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A UNIVERSAL FLOW INVARIANT IN QUANTUM FIELD THEORY

Damiano Anselmi
Dipartimento di Fisica, Università di Pisa, via F. Buonarroti 2, 56126 Pisa, Italia

Abstract

A flow invariant is a quantity depending only on the UV and IR conformal fixed points and not on the flow connecting them. Typically, its value is related to the central charges a and c. In classically-conformal field theories scale invariance is broken by quantum effects and the flow invariant $a_{\rm UV} - a_{\rm IR}$ is measured by the area of the graph of the beta function between the fixed points. There exists a theoretical explanation of this non-trivial fact. On the other hand, when scale invariance is broken at the classical level, it is empirically known that the flow invariant equals $c_{\rm UV} - c_{\rm IR}$ in massive free-field theories, but a theoretical argument explaining why it is so is still missing. A number of related open questions are answered here. A general formula of the flow invariant is found, which holds also when the stress tensor has improvement terms. The conditions under which the flow invariant equals $c_{\rm UV} - c_{\rm IR}$ are identified. Several non-unitary theories are used as a laboratory, but the conclusions are general and an application to the Standard Model is addressed. The analysis of the results suggests some new minimum principles, which might point towards a new understanding of quantum field theory.

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1 Introduction

Anomalies are among the most powerful tools for the investigation of quantum field theory beyond the perturbative expansion. The Adler-Bardeen theorem [1] and the 't Hooft anomaly matching conditions [2] give exact information about the strongly interacting limit of the theory. The trace anomaly encodes the beta function and therefore the renormalization-group flow. Moreover, various anomalies are computable to high orders in perturbation theory, with a limited effort, and in various models, combining the Adler-Bardeen theorem with supersymmetry, they are computable exactly to all orders [3, 4]. All-order formulas are available also in non-supersymmetric theories, for example the formula for the RG flow of the anomaly called a [5, 6].

The quantity a is one of the two anomaly coefficients of the trace Θ of the stress tensor in external gravity. It multiplies the Euler density. The other anomaly, called c, is the coefficient of the square of the Weyl tensor. (This terminology was introduced in [3, 7]). The quantity c is also the coefficient of the stress-tensor two-point function. The definition of a and c by means of the trace anomaly in external gravity is meaningful only in even dimensions. In odd dimensions, the trace anomaly in external gravity is not useful, but c can still be defined by means of the stress-tensor two-point function.

In two dimensions, the difference $c_{\mathrm{UV}} - c_{\mathrm{IR}}$ between the critical values of the unique central charge is related to the correlator $\langle \Theta\, \Theta \rangle$ [8, 9]. Precisely, the formula reads

$$c_{\mathrm{UV}} - c_{\mathrm{IR}} = 3\pi \int \mathrm{d}^2 x\, |x|^2 \, \langle \Theta(x)\, \Theta(0) \rangle. \tag{1.1}$$

This expression is an example of flow invariant, that is to say a quantity defined as the integral of a correlator along the flow[1], whose value depends only on the end points of the flow. The purpose of this paper is to investigate the existence of a universal flow invariant in four dimensions.

The two-dimensional property does not generalize immediately to higher dimensions, where the issues are more involved. A priori, the correlator $\langle \Theta\, \Theta \rangle$ should not know about either a or c, in dimension greater than two. In reality, it knows about both. For example, in marginally relevant flows, generated by the dynamical scale μ in classically conformal quantum field theories (strictly renormalizable at the quantum level), a theoretical argument [5], based on a physical principle, shows that the integral $\int \mathrm{d}^n x\, |x|^n \, \langle \Theta(x)\, \Theta(0) \rangle$ is proportional to $a_{\mathrm{UV}} - a_{\mathrm{IR}}$. On the other hand, empirical evidence suggests that the same integral is relevant also when conformality is broken at the classical level, such as in the presence of masses or super-renormalizable parameters (relevant flows). The integral, however, is proportional to $c_{\mathrm{UV}} - c_{\mathrm{IR}}$ in massive free-field theories [10]. These facts suggest that the integral $\int \mathrm{d}^n x\, |x|^n \, \langle \Theta(x)\, \Theta(0) \rangle$ is the basic ingredient for the constrution of the universal flow invariant in four dimensions, but a number of puzzles are raised, which are considered in this paper.

In the study of flow-invariants in four dimensions, marginally relevant and relevant flows exhibit crucially different properties. In the case of a marginally relevant flow, the trace of the

[1] In (1.1) and in the other flow integrals appearing in the paper, the integrand is taken at distinct points. One can exclude a circle of radius ϵ centered in $x = 0$ and take the limit $\epsilon \to 0$ after the integration.

stress tensor is an evanescent operator; in a more general relevant flow it is not. For example, in Yang-Mills theory or massless QCD, we have

$$\Theta = \frac{1}{4}\varepsilon F_{\mathrm{B}}^2 = \frac{1}{4}\beta F_{\mathrm{R}}^2,$$

where $\beta = \partial \ln \alpha / \partial \ln \mu$, $\varepsilon = 4 - n$ and F_{B}^2, F_{R}^2 denote the bare and renormalized operators, respectively. Classical conformal invariance means that Θ is generated only by quantum effects. In the presence of super-rinormalizable interactions or masses, we have additional terms of the form

$$\Theta = -m^2\phi^2 - m\bar{\psi}\psi,$$

and Θ is nonzero at the classical level.

This might not seem an important difference, at first sight. However, the arguments of [5] depend crucially on the evanescence of the operator Θ. This very evanescence guarantees that the induced action for the conformal factor is convergent. To introduce the research of this paper, it is compulsory to recall those arguments in some detail. Then I list the open questions and describe the answers found in this paper. I also give some examples of physical applications.

1.1 The intrinsic difference between marginally relevant and strictly relevant flows

I consider the four-dimensional case, for simplicity. To study correlators of the stress tensor, the theory is embedded in an external gravitational field. Here we are interested in the dependence on the conformal factor ϕ of the metric $g_{\mu\nu}$ and set $g_{\mu\nu} = \mathrm{e}^{2\phi}\delta_{\mu\nu}$. The induced action for the conformal factor reads at the critical points

$$S_{\mathrm{E}}[\phi] = \frac{1}{180}\frac{1}{(4\pi)^2}\int \mathrm{d}^4x \left\{a_*(\Box\phi)^2 - (a_* - a'_*)\left[\Box\phi + (\partial_\mu\phi)^2\right]^2\right\}. \tag{1.2}$$

where a_* and a'_* are defined by the trace anomaly:

$$\Theta = \frac{1}{90(4\pi)^2}\left[a_*\mathrm{e}^{-4\phi}\Box^2\phi + \frac{1}{6}(a_* - a'_*)\Box R\right].$$

Off-criticality, the induced action has additional non-local contributions. In particular, we consider the Θ-two-point function $\langle\Theta(x)\,\Theta(0)\rangle$. The off-critical structure of the correlator depends on whether Θ is evanescent or not.

In a classically-conformal quantum field theory, where Θ is evanescent, the correlator has the form [5]

$$\langle\Theta(x)\ \Theta(0)\rangle = \frac{1}{180\pi^2}\frac{1}{(4\pi)^2}\Box^2\left(\frac{\beta^2(t)\tilde{f}(t)}{|x|^4}\right), \tag{1.3}$$

where $t = \ln|x|\mu$. This correlator is convergent for $|x| \to 0$. The perturbative expansion of (1.3) has the form

$$\langle\Theta(x)\ \Theta(0)\rangle = \Box^2\sum_k a_k(g)\left(\frac{t^k}{|x|^4}\right), \tag{1.4}$$

where g is the coupling constant. The coefficients $a_k(g)$ are of order $g^{2(n+1)}$. The poles for $|x| \to 0$ are infinitely many, classified by the powers $t^k/|x|^4$. For example, $1/|x|^4$ has a simple pole in the limit $|x| \to 0$, $\ln(|x|\mu)/|x|^4$ has a double and a simple pole, and so on. However, the poles resum together into the beta function, which carries an additional zero for $|x| \to 0$. It can be shown that this zero produces the desired convergence [5].

On the other hand, if Θ is not evanescent, such as in the case of a massive free scalar field φ, then the correlator has the form

$$\langle \Theta(x)\ \Theta(0) \rangle = \frac{m^6}{16\pi^4 |x|^2} K_1^2(m|x|),$$

where K_1 is the modified Bessel function. There is just one pole, $1/(8\pi^4)\ m^4/|x|^4$, for $|x| \to 0$. Therefore, no cancellation can occur. We see that in the class of problems we are considering, classically-conformal quantum field theories (such as massless QCD) are less divergent than massive free-field theories!

In conclusion, the induced action for the conformal factor is convergent in classically-conformal field theories, but not in the theories violating conformality at the classical level.

The reason why the convergence of the induced action is crucial for the arguments of ref. [5] can be summarized as follows. In complete generality, the bosonic terms of the classical action of a quantum field theory should be positive-definite in the Euclidean framework for the functional integral to make sense. The fermionic terms have a universal form, in unitary theories. On physical grounds, we expect that, in a physically acceptable theory, the bosonic part of the generating functional Γ of the one-particle irreducible diagrams, which we call the quantum action, be bounded from below. Shifting Γ of a constant amount, we can conveniently say that the quantum action is positive definite. Since, however, the coupling constants run, it is more precise to say that the quantum action Γ is positive-definite throughout the RG flow if and only if Γ is positive-definite at a given energy.

Analysing a few simple examples, it is easy to get convinced that this positivity property, which is spoiled in general by the regularization, is recovered thanks to the renormalization procedure, and in particular the running of coupling constants. Actually, renormalization can be viewed as the unique algorithm which restores the mentioned positivity property by means of local couterterms.

These considerations apply to the dependence of Γ on the dynamical fields. In the presence of external fields, the positivity property is in general violated, unless new dynamical parameters are introduced, associated with the couplings to the external fields. The only possibility for the positivity property to hold, in the presence of external sources, without adding new parameters to the theory, is that the induced action be by itself convergent in the presence of those sources. This happens if the sources are coupled to evanescent operators.

The conformal factor ϕ is the external field coupled to the trace Θ. $\Gamma[\phi]$ is assured to be convergent in classically-conformal quantum field theories, by the very evanescence of Θ. Instead, $\Gamma[\phi]$ is not convergent in the presence of masses or super-rinormalizable interactions.

For this reason, the positivity property stated above applies only to the classically-conformal quantum field theories.

The last step of the argument of [5], which I do not repeat here, was to prove that the mentioned positivity property implies the formula

$$\Delta a = \frac{15}{2}\pi^2 \int \mathrm{d}^4x \, |x|^4 \, \langle \Theta(x) \, \Theta(0) \rangle \tag{1.5}$$

in four dimensions. For the generalization of the argument to arbitrary even dimensions the reader should refer to ref. [6].

I claim that the argument of [5] is a physical proof of the irreversibility of the marginally relevant flows and of formula (1.5).

A non-trivial by-product of the analysis is that the marginally relevant and the strictly relevant flows have an intrinsically different nature. The investigation of this paper is useful to understand the nature of this difference better.

1.2 Open questions

It is still not known how to generalize the theoretical argument just recalled when classical conformality is violated. The knowledge we have at present in this domain is only empirical [11]. We know that the flow invariant of [5] is proportional to $\Delta c = c_{\mathrm{UV}} - c_{\mathrm{IR}}$ and not $\Delta a = a_{\mathrm{UV}} - a_{\mathrm{IR}}$, in massive free-field theories. A natural implication of this fact is the definition of "$c = a$" theories [11], which have interesting properties in this context (see below). Yet, several issues remain open. Some of the most important questions are:

i) What is the universal expression of the flow invariant?

ii) Why is the flow invariant, which is equal to Δa in marginally relevant flows, equal to Δc in massive free-field theories? What is it in general?

iii) What happens when the theory contains several scales (μ, masses, super-rinormalizable parameters, etc.)? Should the various scales be related to one another in a special way, defining "the" flow, or should they remain arbitrary, the result not depending on their relative values?

I call "parameter" the coefficient λ_a of the deformation $\mathcal{L} \to \mathcal{L} + \lambda_a \mathcal{O}_a$ of the lagrangian $\mathcal{L}$. A dimensionful parameter is a parameter with non-vanishing classical dimension in units of mass. In this definition, a theory can have many scales, one for every dimensioned parameter λ_a, plus the dynamical scale μ. For the purposes of this paper, it is necessary to keep these scales distinguished. I do not write, for example, $\lambda_a = \Lambda^{d_a} g_a$, d_a being the classical dimension of λ_a, to define a unique scale Λ and several dimensionless parameters g_a, unless a special mechanism, such as the spontaneous symmetry breaking, provides such relationships.

The investigations of this paper answer some open questions and address the other ones. While several properties are understood better, other new features emerge. A universal expression of the flow invariant is suggested by the results. This answers question i and corrects a previously proposed expression (see [5], section 2.3), in the presence of improvement terms for the stress tensor. The final formula is (8.1).

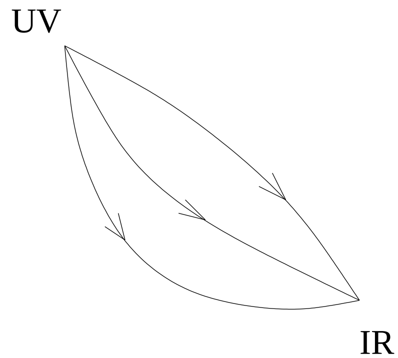

Figure 1: Different flows connecting the same fixed points.

The issue of universality of this flow invariant stands as follows. There is evidence to claim universality in the following two subclasses of quantum field theories:

a) marginally relevant flows;

b) relevant flows with $\Delta a = \Delta c$, in particular flows connecting conformal fixed points with $c = a$.

In these cases, problem *ii*) does not show up. But the flow invariant is expected to be useful also when $\Delta a \neq \Delta c$. I explain how, after describing the flow invariant more precisely.

The formula of the flow invariant is made of three ingredients. First, a flow integral of the type $\int \mathrm{d}^n x \, |x|^n \, \langle \Theta(x) \, \Theta(0) \rangle$. As a second step, this integral is minimized over the space of improved stress tensors. The result of this minimization will still be called "flow integral", with an abuse of language, although in reality it is a combination of flow integrals (see (7.1)). Finally, the formula of the flow invariant is made of a minimization in the space of trajectories relating the dimensioned parameters of the theory.

The analysis of the results suggests a non-trivial answer to question *iii*. In the presence of several scales the value of the flow integral does depend on the relations among them. The predicted value of the invariant (which is Δc, in our cases) is the minimum of the flow integral over all trajectories in the space of the dimensioned parameters of the theory. Assuming that the dimensioned parameters of the theory are the dynamical scale μ and some masses, or super-rinormalizable parameters, $m_1, \ldots, m_k$, a trajectory is, for example, a set of functions $m_1(\mu), \ldots, m_k(\mu)$. Let $\bar{m}_1(\mu), \ldots, \bar{m}_k(\mu)$ denote the trajectory which minimises the flow integral. Along $\bar{m}_1(\mu), \ldots, \bar{m}_k(\mu)$, the value of the flow integral equals Δc. Along all other trajectories, it is bigger than Δc.

Observe that the functions $m(\mu)$, or $\bar{m}(\mu)$ do not refer to the running of the masses, but fix the reference values of the masses, which are no longer arbitrary, but depend themselves on the dynamical scale μ. The complete μ dependence of the masses will be the result of the combination of the usual running plus this relation. For example, the special trajectory $\bar{m}(\mu)$ might be used to relate the value of the Higgs vev in the Standard Model to the dynamical scale μ at a conventional reference energy. In the special trajectory $\bar{m}(\mu)$, the Higgs mass is unambiguously fixed in terms of $\Lambda_{\rm QCD}$. In these considerations, I am assuming that the Standard Model interpolates between well-defined UV and IR fixed points.

In conclusion, the use of the flow invariant in quantum field theories with $\Delta a \neq \Delta c$ is that it selects, by means of a minimum principle, priviledged flows among the set of flows connecting the same fixed points.

We will find no clue to solve mystery *ii*. The investigation of this challenging problem is left for the future. It might be useful, for this purpose, to reconsider the issues studied here from the point of view of the exact renormalization-group flow approach á la Wilson. I expect that, in the wilsonian framework, the difference between marginally relevant and truly relevant flows can be re-interpreted in the context of a deeper and more complete understanding of the problem. In this paper, I use only techniques of (resummed) perturbation theory.

Let us mention once again that in two dimensions the situation is much simpler, because the flow invariant (1.1) does not depend of the dimensioned parameters of the theory and there is a unique central charge. The subclass of four-dimensional theories which best share the properties of two-dimensional theories is the subclass with $\Delta c = \Delta a$ [11].

1.3 Applications and physical predictions

The mentioned minimizations, and other properties studied in the paper, suggest some new minimum principles, which might be important properties of quantum field theory. For example, they might help us reformulating quantum field theory in a framework where the renormalization-group trajectory connecting two fixed points is obtained by minimising some action (see the introduction of ref. [12]). The idea is that a more sophisticated formulation could better help us solve the open issues of quantum field theory.

The investigation of flow invariants and quantum irreversibility, which is a pretty theoretical research domain of quantum field theory, allows us to make various types of quantitative predictions. For example, in massless QCD, which is classically conformally invariant, Δa should be equal to

$$\Delta a = \frac{15}{2}\pi^2 \int \mathrm{d}^4x \, |x|^4 \, \langle \Theta(x) \, \Theta(0) \rangle = 62(N_c^2 - 1) + 11 N_c N_f - N_f^2 + 1,$$

where N_c is the number of colours and N_f is the number of quarks in the fundamental representation. Lattice simulations in massless QCD with fermions are difficult to perform, but not impossible. It would be extremely interesting to test the prediction written above, because it crucially depends of the field content of the IR limit of the theory. This check would give an indirect evidence that the low-energy limit of the theory contains just $N_f^2 - 1$ massless pions, as we expect. Other non-perturbative checks of the formula for Δa are possible in the context of the AdS/CFT correspondence [13], but the computation is involved. The simple flows of ref. [14] are a good laboratory for this test.

Other types of physical predictions are available. The minimum principle found here, which determines the priviledged trajectory $\bar{m}(\mu)$, might have an important physical meaning, besides the mathematical one. Suppose that the theories of nature are so constrained that they all lie on the special trajectories $\bar{m}(\mu)$. Then, all the dimensioned physical parameters are uniquely fixed in terms of the dynamical scale μ, in all the theories of nature, and we can relate the mass of the Higgs particle to μ and therefore predict the Higgs mass. This calculation is complicated by the fact that it involves an integral throughout the flow. Because of its intrinsically non-perturbative nature, this relation might generate very large or very small numbers, possibly relevant for the hierarchy problem. Work is in progress in this direction.

1.4 Organization of the paper

In section 2 I motivate the use of non-unitary theories for our investigations. The precise formula for the flow invariant in the presence of improvement terms for the stress-tensor will be constructed in two successive steps. First, in section 3, I recall the candidate formula proposed in sect. 2.3 of [5] and explain its properties. Then, in sections 4, 5 and 6 I present checks in various higher-derivative models. In section 4 I consider an example where improvement terms are irrelevant, but the two fixed points are connected by a one-parameter family of trajectories. The flow invariant does not depend on the trajectory and is equal to Δc, as expected. In sections 5 and 6 I present checks where the improvement terms are crucial. I show that the unimproved formula of the flow invariant does not work, and that the candidate generalized formula gives a result “close” to the prediction. The discrepancy is resolved in section 7, where the correct formula of the flow invariant is found, and associated with a minimum principle in the space of improvement terms. It is then shown that another minimum principle (the one described above) defines a priviledged trajectory connecting the two conformal fixed points. Along this priviledged trajectory the value of the flow integral is equal to Δc, in agreement with the prediction, while along any other trajectory it is greater than Δc. The second minimum principle is then discussed in detail. Several calculations are done numerically, but the final form of the flow invariant is simpler than the first proposal and its value can be worked out exactly in many cases.

2 The use of non-unitary theories

As a laboratory, I study several higher-derivative theories. These theories are not physical because not unitary. Nevertheless, they are useful for the investigation of a variety of general properties of quantum field theory. The idea is that the flow invariant is meaningful in all mathematically well-defined theories, even if non-unitary, and that a lot can be learned by enlarging the set of theories on which we can work. By “mathematically well-defined theories” I mean renormalizable theories such that the bosonic contributions to the action are positive-definite in the Euclidean framework. Conditions for the fermionic contributions are more tricky and will be discussed later.

Mathematically well-defined theories are not necessarily unitary and even if the bosonic part of the action is positive definite, they can violate reflection positivity. It is easy to write negative-definite two-point correlators and often, the central charges a and c are themselves negative [15]. Positive-definitness of the action assures, in particular, that the propagators have no poles in the Euclidean framework, so that Feynman diagrams are well-defined.

For the moment, I assume that the fermions, if present, have the Dirac action $\bar{\psi}(\not{D}+m(x))\psi$, where $m(x)$ is real and eventually field dependent. It is not straightforward to set conditions for acceptable non-unitary theories containing fermions. A specific model will be analysed in the paper.

Reflection positivity is crucial in investigations about the c-theorem [8]. On the other hand, reflection positivity is less crucial in investigations about flow invariants.

Typically, in unitary theories a flow invariant is the integral of a positive function along the flow. The positive integrand is a two-point correlator. More generally, the flow invariant can be a positive combination of integrals of this type. Consequently, the value of a flow invariant is positive in unitary theories. By definition, its value does not depend on the path connecting the two fixed points, but only on the fixed points themselves. Moreover, it is additive. Concluding, the value of a flow invariant has the form

$$\Delta_{\mathrm{UV}} - \Delta_{\mathrm{IR}},$$

for some quantity Δ, which we can call "central charge" of the conformal fixed point. In various cases it can be identified with a or c. Conformal field theories might have more than one central charge and there might exist several flow invariants.

Under the conditions just stated, a unitary theory will satisfy the inequality

$$\Delta_{\mathrm{UV}} - \Delta_{\mathrm{IR}} \geq 0, \tag{2.1}$$

or "c-theorem". A non-unitary theory is allowed to violate this inequality. The integrand of the flow invariant, which is, as we have said, a two-point correlator, can be negative.

Since, however, our interest is not to test an inequality such as (2.1), but flow invariance, and to possibly classify all flow invariants, it does not matter to us whether the inequality (2.1) is satisfied or not. The tests presented here confirm that the flow invariant is generically well-defined in the matematically well-defined theories, even if they are non-unitary, modulo some conditions on the higher-derivative fermions.

Renormalizability puts formidable constraints on the set of allowed physical theories, which are often not enough numerous, or not sufficiently simple, to make "quantum field theoretical experiments" of the type that we are going to present here. For example, improvement terms for the stress tensor are important in theories containing scalar fields, such as the Standard Model. However, the improvement terms affect the flow invariant above four loops in the $\lambda\varphi^4$ theory [5] and it is often difficult to perform calculations to high orders in perturbation theory. Extending the set of theories to the mathematically, but not necessarily physically, well-defined ones, in the sense specified above, we can enlarge the laboratory of theories enormously, in arbitrary dimensions. This laboratory includes also classically conformal higher-spin theories [15], which sometimes have conformal windows and can be coupled to external gravity. A variety of new checks and "theoretical experiments" are available and calculations are easier.

Moreover, higher-derivative theories and other non-unitary theories, have a number of non-trivial mathematical applications. For example, they can be studied to investigate the properties of the "pondered" Euler density constructed in [6], or the special invariant appearing in the trace anomaly of the so-called "$c = a$ theories", constructed in [11]. In six dimensions this special invariant matches with a particular combination pointed out by Bonora, Pasti and Bregola in ref. [16], and in higher even dimensions agrees with the Henningson-Skenderis construction [17]. Quantum field theory is the most powerful algorithm to study these mathematical properties.

The results presented here and in [5, 6, 11] suggest that quantum field theory has a number of unforeseen, interesting properties, which can be investigated exactly, without restricting to particularly symmetric theories. This, in spite of the usual lore that the open problems of quantum field theory are too difficult.

The claimed difficulty has often been advocated as a motivation to state that exact properties of quantum field theory should be studied using more powerful, "non-perturbative" methods, such as those suggested by string theory. While it is certainly true that the string theoretical methods can improve and extend our knowledge of quantum field theory, it is also doubtless that their power is limited. Typically, these methods cover a very special subclass of quantum field theories, having some peculiar symmetry or related to one another by some sort of duality. Not unfrequently, the physically interesting theories are excluded. For example, the AdS/CFT correspondence is useful to study conformal theories with $c = a$ [17] and flows interpolating between them [18]. The $c = a$ conformal field theories and flows are certainly interesting [11], but physically relevant theories (QCD, Standard Model) do not belong to this class [12].

Instead, the spirit of the investigation presented here is that the knowledge coming from the analysis of a variety of physically uninteresting theories, then applies also to the physically interesting ones. For example, we are going to find the precise, general formula of the flow invariant in the presence of improvement terms for the stress tensor, which can be applied to the Standard Model. We could not infer this formula directly from the Standard Model.

In conclusion, the investigation of the mathematically well-defined quantum field theories is well motivated.

3 First step towards the universal flow invariant

The central charge c_n in n dimensions is normalized so that it equals one for a real free scalar field. In even dimensions, we normalize the central charge a_n so that the $c = a$ theories defined in ref. [11] are those satisfying the condition $c_n = a_n$. The two relevant terms of the trace anomaly in external gravity read

$$\Theta = \frac{(\frac{n}{2})!}{(4\pi)^{n/2}\,(n+1)!}\left[\frac{a_n}{2^{n/2-1}\,n}\,\mathrm{G}_n - c_n\frac{n-2}{4(n-3)}W\Box^{n/2-2}W + \cdots,\right]$$

where

$$\mathrm{G}_n = \mathrm{G}_n = (-1)^{\frac{n}{2}}\varepsilon_{\mu_1\nu_1\cdots\mu_{\frac{n}{2}}\nu_{\frac{n}{2}}}\varepsilon^{\alpha_1\beta_1\cdots\alpha_{\frac{n}{2}}\beta_{\frac{n}{2}}}\prod_{i=1}^{\frac{n}{2}}R^{\mu_i\nu_i}_{\alpha_i\beta_i}$$

is the Gauss-Bonnet integrand and W is the Weyl tensor. Observe that the a_n-normalization chosen in [6, 11] was such that $\Theta = a_n\,\mathrm{G_n} + \cdots$.

In both even and odd dimensions, the central charge c can be defined by the stress-tensor two-point function. Using the notation of ref. [19], we have:

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\rangle = c_n \frac{(n/2)!(n/2-1)!}{(2\pi)^n(n+1)!}\prod^{(2)}_{\mu\nu,\rho\sigma}\Box^{n/2-2}\left(\frac{1}{|x|^n}\right), \tag{3.1}$$

where the spin-2 projection operator $\prod^{(2)}$ is given by

$$\prod^{(2)}_{\mu\nu,\rho\sigma} = \frac{1}{2}(\pi_{\mu\rho}\pi_{\nu\sigma}+\pi_{\mu\sigma}\pi_{\nu\rho}) - \frac{1}{n-1}\pi_{\mu\nu}\pi_{\rho\sigma}, \qquad\qquad \pi_{\mu\nu} = \partial\mu\partial\nu - \Box\delta_{\mu\nu}.$$

In odd dimensions, we understand that

$$\Box^{n/2-2}\left(\frac{1}{|x|^n}\right) = \frac{2^{n-3}(n-3)!}{(n-2)\,|x|^{2n-4}}.$$

On the other hand, a consistent definition of the central charge a_n in odd dimensions is still missing.

The flow invariant proposed in sect. 2.3 of [5], generalized to many improvement terms, reads

$$\Sigma_n = \int \mathrm{d}^n x\,|x|^n \frac{\det\left[\begin{array}{cc}\langle\Theta(x)\,\Theta(0)\rangle & \langle\Theta(x)\,\mathcal{O}_j(0)\rangle\\ \langle\mathcal{O}_i(x)\,\Theta(0)\rangle & \langle\mathcal{O}_i(x)\,\mathcal{O}_j(0)\rangle\end{array}\right]}{\det\left[\langle\mathcal{O}_i(x)\,\mathcal{O}_j(0)\rangle\right]} \equiv \int \mathrm{d}^n x\,|x|^n\,\sigma(x). \tag{3.2}$$

Here $\mathcal{O}_i$, $i = 1,\ldots k$, are the traces of the improvement terms for the stress tensor. This formula was suggested from considerations about the scheme independence and flow invariance in the φ^4-theory in four dimensions, which has a single improvement term ($k = 1$).

The improvement terms $\mathcal{O}_i$ are classified as follows. We assume that the stress tensor $T_{\mu\nu}$ is traceless at the critical points. We consider all local, dimension n, symmetric, identically conserved operators $\Delta T_{\mu\nu}$, which vanish at the critical points. Clearly, the "improved" operators of the form $T_{\mu\nu} + \Delta T_{\mu\nu}$ are equally acceptable stress tensors.

Translation invariance implies that $\partial\mu T_{\mu\nu}$ is finite, but not necessarily that $T_{\mu\nu}$ is finite [20]. It can be inferred that $T_{\mu\nu}$ is finite only if there exists no identically conserved local operator $\Delta T_{\mu\nu}$. In the $\lambda\varphi^4$-theory in four dimensions such an operator exists, and is equal to $h(\lambda)(\partial\mu\partial\nu - \delta_{\mu\nu}\Box)(\varphi^2)$, where $h(\lambda)$ is an arbitrary function of λ, vanishing at the fixed points. The bare and renormalized stress tensors are related by

$$T^R_{\mu\nu} = T^B_{\mu\nu} + A(\lambda)(\partial\mu\partial\nu - \delta_{\mu\nu}\Box)(\varphi^2)^B$$

where A is possibly divergent. Similarly, we have, for the traces, $\Theta^R = \Theta^B - 3A\Box(\varphi^2)^B$. We have therefore $\mathcal{O} = \Box(\varphi^2)$ and

$$\sigma = \langle\Theta\ \Theta\rangle - \frac{\left(\langle\Theta\ \Box\varphi^2\rangle\right)^2}{\langle\Box\varphi^2\ \Box\varphi^2\rangle} \tag{3.3}$$

Observe that, instead, the massive free scalar field admits no improvement term. The operator $\Box(\varphi^2)$ cannot be multiplied by a dimensionless function, vanishing at criticality, because there exists no dimensionless parameter in the theory. If we make a redefinition of the form $T_{\mu\nu} \rightarrow T_{\mu\nu} - \alpha/3\,(\partial\mu\partial\nu - \delta_{\mu\nu}\Box)(\varphi^2)$ (and consequently $\Theta \rightarrow \Theta + \alpha\Box(\varphi^2)$), then α has to be constant. That means, however, that the shift $\alpha\Box(\varphi^2)$ cannot vanish at the critical points, where, instead, the condition $\Theta = 0$ defines the trace (and so the stress tensor) uniquely. Therefore, the stress tensor is unique also off-criticality for the massive free scalar field.

As a third example, useful for the investigation of this paper, let us consider a higher-derivative massive free scalar field φ with lagrangian $1/2\,[(\Box\varphi)^2 + m^4\varphi^2]$. In this theory, φ has dimension $(n-4)/2$. We can set $\Delta T_{\mu\nu} = m^2(\partial\mu\partial\nu - \delta_{\mu\nu}\Box)(\varphi^2)$. The coefficient m^2 assures that the improvement term disappears at criticality. We have therefore $\mathcal{O} = m^2\Box(\varphi^2)$. Nevertheless, we will see below that the flow invariant does not depend on the coefficient of the improvement operator, and so we can just take $\mathcal{O} = \Box(\varphi^2)$.

The integrand $\sigma(x)$, and so Σ_n, are independent of the choice of $T_{\mu\nu}$ among the set of improved stress tensors $T_{\mu\nu} + \Delta T_{\mu\nu}$. On Θ, this means invariance under the transformation

$$\Theta \rightarrow \Theta + a_i\mathcal{O}_i, \qquad \mathcal{O}_i \rightarrow B_{ij}\mathcal{O}_j, \tag{3.4}$$

a_j and B_{ij} being arbitrary constants.

Equivalently, the meaning of the invariance (3.4) is that Σ_n and $\sigma(x)$ are independent of the coupling of the theory to the gravitational background. At the critical points the non-minimal couplings are fixed uniquely by conformal invariance, but at intermediate energies they affect the stress tensor and Θ. The non-minimal couplings are in one-to-one correspondence with the improvement terms of the stress tensor. For example, in the $\lambda\varphi^4$-theory embedded in external gravity, the non-minimal coupling is $R\varphi^2$. By definition, the non-minimal couplings disappear in the flat limit. Therefore, the flatspace theory should be insensitive to them.

The integrand $\sigma(x)$ has other relevant invariance properties. In particular, it was shown in sect. 2.3 of [5] that (3.3) is scheme independent and that it dependes on the running coupling $\lambda(1/|x|)$, but not on the reference value $\lambda(\mu)$ of the coupling. The origin of the possible scheme dependende is the renormalization mixing between the stress tensor and its improvement term. The scheme dependence cancels out in the combination $\sigma(x)$.

The matrix B in (3.4) is the most general real matrix, because there is no canonical way of normalizing the improvement operators. For example, (3.3) is independent on the normalization of the operator $\Box\varphi^2$.

The invariance (3.4) can be proved as follows. In the first case, let $a_i = 0$. It is easy to see that the redefinitions $\mathcal{O}'_i = B_{ij}\mathcal{O}_j$ produce

$$\sigma' = \frac{\det\left[\begin{pmatrix}1 & 0\\ 0 & B\end{pmatrix}\begin{pmatrix}\langle\Theta\,\Theta\rangle & \langle\Theta\,\mathcal{O}_j\rangle\\ \langle\mathcal{O}_i\,\Theta\rangle & \langle\mathcal{O}_i\,\mathcal{O}_j\rangle\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & B^t\end{pmatrix}\right]}{(\det B)^2\,\det\left[\langle\mathcal{O}_i\,\mathcal{O}_j\rangle\right]} = \sigma,$$

since the determinants of the matrix B simplify. Here B^t denotes the transpose of B.

Now, let us take $B = 0$ and $a_i \neq 0$. To prove invariance, we rewrite σ by expanding the numerator determinant along the first row and the first column:

$$\sigma = \langle \Theta\ \Theta \rangle + \sum_{i,j=1}^{k} (-1)^{i+j+1} \langle \Theta\, \mathcal{O}_j \rangle \langle \mathcal{O}_i\, \Theta \rangle \, \frac{\det N_{ij}}{\det N},$$

where N denotes the matrix with entries $\langle \mathcal{O}_i\, \mathcal{O}_j \rangle$ and N_{ij} is the minor obtained by suppressing the i^{th} row and the j^{th} column. Defining $\Theta' = \Theta + a_i \mathcal{O}_i$ we can readily show $\sigma' = \sigma$ by means of the well known identity

$$\sum_{i=1}^{k} \langle \mathcal{O}_k\, \mathcal{O}_i \rangle \ \det N_{ij}\, (-1)^{i+j+1} = -\delta_{kj}\ \det N.$$

We have recalled in the introduction that the flow invariant is proportional to Δa in marginally relevant flows. The proportionality coefficient can be read from [6]. In the normalization used here, we have

$$\Sigma_n = \Delta a_n \, \frac{\Gamma(n/2+1)}{\pi^{n/2}\,(n+1)}.$$

Instead, we know that the flow invariant is proportional to $\Delta c_n = c_{n\,\mathrm{UV}} - c_{n\,\mathrm{IR}}$ is unitary, massive free-field theories. The proportionality coefficient can be read from [11]:

$$\Sigma_n = \Delta c_n \, \frac{\Gamma(n/2+1)}{\pi^{n/2}\,(n+1)}. \tag{3.5}$$

The theories studied in this paper are free higher-derivative massive bosonic and fermionic theories. We therefore expect that the prediction (3.5) applies to our case.

However, the results do not confirm this prediction. Later we will see that this failure has two reasons.

First, the proposed formula for the flow invariant is not correct in the presence of improvement terms for the stress tensor. The point is that the flow invariant is not uniquely fixed by the symmetry (3.4). The integral Σ_n is invariant under an enlarged symmetry, of the same form as (3.4), but with point-dependent a_i and B_{ij}. This is an unnecessary requirement, since it is associated with non-local non-minimal coulings to external gravity. The flow invariant has certainly to be invariant under (3.4), with constant a_i and B_{ij}, but it need not be the integral of an invariant integrand $\sigma(x)$.

Second, the correct flow integral is not always equal to the right-hand side of (3.5), but this happens in a special trajectory connecting the UV and IR fixed points.

For pedagogical reasons, and taking into account of the empirical nature of the investigations of this paper, I present the results as I have found them, starting from the more intuitive, although incorrect, formulations of the prediction and the flow invariant, and letting the correct formulas emerge from the analysis of the results.

4 First test of the prediction

As a first test, we consider a higher-derivative theory of fermions with lagrangian

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ - m_1)(\partial\!\!\!/ + m_2)\psi.$$

This theory has no improvement term. We have a one-parameter family of relevant flows connecting the same pair of conformal fixed points, the parameter being m_1/m_2.

The stress tensor reads

$$T_{\mu\nu} = \frac{1}{4}\left[\bar{\psi}\gamma\mu\overleftrightarrow{\partial\nu}(\partial\!\!\!/ + m_2)\psi + \bar{\psi}\gamma\nu\overleftrightarrow{\partial\mu}(\partial\!\!\!/ + m_2)\psi - \bar{\psi}(\overleftarrow{\partial\!\!\!/} + m_1)\gamma\mu\overleftrightarrow{\partial\nu}\psi - \bar{\psi}(\overleftarrow{\partial\!\!\!/} + m_1)\gamma\nu\overleftrightarrow{\partial\mu}\psi\right]$$
$$-\frac{1}{2}\delta_{\mu\nu}\left(\bar{\psi}(\partial\!\!\!/ - m_1)(\partial\!\!\!/ + m_2)\psi + \bar{\psi}(\overleftarrow{\partial\!\!\!/} - m_1)(\overleftarrow{\partial\!\!\!/} + m_2)\psi\right).$$

where we have kept the terms proportional to the field equations.

The trace is

$$\Theta = -\bar{\psi}\overleftarrow{\partial\!\!\!/}\,\partial\!\!\!/\psi + m_1 m_2\bar{\psi}\psi.$$

Observe that the trace is non-vanishing in the massless limit. Nevertheless, the theory is conformal at $m_1 = m_2 = 0$, since the operator $\bar{\psi}\overleftarrow{\partial\!\!\!/}\,\partial\!\!\!/\psi$ has a vanishing two-point function: $\langle\Theta(x)\,\Theta(0)\rangle = 0$ at $x \neq 0$ and $m_1 = m_2 = 0$. This can be shown immediately using the massless field equations. Moreover, the correlator $\langle T_{\mu\nu}(x)\,\Theta(0)\rangle$ is also vanishing at $x \neq 0$, since each term in the stress tensor contains either $\partial\!\!\!/\psi$ or $\bar{\psi}\overleftarrow{\partial\!\!\!/}$. This is a peculiarity of higher-derivative theories, which violate reflection positivity. A nonvanishing operator can have a vanishing two-point function. In a physical reduction of the theory, admitting that it exists, such operators should be projected away, presumably in a cohomological sense.

At $m_1 = m_2 = 0$ the central charge c_n is positive and twice the one of ordinary fermions:

$$c_n = 2^{[n/2]}(n-1),$$

where $[n/2]$ denotes the integral part of $n/2$. Our prediction reads therefore

$$\Sigma_n = \int \mathrm{d}^n x\,|x|^n\langle\Theta(x)\,\Theta(0)\rangle = \frac{2^{[n/2]}(n-1)\Gamma(n/2+1)}{\pi^{n/2}(n+1)}. \tag{4.1}$$

I have checked this prediction in six, eight and ten dimensions, for various values of m_1/m_2, performing the integral Σ_n has numericaly in the momentum space. When $m_1 = m_2 = m$ it is easy to compute the flow integral exactly. We have

$$\langle\Theta(x)\,\Theta(0)\rangle = 2^{[n/2]+1}m^2(G_n'^2(x) - m^2G_n^2(x)), \qquad G_n(x) = -\frac{1}{2\pi}\left(\frac{m}{2\pi|x|}\right)^{n/2-1}K_{n/2-1}(m|x|),$$

K denoting the modified Bessel function, and

$$\int \mathrm{d}^n x\,|x|^n\,m^2(G_n'^2 - m^2G_n^2) = \frac{(n-1)\Gamma(n/2+1)}{2\,\pi^{n/2}(n+1)},$$

which, inserted into the left-hand side of (4.1), produces the correct result. The numerical results confirm the prediction for arbitrary values of m_1/m_2.

5 Second test of the prediction

We now consider the scalar theory defined by the lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\Box\varphi)^2 + \beta m^2(\partial\alpha\varphi)^2 + m^4\varphi^2\right].$$

The stress tensor can be fixed in the following way. At $m = 0$ $T_{\mu\nu}$ is conserved and traceless. There is a unique expression, up to an overall constant, satisfying this condition. In six dimensions this expression was found in ref. [15]:

$$\begin{aligned} T_{\mu\nu} = h\Big\{ & \frac{3}{4}\partial_\mu\partial_\alpha\varphi\,\partial_\nu\partial_\alpha\varphi - \frac{3}{2}\Box\varphi\,\partial_\mu\partial_\nu\varphi + \partial_\nu\Box\varphi\,\partial_\mu\varphi + \partial_\mu\Box\varphi\,\partial_\nu\varphi - \frac{1}{2}\partial_\mu\partial_\nu\partial_\alpha\varphi\,\partial_\alpha\varphi \\ & -\frac{1}{4}\varphi\,\Box\partial_\mu\partial_\nu\varphi + \delta_{\mu\nu}\left[-\frac{1}{4}\partial_\alpha\Box\,\varphi\partial_\alpha\varphi - \frac{1}{8}\left(\partial_\alpha\partial_\beta\varphi\right)^2 + \frac{1}{4}\left(\Box\varphi\right)^2\right]\Big\} \end{aligned}$$

To fix the overall constant, it is sufficient to inspect the stress tensor up to total derivatives:

$$T_{\mu\nu} = -\frac{5}{2}h\varphi\,\partial\mu\partial\nu\Box\varphi + \text{tot. ders.} = \frac{2}{\sqrt{g}}\,\frac{\delta}{\delta g^{\mu\nu}}\int\frac{1}{2}\sqrt{g}(\partial\rho\partial\sigma\varphi)(\partial\alpha\partial\beta\varphi)g^{\rho\sigma}\,g^{\alpha\beta}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} + \text{tot. ders.}$$

This gives $h = -4/5$. Finally, the contributions of the mass operators can be straightforwardly calculated from the embedding in external gravitaty.

The stress-tensor two-point function, at $m = 0$ in $n = 6$, can be read from ref. [15]:

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\rangle = -\frac{1}{5376\pi^6}\prod^{(2)}_{\mu\nu,\rho\sigma}\Box\left(\frac{1}{|x|^6}\right) = c_6\frac{2\cdot 3!}{2^6\cdot 7!\cdot\pi^6}\prod^{(2)}_{\mu\nu,\rho\sigma}\Box\left(\frac{1}{|x|^6}\right).$$

This gives the central charge

$$c_6 = -5.$$

The negative sign of the central charge signals the presence of ghosts in the theory.

The prediction (3.5) finally reads

$$\Sigma_6 = -\frac{30}{7\pi^3}.$$

We can generalize the calculations and the prediction to arbitrary dimensions. The stress tensor is easily found to be

$$\begin{aligned} T_{\mu\nu} = & -\frac{n+2}{2(n-1)}(\partial_\nu\Box\varphi\,\partial_\mu\varphi + \partial_\mu\Box\varphi\,\partial_\nu\varphi) + \frac{n(n+2)}{2(n-1)(n-2)}\Box\varphi\,\partial_\mu\partial_\nu\varphi + \frac{2}{n-1}\partial_\mu\partial_\nu\partial_\alpha\varphi\,\partial_\alpha\varphi \\ & -\frac{2n}{(n-1)(n-2)}\partial_\mu\partial_\alpha\varphi\,\partial_\nu\partial_\alpha\varphi + \frac{n-4}{2(n-1)}\varphi\,\Box\partial_\mu\partial_\nu\varphi + \delta_{\mu\nu}\left[\frac{1}{n-1}\partial_\alpha\Box\varphi\,\partial_\alpha\varphi\right. \\ & \left. +\frac{2}{(n-1)(n-2)}\left(\partial_\alpha\partial_\beta\varphi\right)^2 - \frac{n+2}{2(n-1)(n-2)}\left(\Box\varphi\right)^2 - \frac{n-4}{2(n-1)}\varphi\,\Box^2\varphi - \frac{m^4}{2}\varphi^2\right] \\ & +\beta m^2\left(\partial\mu\varphi\,\partial\nu\varphi - \frac{\delta_{\mu\nu}}{2}(\partial\alpha\varphi)^2\right) - \alpha m^2(\partial\mu\partial\nu - \Box\delta_{\mu\nu})(\varphi^2). \end{aligned}$$

The result can be derived also from the complete coupling to gravity, which is known in this case. We have

$$\mathcal{L}=\frac{1}{2}\sqrt{g}\left(\varphi\Delta_4\varphi+\beta m^2(\partial\mu\varphi)(\partial\nu\varphi)g^{\mu\nu}+\alpha Rm^2\varphi^2+m^4\varphi^2\right), \tag{5.1}$$

where (see for example [21])

$$\begin{aligned}\Delta_4 = {} & \nabla^2\nabla^2+\nabla\mu\left[\frac{4}{n-2}R^{\mu\nu}-\frac{n^2-4n+8}{2(n-1)(n-2)}g^{\mu\nu}R\right]\partial\nu-\frac{n-4}{4(n-1)}\nabla^2R\\ & -\frac{n-4}{(n-2)^2}R_{\mu\nu}R^{\mu\nu}+\frac{(n-4)(n^3-4n^2+16n-16)}{16(n-1)^2(n-2)^2}R^2.\end{aligned} \tag{5.2}$$

The stress tensor can be obtained by direct differentiation of the lagrangian embedded in the external gravitational field. The result agrees with the previous calculation and ref. [15].

A lengthy calculation gives

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\rangle=-\frac{(n-4)(n^2-16)\left[\Gamma(n/2-2)\right]^2}{2^{n+4}\pi^n(n^2-1)\Gamma(n-2)}\prod\nolimits^{(2)}_{\mu\nu,\rho\sigma}\Box^{n/2-2}\left(\frac{1}{|x|^n}\right)$$

and therefore the central charge

$$c_n=-\frac{2(n+4)}{n-2},$$

which agrees with the known results for $n=6$ and $n=4$. The value for $n=4$ can be found in [15], formula (2.4), where an additional factor 1/120 is included, in the more conventional four-dimensional normalization. Here, instead, we have normalized c to be always 1 for a free real scalar field.

The prediction (3.5) reads

$$\Sigma_n=-\frac{2(n+4)\Gamma(n/2+1)}{(n+1)(n-2)\,\pi^{n/2}}. \tag{5.3}$$

We now proceed to check this prediction.

The propagator is a convolution of two Bessel functions. Precisely,

$$G_{n,r}(x)\equiv\int\frac{\mathrm{d}^np}{(2\pi)^n}\frac{\mathrm{e}^{ipx}}{(p^2+m^2\gamma_+^2)(p^2+m^2\gamma_-^2)}=\frac{1}{(2\pi)^{n/2}\,x^{n-4}}\int_0^\infty\frac{t^{n/2}\,J_{n/2-1}(t)\,\mathrm{d}t}{(t^2+rm^2x^2)\left(t^2+\frac{1}{r}m^2x^2\right)},$$

where $\gamma_-=1/\gamma_+$ and $\beta=\gamma_+^2+\gamma_-^2$ and $\gamma_+^2=r$. The propagator can be written more explicitly in two classes of cases: in odd dimensions, for any value of r, and in even dimensions, for $r=1$. For example in three, five, seven and nine dimensions we have

$$\begin{aligned}G_{3,r}(x) &= \frac{\mathrm{e}^{-m|x|\gamma_-}-\mathrm{e}^{-m|x|\gamma_+}}{4\pi m^2|x|(\gamma_+^2-\gamma_-^2)} && \to\frac{\mathrm{e}^{-m|x|}}{8\pi m},\\ G_{5,r}(x) &= \frac{(1+m|x|\gamma_-)\,\mathrm{e}^{-m|x|\gamma_-}-(1+m|x|\gamma_+)\,\mathrm{e}^{-m|x|\gamma_+}}{8\pi^2m^2|x|^3(\gamma_+^2-\gamma_-^2)} && \to\frac{\mathrm{e}^{-m|x|}}{16\pi^2|x|},\end{aligned}$$

$$G_{7,r}(x) = \frac{(3+3m|x|\gamma_- + m^2|x|^2\gamma_-^2)\,\mathrm{e}^{-m|x|\gamma_-} - (3+3m|x|\gamma_+ + m^2|x|^2\gamma_+^2)\,\mathrm{e}^{-m|x|\gamma_+}}{16\pi^3 m^2|x|^5(\gamma_+^2-\gamma_-^2)} \to \frac{(1+m|x|)\,\mathrm{e}^{-m|x|}}{32\pi^3|x|^3},$$

$$G_{9,r}(x) = \frac{(15+15m|x|\gamma_- + 6m^2|x|^2\gamma_-^2 + m^3|x|^3\gamma_-^3)\,\mathrm{e}^{-m|x|\gamma_-}}{32\pi^4 m^2|x|^7(\gamma_+^2-\gamma_-^2)} - \frac{(15+15m|x|\gamma_+ + 6m^2|x|^2\gamma_+^2 + m^3|x|^3\gamma_+^3)\,\mathrm{e}^{-m|x|\gamma_+}}{32\pi^4 m^2|x|^7(\gamma_+^2-\gamma_-^2)} \to \frac{(3+3m|x|+m^2x^2)\,\mathrm{e}^{-m|x|}}{64\pi^4|x|^5}.$$

On the right hand sides of the arrows, the expressions of the propagators for $r=\gamma_+=\gamma_-=1$ are reported.

For $r=1$ the function $G_{n,1}(x)$ can be written, in arbitrary dimensions, by means of a modified Bessel function:

$$G_{n,1}(x) = \frac{1}{2(2\pi)^{n/2}}\left(\frac{m}{|x|}\right)^{n/2-2} K_{n/2-2}(m|x|). \tag{5.4}$$

The trace of the stress tensor reads

$$\Theta = -2m^4\varphi^2 - \frac{n-2}{2}\beta m^2(\partial\alpha\varphi)^2 - \frac{n-4}{2}\beta m^2\varphi\,\Box\varphi + \alpha(n-1)m^2\Box(\varphi^2),$$

using the field equations $\Box^2\varphi - \beta m^2\Box\varphi + m^4\varphi = 0$. Recalling that Σ_n is invariant under the redefinition $\Theta\to\Theta+\gamma\Box(\varphi^2)$, with γ arbitrary, we can use the following simplified expression for Θ:

$$\Theta' = -2m^4\varphi^2 + \beta m^2\varphi\,\Box\varphi.$$

Observe that this expression does not contain an explicit dependence on the dimension.

We find

$$\begin{aligned}\sigma(x) &= \langle\Theta'(x)\,\Theta'(0)\rangle - \frac{\left(\langle\Theta'(x)\,\Box\varphi^2(0)\rangle\right)^2}{\langle\Box\varphi^2(x)\,\Box\varphi^2(0)\rangle}\\ &= 8m^8G^2 + \beta^2m^4\left(G\Box^2G + (\Box G)^2\right) - 8\beta m^6 G\Box G - 2m^4\frac{\left[\Box\left(2m^2G^2-\beta G\Box G\right)\right]^2}{\Box^2G^2}.\end{aligned}$$

The numerical results are summarized in tables 1 and 2, where the ratio R between the calculated value of the flow integral and the predicted value (5.3) is reported.

In table 1 we show the results for $r=1$. In 4 and 5 dimensions, the interand $\sigma(x)$ is singular, and Σ is ill-defined. This might either mean that the expression of the flow invariant is not correct, or that it does not make sense to apply the theory of flow invariants to non-unitary theories. The integral Σ is well-defined in all other dimensions. The results show a discrepancy with respect to the prediction. The discrepancy becomes smaller in higher dimensions. This suggests that the study of flow invariants in non-unitary theories cannot be completely devoid of meaning, but presumably the expression of the flow invariant is not precise. The correct expression is found in section 7.

n	R	n	R	n	R
4	sing.	10	1.14836	70	1.00304
5	sing.	12	1.09797	100	1.00152
6	2.12370	15	1.06120	150	1.00069
7	1.41636	20	1.03440	200	1.00039
8	1.26673	30	1.01564	300	1.00017
9	1.19275	45	1.00716	500	1.00006

Table 1: Ratio between calculated and predicted values of Σ_n for $r = 1$ in various dimensions n.

γ_+	R_7	R_9
1	1.41636	1.19275
2	1.43223	1.18973
3	1.44277	1.18614
4	1.44711	1.18570
6	1.44934	1.18561
8	1.44977	1.18560
10	1.44989	1.18560
20	1.44997	1.18560
50	1.44997	1.18560

Table 2: Ratio between calculated and predicted values of Σ_n in dependence of m_1/m_2 in 7 and 9 dimensions.

In table 2 we report results in 7 and 9 dimensions, for various values of $r = \gamma_+^2$, to check if Σ depends on the path connecting the UV and IR fixed points. We see that the result does depend on γ_+, and this dependence becomes smaller for high values of γ_+. This is another confirmation that the proposed formula for Σ cannot be correct, but also that it must be somewhat close to the correct expression.

6 Third test of the prediction

In the third test, we consider the fermionic higher-derivative theory with lagrangian

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + m_1)(\partial\!\!\!/ + m_2)(\partial\!\!\!/ + m_3)\psi.$$

The stress tensor can be computed with the method of section 5. This problem, in four dimensions, was also considered in [15]. The formula (2.5) of [15], however, contains a mistake, since the term $\bar{\psi}\overleftarrow{\partial\!\!\!/}(\gamma_\mu \overleftrightarrow{\partial_\nu} + \gamma_\nu \overleftrightarrow{\partial_\mu})\partial\!\!\!/\psi$ was neglected. The expression given there does not agree with the conformally-invariant coupling of the theory to external gravity. We repeat the derivation here in arbitrary dimensions.

The stress tensor is found by writing the most general linear combination of terms, with free parameters as coefficients. The coefficients can be fixed in three steps. In the first step, conservation is imposed up to terms proportional to the field equations. In the second step, the condition of vanishing trace is imposed up to terms proportional to the field equations. After the first two steps two undetermined parameters survive. In the final step, the expression of the stress tensor up to total derivatives is considered, and matched with the result obtained by differentiating the action with respect to the metric tensor. The non-minimal couplings can be neglected in this procedure. We find,

$$T_{\mu\nu} = e^a_{\{\mu} \frac{\delta \mathcal{L}}{\delta e^a_{\nu\}}}\Bigg|_{\rm flat} = e^a_{\{\mu} \frac{\delta}{\delta e^a_{\nu\}}} \left(e\bar{\psi} D\!\!\!\!/^3 \psi\right)\Bigg|_{\rm flat} + \text{t.d.} = \bar{\psi}\left[2\partial\mu\partial\nu\partial\!\!\!/ + \frac{1}{2}(\gamma\mu\partial\nu + \gamma\nu\partial\mu)\Box\right]\psi + \text{t.d.}$$

where "t.d." means total derivatives. We have two terms and therefore two new conditions. Consequently, the third step fixes the surviving parameters and gives a unique answer.

Observe that, using this procedure it is not necessary to know the complete expression of the conformally-invariant coupling to external gravity, i.e. the analogue of formulas (5.1) and (5.2). Very likely, the result (6.1) is sufficient to determine the complete conformally-invariant coupling. These considerations will not be pursued here.

In the massless theory, the stress tensor reads

$$\begin{aligned} T_{\mu\nu} =& \frac{1}{4}\left(\bar{\psi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\Box\psi - \Box\bar{\psi}(\gamma_\mu\overleftarrow{\partial_\nu} + \gamma_\nu\overleftarrow{\partial_\mu})\psi\right) - \frac{1}{4}\bar{\psi}\overleftarrow{\partial\!\!\!/}(\gamma_\mu\overleftrightarrow{\partial_\nu} + \gamma_\nu\overleftrightarrow{\partial_\mu})\partial\!\!\!/\psi \\ &+ \frac{n+2}{4(n-2)}\left(\Box\bar{\psi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\psi - \bar{\psi}(\gamma_\mu\overleftarrow{\partial_\nu} + \gamma_\nu\overleftarrow{\partial_\mu})\Box\psi\right) + \frac{1}{(n-2)}\partial_\alpha\bar{\psi}(\gamma_\mu\overleftrightarrow{\partial_\nu} + \gamma_\nu\overleftrightarrow{\partial_\mu})\partial_\alpha\psi \\ &+ \frac{1}{(n-1)(n-2)}(\partial_\mu\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\partial_\nu\psi + \partial_\nu\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\partial_\mu\psi) - \frac{n}{(n-1)(n-2)}(\bar{\psi}\overleftarrow{\partial\!\!\!/}\partial_\mu\partial_\nu\psi - \partial_\mu\partial_\nu\bar{\psi}\partial\!\!\!/\psi) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{n-1}(\bar{\psi}\partial\!\!\!/\partial_\mu\partial_\nu\psi-\partial_\mu\partial_\nu\bar{\psi}\overleftarrow{\partial\!\!\!/}\psi)+\frac{n}{(n-1)(n-2)}\delta_{\mu\nu}(\bar{\psi}\overleftarrow{\partial\!\!\!/}\Box\psi-\Box\bar{\psi}\partial\!\!\!/\psi)\\
&-\frac{2}{(n-1)(n-2)}\delta_{\mu\nu}\partial_\alpha\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\partial_\alpha\psi+\frac{1}{n-1}\delta_{\mu\nu}(\bar{\psi}\Box\partial\!\!\!/\psi-\Box\bar{\psi}\overleftarrow{\partial\!\!\!/}\psi),
\end{aligned}
\tag{6.1}
$$

The term proportional to the free-field equations can be fixed by imposing conservation in the massive case (see below).

The mass-independent part of the stress tensor contributes to the trace with

$$
\Theta=\frac{3}{2}(\bar{\psi}\Box\partial\!\!\!/\psi-\Box\bar{\psi}\partial\!\!\!/\psi).
$$

The central charge c_n is found from the two-point function (3.1). The result is

$$
c_n=-2^{[n/2]-1}\,\frac{n^2+n-18}{n-2}
$$

The prediction (3.5) reads in this case

$$
\Sigma_n=-\frac{2^{[n/2]-1}(n^2+n-18)\,\Gamma(n/2+1)}{\pi^{n/2}(n+1)(n-2)}.
\tag{6.2}
$$

In the massive case, the other contributions to the stress tensor can be written using the formulas for the theories $\bar{\psi}\Box\psi$ and $\bar{\psi}\partial\!\!\!/\psi$. We have

$$
\begin{aligned}
\Delta T_{\mu\nu}=&\frac{1}{4}(m_1+m_2+m_3)\left(\bar{\psi}\gamma\mu\overleftrightarrow{\partial\nu}\partial\!\!\!/\psi+\bar{\psi}\gamma\nu\overleftrightarrow{\partial\mu}\partial\!\!\!/\psi-\bar{\psi}\overleftarrow{\partial\!\!\!/}\gamma\mu\overleftrightarrow{\partial\nu}\psi-\bar{\psi}\overleftarrow{\partial\!\!\!/}\gamma\nu\overleftrightarrow{\partial\mu}\psi\right)\\
&+\frac{1}{4}(m_1m_2+m_1m_3+m_2m_3)\left(\bar{\psi}\gamma\mu\overleftrightarrow{\partial\nu}\psi+\bar{\psi}\gamma\nu\overleftrightarrow{\partial\mu}\psi\right)
\end{aligned}
$$

plus a couple of terms proportional to the field equations, which we can omit. In the end, we find the following trace:

$$
\begin{aligned}
\Theta=&-(m_1+m_2+m_3)(\bar{\psi}\Box\psi+\bar{\psi}\overleftarrow{\partial\!\!\!/}\partial\!\!\!/\psi+\Box\bar{\psi}\psi)\\
&-(m_1m_2+m_1m_3+m_2m_3)(\bar{\psi}\partial\!\!\!/\psi-\bar{\psi}\overleftarrow{\partial\!\!\!/}\psi)-3m_1m_2m_3\bar{\psi}\psi.
\end{aligned}
$$

The improvement operator is $\mathcal{O}=\Box(\bar{\psi}\psi)$.

We consider the case $m_1=m_2=-m_3\equiv m$, for simplicity, where the porpagator is

$$
\langle\bar{\psi}(x)\,\psi(0)\rangle=mG_{n,1}(x)-\frac{x\!\!\!/}{|x|}G'_{n,1}(x)
$$

and $G_{n,1}(x)$ is the same as in (5.4).

The results are reported table 3 and are very similar to those of the scalar theory studied in section 5. Again, there is a discrepancy with respect to the prediction. The discrepancy, however, decreases when the space-time dimension increases. In dimension 500, we have a 0.006% discrepancy.

n	R	n	R	n	R
4	sing.	10	1.14773	70	1.00308
5	2.36284	12	1.09900	100	1.00153
6	1.60279	15	1.06230	150	1.00069
7	1.36553	20	1.03511	200	1.00039
8	1.25320	30	1.01593	300	1.00017
9	1.18887	45	1.00726	500	1.00006

Table 3: True versus predicted value in various dimensions.

7 The solution of the puzzle

The numerical results presented in the previous sections do not reproduce the predictions exactly. Sometimes, such as for $n = 4, 5$ in the scalar theory of section 5 and for $n = 4$ in the fermion theory of section 6, the flow integral is ill-defined. This happens because the correlators $\langle \Box\varphi^2 \, \Box\varphi^2 \rangle$ or $\langle \Box\bar{\psi}\psi \, \Box\bar{\psi}\psi \rangle$ vanish in one or more points. We know that, on the other hand, in physical, unitary quantum field theories, these correlators are positive.

We have already discussed that there are good reasons to believe that the flow invariant is meaningful in the mathematically well-defined theories. The very fact that the results of the previous sections are in most cases close to the predictions, althought not equal to those, supports this consideration. The point is that we have not used the correct formula for the flow invariant. The solution is as follows.

We have required that $\sigma(x)$ be invariant under the symmetry (3.4), but this requirement is too strong. Indeed, we just need that the integral of $\sigma(x)$ be invariant under this symmetry. The correct formula reads

$$\sigma_n = \int \mathrm{d}^n x \, |x|^n \langle \Theta(x) \, \Theta(0) \rangle - M^t N^{-1} M, \tag{7.1}$$

where M_i and N_{ij} are the vector and matrix defined by

$$M_i = \int \mathrm{d}^n x \, |x|^n \langle \Theta(x) \, \mathcal{O}_i(0) \rangle, \qquad N_{ij} = \int \mathrm{d}^n x \, |x|^n \langle \mathcal{O}_i(x) \, \mathcal{O}_j(0) \rangle$$

The invariant σ_n is more general than Σ_n, since it satisfies nothing more that the minimum symmetry requirements. The invariance of (7.1) under (3.4) is easy to prove and I leave this as an exercise for the reader.

We have produced two expressions, σ_n and Σ_n, which are both invariant under (3.4). This means that the symmetry (3.4) does not fix the invariant uniquely and that we need a more powerful principle. The answer is a minimum principle stating that

$$\sigma_n = \min_a \int \mathrm{d}^n x \, |x|^n \, \langle \Theta_a(x) \, \Theta_a(0) \rangle, \tag{7.2}$$

where

$$\Theta_a = \Theta + a_i \mathcal{O}_i,$$

and a_i are arbitrary constants. Denoting by $\bar{a}_i$ the constants minimizing the expression above, we find the solution

$$\bar{a}_i = -(N^{-1})_{ij} M_j,$$

whence the result (7.1) follows.

Formula (7.2) means that we have to minimize the integral of $\langle \Theta\, \Theta \rangle$ in the entire space of improved Θ's. In unitary theories, there always exist a non-negative minimum, since $\langle \Theta_a(x)\, \Theta_a(0) \rangle \geq 0$. The minimum is zero if and only if the theory is conformal.

Indeed, in a unitary theory, the condition of criticality in the presence of improvement terms is not defined by $\Theta = 0$, which is meaningless, but by the equality $\sigma_n = 0$, which is equivalent to say that there exist constants $\bar{a}_i$ such that

$$\Theta = -\bar{a}_i\, \mathcal{O}_i.$$

The proof of this fact is straightforward. Putting $\sigma_n = 0$ in expression (7.2) and using the fact that $\langle \Theta_a\, \Theta_a \rangle \geq 0$, we see that there exist constants $\bar{a}_i$ such that $\langle \Theta_{\bar{a}}(x)\, \Theta_{\bar{a}}(0) \rangle \equiv 0$. In a unitary theory this means that the operator $\Theta_{\bar{a}}$ vanishes, which implies the statement.

We can prove that in unitary theories $\sigma_n \geq \Sigma_n$. For simplicity, we consider the case of a single improvement operator $\mathcal{O}$. The proof is a standard application of the Schwarz-Hölder inequality. We have, for arbitrary functions f and g and an arbitrary constant a,

$$0 \leq \int (f + ag)^2 \, \mathrm{d}\mu = \int f^2 \, \mathrm{d}\mu + 2a \int fg \, \mathrm{d}\mu + a^2 \int g^2 \, \mathrm{d}\mu, \tag{7.3}$$

where $\mathrm{d}\mu$ denotes the integration measure. Since (7.3) holds for arbitrary a, the discriminant must be negative, or zero:

$$\left(\int fg \, \mathrm{d}\mu \right)^2 \leq \left(\int f^2 \, \mathrm{d}\mu \right) \left(\int g^2 \, \mathrm{d}\mu \right).$$

In unitary theories, $\langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle > 0$ and choosing

$$g(x) = \sqrt{\langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle}, \qquad f(x) = \frac{1}{g(x)} \langle \Theta(x)\, \mathcal{O}(0) \rangle,$$

and $\mathrm{d}\mu = \mathrm{d}^n x\, |x|^n$, we have

$$\frac{\left(\int \mathrm{d}^n x\, |x|^n \, \langle \Theta(x)\, \mathcal{O}(0) \rangle \right)^2}{\int \mathrm{d}^n x\, |x|^n \, \langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle} \leq \int \mathrm{d}^n x\, |x|^n \, \frac{\left(\langle \Theta(x)\, \mathcal{O}(0) \rangle \right)^2}{\langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle}.$$

Adding $-\int \mathrm{d}^n x\, |x|^n \, \langle \Theta(x)\, \Theta(0) \rangle$ to both sides, we get the desired result, $\sigma_n \geq \Sigma_n$.

This conclusion is quite reasonable. The integrand $\sigma(x)$ of Σ_n is the minimum value of the correlator $\langle \Theta_a(x)\, \Theta_a(0) \rangle$. However, the value $\bar{a}$ at which the correlator $\langle \Theta_a(x)\, \Theta_a(0) \rangle$ (not its integral) is minimum, is point-dependent, $\bar{a} = \bar{a}(x)$. Therefore $\Theta_{\bar{a}}$ is non-local. If we allow a to be point-dependent, we are minimizing over a much larger space and therefore we get a smaller value $\Sigma_n \leq \sigma_n$. The mentioned non-locality is the ultimate reason why Σ_n cannot be the correct expression for the flow invariant.

In non-unitary theories, $\langle \mathcal{O}(x)\, \mathcal{O}(0)\rangle$ does not have, in general, a definite sign. If, however, it happens that it is identically negative, then we can repeat the above argument by replacing the correlator $\langle \mathcal{O}(x)\, \mathcal{O}(0)\rangle$ with its absolute value, in the definition of the function g. It is easy to verify that we get the inversed inequality $\sigma_n \leq \Sigma_n$. In the bosonic model considered in section 5, the correlator $\langle \mathcal{O}(x)\, \mathcal{O}(0)\rangle$ is identically positive. This means that the numerical value of Σ_n should be smaller than the predicted value σ_n. Since both are negative, this implies $R \geq 1$. This is always verified: see tables 1 and 2. In the fermionic model of section 6 the situation is similar, when $m_1 = m_2 = -m_3$, as we see from table 3.

In conclusion, the prediction (3.5) should be replaced by

$$\sigma_n = \Delta c_n \frac{\Gamma(n/2+1)}{\pi^{n/2}\,(n+1)}. \tag{7.4}$$

Observe that the new formula (7.1) is easier to compute, since each separate integral can be reduced to elementary integrals of rational functions in momentum space.

We now reconsider the bosonic model of section 5 and test the new prediction in the case $m_1 = m_2 = m$.

We have

$$\sigma_n = \alpha_n - \frac{\beta_n^2}{\gamma_n},$$

where

$$\alpha_n = \int \mathrm{d}^n x\, |x|^n \, \langle \Theta(x)\, \Theta(0)\rangle = \int \frac{\mathrm{d}^n p}{(2\pi)^n} (-1)^{n/2} \frac{(4m^4p^4 + 16m^4p^2 + 8m^4) I_n(p) + 4m^4p^2 J_n(p)}{(p^2+m^2)^2},$$

$$\beta_n = 2n(n-1) \int \mathrm{d}^n x\, |x|^{n-2} \, \langle \Theta(x)\, \varphi^2(0)\rangle = -8m^4 n(n-1) \int \frac{\mathrm{d}^n p}{(2\pi)^n} (-1)^{n/2-1} \frac{I_n^{(2)}(p)}{(p^2+m^2)},$$

$$\gamma_n = 4n(n-1)(n-2)^2 \int \mathrm{d}^n x\, |x|^{n-4} \, \langle \varphi^2(x)\, \varphi^2(0)\rangle = 8n(n-1)(n-2)^2 \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{(-1)^{n/2} I_n^{(1)}(p)}{(p^2+m^2)^2},$$

and

$$I_n(p) \equiv \left(\frac{\partial^2}{\partial p^2}\right)^{n/2} \frac{1}{(p^2+m^2)^2}, \qquad J_n(p) \equiv \left(\frac{\partial^2}{\partial p^2}\right)^{n/2} \frac{p^2}{(p^2+m^2)^2},$$

$$I_n^{(1)}(p) \equiv \left(\frac{\partial^2}{\partial p^2}\right)^{n/2-2} \frac{1}{(p^2+m^2)^2}, \qquad I_n^{(2)}(p) \equiv \left(\frac{\partial^2}{\partial p^2}\right)^{n/2-1} \frac{1}{(p^2+m^2)^2}.$$

These expressions can be worked out with a certain algebraic effort:

$$I_n(p) = 2^{n-1} n! (-1)^{n/2} (m^2)^{n/2-2} \frac{p^4(n/2-1) - 2m^2p^2(n/2+1) + m^4(n/2+1)}{(p^2+m^2)^{n+2}},$$

$$J_n(p) = 2^n n! (-1)^{n/2+1} (m^2)^{n/2-1} \frac{p^4(n/4) - m^2p^2(n/2+1) + m^4(n/4)}{(p^2+m^2)^{n+2}},$$

$$I_n^{(1)}(p) = 2^{n-4} (n-3)! (-1)^{n/2} (m^2)^{n/2-2} \frac{1}{(p^2+m^2)^{n-2}},$$

$$I_n^{(2)}(p) = 2^{n-2} (n-2)! (-1)^{n/2} (m^2)^{n/2-2} \frac{p^2(n/2-1) - m^2(n/2)}{(p^2+m^2)^{n}}.$$

γ_+	R_7	R_9
1	1.000000	1.000000
2	0.924873	0.906629
3	0.778893	0.742547
4	0.681417	0.640275
6	0.580400	0.537945
8	0.530913	0.488573
10	0.501515	0.459344
20	0.440888	0.399087
50	0.397027	0.355388

Table 4: Ratio between calculated and predicted values of σ_n in dependence of m_1/m_2 in 7 and 9 dimensions.

The integrals are elementary and give

$$\alpha_n = \frac{2\Gamma(n/2+1)}{(n+1)\pi^{n/2}}, \qquad \beta_n = -\frac{2\Gamma(n/2+1)}{m^2\pi^{n/2}}, \qquad \gamma_n = \frac{(n-2)\Gamma(n/2+1)}{m^4\pi^{n/2}}$$

Finally, we find

$$\sigma_n = -\frac{2(n+4)\Gamma(n/2+1)}{(n+1)(n-2)\pi^{n/2}},$$

in agreement with the prediction.

We now continue the analysis of the scalar model of section 5, but set $m_1 \neq m_2$. We find an unexpected behavior. It would be natural to expect that the value of the flow integral does not depend of the ratio m_1/m_2, i.e. on the particular trajectory connecting the same UV and IR fixed points (see point *iii* of sect. 1.2). It turns out, however, that this is not the case. The value of the flow integral does know about the trajectory. It is minimal and equal to (7.4) on a priviledged trajectory, which is, in our case, precisely the trajectory with $m_1 = m_2$.

I have computed σ_n numerically for various values of m_1/m_2 in 7 and 9 dimensions. The results, normalized to the predicted value appearing on the right-hand side of (7.4), are shown in table 4.

We see that the ratio R is not constant, but decreases when the m_1/m_2 departs from 1 (the result is clearly invariant under $m_1/m_2 \to m_2/m_1$). Recalling that the value of the integral is negative, we conclude that the flow integral is minimal, and equal to the prediction, for $m_1 = m_2$.

As a further check, we can compute the flow integral σ exactly in p-space for $n = 4$, as a function of $m_1^2/m_2^2 = r^2$. The result reads

$$\sigma(r) = -6\,\frac{(r^2-1)^2(3r^4-26r^2+3)+(r^8+18r^6-18r^2-1)\ln r^2 - 10r^2(r^4+1)\ln^2 r^2}{5\pi^2(r^2-1)^3(r^2\ln r^2+\ln r^2-2r^2+2)}$$

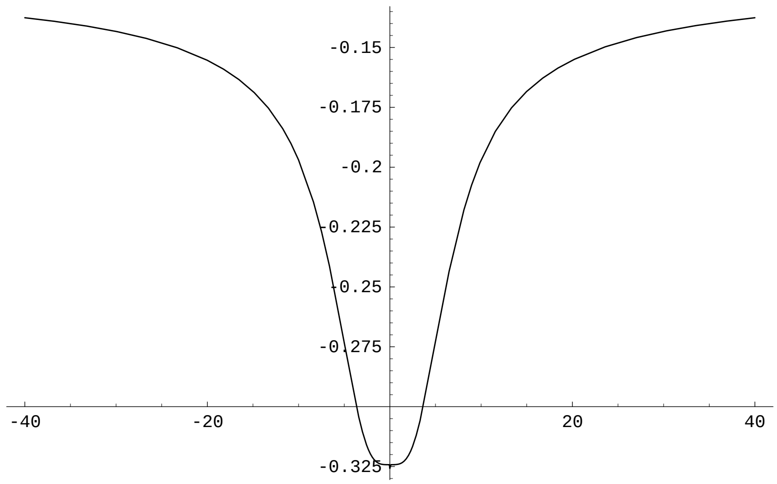

Figure 2: Plot of σ_n for the bosonic model of sect. 5 in $n = 4$.

and is plotted in figure 2, with $x = \ln r^2$ in the abscissa. The minimum is at $r = 1$, where it equals the expected value $\sigma_4 = -16/(5\pi^2)$. The maximum value, for $r = 0$ and $r = \infty$, is $-6/(5\pi^2)$.

An exact calculation can also be done in the fermion model of section 6, in the case $m_1 = m_2 = -m_3 = m$. Since, however, it is rather lengthy, I have preferred to check the prediction numerically. In all cases the agreement $R = 1$ has been found, up to the sixth decimal figure. The results are given in table 5, normalized to the prediction P_n, that is to say the right-hand side of (6.2).

The situation is more complicated, when we consider other trajectories than $m_1 = m_2 = -m_3$. Taking $m_1 = m_2$, the integral of (7.1) is a function of m_3, plotted in figure 3. We see that

n	$-\alpha_n/P_n$	$-\beta_n/P_n$	$-\gamma_n/P_n$	σ_n/P_n	n	$-\alpha_n/P_n$	$-\beta_n/P_n$	$-\gamma_n/P_n$	σ_n/P_n
4	9	10	10	1	20	2.55224	31.9701	287.731	1
5	3	6	9	1	30	2.67105	51.3947	719.526	1
6	5/2	7	14	1	45	2.76608	80.9708	1740.87	1
7	2.36842	8.42105	21.0526	1	70	2.84249	130.645	4441.92	1
8	7/3	10	30	1	100	2.88693	190.459	9332.51	1
9	7/3	35/3	245/6	1	150	2.92312	290.311	21483.	1
10	2.34783	13.3913	53.5652	1	200	2.94176	390.235	38633.2	1
12	2.3913	16.9565	84.7826	1	300	2.96079	590.158	87933.5	1
15	2.45946	22.4865	146.162	1	500	2.97629	990.095	246534.	1

Table 5: Results for the fermionic model of section 6 with $m_1 = m_2 = -m_3$.

the point $m_3 = -1$ is an extremum, but not a minimum of σ_4 (a maximum, in this case) and that a minimum does not exist. There is, instead, a singularity for $m_3 = -0.683155$ and there exist other extrema beyond the singular point. One of these is $m_1 = m_2 = m_3$, for which the ratio σ_4/P_4 equals -135, a value which does not have a clear interpretation. The singularity visible in figure 3 is due to a zero of γ_4 and is a clear sign that certain non-unitary theories can be sufficiently bad to give unexpected problems. It might be wise to restrict the set of non-unitary theories to the purely bosonic ones, where there exists a notion of positive-definiteness for the action. Eventually, we can include the supersymmetric non-unitary theories having a positive-definite bosonic action. We naturally expect that the boson-fermion pairing imposed by supersymmetry forces the fermionic sector of the theory to be also well-behaved.

I have checked that $m_1 = m_2 = -m_3$ and $m_1 = m_2 = m_3$ are extrema in the full space of the parameters m_1, m_2 and m_3. I have also extended the check to various dimensions other than 4. I report here only a few results, obtained numerically. In particular, at the point $m_1 = m_2 = -m_3 = 1$, we have

$$\frac{\partial(\alpha_4/P_4)}{\partial m_{1,2}} = -20, \qquad \frac{\partial(\beta_4/P_4)}{\partial m_{1,2}} = 0, \qquad \frac{\partial(\gamma_4/P_4)}{\partial m_{1,2}} = 20,$$

$$\frac{\partial(\sigma_4/P_4)}{\partial m_{1,2}} = \frac{\partial(\alpha_4/P_4)}{\partial m_{1,2}} - 2\frac{\beta_4}{\gamma_4}\frac{\partial(\beta_4/P_4)}{\partial m_{1,2}} + \frac{\beta_4^2}{\gamma_4^2}\frac{\partial(\gamma_4/P_4)}{\partial m_{1,2}} = 0,$$

$$\frac{\partial(\alpha_4/P_4)}{\partial m_3} = -40, \qquad \frac{\partial(\beta_4/P_4)}{\partial m_3} = -10, \qquad \frac{\partial(\gamma_4/P_4)}{\partial m_3} = 20, \qquad \frac{\partial(\sigma_4/P_4)}{\partial m_3} = 0,$$

in dimension 4 and

$$\frac{\partial(\alpha_6/P_6)}{\partial m_{1,2}} = -\frac{14}{3}, \qquad \frac{\partial(\beta_6/P_6)}{\partial m_{1,2}} = 0, \qquad \frac{\partial(\gamma_6/P_6)}{\partial m_{1,2}} = \frac{56}{3}, \qquad \frac{\partial(\sigma_6/P_6)}{\partial m_{1,2}} = 0,$$

$$\frac{\partial(\alpha_6/P_6)}{\partial m_3} = -\frac{28}{3}, \qquad \frac{\partial(\beta_6/P_6)}{\partial m_3} = -7, \qquad \frac{\partial(\gamma_6/P_6)}{\partial m_3} = \frac{28}{3}, \qquad \frac{\partial(\sigma_6/P_6)}{\partial m_3} = 0,$$

in dimension 6.

8 Conclusions

Collecting the information obtained so far, the final formula of the flow invariant reads in unitary theories

$$\sigma_n = \min_{m(\mu)} \min_a \int \mathrm{d}^n x \, |x|^n \, \langle \Theta_a(x) \, \Theta_a(0) \rangle, \tag{8.1}$$

where the minimization is performed both in the space of improved stress tensors and in the space of trajectories $m(\mu)$ relating the dimensioned parameters of the theory. The invariant is universal and proportional to Δa in classically-conformal theories (marginally-relevant flows), where there is no need to minimize over the set of trajectories $m(\mu)$, because there is only the dynamical scale μ. It is proportional to Δc in flows of strictly relevant operators, where μ is absent. There is evidence to claim that the flow invariant is universal in the class of flows with $\Delta a = \Delta c$. When $\Delta a \neq \Delta c$ the formula is still expected to give a characterization of the flow, in

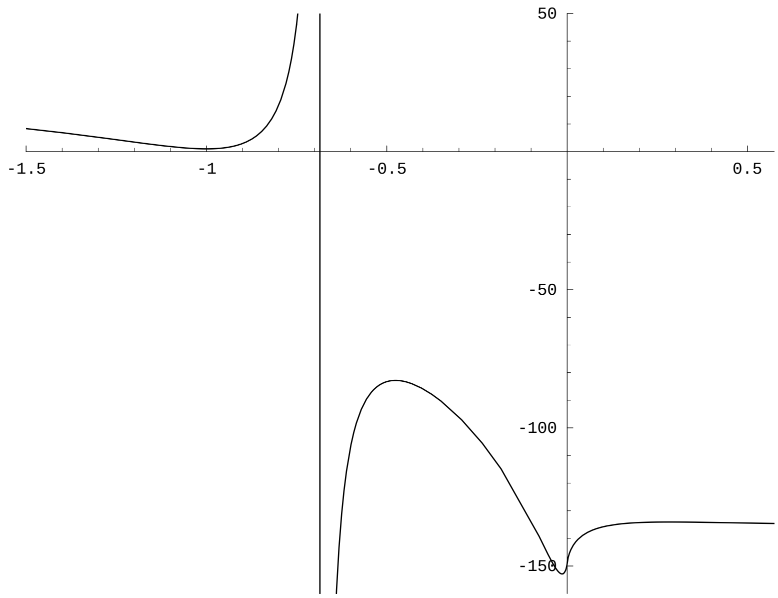

Figure 3: Plot of σ_4/P_4 versus m_3 for the fermionic model of sect. 6 in $n = 4$, with $m_1 = m_2$.

the sense that the minimum principle selects priviledged flows among the flows connecting the same fixed points. These flows are special, because in the examples considered in this paper the integral appearing in σ_n, calculated along the priviledged trajectory, equals the predicted value Δc.

Several problems remain open. The relation between the flow invariant and the central charges a and c has to be clarified in the general flows (see point *ii* of sect. 1.2). A possible way to shed light on the open questions might be to reconsider the issues studied here from the point of view of the wilsonian exact renormalization-group approach. Here I have studied mostly gaussian theories, but the results are expected to be more general, since they include all the cases treated so far [3, 4, 5, 11].

In non-unitary theories with a positive-definite action, we have found that the minimum still exists. Nevertheless, the condition of positive-definiteness of the action is meaningful only for bosonic theories and it is not straightforward to define the acceptable non-unitary fermionic theories. We have found that in such theories the minimum might not exist, and the prediction holds at an extremum of the flow integral of (8.1). Singularities might limit the acceptable region. Typically, other extrema exist outside of it, but the value of the integral (8.1) in those points does not seem to have a clear interpretation. The privileged trajectory is $m_1 = m_2$ in the bosonic model of section 5, and $m_1 = m_2 = -m_3$ in the fermionic model of section 6.

Our initial expectations have been updated in two relevant points. First, it is now clear how to define the flow invariant in the presence of improvement terms for the stress tensor. A natural minimum principle selects the correct formula. Second, the flow invariant is not independent of the trajectory in the space of dimensioned parameters. This might also suggest that the name "flow invariant", conceived on the basis of the initial expectations (that the integral (7.1) was completely independent on the trajectory) is not completely justified. Equivalently, we can say that flow invariance is trivial, since the minimum σ_n of (8.1) is obviously independent of the trajectory, given that it is obtained by minimizing (7.1) over all trajectories. I expect that the form (8.1) of the flow invariant applies also to the Standard Model, in which case we might be able to relate the Higgs mass to $\Lambda_{\rm QCD}$. Finally, it can be speculated that a more sophisticated invariant might be written, by giving a suitable weight to each trajectory and functionally integrating over all trajectories, instead of minimising.

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