace anomaly c. I study the conformally flat background in detail and briefly describe this generalization.

The sum rules consist of flow integrals of the Θ-correlators in coordinate space, multiplied by polynomials of degree four in the coordinates. Every flow integral is equal to a linear combination of Δa and $\Delta a'$. Combining the sum rules, it is possible to eliminate $\Delta a'$ and obtain flow invariants, in particular sum rules for Δa. A flow invariant is a (multiple) flow integral of a correlator, whose value depends only on the extrema of the flow. Checks of the sum rules in massive theories are presented in detail. Further calculations, performed recently [13, 14], show explicitly that the flow-dependence of $\Delta a'$ cancels out in the sum rules for Δa, which therefore appear to be meaningful.

I discuss the possible applications of the sum rules to the problem of the irreversibility of the RG flow, comparing different, equivalent sum rules for Δa. These do not appear to imply the irreversibility of the RG flow in a straightforward way. In particular, I discuss certain difficulties to apply Osterwalder-Schrader positivity [16].

Finally, I describe the meaning of the results of [17] in the new framework. In [17], it was shown that a physical principle, suggested by the properties of renormalization, implies a certain sum rule for Δa in unitary, classically conformal theories. The formula of [17] involves only the Θ-two-point function. The idea of [17] can be collected into a "dynamical" vanishing sum rule for the Θ-four-point function, not contained in the set of "kinematic" sum rules worked out here.

The approach of this paper can be considered alternative, if not competing, with those of [8] and [5, 6, 15]. At the moment, it is not clear which approach is more convenient for practical computations.

The construction of this paper generalises to arbitrary even dimensions [14].

The paper is organised as follows. In sect. 2, I introduce the notation and the general framework for the gravitational embedding and discuss the convergence of the flow integrals. In sect. 3, I study the UV and IR limits of the Θ-correlators and derive the sum rules. I comment on the possible scheme dependence of certain flow integrals and the scheme independence of the sum rules. In sect. 4, I derive flow invariants from the sum rules. In sect. 5, I give explicit examples and checks of the formulas. In sect. 6, I show that most sum rules are consequences of simple algebraic symmetry properties of the integrals, plus the property that an integrated-trace insertion is a scale derivative. In sect. 7, I write explicit sum rules for Δa, focusing, in particular, on unitary, classically conformal quantum field theories. I discuss the relevance of these formulas for the issue of the irreversibility of the RG flow. I also comment on the relation between the results of the present paper and those of [17] and on the difficulties to apply OS positivity. Section 8 contains the conclusions.

2 Preliminaries

In this section I describe the gravitational embedding and the regularization technique. I also comment on the convergence of correlators.

Gravitational embedding. The embedded theory is renormalizable [5, 6]. I add bare lagrangian terms of the form $\Gamma_0[g_{\mu\nu}] = \Gamma_a[g_{\mu\nu}] + \Gamma_b[g_{\mu\nu}]$, where

$$\Gamma_a[g_{\mu\nu}] = \int \mathrm{d}^n x\ \sqrt{g}\left[a_0 W^2 + b_0 \mathrm{G} + c_0 R^2\right], \qquad \Gamma_b[g_{\mu\nu}] = \int \mathrm{d}^n x\ \sqrt{g}\left[M_0^2 R + \Lambda_0\right]. \tag{2.1}$$

I use the dimensional-regularization technique in the Euclidean framework. The space-time dimension is $n = 4 - \varepsilon$. A further term, $\int \mathrm{d}^n x\,\sqrt{g}\Box R$, can be omitted in Γ_a, since $\sqrt{g}\Box R$ is a total derivative in every n. The integral $\int \mathrm{d}^n x\,\sqrt{g}\mathrm{G}$ is kept, since G reduces to the Euler density only in four dimensions. The expression of G in n dimensions is equal to the one given in the previous section. The integral $\int \mathrm{d}^n x\,\sqrt{g}W^2$ is conformal invariant only in four dimensions. The coefficients a_0, b_0, c_0, M_0 and Λ_0 are independent (bare) parameters, which appropriately reabsorb the divergences. In classically conformal theories (e.g. massless QED), M_0 and Λ_0 are absent [5, 6]. I have separated the "dimensionless divergences" $\Gamma_a[g_{\mu\nu}]$ from the "dimensioned divergences" $\Gamma_b[g_{\mu\nu}]$ for later convenience.

The induced action for the external metric is defined as

$$\Gamma[g_{\mu\nu}] \equiv \Gamma_0[g_{\mu\nu}]\ + \Gamma'[g_{\mu\nu}] = \Gamma_0[g_{\mu\nu}] - \ln \int [\mathrm{d}\varphi]\ \exp\left(-S[\varphi, g_{\mu\nu}]\right), \tag{2.2}$$

where φ collectively denotes the dynamical fields of the theory and $S[\varphi, g_{\mu\nu}]$ is the action embedded in the external metric $g_{\mu\nu}$.

A great simplification occurs, if the background metric is restricted to be conformally flat, $g_{\mu\nu} = \delta_{\mu\nu}\ \mathrm{e}^{2\phi}$. The conformal factor ϕ couples to the trace Θ of the stress tensor. It can be proved that the $\Gamma_a[\phi]$ is finite. Its $\varepsilon \to 0$ limit is precisely the UV expression of the Riegert action for conformally flat metrics.

More precisely, we can distinguish two classes of theories: the classically conformal theories and the theories which are not conformal at the classical level.

In classically conformal theories, the quartic, cubic, quadratic and linear diverges can be canonically set to zero. This means that $\Gamma_b[\phi]$ is absent. Moreover, it can be proved that the logarithmic divergences of the Θ-correlators resum to zero. This convergence property is known for the two-point correlator of Θ (see for example [17] and sect. 1.1 of [12]). In this paper, it will be proved in complete generality (it is a consequence of the convergence of the sum rules). The convergence of the sum rules proves also that the action $\Gamma[\phi]$ is fully convergent off criticality in the classically conformal theories.

If the theory is not conformal at the classical level (e.g. some field are massive), $\Gamma_b[\phi]$ remains divergent. Nevertheless, this term does not affect the sum rules. It can be projected away imposing certain restrictions on the test functions. The argument mentioned above to show that $\Gamma_a[\phi]$ has a finite $\varepsilon \to 0$ limit applies also to this case.

The induced action for the conformal factor ϕ (the $\varepsilon \to 0$ limit of $\Gamma_a[\phi]$) reads at criticality [9, 10, 11]

$$\Gamma^*[\phi] = \frac{1}{8\pi^2}\int \mathrm{d}^4x \left\{ a_*(\Box\phi)^2 + (a'_* - a_*)\left[\Box\phi + (\partial_\mu\phi)^2\right]^2 \right\}. \tag{2.3}$$

This is the specialisation of the Riegert action [9] to conformally flat metrics (see also [17]).

I stress that I specialise *ab initio* to conformally flat metrics $\delta_{\mu\nu}\mathrm{e}^{2\phi}$, and use the dimensional-regularization technique. This strategy is convenient for the purposes of this paper, but has a little drawback: since $\Gamma^*[\phi]$ is local, the critical values of the coefficient a are not calculable in this framework. The coefficient a can be calculated at criticality in the following two cases: when the dimensional-regularization technique is used, but the background metric $g_{\mu\nu}$ is kept generic – then a multiplies a non-local term of the stress-tensor three-point function and a pole of $\Gamma_a[g_{\mu\nu}]$; when a different regularization technique (e.g. Pauli-Villars) is used – then it is possible to use directly the conformally flat metric. In practice, in the framework of this paper, a behaves like a', since both terms of $\Gamma^*[\phi]$ are on the same footing. This has no effect on the calculations of this paper, which are about the differences $\Delta a = a_{\mathrm{UV}} - a_{\mathrm{IR}}$ and $\Delta a' = a'_{\mathrm{UV}} - a'_{\mathrm{IR}}$.

Summarising, we can write

$$\Gamma[\phi] = \Gamma^*[\phi] + \Gamma_b[\phi] - \ln\int[\mathrm{d}\varphi]\,\exp\left(-S[\varphi,\phi]\right).$$

It will be proved later that this $\Gamma^*[\phi]$ is precisely $\Gamma^{\mathrm{UV}}[\phi]$, which means expression (2.3) with $a_*, a'_* \to a_{\mathrm{UV}}, a'_{\mathrm{UV}}$.

3 Derivation of the sum rules

In this section, I derive the sum rules for Θ-correlators.

Let $T_{\mu\nu} = 2/\sqrt{g}\ \delta S/\delta g^{\mu\nu}$ be the stress tensor, Θ its trace and $\overline{\Theta} = \sqrt{g}\Theta = -\delta S/\delta\phi$. The functional derivatives $\delta^{(k)}\Gamma[\phi]/(\delta\phi(x_1)\cdots\delta\phi(x_k))$ of the induced action Γ restricted to the conformal factor ϕ are denoted by $\Gamma_{x_1\cdots x_k}$. A similar notation is used for $\Gamma'_{x_1\cdots x_k}$ and the functional derivatives of $\overline{\Theta}(x)$.

I begin with the relations between the functional derivatives of Γ' and the $\overline{\Theta}$-correlators. Successively differentiating, we have $\Gamma'_x = -\langle\overline{\Theta}(x)\rangle$ and

$$\Gamma'_{x_1x_2}[\phi] = -\langle\overline{\Theta}(x_1)\,\overline{\Theta}(x_2)\rangle - \langle\overline{\Theta}_{x_2}(x_1)\rangle,$$

$$
\begin{aligned}
\Gamma'_{x_1x_2x_3}[\phi] = &-\langle\overline{\Theta}(x_1)\,\overline{\Theta}(x_2)\,\overline{\Theta}(x_3)\rangle - \langle\overline{\Theta}_{x_3}(x_1)\,\overline{\Theta}(x_2)\rangle - \langle\overline{\Theta}(x_1)\,\overline{\Theta}_{x_3}(x_2)\rangle \\
&- \langle\overline{\Theta}_{x_2}(x_1)\,\overline{\Theta}(x_3)\rangle - \langle\overline{\Theta}_{x_2x_3}(x_1)\rangle, \\
\Gamma'_{x_1x_2x_3x_4}[\phi] = &-\langle\overline{\Theta}(x_1)\,\overline{\Theta}(x_2)\,\overline{\Theta}(x_3)\,\overline{\Theta}(x_4)\rangle - \sum_{\{6\}}\langle\overline{\Theta}_{x_l}(x_i)\,\overline{\Theta}(x_j)\,\overline{\Theta}(x_k)\rangle - \sum_{\{4\}}\langle\overline{\Theta}_{x_kx_l}(x_i)\,\overline{\Theta}(x_j)\,\rangle \\
&- \sum_{\{3\}}\langle\overline{\Theta}_{x_k}(x_i)\,\overline{\Theta}_{x_l}(x_j)\,\rangle - \langle\,\overline{\Theta}_{x_2x_3x_4}(x_1)\rangle,
\end{aligned}
\tag{3.1}
$$

etc. The notation is as follows. In the expression $\overline{\Theta}_{x_jx_k\cdots x_l}(x_i)$, it is understood that $i < j, k, \cdots, l$. The number in curly brackets is the number of ways to distribute the indices with this constraint. The symbol $\langle\cdots\rangle$ denotes the connected components of the correlators.

Criticality. Using (2.3), we find that, at criticality, $\Gamma^*_x[0] = 0$ and $\Gamma^*_{x_1\cdots x_k}[\phi] = 0$ for $k > 4$. With the help of test functions u, we find that the other functional derivatives of Γ satisfy

$$
\begin{aligned}
\int \mathrm{d}^4x\ u(x)\ \Gamma^*_{x0}[0] &= \frac{1}{4\pi^2}a'_*\Box^2 u(0), \\
\int \mathrm{d}^4x\ \mathrm{d}^4y\ u(x,y)\ \Gamma^*_{xy0}[0] &= \frac{a_* - a'_*}{\pi^2}\left[\Box^x\Box^y - (\partial^x\cdot\partial^y)^2\right]\ u(0,0), \\
\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,u(x,y,z)\,\Gamma^*_{xyz0}[0] &= \frac{a_* - a'_*}{\pi^2}\left[\Box^x\partial^y\cdot\partial^z + 2\,\partial^x\cdot\partial^y\,\partial^x\cdot\partial^z + \text{cycl.perms.}\right]u(0,0,0).
\end{aligned}
\tag{3.2}
$$

The last argument of $\Gamma^*_{x_1\cdots x_k}$ is set to zero using translational invariance.

Off-criticality. Off-criticality, the correlators (or combinations of correlators) $\Gamma_{x_1\cdots x_k}$ depend on the energy scale. We want to study the UV and IR limits of $\Gamma_{x_1\cdots x_k}$ and relate them to (3.2). Using suitable test functions (satisfying some restrictions explained below), we have

$$
\lim_{\mathrm{UV\,(IR)}}\int\prod_{i=1}^{k}\mathrm{d}^4x_i\ u(x_1,\cdots,x_k)\ \Gamma_{x_1\cdots x_k0} = \int\prod_{i=1}^{k}\mathrm{d}^4x_i\ u(x_1,\cdots,x_k)\ \Gamma^{\mathrm{UV\,(IR)}}_{x_1\cdots x_k0}. \tag{3.3}
$$

It is understood that, after taking the ϕ-derivatives of Γ, ϕ is set to zero.

The UV and IR limits are defined as follows. Let μ be the dynamical scale. I collectively denote the dimensionful parameters of the theory with m. Concretely, in QCD we can take Λ_{QCD} and the quark masses (or an equivalent number of independent hadron masses). After the replacements $\mu, m \to \lambda\mu, \lambda m$ in $\Gamma_{x_1\cdots x_n}$, the UV and IR limits are $\lambda \to 0$ and $\lambda \to \infty$, respectively.

The terms of the form $\Gamma_b[\phi]$ contained in $\Gamma[\phi]$ can be projected away with a clever choice of the test functions u. The limits (3.3) exist if the theory interpolates between well-defined IR and UV conformal fixed points, which I assume. The difference between the UV and IR values of the central charges a and a' can then be expressed by certain integrals. These are assured to be convergent by the very same existence and finiteness of $\Gamma[\phi]$ at criticality, where it is equal to (2.3).

Sum-rule generator. I rescale μ and m by a factor λ and denote the rescaled correlators by $\Gamma^{\lambda}_{x_1\cdots x_k}$. Making a change of variables $x_i \to x_i/\lambda$ in the integrals, we get expressions of the form

$$\int \prod_{i=1}^{k} \mathrm{d}^4 x_i \; u(x_1, \cdots, x_k) \; \Gamma^{\lambda}_{x_1 \cdots x_k 0} = \lambda^4 \int \prod_{i=1}^{k} \mathrm{d}^4 x_i \; u(x_1/\lambda, \cdots, x_k/\lambda) \; \Gamma_{x_1 \cdots x_k 0}.$$

As anticipated above, we have to impose certain conditions on the test functions u. At $\{x_i\} = \{0\}$, we demand that u vanishes together with its first three derivatives with respect to all coordinates:

$$u(0) = \partial_i u(0) = \partial_i \partial_j u(0) = \partial_i \partial_j \partial_k u(0) = 0, \tag{3.4}$$

where $\partial_i = \partial/\partial x_i$. These conditions cancel out the quartic, cubic, quadratic and linear divergences from the integrals, and project $\Gamma_b[\phi]$ away. From now on, I will always omit the irrelevant terms $\Gamma_b[\phi]$.

To implement the above conditions more explicitly, I take a test function of the form

$$u(x_1, \cdots, x_k) = \sum_{\{k_i\}} \prod_{i=1}^{4} \frac{1}{k_i!} \left(x_i \cdot \partial_i\right)^{k_i} U(x_1, \cdots, x_k), \tag{3.5}$$

where the sum runs over all sets of $k_i = 0, 1, 2, 3$ or 4, such that $\sum_{i=1}^{k} k_i = 4$. The fourth derivatives of u in the origin coincide with those of U: $\partial_i \partial_j \partial_k \partial_l u(0) = \partial_i \partial_j \partial_k \partial_l U(0)$. With the parametrisation (3.5), it is sufficient to demand that the function U be regular and bounded. We have in the end

$$\int \prod_{i=1}^{k} \mathrm{d}^4 x_i \; \sum_{\{k_i\}} \prod_{i=1}^{4} \frac{1}{k_i!} \left(x_i \cdot \partial_i\right)^{k_i} U(x_1/\lambda, \cdots, x_k/\lambda) \; \Gamma_{x_1 \cdots x_k 0}.$$

In the IR limit $\lambda \to \infty$, the result is

$$\sum_{\{k_i\}} \prod_{i=1}^{4} \frac{1}{k_i!} \left(\partial_i^{\mu_i}\right)^{k_i} u(0) \int \prod_{i=1}^{k} \mathrm{d}^4 x_i \; \left(x_i^{\mu_i}\right)^{k_i} \; \Gamma_{x_1 \cdots x_k 0}.$$

Here the expression $\left(x_i^{\mu_i}\right)^{k_i} \left(\partial_i^{\mu_i}\right)^{k_i}$ stands for $\left(x_i \cdot \partial_i\right)^{k_i}$. Equation (3.3) gives then

$$\int \prod_{i=1}^{k} \mathrm{d}^4 x_i \; u(x_1, \cdots, x_k) \; \Gamma^{\mathrm{IR}}_{x_1 \cdots x_k 0} = \sum_{\{k_i\}} \prod_{i=1}^{4} \frac{1}{k_i!} \left(\partial_i^{\mu_i}\right)^{k_i} u(0) \int \prod_{i=1}^{k} \mathrm{d}^4 x_i \; \left(x_i^{\mu_i}\right)^{k_i} \; \Gamma_{x_1 \cdots x_k 0}. \tag{3.6}$$

This proves that the integrals on the right-hand side converge. Being this true for arbitrary u, the logarithmic divergences of $\Gamma_{x_1 \cdots x_k 0}$ resum to zero. From (3.1), we conclude that the logarithmic divergences of every $\overline{\Theta}$-correlator also vanish, after resummation.

The UV limit $\lambda \to 0$ can be studied as follows. The local part of $\Gamma_{x_1 \cdots x_k}$ (terms of the form $\prod_{i=2}^{k} \partial^{k_i} \delta(x_1 - x_i)$ with $\sum_{i=1}^{n} k_i = 4$) is invariant under the λ rescaling. This contribution survives and gives terms proportional to $\partial^4 U(0) = \partial^4 u(0)$. Instead, the non-local part of $\Gamma_{x_1, \cdots, x_k}$ multiplies $\partial^4 U(x/\lambda) \to \partial^4 U(\infty) = 0$ and the same integral as in (3.6), which I have just proved to be convergent. We conclude from (3.3) that the local part of $\Gamma_{x_1 \cdots x_k}$ is just $\Gamma^{\mathrm{UV}}_{x_1 \cdots x_k}$.

Equation (3.6) is the master equation generating the sum rules we are going to study.

Sum rule for the two-point function. With $k=1$, we can read the sum rule for the two-point function. We have, from (3.2) and (3.6),

$$\frac{1}{4\pi^2}a'_{\mathrm{IR}}\Box^2 u(0)=\frac{1}{4!}\partial^\mu\partial^\nu\partial^\rho\partial^\sigma u(0)\int \mathrm{d}^4x\ x^\mu x^\nu x^\rho x^\sigma\ \Gamma_{x0}=\frac{1}{192}\Box^2 u(0)\int \mathrm{d}^4x\ |x|^4\ \Gamma_{x0}. \quad (3.7)$$

In the last step, I have used the fact that Γ_{x0} depends only on $|x|$. Furthermore, writing $\Gamma=\Gamma^{\mathrm{UV}}+\Gamma'$ (omitting the irrelevant term Γ_b), we have

$$\Gamma_{x0}=\Gamma'_{x0}+\frac{1}{4\pi^2}a'_{\mathrm{UV}}\Box^2\delta(x)=-\left\langle\Theta(x)\,\Theta(0)\right\rangle+\frac{1}{4\pi^2}a'_{\mathrm{UV}}\Box^2\delta(x). \quad (3.8)$$

We then recover the known sum rule for the central charge a' [17]:

$$\Delta a'\equiv a'_{\mathrm{UV}}-a'_{\mathrm{IR}}=\frac{\pi^2}{48}\int \mathrm{d}^4x\ |x|^4\ \left\langle\Theta(x)\,\Theta(0)\right\rangle. \quad (3.9)$$

I recall that the Θ-correlators (encoded in Γ') do not include local contributions (which are encoded in Γ^{UV}).

We can now repeat this procedure to extract the sum rules for the many-point functions of Θ. I begin with the three-point function.

Sum rules for the three-point function. Using straightforward identities such as

$$\int \mathrm{d}^4y\ y^\rho\ \Gamma_{xy0}=\frac{x^\rho}{x^2}\int \mathrm{d}^4y\ \ (x\cdot y)\ \Gamma_{xy0},$$
$$\int \mathrm{d}^4y\ y^\rho y^\sigma\ \Gamma_{xy0}=\frac{1}{3x^2}\int \mathrm{d}^4y\ \left[\delta^{\rho\sigma}\left(x^2y^2-(x\cdot y)^2\right)+\frac{x^\rho x^\sigma}{x^2}\left(4\,(x\cdot y)^2-x^2y^2\right)\right]\ \Gamma_{xy0},$$

we obtain

$$\begin{aligned}\int \mathrm{d}^4x\ \mathrm{d}^4y\ u(x,y)\ \Gamma^{\mathrm{IR}}_{xy0}=\frac{1}{576}\int \mathrm{d}^4x\ \mathrm{d}^4y\ \Gamma_{xy0}\ &\Big\{3\left(|x|^4\Box^x\Box^x+|y|^4\Box^y\Box^y\right)\\ &+2\left[\left(5x^2y^2-2\,(x\cdot y)^2\right)\Box^x\Box^y+2\left(4\,(x\cdot y)^2-x^2y^2\right)(\partial^x\cdot\partial^y)^2\right]\\ &+12\,(x\cdot y)\left(x^2\Box^x+y^2\Box^y\right)\partial^x\cdot\partial^y\Big\}\,u(0).\end{aligned}$$

Comparing (3.6) with (3.2), we have the formulas

$$\int \mathrm{d}^4x\ \mathrm{d}^4y\ |x|^4\ \Gamma_{xy0}=\int \mathrm{d}^4x\ \mathrm{d}^4y\ |y|^4\ \Gamma_{xy0}=0, \quad (3.10)$$

$$\int \mathrm{d}^4x\ \mathrm{d}^4y\ x^2\,(x\cdot y)\ \Gamma_{xy0}=\int \mathrm{d}^4x\ \mathrm{d}^4y\ y^2\,(x\cdot y)\ \Gamma_{xy0}=0, \quad (3.11)$$

$$\frac{\pi^2}{48}\int \mathrm{d}^4x\ \mathrm{d}^4y\ x^2y^2\ \Gamma_{xy0}=a_{\mathrm{IR}}-a'_{\mathrm{IR}}, \quad (3.12)$$

$$-\frac{\pi^2}{24}\int \mathrm{d}^4x\ \mathrm{d}^4y\ \ (x\cdot y)^2\ \Gamma_{xy0}=a_{\mathrm{IR}}-a'_{\mathrm{IR}}. \quad (3.13)$$

As before, we can isolate the local contributions (Γ^{UV}) from the rest and get

$$\int \mathrm{d}^4x\ \mathrm{d}^4y\ |x|^4\ \Gamma'_{xy0} = 0, \tag{3.14}$$

$$\int \mathrm{d}^4x\ \mathrm{d}^4y\ x^2\,(x\cdot y)\ \Gamma'_{xy0} = 0, \tag{3.15}$$

$$\frac{\pi^2}{48}\int \mathrm{d}^4x\ \mathrm{d}^4y\ x^2y^2\ \Gamma'_{xy0} = \Delta a' - \Delta a, \tag{3.16}$$

$$\int \mathrm{d}^4x\ \mathrm{d}^4y\ \ \left[x^2y^2 + 2\,(x\cdot y)^2\right]\ \Gamma'_{xy0} = 0. \tag{3.17}$$

Sum rules for the four-point function. The derivation of the sum rules for correlators with more Θ-insertions is entirely similar. With the four-point function, we have the following set of identically vanishing relations:

$$\int \mathrm{d}^4x\ \mathrm{d}^4y\ \mathrm{d}^4z\ P_4(x,y,z)\ \Gamma_{xyz0} = 0, \tag{3.18}$$

where $P_4(x,y,z)$ is one of the following monomials: $|x|^4, x^2\,(x\cdot y)\,, x^2y^2, (x\cdot y)^2$, and permutations. The other sum rules for the four-point function read

$$\int \mathrm{d}^4x\ \mathrm{d}^4y\ \mathrm{d}^4z\ u(x,y,z)\ \Gamma^{\mathrm{IR}}_{xyz0} = \frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y\ \mathrm{d}^4z\ \Gamma_{xyz0}\ \left(x^\mu x^\nu y^\rho z^\sigma \partial^x_\mu\partial^x_\nu\partial^y_\rho\partial^z_\sigma + \text{perms.}\right) u(0).$$

Using

$$\begin{aligned}\int \mathrm{d}^4y\,\mathrm{d}^4z\,y^\rho z^\sigma\,\Gamma_{xyz0} = \frac{1}{3x^2}\int \mathrm{d}^4y\,\mathrm{d}^4z\,\Gamma_{xyz0}\ \Big\{&\delta^{\rho\sigma}\left[x^2\,(y\cdot z) - (x\cdot y)\,(x\cdot z)\right]\\ &+\frac{x^\rho x^\sigma}{x^2}\left[4\,(x\cdot y)\,(x\cdot z) - x^2\,(y\cdot z)\right]\Big\},\end{aligned}$$

we finally arrive at

$$\Delta a' - \Delta a = \frac{\pi^2}{48}\int \mathrm{d}^4x\ \mathrm{d}^4y\ \mathrm{d}^4z\ \ (x\cdot y)\,(x\cdot z)\ \ \Gamma'_{xyz0}, \tag{3.19}$$

$$0 = \int \mathrm{d}^4x\ \mathrm{d}^4y\ \mathrm{d}^4z\ \ \left[x^2\,(y\cdot z) - (x\cdot y)\,(x\cdot z)\right]\ \ \Gamma'_{xyz0}. \tag{3.20}$$

Sum rules for the correlators with a higher number of insertions. We have

$$\int \mathrm{d}^4x_1\ \cdots\ \mathrm{d}^4x_n\ P_4(x_1,\cdots,x_n)\ \Gamma_{x_1\cdots x_n0} = 0 \tag{3.21}$$

for $n \geq 4$ and an arbitrary degree-four polynomial $P_4(x_1,\cdots,x_n)$.

Sum rules for Δc. In a similar way, from (1.1) it is possible to derive sum rules for the trace anomaly c, keeping the background metric generic. The Δc sum rules are expressed in terms of more complicated flow integrals, of the form

$$\int \prod_{i=1}^{k} \mathrm{d}^4x_i\ P_4^{\mu_1\nu_1\cdots\mu_k\nu_k}(x_1,\cdots,x_k)\ \frac{\delta^{(k+1)}\Gamma}{\delta g_{\mu_1\nu_1}(x_1)\cdots\delta g_{\mu_k\nu_k}(x_k)\ \delta\phi(0)},$$

where now P_4 is a polynomial tensor of degree 4 in the coordinates. As usual, it is understood that the Γ-derivatives in the integrand are evaluated in flat space. These flow integrals involve correlators containing intertions of one trace Θ and an arbitrary number of stress tensors $T_{\mu\nu}$, plus their derivatives with respect to the metric, such as $\delta\Theta/\delta g_{\mu\nu}$ and $\delta T_{\rho\sigma}/\delta g_{\mu\nu}$.

Scheme independence of the sum rules. I now show that the sum rules are scheme independent. The sum rules are made of flow integrals of correlators which contain insertions of the operator Θ and the ϕ-derivatives of Θ, as can be read in (3.1). These operators are finite. In a unitary, renormalizable quantum field theory, Θ has the form

$$\Theta = -m_s^2\varphi^2 - m_f\ \overline{\psi}\psi - \frac{\beta_\alpha}{4\alpha}F_{\mu\nu}^{a\ 2} - \beta_\lambda\ \varphi^4 - \beta_g\ \varphi\overline{\psi}\psi, \tag{3.22}$$

up to terms proportional to the field equations. In a generic background ϕ, the ϕ-derivatives of Θ discriminate the mass terms from the terms proportional to the beta functions. However, $m_s^2\varphi^2$ and $m_f\ \overline{\psi}\psi$ are themselves finite (by definition, the renormalization constants of m_s and m_f compensate the renormalization constants of φ^2 and $\overline{\psi}\psi$, respectively). Therefore, both Θ and its ϕ-derivatives are finite.

Now, the correlators of gauge-invariant, finite operators are scheme independent at distinct points, after the perturbative series is resummed. However, the flow integrals are sensitive also to the contributions of the coinciding points, which, in principle, can be responsible of a scheme dependence. We have to show that the contributions of the coinciding points do not spoil the scheme independence of the sum rules.

To have control on the scheme effects, it is convenient to work with the effective action Γ, rather than the separate correlators. The scheme dependence of Γ is classified by the set of arbitrary finite local terms that can be added to Γ. These have the same form as Γ_a and Γ_b in (2.1), with finite coefficients. As far as the terms of type Γ_a are concerned, $\int\sqrt{g}W^2$ is zero on conformally flat metrics, $\int\sqrt{g}\mathrm{G}$ vanishes in four dimension (because $\sqrt{g}\mathrm{G}$ is a total derivative) and $\int\sqrt{g}R^2$ is responsible for the scheme dependence of a'. The quantity a' can be shifted by an arbitrary constant, independent of the energy. This ambiguity disappears in the difference $\Delta a'$, which however remains dependent on the flow connecting the two fixed points. On the other hand, the terms of the type Γ_b are projected away from the sum rules, as I have previously remarked.

In the sum rules, the combinations $\Gamma'_{x_1\cdots x_k}$ appear, rather than the separate correlators. The objects $\Gamma'_{x_1\cdots x_k}$ are combinations of correlators such that the scheme dependences mutually cancel. Note that the "prime" in $\Gamma'_{x_1\cdots x_k}$ (see (2.2)) projects away also Γ_a and Γ_b. For this reason, the sum rules are scheme independent.

In more detail, we see from (3.1) that $\Gamma'_{x_1\cdots x_k}$ is the sum of $-\langle\overline{\Theta}(x_1)\cdots\overline{\Theta}(x_k)\rangle$ plus correlators containing insertions of the ϕ-derivatives $\overline{\Theta}_{x_1\cdots x_j}(x_{j+1})$. The correlator $\langle\overline{\Theta}(x_1)\cdots\overline{\Theta}(x_k)\rangle$ is scheme independent at distinct points, but scheme dependent at coinciding points. The correlators containing insertions of $\overline{\Theta}_{x_1\cdots x_l}(x_{l+1})$ contribute only when some points coincide. Their scheme depencence compensates the scheme dependence of $\langle\overline{\Theta}(x_1)\cdots\overline{\Theta}(x_k)\rangle$, so that the sum $\Gamma'_{x_1\cdots x_k}$ is everywhere scheme independent. In practical computations, it might be necessary to treat each correlator of $\Gamma'_{x_1\cdots x_k}$ separately. This can lead to intermediate scheme dependent expressions. It might also be practically difficult to separate Γ from Γ', especially non-perturbatively, while there is complete control on the (eventually resummed) perturbative calculations.

Other approaches to the study of the scheme dependence in the stress-tensor correlators can be found in [8] and [15].

4 Flow invariants

The sum rules of the previous section allow us to construct flow invariants. A flow invariant is a quantity defined as the integral of a correlation function along the RG flow, such that its value depends only on the fixed points of the flow. Flow invariants are useful to characterise RG flows. Hopefully, they can make some computations easier. For example, it might be possible to compute $a_{\rm IR}$ along more convenient flows connecting the same fixed points.

We know that a' does not have an unambiguous meaning at criticality, but a has. Consequently, Δa depends only on the end points of the flow, while $\Delta a'$ can depend on the particular flow connecting the end points. Explicit calculations [12] prove that there are models in which the flow dependence of $\Delta a'$ is non-trivial. In ref. [12], flow invariance was recovered by minimising the flow integral (3.9) over the trajectories connecting the same fixed points. With the knowledge gained in the present paper, we can write other universal flow invariants, and in particular sum rules for Δa, eliminating $\Delta a'$ from the identities of the previous section. Examples of Δa sum rules, to be used in the next sections, are *i*) the difference between (3.9) and (3.16) and *ii*) the difference between (3.9) and (3.19). Recent calculations [13] have verified that the flow dependence of $\Delta a'$ does cancel out in the sum rules for Δa. Further results supporting this conclusion will be published soon [14]. Therefore, the sum rules for Δa are non-trivial examples of flow invariants.

Other examples of flow invariants are (3.14), (3.15), (3.17), (3.18) (with the appropriate P_4's), (3.20) and (3.21).

I stress once again that the sum rules are completely general. They hold (at the perturbative level and at the non-perturbative level) in every renormalizable quantum field theory interpolating between conformal UV and IR fixed points. The theory need not be unitary, so also the higher-derivative theories treated in [12] are included. In the φ^4-theory the stress tensor admits an improvement term and therefore an arbitrary parameter (see [6], p. 189). This paramenter can be fixed with the minimum principle of [12, 13].

The construction of this paper naturally extends to quantum field theory in arbitrary even dimensions [14], where renormalizable theories are mostly non-unitary.

5 Examples and checks

To check the sum rules, I illustrate some examples.

Massive scalar field. The action in external gravity is

$$S=\frac{1}{2}\int \mathrm{d}^4x\ \sqrt{g}\left\{g^{\mu\nu}\partial_\mu\varphi\ \partial_\nu\varphi+\frac{1}{6}R\varphi^2+m^2\varphi^2\right\},$$

fixed by the requirement that it generates the improved stress tensor at $m=0$. Focusing on the conformal factor ϕ and eliminating a total derivative, we can simplify the action and write

$$S=\frac{1}{2}\int \mathrm{d}^4x\left\{\ \left[\partial_\mu\left(\varphi\ \mathrm{e}^\phi\right)\right]^2+m^2\varphi^2\mathrm{e}^{4\phi}\right\}.$$

This gives

$$\overline{\Theta}=-\frac{\delta S}{\delta\phi}=-2m^2\varphi^2\mathrm{e}^{4\phi}+\left(\varphi\ \mathrm{e}^\phi\right)\Box\left(\varphi\ \mathrm{e}^\phi\right).$$

To further simplify some formulas, it is useful to subtract a term proportional to the field equations, and define

$$\widetilde{\Theta}=\overline{\Theta}+\varphi\frac{\delta S}{\delta\varphi}=-m^2\varphi^2\mathrm{e}^{4\phi}. \tag{5.1}$$

We have

$$\Gamma'_x=-\left\langle\widetilde{\Theta}(x)\right\rangle,$$

since the additional term integrates by parts to zero. We take another functional derivative:

$$\Gamma'_{xy}=-\left\langle\widetilde{\Theta}(x)\ \overline{\Theta}(y)\right\rangle-\left\langle\frac{\delta\widetilde{\Theta}(x)}{\delta\phi(y)}\right\rangle.$$

In the first term, we insert $\widetilde{\Theta}$ at the place of $\overline{\Theta}$. The additional term $\varphi\delta S/\delta\varphi$ can be integrated by parts. We obtain

$$\begin{aligned}\Gamma'_{xy}&=-\left\langle\widetilde{\Theta}(x)\ \widetilde{\Theta}(y)\right\rangle+\left\langle\left(\varphi(y)\frac{\delta}{\delta\varphi(y)}-\frac{\delta}{\delta\phi(y)}\right)\widetilde{\Theta}(x)\right\rangle\\&=-\left\langle\widetilde{\Theta}(x)\ \widetilde{\Theta}(y)\right\rangle-2\delta(x-y)\left\langle\widetilde{\Theta}(x)\right\rangle.\end{aligned}\tag{5.2}$$

Iterating this procedure, we have

$$\begin{aligned}\Gamma'_{xyz}=&-\left\langle\widetilde{\Theta}(x)\ \widetilde{\Theta}(y)\ \widetilde{\Theta}(z)\right\rangle-2\delta(x-y)\left\langle\widetilde{\Theta}(x)\ \widetilde{\Theta}(z)\right\rangle-2\delta(y-z)\left\langle\widetilde{\Theta}(y)\ \widetilde{\Theta}(x)\right\rangle\\&-2\delta(z-x)\left\langle\widetilde{\Theta}(z)\ \widetilde{\Theta}(y)\right\rangle-4\delta(x-y)\delta(x-z)\left\langle\widetilde{\Theta}(x)\right\rangle,\end{aligned}\tag{5.3}$$

and so on.

We know, from the free-field values given in the introduction, that $\Delta a=1/360$. Moreover, (5.1) and (3.9) give $\Delta a'=\Delta c=1/120$ [18].

I begin the checks with (3.14). The terms of Γ'_{xyz} containing two $\widetilde{\Theta}$-insertions give either the integral (3.9) or zero. We get the prediction

$$\begin{aligned}\Delta a'&=\frac{\pi^2m^6}{192}\int \mathrm{d}^4x\ \mathrm{d}^4y\ |x|^4\ \left\langle\varphi^2(x)\ \varphi^2(y)\ \varphi^2(0)\right\rangle\\&=\frac{\pi^2m^6}{24}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^2}\left(\frac{\partial^2}{\partial p^2}\right)^2\frac{1}{p^2+m^2}.\end{aligned}$$

The calculation is straightforward in momentum space and verifies the prediction. The check of (3.15) proceeds similarly.

The sum rule (3.16) can be converted to

$$3\Delta a' - \Delta a = \frac{\pi^2 m^6}{48} \int \mathrm{d}^4 x \ \mathrm{d}^4 y \ x^2 y^2 \ \left\langle \varphi^2(x) \ \varphi^2(y) \ \varphi^2(0) \right\rangle .$$

This can be verified immediately. The sum rule (3.17) becomes

$$3\Delta a' + \Delta a = \frac{\pi^2 m^6}{24} \int \mathrm{d}^4 x \ \mathrm{d}^4 y \ \ (x \cdot y)^2 \ \ \left\langle \varphi^2(x) \ \varphi^2(y) \ \varphi^2(0) \right\rangle$$

and is also verified.

For the sum rules involving the four-point function, it is necessary to differentiate (5.3) once more and then insert it into (3.18), (3.19) and (3.20). Using (3.19), we get, for example,

$$9\Delta a' - 5\Delta a = \frac{\pi^2}{96} \int \mathrm{d}^4 x \ \mathrm{d}^4 y \ \mathrm{d}^4 z \ x^2 \,(y-z)^2 \ \ \left\langle \widetilde{\Theta}(x) \ \widetilde{\Theta}(y) \, \widetilde{\Theta}(z) \, \widetilde{\Theta}(0) \right\rangle . \tag{5.4}$$

The check of this identity, which is verified, requires a non-trivial amount of work, always in momentum space. Similarly, using (3.20) we arrive at

$$3\Delta a' + \Delta a = \frac{\pi^2}{96} \int \mathrm{d}^4 x \ \mathrm{d}^4 y \ \mathrm{d}^4 z \ \ [x \cdot (y-z)]^2 \ \left\langle \widetilde{\Theta}(x) \ \widetilde{\Theta}(y) \, \widetilde{\Theta}(z) \, \widetilde{\Theta}(0) \right\rangle . \tag{5.5}$$

The check of the identities and sum rules with more $\widetilde{\Theta}$-insertions are left to the reader.

Massive fermion. We have the action

$$S = \int \mathrm{d}^4 x \ \left[\frac{1}{2} \left(\overline{\psi} \ \mathrm{e}^{3\phi/2} \right) \overleftrightarrow{\partial\!\!\!/} \left(\psi \ \mathrm{e}^{3\phi/2} \right) + m \ \overline{\psi} \psi \ \mathrm{e}^{4\phi} \right] .$$

As above, we can define a $\widetilde{\Theta}$ subtracting the field equations from $\overline{\Theta}$:

$$\widetilde{\Theta} \equiv -\frac{\delta S}{\delta \phi} + \frac{3}{2} \ \overline{\psi} \ \frac{\delta_l S}{\delta \overline{\psi}} + \frac{3}{2} \frac{\delta_r S}{\delta \psi} \psi = -m \ \overline{\psi} \psi \ \mathrm{e}^{4\phi} . \tag{5.6}$$

We find

$$\Gamma'_x = -\left\langle \widetilde{\Theta}(x) \right\rangle , \qquad\qquad \Gamma'_{xy} = -\left\langle \widetilde{\Theta}(x) \ \widetilde{\Theta}(y) \right\rangle - \delta(x-y) \left\langle \widetilde{\Theta}(x) \right\rangle ,$$

etc. For example, we find the prediction

$$2\Delta a' - \Delta a = \frac{\pi^2}{48} m^3 \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, x^2 \, y^2 \ \left\langle \overline{\psi}\psi(x) \, \overline{\psi}\psi(y) \, \overline{\psi}\psi(0) \right\rangle .$$

As usual, the integral can be more easily calculated in momentum space, and gives 5/72. This agrees with the prediction, since $\Delta a = 11/360$ and $\Delta a' = \Delta c = 1/20$ [18]. I leave the remaining checks to the reader.

Yang-Mills theory with massless fermions. We have to keep the dimension n different from 4. The action in the ϕ-background reads

$$S = \int \mathrm{d}^4 x \ \left[\frac{1}{4\alpha} F_{\mu\nu}^{a\ 2} \ \mathrm{e}^{-\varepsilon\phi} + \frac{1}{2} \left(\overline{\psi}^{\,i} \ \mathrm{e}^{(3-\varepsilon)\phi/2} \right) \overleftrightarrow{D\!\!\!\!/}_{ij} \left(\psi^j \ \mathrm{e}^{(3-\varepsilon)\phi/2} \right) \right] .$$

All quantities are bare. We define

$$\widetilde{\Theta} \equiv -\frac{\delta S}{\delta \phi} + \frac{3-\varepsilon}{2}\,\overline{\psi}\,\frac{\delta_l S}{\delta \overline{\psi}} + \frac{3-\varepsilon}{2}\frac{\delta_r S}{\delta \psi}\psi = \frac{\varepsilon}{4\alpha} F_{\mu\nu}^{a\ 2}\ \mathrm{e}^{-\varepsilon\phi} \tag{5.7}$$

and get

$$\Gamma'_x = -\left\langle \widetilde{\Theta}(x) \right\rangle, \qquad\qquad \Gamma'_{xy} = -\left\langle \widetilde{\Theta}(x)\ \widetilde{\Theta}(y) \right\rangle + \varepsilon\delta(x-y)\left\langle \widetilde{\Theta}(x) \right\rangle, \tag{5.8}$$

etc. The second term in Γ'_{xy} is negligible, because it is multiplied by ε. Similar terms are negligible in $\Gamma'_{x_1\cdots x_k}$.

The proof that these evanescent contact terms are negligible can be done as follows. Let us consider, for example, the term $\varepsilon\delta(x_1-x_2)\left\langle \widetilde{\Theta}(x_2)\cdots\widetilde{\Theta}(x_k) \right\rangle$ in $\Gamma'_{x_1\cdots x_k}$. Inserted into a flow integral for the k^{th}-point function of $\widetilde{\Theta}$, this term, after integration over x_1, produces a convergent flow integral for the $(k-1)^{\mathrm{th}}$-point function, multiplied by ε. Clearly, the ε-factor kills this contribution.

Concluding, we can write

$$\Gamma'_{x_1\cdots x_k} = -\left\langle \widetilde{\Theta}(x_1)\cdots\widetilde{\Theta}(x_k) \right\rangle + \mathcal{O}(\varepsilon). \tag{5.9}$$

Similar arguments show that the contact terms of the correlators $\left\langle \widetilde{\Theta}(x_1)\cdots\widetilde{\Theta}(x_k) \right\rangle$ are themselves evanescent and do not contribute to the sum rules. This can be seen from the operator-product expansion of two Θs.

At the level of renormalized operators, I recall that, in flat space, [5]

$$\widetilde{\Theta} = -\frac{\widehat{\beta}(\alpha)}{4\alpha}\left[F_{\mu\nu}^{a\ 2}\right] + \frac{1}{2}\gamma[\overline{\psi}\,\overleftrightarrow{\not{D}}\psi] \equiv \widetilde{\Theta}' + \frac{1}{2}\gamma[\overline{\psi}\,\overleftrightarrow{\not{D}}\psi],$$

γ denoting the anomalous dimension of the fermions. Here the beta function is defined as $\widehat{\beta} = \mathrm{d}\ln\alpha/\mathrm{d}\ln\mu = \beta - \varepsilon$.

In the correlator (5.9), we can freely replace $\widetilde{\Theta}$ with $\widetilde{\Theta}'$, since the term proportional to the fermion-field equation gives no contribution. This can be proved as follows. First, notice that the renormalized and bare operators $\overline{\psi}\,\overleftrightarrow{\not{D}}\psi$ coincide. The insertions of $\overline{\psi}\,\overleftrightarrow{\not{D}}\psi$ inside the correlators can be integrated by parts. This generates insertions of objects of the form

$$\overline{\psi}\,\frac{\delta_l A}{\delta\overline{\psi}} + \frac{\delta_r A}{\delta\psi}\psi, \tag{5.10}$$

with A equal to $\widetilde{\Theta}$, $\widetilde{\Theta}'$ or $\overline{\psi}\,\overleftrightarrow{\not{D}}\psi$. For $A = \widetilde{\Theta}$, (5.10) is zero. This is easily seen from the bare expression (5.7) of $\widetilde{\Theta}$. This fact proves also that, for $A = \widetilde{\Theta}'$, (5.10) is proportional to $\overline{\psi}\,\overleftrightarrow{\not{D}}\psi$, times a delta function. This is trivially true also for $A = \overline{\psi}\,\overleftrightarrow{\not{D}}\psi$. So, the integration by parts of the $\overline{\psi}\,\overleftrightarrow{\not{D}}\psi$-insertion returns an expression which contains at least another $\overline{\psi}\,\overleftrightarrow{\not{D}}\psi$-insertion. Then, the procedure can be iterated, with a new integration by parts. In the end, we get zero. In conclusion, a correlator with an arbitrary number of insertions of $\widetilde{\Theta}'$ and at least one insertion of $\overline{\psi}\,\overleftrightarrow{\not{D}}\psi$, is equal to zero.

Formula (5.9) is valid for all classically conformal quantum field theories with a finite stress tensor. In the φ^4-theory, where the stress tensor admits an improvement term, the finite stress tensor can be fixed with the method of [12].

6 Self-consistency of the sum rules

The sum rules have been obtained imposing that the Θ-correlators tend to the prescribed UV and IR limits, fixed by the Riegert action for conformally flat metrics. The Riegert action is obtained by integrating the trace anomaly with respect to the conformal factor. In turn, the form of the trace anomaly is fixed by dimensional counting and general covariance.

This suggests that the sum rules are mainly consequences of the general covariance of the gravitational embedding. In this section, I prove that it is not so. Most sum rules are related to one another by symmetry properties of the integrands. All vanishing sum rules follow from the property that an integrated-trace insertion is equal to the scale derivative of the correlator. General covariance imposes only one relation among the sum rules. This helps clarifying which sum rules have a truly non-trivial content.

Equivalent polynomials. The idea is the following. A sum rule has the general form

$$\int \mathrm{d}^4x_1 \ \cdots \ \mathrm{d}^4x_k \ P_4(x_1,\cdots,x_k)\ \Gamma'_{x_1\cdots x_k 0} = f\Delta a + g\Delta a', \tag{6.1}$$

with $(f,g)=(-1,1)$, $(0,1)$ or $(0,0)$. The flow integral of (6.1) can be rewritten as

$$\lim_{V\to\infty}\frac{1}{V}\int_V \mathrm{d}^4x_1 \ \cdots \ \mathrm{d}^4x_k \ \mathrm{d}^4x_{k+1}\ P_4(x_1-x_{k+1},\cdots,x_k-x_{k+1})\ \Gamma'_{x_1\cdots x_k x_{k+1}},$$

where the integrals are restricted to a finite volume V. Using translational invariance of $\Gamma'_{x_1\cdots x_k x_{k+1}}$, we can set $x_{k+1}=0$ in the integrand. The x_{k+1}-integration factorizes and, in the limit $V\to\infty$, it simplifies the factor $1/V$. Expression (6.1) is therefore recovered.

Translational invariance can be used to set any of the coordinates x_i to zero. This generates equivalent sum rules. We can define an equivalence relation $\sim$ among the polynomials P_4:

$$P_4(x_1,\cdots,x_k)\sim P_4(x_1-x_i,\cdots,x_{i-1}-x_i,-x_i,x_{i+1}-x_i,\cdots,x_k-x_i) \tag{6.2}$$

for every i. Eq. (6.2) is found setting $x_i=0$ and renaming x_{k+1} as x_i. Of course, equivalent polynomials are also those obtained from $P_4(x_1,\cdots,x_k)$ by permuting $x_1,\cdots,x_k$.

The case $k=2$. For example, if we take $k=2$, we get

$$\begin{array}{c}(x\cdot y)^2 \ \sim \ [(x-y)\cdot y]^2, \qquad |x|^4 \ \sim \ |x-y|^4, \qquad x^2y^2 \ \sim \ (x-y)^2y^2, \\ x^2\,(x\cdot y) \ \sim \ (x-y)^2\,(y-x)\cdot y \ \sim \ y^2\,x\cdot(x-y)\ .\end{array}$$

Working out these equivalence relations in more detail, it is easily seen that they reduce to two independent relations, namely

$$3|x|^4 \ \sim \ 6x^2\,(x\cdot y) \ \sim \ 4\,(x\cdot y)^2+2x^2y^2. \tag{6.3}$$

Therefore, the vanishing sum rules (3.14), (3.15) and (3.17) are equivalent. Concluding, there exists a unique vanishing sum rule for the three-point function of Θ, say (3.14), generated by $P_4(x,y)=|x|^4$.

Observe that since $P_4(x,y)=|x|^4$ does not depend on y, in (3.14) we can single out the insertion of an integrated trace. The sum rule (3.14) follows from the properties of this insertion, which I now derive in generality. For simplicity, I assume that the super-renormalizable parameters in the theory are just the masses. The derivation can be immediately extended to theories with other super-renormalizable parameters.

The generator of vanishing sum rules. If $S=\int \mathrm{d}^n x\,\mathcal{L}$ is the action and $\mathcal{L}$ the lagrangian, we define

$$\overline{\Theta}=-\frac{\delta S}{\delta\phi},\qquad \widetilde{\Theta}=-\frac{\tilde{\delta}S}{\tilde{\delta}\phi},\qquad \widehat{\Theta}=-\frac{\hat{\delta}S}{\hat{\delta}\phi},$$

where

$$\begin{aligned}\frac{\tilde{\delta}A}{\tilde{\delta}\phi}&\equiv\frac{\delta A}{\delta\phi}-\left(1-\frac{\varepsilon}{2}\right)\sum_{\varphi}\varphi\frac{\delta S}{\delta\varphi}-\frac{3-\varepsilon}{2}\sum_{\psi}\left(\overline{\psi}\,\frac{\delta_l A}{\delta\overline{\psi}}+\frac{\delta_r A}{\delta\psi}\psi\right),\\ \frac{\hat{\delta}A(x)}{\hat{\delta}\phi(y)}&\equiv\frac{\tilde{\delta}A(x)}{\tilde{\delta}\phi(y)}-\delta(x-y)\sum_m m\frac{\partial A(x)}{\partial m}.\end{aligned}$$

Here, φ, ψ and m denote collectively the scalar fields, fermions and masses of the theory. As illustrated in the previous section, $\overline{\Theta}$ contains three kinds of terms: field equations of the scalars and fermions; mass operators; evanescent terms. The operator $\widetilde{\Theta}$ contains only mass terms and evanescent terms. The operator $\widehat{\Theta}$ contains just the evanescent terms.

Integrating

$$\Gamma'_x=\frac{\delta\Gamma'}{\delta\phi(x)}=-\left\langle\widetilde{\Theta}(x)\right\rangle=-\left\langle\widehat{\Theta}(x)\right\rangle+\sum_m m\left\langle\frac{\partial\mathcal{L}(x)}{\partial m}\right\rangle$$

over x, we obtain

$$\int \mathrm{d}^n x\,\Gamma'_x=-\left\langle\int \mathrm{d}^n x\,\widehat{\Theta}(x)\right\rangle+\sum_m m\left\langle\frac{\partial S}{\partial m}\right\rangle. \tag{6.4}$$

A classical argument [19, 20], which applies unchanged at $\phi\neq 0$, can express the insertion of the integrated $\widehat{\Theta}$ in terms of a μ-derivative, namely

$$\left\langle\int \mathrm{d}^n x\,\widehat{\Theta}(x)\right\rangle=-\mu\frac{\partial\Gamma'}{\partial\mu}. \tag{6.5}$$

This is proved observing that, at the bare level, Γ depends on μ only because the bare coupling (say, the gauge coupling α) is dimensionful away from 4 dimensions. For a gauge coupling, we have $\alpha=\alpha'\mu^\varepsilon$, with α' dimensionless. Then, the μ-derivative of Γ can be written as

$$\mu\frac{\partial\Gamma'}{\partial\mu}=\varepsilon\alpha\frac{\partial\Gamma'}{\partial\alpha}=-\left\langle\int \mathrm{d}^n x\,\frac{\varepsilon}{4\alpha}F^2\,\mathrm{e}^{-\varepsilon\phi}\right\rangle=-\left\langle\int \mathrm{d}^n x\,\widehat{\Theta}(x)\right\rangle.$$

It is easy to extend the calculation to φ^4-interactions and Yukawa couplings. Then, we can rewrite (6.4) as

$$\int \mathrm{d}^n x\, \Gamma'_x = \sum_m m \frac{\partial \Gamma'}{\partial m} + \mu \frac{\partial \Gamma'}{\partial \mu}. \tag{6.6}$$

This is the generator of all vanishing sum rules.

Vanishing sum rules with insertions of an integrated trace. Taking k ϕ-derivatives of (6.6), we get

$$\int \mathrm{d}^n x\, \Gamma'_{x x_1 \cdots x_k} = \left(\mu \frac{\partial}{\partial \mu} + \sum_m m \frac{\partial}{\partial m} \right) \Gamma'_{x_1 \cdots x_k}.$$

Multiplying by an arbitrary degree-four polynomial $P_4(x_1, \cdots, x_{k-1})$, integrating over $x_1, \cdots, x_{k-1}$, and setting $x_k = 0$, we obtain

$$\int \mathrm{d}^n x \prod_{i=1}^{k-1} \mathrm{d}^n x_i\, P_4(x_1, \cdots, x_{k-1})\, \Gamma'_{x x_1 \cdots x_{k-1} 0}$$
$$= \left(\mu \frac{\partial}{\partial \mu} + \sum_m m \frac{\partial}{\partial m} \right) \int \prod_{i=1}^{k-1} \mathrm{d}^n x_i\, P_4(x_1, \cdots, x_{k-1})\, \Gamma'_{x_1 \cdots x_{k-1} 0}.$$

The integrals are equal to linear combinations of Δa and $\Delta a'$. These are dimensionless quantities and, in particular, they are annihilated by the scale-derivative operator $\mu\, \partial / \partial \mu + \sum_m m\, \partial / \partial m$ [1]. We conclude that

$$\int \mathrm{d}^n x \prod_{i=1}^{k-1} \mathrm{d}^n x_i\, P_4(x_1, \cdots, x_{k-1})\, \Gamma'_{x x_1 \cdots x_{k-1} 0} = 0 \tag{6.7}$$

for all k's and all P_4's. Since Γ_{UV} satisfies (6.7), we can replace Γ' with Γ in (6.7).

Formula (6.7) implies the vanishing sum rules which contain at least one integrated-trace insertion. The other vanishing sum rules can be obtained from (6.7), by applying the equivalence relations (6.2). For $k = 2$, this proves (3.14), (3.15) and (3.17).

The cases $k = 3, 4$ and higher.

For $k = 3$, there are only two polynomials for which (6.7) does not apply: these are $(x \cdot y)\,(x \cdot z)$ and $x^2\,(y \cdot z)$. Using (6.2) and (6.7), we get

$$(x \cdot y)\,(x \cdot z) \ \sim \ (x - y) \cdot y\, (x - y) \cdot (y - z) \ \sim \ x^2\,(y \cdot z)\,,$$

which proves (3.20). For $k = 4$, the only polynomial for which (6.7) does not apply is $(x \cdot y)\,(z \cdot w)$. We have

$$(x \cdot y)\,(z \cdot w) \ \sim \ -\,(x \cdot y)\,(z \cdot w) \ \sim \ 0.$$

For $k > 4$, (6.7) applies to all polynomials.

In conclusion, all vanishing sum rules can be derived from (6.7) and (6.2). Three non-vanishing sum rules are left: one for $k = 2$, one for $k = 3$ and one for $k = 4$. However, the independent quantities are just two: Δa and $\Delta a'$. The relation among the three non-vanishing flow integrals is due to the general covariance of the gravitational embedding.

[1] Observe that it is not necessary to assume that $\Delta a'$ is independent of the dimensionful parameters m and μ. Indeed, it was explicitly demonstrated in [12] that $\Delta a'$ does depend on the ratios among them.

7 On the irreversibility of the RG flow

In this section, I discuss the issue of the irreversibility of the RG flow and the possible relevance of the sum rules found here.

In [17], I have shown that a physical principle, precisely the statement that in unitary, classically conformal quantum field theories, the induced action $\Gamma[\phi]$ is positive definite at every energy, if it is positive definite at some energy, implies

$$\Delta a = \Delta a' \geq 0. \tag{7.1}$$

The physical principle was suggested by the consideration that only divergences can be responsible for a violation. However, the very evanescence of $\widehat{\Theta}$ makes $\Gamma[\phi]$ divergent-free. Equality (7.1) has been checked to the fourth-loop order in perturbation theory.

Independently of the arguments of [17], sum rules for Δa can be written using the formulas derived in the present paper. For example, we have

$$\begin{aligned}\Delta a &= -\frac{\pi^2}{48}\int \mathrm{d}^4x\,|x|^4\,\Gamma'_{x0} - \frac{\pi^2}{48}\int \mathrm{d}^4x\,\mathrm{d}^4y\,x^2\,y^2\,\Gamma'_{xy0} \\ &= -\frac{\pi^2}{48}\int \mathrm{d}^4x\,|x|^4\,\Gamma'_{x0} - \frac{\pi^2}{48}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,\,(x\cdot y)\,(x\cdot z)\,\Gamma'_{xyz0}.\end{aligned}$$

The relations between $\Gamma'_{x_1\cdot x_k}$ and the Θ-correlators are read from (3.1). In classically conformal field theories, the formulas can be simplified using (5.9).

The equality (7.1) of Δa and $\Delta a'$ amounts to an additional, "dynamical" vanishing rule, not contained in the set of "kinematic" vanishing rules of section 6. For example,

$$\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,\,(x\cdot y)\,(x\cdot z)\left\langle\widehat{\Theta}(x)\,\widehat{\Theta}(y)\,\widehat{\Theta}(z)\,\widehat{\Theta}(0)\right\rangle = 0,$$

in classically conformal theories. The evaluation of this integral in perturbation theory appears to be non-trivial and I am forced to postpone this to a future investigation.

On the application of Osterwalder-Schrader positivity. The next question is how to apply Osterwalder-Schrader positivity [16], which states that the expression

$$\sum_{k,m}\int f_k(x_1,\cdots,x_k)\,\bar{f}_m(\theta y_1,\cdots,\theta y_m)\,\left\langle\widehat{\Theta}(x_1)\cdots\widehat{\Theta}(x_k)\,\widehat{\Theta}(y_1)\cdots\widehat{\Theta}(y_k)\right\rangle, \tag{7.2}$$

is non-negative, for every set of functions $f_k(x_1,\cdots,x_k)$, vanishing unless $x_1^0 > \cdots > x_k^0 > 0$. The integral is in $\mathrm{d}^4x_1\cdots\mathrm{d}^4x_k\,\mathrm{d}^4y_1\cdots\mathrm{d}^4y_m$ and $\theta(x^0,x^1,x^2,x^3) = (-x^0,x^1,x^2,x^3)$. The positivity condition holds for every choice of the x^0-axis.

Application of OS positivity to flow integrals of the four-point function is not straightforward. Here I discuss the main difficulties.

The conditions on the functions f exclude the coincident points of the correlators from the integrals. The correlators might contain contact, semi-local terms (the local terms, instead, are contained in Γ^{UV}). The semi-local terms are cut away from the OS condition, but contribute to the sum rules of the previous sections.

It is possible to write equivalent sum rules for Δa, such that the polynomial appearing in the flow integral of the four-point function is positive. This, however, is not sufficient to apply OS positivity, whose formulation is intrinsically non covariant, because of the choice of a "time" axis and the condition that the functions f_k vanish when some of their arguments have non-positive "times".

I conclude that there does not appear to be a straightforward way to apply OS positivity and prove the irreversibility of the RG flow from the kinematic sum rules of this paper.

8 Conclusions

I have derived general sum rules for the anomalies a and a', expressing the differences $\Delta a = a_{\mathrm{UV}} - a_{\mathrm{IR}}$ and $\Delta a' = a'_{\mathrm{UV}} - a'_{\mathrm{IR}}$ as multi-flow integrals of correlators containing insertions of the trace of the stress tensor. Universal flow invariants are constructed by eliminating $\Delta a'$. The sum rules hold in the most general renormalizable quantum field theory (unitary or not), interpolating between UV and IR conformal fixed points. All vanishing sum rules can be derived from simple symmetry properties, combined with the fact that an integrated-trace insertion is equal to a scale derivative. The statements of [17] can be collected into a further, "dynamical" vanishing sum rule, not contained in the set of "kinematic" formulas found here. Application of Osterwalder-Shrader positivity to flow integrals of the four-point function is not immediate. The approach developed here can be naturally generalized to write sum rules for Δc.

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References

[1] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, Nonperturbative formulas for central functions in supersymmetric gauge theories, Nucl. Phys. B 526 (1998) 543 and hep-th/9708042.

[2] D. Anselmi, J. Erlich, D.Z. Freedman and A.A. Johansen, Positivity constraints on anomalies in supersymmetric theories, Phys. Rev. D57 (1998) 7570 and hep-th/9711035.

[3] J.L. Cardy, Is there a c theorem in four dimensions?, Phys. Lett. B215 (1988) 749;

J.L. Cardy, Central charge and the universal combinations of amplitudes in two-dimensional theories away from criticality, Phys. Rev. Lett. 60 (1988) 2709.

[4] A.B. Zamolodchikov, "Irreversibility" of the flux of the renormalization group in a 2D field theory, JETP Lett. 43 (1986) 730.

[5] S.J. Hathrell, Trace anomalies and QED in curved space, Ann. Phys. (NY) 142 (1982) 34.

[6] S.J. Hathrell, Trace anomalies and $\lambda\phi^4$ theory in curved space, Ann. Phys. (N.Y.) 139 (1982) 136.

[7] D. Anselmi, Central functions and their physical implications, JHEP 9805:005 (1998) and hep-th/9702056.

[8] A. Cappelli, R. Guida and N. Magnoli, Exact consequences of the trace anomaly in four dimensions, hep-th/0103237.

[9] R.J. Riegert, A non-local action for the trace anomaly, Phys. Lett. B 134 (1984) 56.

[10] E.S. Fradkin and A.A. Tseytlin, Conformal anomaly in Weyl theory and anomaly free superconformal theory, Phys. Lett. B 134 (1984) 187.

[11] I. Antoniadis and E. Mottola, 4-D quantum gravity in the conformal sector, Phys. Rev. D45 (1992) 2013.

[12] D. Anselmi, A universal flow invariant in quantum field theory, hep-th/0101088. To appear in Class. and Quantum Grav.

[13] D. Anselmi, A note of the improvement ambiguity of the stress tensor and the critical limits of correlation functions, hep-th/0110292.

[14] D. Anselmi and G. Festuccia, work in progress.

[15] I. Jack and H. Osborn, Analogs for the c theorem for four-dimensional renormalizable field theories, Nucl. Phys. B 341(1990) 647.

[16] K. Osterwalder and R. Shrader, Axioms for Euclidean Green functions. 1, Commun. Math. Phys. 31 (1973) 83.

K. Osterwalder and R. Shrader, Axioms for Euclidean Green functions. 2, Commun. Math. Phys. 42 (1975) 281.

[17] D. Anselmi, Anomalies, unitarity and quantum irreversibility, Ann. Phys. (NY) 276 (1999) 361 and hep-th/9903059 .

[18] D. Anselmi, Towards the classification of conformal field theories in arbitrary even dimension, Phys. Lett. B 476 (2000) 182 and hep-th/9908014.

[19] S.L. Adler, J.C. Collins and A. Duncan, Energy-momentum-tensor trace anomaly in spin 1/2 electrodynamics, Phys. Rev. D15 (1977) 1712.

[20] N.K. Nielsen, The energy momentum tensor in a nonabelian quark gluon theory, Nucl. Phys. B 120 (1977) 212.