

# Anomalies, Unitarity and Quantum Irreversibility

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## Abstract

The trace anomaly in external gravity is the sum of three terms at criticality: the square of the Weyl tensor, the Euler density and  $\square R$ , with coefficients, properly normalized, called  $c$ ,  $a$  and  $a'$ , the latter being ambiguously defined by an additive constant. Considerations about unitarity and positivity properties of the induced actions allow us to show that the total RG flows of  $a$  and  $a'$  are equal and therefore the  $a'$ -ambiguity can be consistently removed through the identification  $a' = a$ . The picture that emerges clarifies several long-standing issues. The interplay between unitarity and renormalization implies that the flux of the renormalization group is irreversible. A monotonically decreasing  $a$ -function interpolating between the appropriate values is naturally provided by  $a'$ . The total  $a$ -flow is expressed non-perturbatively as the invariant (i.e. scheme-independent) area of the graph of the beta function between the fixed points. We test this prediction to the fourth loop order in perturbation theory, in QCD with  $N_f \lesssim 11/2 N_c$  and in supersymmetric QCD. There is agreement also in the absence of an interacting fixed point (QED and  $\varphi^4$ -theory). Arguments for the positivity of  $a$  are also discussed.

## 1 Introduction

Anomalies are often calculable to high orders in perturbation theory with a relatively moderate effort. Sometimes they are calculable exactly to all orders.

Much of the present knowledge about the low-energy limit of asymptotically free quantum field theories comes from anomalies, via the Adler–Bardeen theorem [1]. Conserved axial currents have one-loop exact anomalies and the ‘t Hooft anomaly matching conditions [2] put constraints on the low-energy limit of the theory.

A second class of anomalies, related to the stress tensor, called *central charges*, do not satisfy the Adler–Bardeen theorem. Nevertheless, they obey various positivity constraints, which also put restrictions on the low-energy limit of the theory.

Other remarkable positivity constraints are those obeyed by the spectrum of anomalous dimensions of the quantum conformal algebra, i.e. the algebra generated by the operator product expansion of the stress tensor. Applications of the Nachtmann theorem [3] reveal non-trivial properties of strongly coupled conformal field theories [4, 5], especially in the presence of supersymmetry, where the algebraic structure simplifies considerably.

Furthermore, in supersymmetric theories, the two classes of anomalies mentioned above, axial and trace, are related to each other, and the Adler–Bardeen theorem can be used to compute the exact IR values of the central charges in the conformal window [6]. The consequent large class of restrictions on the low-energy limit of the theory can be studied explicitly [7].

There are positivity properties of the central charges that have not been rigorously proved, yet. The purpose of this paper is to clarify certain long-standing issues in this context.

The trace anomaly of the energy-momentum tensor is deeply related to the renormalization group flow. There is empirical evidence [6, 7] that a central charge, called  $a$ , is positive and takes greater values in the UV than in the IR:  $a_{\text{UV}} \geq a_{\text{IR}} \geq 0$ . The quantity  $a$  is interpreted as the number of massless degrees of freedom of the theory. This means that the flux of the renormalization group is irreversible.

We call this notion *quantum irreversibility*, to distinguish it from time irreversibility, proper of thermodynamics and statistical mechanics.

A first suggestion in favour of this idea comes from two-dimensional quantum field theory, where Zamolodchikov [8] proved that the central extension  $c$  of the stress tensor operator product algebra is positive and monotonically decreasing along the renormalization group flow.

A four-dimensional generalization of this property is, however, more difficult to prove. In four dimensions, for example, the set of candidate central charges is richer, and among them there is also the central extension  $c$  of the operator product algebra. Various proposals for the good candidate have appeared in the literature, as well as attempts to prove the irreversibility property. We do not review the history of this research here, but one proposal, due to Cardy [9], deserves special mention, since the results of [6] were able to reject all the other candidates, in particular the central extension  $c$  of the OPE algebra. At the same time, the impressive amount of evidence in favour of the “ $a$ -theorem” [6, 7] convinced many people that quantum irreversibility was true.

In this paper we reconsider the matter under a different viewpoint. We present a general picture that clarifies various issues related with the trace anomaly and unifies some notions that have been so far considered unrelated. Quantum irreversibility is intrinsically contained in this picture, and the outcome is an explicit non-perturbative formula for the total flow of  $a$  that can be tested successfully to the fourth loop order in perturbation theory. This formula gives an intuitive (and geometrical) picture of quantum irreversibility, measured by the area of the graph of the beta function.

In the rest of the introduction we give the basic guidelines of our arguments, anticipate some applications, and explain how the paper is structured. The presentation is meant to be self-consistent and we take this opportunity to discuss the same issue under different viewpoints.

In four dimensions the trace anomaly operator equation in an external gravitational field can be written in the form

$$\Theta = -\frac{1}{120} \frac{1}{(4\pi)^2} \left[ \tilde{c}(\alpha) W^2 - \frac{1}{3} \tilde{a}(\alpha) G + \frac{2}{9} \tilde{a}'(\alpha) \square R + \beta(\alpha) h(\alpha) R^2 \right] + \frac{1}{4} \beta(\alpha) F^2, \quad (1.1)$$

where  $\alpha$  denotes the renormalized coupling constant, at a certain reference scale  $\mu$ , and we have defined the  $\beta$ -function as

$$\beta(\alpha) = \frac{1}{\alpha} \mu \frac{d\alpha}{d\mu} = \frac{d \ln \alpha}{d \ln \mu}; \quad (1.2)$$

$W$  is the Weyl tensor of the external gravitational field,  $W^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2$ , and  $G = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$  is the Gauss–Bonnet integrand. The stress tensor is  $T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$ ,  $S$  denoting the action in the gravitational background. The curvature conventions are those of refs. [10, 11]. We work partly in the Lorentzian framework and partly in the Euclidean framework.

The last term of (1.1) is written, for concreteness, in the case of Yang–Mills theory. In general it should read  $\sum_i \beta_i \mathcal{O}_i$ , where the sum runs over the set of coupling constants of the theory. In the presence of scalar fields  $\varphi$  there is an additional complication, which has to be treated apart, due to the renormalization mixing between the stress-energy tensor and the operator  $(\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \varphi^2$ ,  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . Arguments and conclusions are valid for the most general renormalizable quantum field theory.

The term  $R^2$  does not contribute at criticality, since it is neither a total derivative nor a conformal-invariant. For this reason one can factorize a  $\beta(\alpha)$  in its coefficient, which multiplies a certain function  $h(\alpha)$ .

The tilde over the functions  $\tilde{c}(\alpha)$ ,  $\tilde{a}(\alpha)$  and  $\tilde{a}'(\alpha)$  is used to remark that these functions, as they appear in the anomaly operator equation, have not a direct physical meaning at a generic energy scale (see the discussion in section 1 of ref. [12] for all details). In particular, they are scheme-dependent. Physical quantities have to be defined via matrix elements of operators rather than the operator equations. An operator equation contains artefacts that disappear in matrix elements. In practice, when inserting eq. (1.1) inside correlators, the contributions coming from the matrix elements of the dynamical operator  $\frac{\beta}{4} F^2$  will restore full scheme independence. Only at criticality, where  $\beta = 0$ , is there no such ambiguity, where  $\tilde{c}_*$

and  $\tilde{a}_*$  have a direct physical meaning. Instead,  $a'_*$  retains a peculiar type of ambiguity that we are going to discuss in detail.

In order to properly interpolate between the UV and IR critical values, one has to define physical (i.e. scheme-independent) *central functions*,  $c(\alpha)$ ,  $a(\alpha)$  and  $a'(\alpha)$ , through matrix elements of operators. In ref. [12] this was done for the function  $c(\alpha)$  and certain “secondary” central charges. In the first part of the paper we extend this analysis to the function  $a'(\alpha)$  and prove that it satisfies the irreversibility property. In the rest of the paper we explain how this interpolating function is also a good central function for  $a(\alpha)$ , so that  $a$  satisfies the irreversibility property as well (the “ $a$ -theorem”, see [6]).

Reflection positivity implies

$$c \geq 0,$$

since  $c$  is the overall constant of the stress tensor two-point function, whose structure is uniquely fixed by conformality at the fixed points. At the OPE level [4, 5]  $c$  represents the central extension of the quantum conformal algebra, which is the reason why we retain the symbol  $c$  for it. It has been shown, even at the non-perturbative level [6] in the conformal window, that the central extension is not, in general, the quantity that monotonically decreases along the RG flow: this is true only in two dimensions. Following [6] we use a different symbol for the decreasing quantity,  $a$ , and speak of “ $a$ -theorem”. At the level of OPE algebra, the interpretation of the quantity  $a$  is different from that in two dimensions [5]: the combination  $1 - a/c$  is indeed a structure constant of the OPE algebra.

There are cases, also in four dimensions, where the central extension does decrease from the UV to the IR, for example when the theory interpolates between conformal fixed points with  $c = a$ . These conformal field theories have a simplified OPE algebra and other nice properties [4, 5].  $c$  decreases in several other particular models, or in part of the conformal window. Examples of this kind are treated in [6, 7]. Nevertheless, this behaviour is not general and so the central extension is not a good counter of the massless degrees of freedom of the system.

The quantity  $a$  has been shown to have the desirable properties, namely

$$a_{\text{UV}} \geq a_{\text{IR}} \geq 0, \tag{1.3}$$

in various concrete models, the most impressive results being the exact formulas of [6] for the conformal windows in supersymmetric theories, applied in [7] to a large variety of cases. Off the conformal window, non-perturbative tests based on general physical grounds in QCD [9] and on various duality conjectures in supersymmetric theories [13] are successful, but less constraining, since  $c$  passes them also. At the rigorous level, both proofs that  $a$  is positive and decreasing along the RG flow have been missing. Positivity of  $a$  passes the tests of [6, 7] within the known conformal windows. The breakdown of this condition typically signals that the IR fixed point does not exist (as in pure N=1 supersymmetric QCD).

Finally, the quantity  $a'$  has remained, up to now, somewhat mysterious, especially at criticality; yet it is simple to prove that it is monotonically decreasing from the UV to the IR. There exists a central function  $a'(t) = a'[\alpha(t)]$  that satisfies the irreversibility property at any

intermediate energy scale,

$$da'(t)/dt \leq 0;$$

nevertheless, as it stands,  $a'$  is not a good counter of degrees of freedom, since it is meaningless at criticality. One of the purposes of our analysis is to clarify the meaning of  $a'$  and its relationship with the other two quantities, in particular the quantity  $a$ .

We see that each of the three quantities has part of the properties that we would like for a single quantity:  $c$  is positive, but not monotonically decreasing;  $a'$  is monotonically decreasing, but ill-defined at criticality;  $a$  has the good properties, but so far only at the empirical level, in the sense that they have not been proved rigorously. It is only by uncovering the deep meaning of each quantity and the interplay among them that a clarifying picture can emerge. We can say that the matter is much simpler in two dimensions, because, in some sense, “ $c = a = a'$ ”. The proof of irreversibility [8] and positivity are straightforward in two dimensions.

In some works [14] one finds arguments in favour of the identification  $a' = 3c$ . This is however an artefact of the regularization scheme (a dimensional continuation preserving conformal invariance in  $d$ -dimensions) and is actually inconsistent. Indeed, if this equality were consistent, it would hold both in the UV limit and the IR limit. However,  $a'$  is monotonically decreasing, while  $c$  has an indefinite behaviour, as proved in [6]. There are many known examples where  $a'_{UV} = 3 c_{UV}$  does not imply  $a'_{IR} = 3 c_{IR}$ .

Therefore, if the ambiguous quantity  $a'$  has to be identified with one of the two unambiguous central charges, or a linear combination of them, it can only be identified with  $a$ . The relative factor can be chosen in such a way that the relation  $a' = a$  has other noticeable properties. In particular the induced action for the conformal factor (the Riegert action [15]) simplifies enormously (it becomes free as in two dimensions).

Considerations about positivity of induced effective actions (absence of negatively normed states) allow us to show that the identification  $a' = a$  is consistent, i.e. that if assumed in the UV limit of the theory it holds also in the IR limit, precisely

$$a_{UV} - a_{IR} = a'_{UV} - a'_{IR}. \quad (1.4)$$

The ambiguity of the quantity  $a'$  can be resolved by fixing  $a'_{UV} = a_{UV}$ . Then (1.4) implies  $a'_{IR} = a_{IR}$ . Therefore, according to the above observations, one can define a monotonically decreasing physical function  $a'[\alpha(t)]$  at all intermediate energies, whose values coincide with the values of  $a$  at the critical points. In this interpretation irreversibility is the result of the interplay between unitarity and renormalization.

The same considerations show that  $a$  is positive throughout the RG flow, once it is positive at some reference energy and since  $a > 0$  in the free field limit, we have  $a \geq 0$  also in the interacting fixed point.

From the identification  $a' = a$ , a simple non-perturbative formula expressing the total RG flow of  $a$  as the invariant area of the graph of the beta function follows (here “invariant” area means scheme-independent). This formula can be checked in perturbation theory, to the fourth loop order included, in all renormalizable models.

Using our results, a notion of “proper” coupling constant  $\bar{\alpha}$  can be defined, which is the coupling constant for which the “Zamolodchikov” metric is constant throughout the renormalization group flow. The total flows of  $a$  and  $a'$  equal the area of the graph of the proper beta function (i.e. the beta function for  $\bar{\alpha}$ ) between the fixed points. This area is quantized in QCD. In general, one can say that a universal unit-area cell is assigned to each massless degree of freedom.

The paper is organized into two main parts. The first part (section 2) is devoted to the interpolating function for  $a'$ , the second part (section 3) to the removal of its ambiguity, through the relationship with  $a$ .

Other implications are presented in the final part of the paper and the conclusions. In particular, we discuss the induced action for the conformal factor (the Riegert action) and show that our identification  $a' = a$  reduces it to a free action at criticality. Moreover, we derive an expression for the vacuum energy  $E_0$ .

A comment on the claimed irreversibility is in order. The statement is about the *intrinsic* irreversible character of the flux of the renormalization group. By intrinsic we mean proper to the dynamical scale  $\mu$  introduced by renormalization. The desired effect has to be suitably “cleaned” from spurious effects of different nature, that can either enhance or spoil the property in a trivial way. For this reason we consider the most general renormalizable theory with no mass parameter. A mass would trivially enhance the theorem, by killing degrees of freedom in the IR, without modifying the UV. Instead, a non-renormalizable interaction would trivially spoil the theorem, by killing degrees of freedom in the UV, without modifying the IR. Even the non-perturbative effects of QCD, such as chiral symmetry breaking, enhance the theorem, so that the crucial region for testing the intrinsic irreversibility of the RG flow is precisely the conformal window.

## 2 The quantity $a'$ : ambiguity, irreversibility, interpolating function

The quantity  $a'$  is known to be ambiguous by an arbitrary additive constant. A regularization technique can often hide this ambiguity and give an apparently unambiguous result for  $a'$ . For example, in [14],  $a'$  is related to  $c$ . A general calculation can be found in [10], with relevant comments in the concluding paragraph. This ambiguity, related to the addition of an arbitrary finite  $\int R^2$ -term in the induced effective action, does not spoil the  $a'$ -physical content completely. For example, the  $a'$ -RG flow is unambiguously defined;  $a'$  is like an additional coupling constant of the theory. Once it is normalized at a reference energy scale, it is fixed at any other energy scale. Positive definiteness of the induced effective action for the conformal factor imposes nevertheless a bound on  $a'$  (see section 3.2).

At criticality formula (1.1) becomes

$$\Theta = -\frac{1}{120} \frac{1}{(4\pi)^2} \left[ c_* W^2 - \frac{1}{3} a_* G + \frac{2}{9} a'_* \square R \right]. \quad (2.5)$$

For a free theory with  $N_s$  real scalar fields,  $N_f$  Dirac fermions, and  $N_v$  vector fields, we have [14]

$$c_{\text{free}} = N_s + 6N_f + 12N_v, \quad a_{\text{free}} = N_s + 11N_f + 62N_v, \quad (2.6)$$

while  $a'_{\text{free}}$  remains for the moment undetermined. For the central charges we use over-all normalizations different from those of ref. [6] and the previous literature, in order to have integer valued quantities at the free critical points.

The operator  $\Theta$  is associated with the conformal factor  $\phi$  of the metric,  $g_{\mu\nu} = e^{2\phi}\delta_{\mu\nu}$ .  $W$  does not depend on  $\phi$ , while the Euler density depends on  $\phi$  quadratically. Instead, the term  $\square R$  contains a linear term in the conformal factor  $\phi$  around flat space,  $R = -6e^{-2\phi}[\square\phi + (\partial_\mu\phi)^2]$ . Therefore, the two-point function of  $\Theta$  is proportional to the number  $a'_*$  in the conformal limit  $\beta = 0$ . Using  $\Theta = -e^{-4\phi}\delta S/\delta\phi$  we have

$$\langle\Theta(x)\Theta(y)\rangle = i\frac{\delta\langle\Theta(x)\rangle}{\delta\phi(y)}\Big|_{\phi=0} = -i\frac{\delta^2 S_{\text{eff}}[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=0} = \frac{i}{90(4\pi)^2}a'_*\square^2\delta(x-y). \quad (2.7)$$

Here  $S_{\text{eff}}[\phi]$  denotes the induced effective action for the conformal factor (it will be calculated explicitly in sect. 3.1). Turning to the Euclidean framework, we get

$$\langle\Theta(x)\Theta(y)\rangle_{\text{E}} = -\frac{\delta^2 S_{\text{E}}[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=0} = -\frac{1}{90(4\pi)^2}a'_*\square^2\delta(x-y).$$

The subscript E denotes correlators and quantities in the Euclidean frame. A positive definite effective action implies  $a' > 0$  and the negative sign in (2.7) is consistent with this. In two dimensions  $\langle\Theta(x)\Theta(0)\rangle = -c\frac{\pi}{3}\square\delta(x)$ .

Using formula (2.7) we see that the quantity  $a'_*$  can be expressed at criticality by the integral

$$a'_* = -\frac{15}{2}\pi^2\int d^4x|x-y|^4\langle\Theta(x)\Theta(y)\rangle_{\text{E}}. \quad (2.8)$$

## 2.1 Generalities about the $a'$ -function

We now consider the off-critical theory. We can define a function  $a'(r)$  of the intermediate energy scale  $1/r$  by restricting the integration over a four-sphere  $S(r, y)$  of radius  $r$  and centred at the point  $y$ , precisely

$$a'(r_2) - a'(r_1) = -\frac{15}{2}\pi^2\int_{S(r_1, y)}^{S(r_2, y)} d^4x|x-y|^4\langle\Theta(x)\Theta(y)\rangle. \quad (2.9)$$

The notation means that the integral is performed in the region contained between the two four-spheres. Unless differently specified, correlators are in the Euclidean framework. Our formula (2.9) does not give the critical values

$$a'_{\text{UV}} = \lim_{r\rightarrow 0} a'(r), \quad a'_{\text{IR}} = \lim_{r\rightarrow\infty} a'(r)$$

of the function  $a'$ , which are related to the ambiguity already mentioned. Nevertheless, we prove in this paper that there is also a universal way to remove it.

For a critical theory we have, from formula (2.7),  $a'_{\text{UV}} = a'_{\text{IR}} = a'(r) = a'_*$ . Off-criticality, the running of  $a'(r)$  is due to the internal term  $\frac{\beta}{4\alpha}F^2$  appearing in the operator anomaly equation (1.1) for  $\Theta$ . Its effect is a non-local term in the correlator  $\langle \Theta(x) \Theta(y) \rangle$ , which we write as

$$\langle \Theta(x) \Theta(y) \rangle = \frac{1}{15\pi^4} \frac{\beta^2[\alpha(t)]f[\alpha(t)]}{|x-y|^8}, \quad \text{for } x \neq y. \quad (2.10)$$

The function  $f(t)$  has finite UV and IR limits. The regularity of the function  $f(t)$  at criticality follows from the very definition of the  $\beta$ -function and the operator  $F^2$ : the coefficients of the operators  $\mathcal{O}_i$  (i.e. the  $\beta$ -functions) are precisely the zeros of  $\Theta$  in the operator equation  $\Theta = \beta_i \mathcal{O}_i$  (for example  $\Theta = \frac{\beta}{4}F^2$ ). The proof can be found in the classical works by Adler, Collins and Duncan [16], Nielsen [17] and Collins, Duncan and Joglekar [18]<sup>1</sup>. From this very fact it follows, among the other things, that at criticality the anomalous dimension of the operator  $F^2$  coincides with the slope  $\beta'_*$  of the  $\beta$ -function [20]. For our purposes, we just need that the function  $f[\alpha(t)]$  be bounded and non-vanishing at both critical points.

Reflection positivity [21] of the correlator (2.10) at  $x \neq y$  points assures that

$$f(t) \geq 0. \quad (2.11)$$

Now, let us insert (2.10) into (2.9). We obtain

$$a'(r) - a'_{\text{UV}} = -\frac{1}{2\pi^2} \int_{S(r,y)} d^4x \frac{\beta^2(t)f(t)}{|x-y|^4} = -\int_{\alpha_{\text{UV}}}^{\alpha(r)} \frac{d\alpha}{\alpha} \beta(\alpha)f(\alpha) = -\int_{-\infty}^{\ln r\mu} dt \beta^2(t)f(t), \quad (2.12)$$

and, for the total flow of the quantity  $a'$ ,

$$a'_{\text{UV}} - a'_{\text{IR}} = \int_{-\infty}^{+\infty} dt \beta^2(t)f(t) = -\int_{\alpha_{\text{UV}}}^{\alpha_{\text{IR}}} \frac{d\alpha}{\alpha} \beta(\alpha)f(\alpha) \geq 0. \quad (2.13)$$

The integral is convergent<sup>2</sup> [23]. Around the UV fixed point (which we assume to be free for concreteness) we have  $\beta \sim -\alpha$ ,  $\alpha \sim -1/t$ ,  $f \sim \text{const.}$ , and  $a'_{\text{UV}} - a'_{\text{IR}} \sim \int dt/t^2$  convergent for large  $t$ . Around the IR fixed point we have  $\beta \sim \beta'_*(\alpha - \alpha_{\text{IR}})$ ,  $\beta'_* > 0$ . Solving the renormalization group equation, we find  $\beta \sim e^{-t\alpha_{\text{IR}}\beta'_*}$  [20]. As expected, convergence is much faster around the IR fixed point, where it is exponential. In ref. [23], Zee shows that convergence holds also when  $\beta \sim (\alpha - \alpha_*)^n$ ,  $n > 1$ .

The monotonically decreasing behaviour of the function  $a'(r)$  is evident:

$$\frac{da'(t)}{dt} = -\beta^2(t)f(t) \leq 0.$$

<sup>1</sup>The existence of trace anomalies was first established by Coleman and Jackiw in ref. [19].

<sup>2</sup> Integrals like (2.8) are the matter-induced gravitational couplings. Specifically,  $\int d^4x \langle \Theta(x) \Theta(0) \rangle$ ,  $\int d^4x |x|^2 \langle \Theta(x) \Theta(0) \rangle$  and  $\int d^4x |x|^4 \langle \Theta(x) \Theta(0) \rangle$  are the induced cosmological constant, the induced Newton constant and an induced higher-derivative coupling, respectively [22]. As they stand, the first two integrals, however, are divergent [9, 23]. No statement can be made about the signs of the induced cosmological constant and the Newton constant [22, 23]. In particular, arguments for irreversibility based on  $\int d^4x \langle \Theta(x) \Theta(0) \rangle$  [9, 24] present several unresolved problems [9].

Summarizing, the integral expressing the total flow is convergent, if there exists an IR fixed point  $\alpha_{\text{IR}} < \infty$  and it can be interpreted as the *area* of the graph spanned by the beta function between the fixed points. Since the  $\beta$ -function depends on the subtraction scheme, while the area in question must be scheme independent, the integral has to be performed with a suitable metric, an “ein-bein” that restores scheme invariance. This metric is precisely the function  $f(\alpha)$ .

**An alternative expression**

For later convenience it is useful to re-express the correlator (2.10) in a slightly different way, namely

$$\langle \Theta(x) \Theta(y) \rangle = \frac{4}{45} \frac{1}{(4\pi)^4} \square^2 \left( \frac{\beta^2[\alpha(t)] \tilde{f}[\alpha(t)]}{|x-y|^4} \right) \quad \text{for } x \neq y. \quad (2.14)$$

The above factorization of  $\square^2$  comes naturally, for example, if one writes the stress tensor two-point function as in formula (1.1) of ref. [12]. One has

$$\beta^2(t) f(t) = \frac{1}{192} \left( \frac{d}{dt} - 2 \right) \left( \frac{d}{dt} - 4 \right)^2 \left( \frac{d}{dt} - 6 \right) \beta^2(t) \tilde{f}(t). \quad (2.15)$$

As far as positivity is concerned, we can safely cross the  $\square^2$  and infer positivity of  $\tilde{f}$  by the positivity of  $f$ . This can be proved via the following general argument. Let  $v[\alpha(t)]$  and  $u[\alpha(t)]$  be functions related by the equation

$$\left( \frac{d}{dt} - n \right) v[\alpha(t)] = u[\alpha(t)],$$

$n$  being a positive integer. Then we can prove that if  $u$  is positive,  $v$  is negative, and vice versa. Indeed, the solution of the differential equation is

$$v[\alpha(t)] = - \int_t^\infty e^{n(t-t')} u[\alpha(t')] dt'.$$

The arbitrary constant is fixed by the requirement that  $nv + u = 0$  (i.e.  $d/dt \equiv 0$ ) at criticality. This equality can be verified for, say,  $t \rightarrow -\infty$  by writing

$$\begin{aligned} v[\alpha(t)] &= \int_0^{-\infty} e^{n\xi} u[\alpha(t-\xi)] d\xi \\ &\xrightarrow{t \rightarrow -\infty} u[\alpha(-\infty)] \int_0^{-\infty} e^{n\xi} d\xi = -\frac{1}{n} u[\alpha(-\infty)]. \end{aligned}$$

The limit  $t \rightarrow +\infty$  is similar. Dependence of  $v$  on  $\alpha(t)$  can be checked by taking the derivative with respect to  $\alpha(t)$  :

$$\frac{dv}{d\alpha} = - \int_0^{-\infty} e^{n\xi} \beta[\alpha(t-\xi)] \frac{\partial u[\alpha(t-\xi)]}{\partial \ln \alpha(t-\xi)} d\xi = -\beta[\alpha(t)] \frac{\partial v}{\partial \ln \alpha(t)}.$$

In the last step we have used the equality

$$\beta[\alpha(t)] \frac{\partial}{\partial \ln \alpha(t)} = \beta[\alpha(s)] \frac{\partial}{\partial \ln \alpha(s)}$$

for any  $t$  and  $s$ .

We finally observe that the equality  $nv + u = 0$ , holding at the fixed points, assures that  $u$  and  $v$  have the same behavior at criticality. In particular,  $u \sim \beta^2$  implies  $v \sim \beta^2$ , which is why we collect a  $\beta^2$  in front of  $\tilde{f}$  in (2.14). We conclude that  $\tilde{f}$  is positive and depends on the running coupling constant.

Some arguments work also for the case  $n = 0$ , if  $u$  tends to zero sufficiently fast at criticality. In that case we can write

$$v[\alpha(t)] = v[\alpha(\infty)] - \int_t^\infty u[\alpha(t')] dt',$$

but there is no unambiguous way to fix the additive constant, so that the sign of the function  $v$  is in general not fixed.

Using (2.15) we can also write

$$a'_{\text{IR}} - a'_{\text{UV}} = - \int_{-\infty}^{+\infty} dt \beta^2(t) \tilde{f}(t), \quad (2.16)$$

since all the terms containing  $\frac{d}{dt}$  integrate straightforwardly to zero.

## 2.2 Interpolation between the critical values

We now study the correlator (2.10) at  $x = y$  as well as  $|x - y| = \infty$ , which we can write in the form

$$\langle \Theta(x) \Theta(y) \rangle = -\frac{1}{90} \frac{1}{(4\pi)^2} \square^2 \left[ a'_{\text{UV}} \delta(x - y) - \frac{1}{2\pi^2} \frac{\beta^2(t) \tilde{f}(t)}{|x - y|^4} - a'_{\text{IR}} \frac{1}{|x - y|^8} \delta\left(\frac{x - y}{|x - y|^2}\right) \right], \quad (2.17)$$

Let us first discuss the singularities at  $x = y$ . It is easy to see that this correlator, in particular the central non-local term, is well defined as a distribution. For the study of the divergent part, we can ignore the overall  $\square^2$ . We have

$$\int d^4x u(x - y) \frac{\beta^2(t) \tilde{f}(t)}{|x - y|^4} < \infty$$

for any regular bounded test function  $u$ . This means that the perturbative divergences sum up and disappear once the cut-off is removed. This situation is common when dealing with anomalies and, in general, evanescent operators [25]. Certainly we can have information from the perturbative divergences before removing the cut-off [10], but the final correlator is convergent.

Note the “ $\delta$ -function at infinity”, that we include formally in (2.17), required by conformal invariance. When the theory is conformal the middle term vanishes and  $a'_{\text{UV}} = a'_{\text{IR}} = a'_*$ , so that

$$\langle \Theta(x) \Theta(y) \rangle = -\frac{1}{90} \frac{1}{(4\pi)^2} a'_* \square^2 \left[ \delta(x-y) - \frac{1}{|x-y|^8} \delta\left(\frac{x-y}{|x-y|^2}\right) \right], \quad (2.18)$$

which is indeed conformal-invariant. Formula (2.17) expresses that  $a'_{\text{UV}}$  and  $a'_{\text{IR}}$  are the small- and large-distance limits of  $a'(r)$ , respectively, and the non-local term interpolates between the two. For a running theory we have necessarily  $a'_{\text{UV}} \neq a'_{\text{IR}}$  (actually  $a'_{\text{UV}} > a'_{\text{IR}}$ ). We would like to describe this interpolation in more detail.

By performing a rescaling  $\mu \rightarrow \lambda\mu$  we can prove the following limits:

$$\lim_{\lambda \rightarrow 0} \frac{1}{2\pi^2} \frac{\beta^2(t + \ln \lambda) \tilde{f}(t + \ln \lambda)}{|x-y|^4} = (a'_{\text{UV}} - a'_{\text{IR}}) \frac{1}{|x-y|^8} \delta\left(\frac{x-y}{|x-y|^2}\right), \quad (2.19)$$

$$\lim_{\lambda \rightarrow \infty} \frac{1}{2\pi^2} \frac{\beta^2(t + \ln \lambda) \tilde{f}(t + \ln \lambda)}{|x-y|^4} = (a'_{\text{UV}} - a'_{\text{IR}}) \delta(x-y). \quad (2.20)$$

Again, these formulas have to be meant in the sense of distributions, so their proofs are worked out by means of a test function. We have

$$\frac{1}{2\pi^2} \int d^4x u(|x-y|) \frac{\beta^2(t + \ln \lambda) \tilde{f}(t + \ln \lambda)}{|x-y|^4} = \int_{-\infty}^{+\infty} dt \beta^2(t) \tilde{f}(t) u(|x-y|/\lambda) \rightarrow (a'_{\text{UV}} - a'_{\text{IR}}) u(1/\lambda)$$

for  $\lambda \rightarrow \infty, 0$ . In the case  $\lambda \rightarrow 0$ , formula (2.19) is recovered. On the other hand, in the limit  $\lambda \rightarrow \infty$  the result  $(a_{\text{UV}} - a_{\text{IR}})u(\infty)$  is also in agreement with (2.20).

Using (2.19) and (2.20) we see that, in the UV limit, (2.17) tends to formula (2.18) with  $a'_* = a'_{\text{UV}}$ . Similarly (2.19) shows that, in the IR limit, (2.17) tends also to formula (2.18) with  $a'_* = a'_{\text{IR}}$ . We have therefore proved that the correlator  $\langle \Theta\Theta \rangle$  interpolates between the UV and IR values of the coefficient of the term  $\square R$  in the trace anomaly operator equation.

### 2.3 Scheme independence in the presence of scalar fields

It is important that the function  $f$  depends only on the running coupling  $\alpha(t)$ , i.e. that it does not depend explicitly on  $\alpha(\mu)$ , as a consequence of the Callan–Symanzik equations and the finiteness of the stress–energy tensor. However, it is well known that in the presence of scalar fields  $\varphi$  the stress–energy tensor is not truly finite [26, 12]. It mixes with the operator  $(\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \varphi^2$  and therefore a proper definition of a physical (i.e. scheme-independent) function depending on the coupling constant at a single energy scale is more subtle. A function of this type was called *central function* in ref. [12]. In the example of the previous section the central function was  $\beta^2 f$ : it is physical since it is defined by a physical correlator ( $f$  alone, instead, is scheme-dependent, since  $\beta$  is). In general, the two-point function of an operator  $\mathcal{O}$  (with canonical dimension  $d$  and, for simplicity, not mixing with other operators) can be written in the form

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{Z^2[\alpha(t), \alpha(\mu), s] A[\alpha(t), s]}{|x-y|^{2d}} = \frac{B[\alpha(t), \alpha(\mu)]}{|x-y|^{2d}}. \quad (2.21)$$

In order to be as explicit as possible, we use a (temporary) heavy notation. The “variable”  $s$  refers to scheme dependence. The renormalization constant  $Z$  depends in general on the values of the coupling constant at two different energy scales:

$$Z = \exp \left( \int_{\alpha(\mu)}^{\alpha(t)} \frac{\gamma(\alpha) d\alpha}{\beta(\alpha)} \right)$$

and on the subtraction scheme, since both  $\beta$  and  $\gamma$  do. Formula (2.21) contains no central function: each function appearing there is either scheme-dependent or depends on the couplings at two different energy scales (which prevents from defining univocal critical limits).

If the operator  $\mathcal{O}$  is a conserved current, then  $Z = 1$  and  $B = B[\alpha(t)]$  has the desired properties. In passing, we note that  $B$  is scheme-independent although the running coupling constant  $\alpha(t)$  depends on the scheme,  $\alpha(t) = \alpha(t, s)$ . The reason is that the function  $B$ , as a function of  $\alpha$ , is also scheme-dependent,  $B = B(\alpha, s)$ , while it is scheme-independent as a function of  $t$ : in  $B(t) = B[\alpha(t, s), s]$ . In the physical correlator the two scheme dependences cancel each other.

If the operator  $\mathcal{O}$  is the stress-energy tensor and scalar fields are present, then we need an additional effort to identify the desired central function. As in ref. [12], the matrix of renormalization constants for the couple  $(\mathcal{O}_1, \mathcal{O}_2) \equiv (\Theta, \square\phi^2)$  of mixing operators is triangular,

$$Z_{ij} = \begin{pmatrix} 1 & \xi \\ 0 & \zeta \end{pmatrix} = Z_{ij}[\alpha(t), \alpha(\mu), s], \quad (2.22)$$

$\zeta$  being the renormalization constant of the mass operator  $\phi^2$ . We have

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle = \frac{1}{|x|^4} Z_{ik}[\alpha(t), \alpha(\mu), s] Z_{jl}[\alpha(t), \alpha(\mu), s] F_{kl}[\alpha(t), s] \equiv \frac{1}{|x|^4} P_{ij}[\alpha(t), \alpha(\mu)], \quad (2.23)$$

where we have exhibited all dependences as in (2.21). Writing

$$P_{ij}[\alpha(t), \alpha(\mu)] \equiv \begin{pmatrix} p & r \\ r & q \end{pmatrix}, \quad F_{kl}[\alpha(t), s] \equiv \begin{pmatrix} k & h \\ h & g \end{pmatrix} \quad (2.24)$$

and combining (2.22), (2.23) and (2.24) we find

$$\frac{\det P}{q} = p - \frac{r^2}{q} = \frac{\det F}{g} = k - \frac{h^2}{g} \equiv \beta^2[\alpha(t)] f[\alpha(t)] \geq 0. \quad (2.25)$$

Now,  $p - r^2/q$  is manifestly scheme-independent, while  $k - h^2/g$  does not depend on  $\alpha(\mu)$ . The equality of the two expressions allows us to conclude that both expressions are scheme-independent and functions of the running coupling  $\alpha(t)$ . This defines the desired central function when scalar fields are present, which we write as  $\beta^2[\alpha(t)] f[\alpha(t)]$ . It has to be inserted into formulas (2.13), (2.12), (2.16), etc., to give the general formula of the  $a'$ -function. Formula (2.25) gives the only invariant of the similarity transformation  $P = ZFZ^t$ , with  $P$  and  $F$  symmetric and  $Z$  triangular of the form (2.22).

We have factorized out a  $\beta^2$ : indeed,  $p \sim \langle \Theta \Theta \rangle$  is proportional to  $\beta^2$ , while  $r \sim \langle \Theta \square \varphi^2 \rangle$  is proportional to  $\beta$ . Finally, the denominator is regular since the function  $g$  is regular at criticality. To see this, one observes that the factor  $\zeta$  in the correlator  $\langle \square \varphi^2 \square \varphi^2 \rangle = q/|x|^8 = \zeta^2 g/|x|^8$  takes care of the eventual anomalous dimension  $\gamma$  of the operator  $\varphi^2$  at criticality ( $\zeta \sim 1/|x|^{2\gamma}$ ), so that  $g$  remains non-vanishing (and positive by applying reflection positivity to this correlator).

Finally, reflection positivity of  $\langle \mathcal{O}_i \mathcal{O}_j \rangle$  assures that the matrix  $P$  is positive-definite. In particular,  $\det P$  is positive. Since  $q$  is also positive,  $f \geq 0$  follows.

The existence of the (unique) invariant (2.25) for the similarity transformation that relates  $P$  and  $F$  by the matrix  $Z$  of (2.22) is an important fact; it assures that scalar fields are included in the treatment. By comparing the calculations done in [11] and [10] one can appreciate the additional amount of effort required by scalar fields.

Our discussion about scheme dependence is an introduction to the notion of ‘‘covariance’’ that we will formulate later on in this context (section 3.5).

### 3 The critical values of the quantity $a'$ : normalization, physical meaning and its relationship with $a$

It is a consequence of our arguments that even if the quantity  $a'$  is to some extent undetermined, its RG group flow is uniquely determined. For example, the difference  $a'_{\text{UV}} - a'_{\text{IR}}$  is the invariant area of the graph of the beta function and the derivative of  $a'$  is expressed in terms of a physical correlator,  $\langle \Theta(x) \Theta(y) \rangle$ . Therefore, the ambiguity of  $a'$  can be at most an additive constant, so that once we normalize it at a reference energy scale (for example in the UV limit) then it is fixed at any energy scale. Perturbative calculations allow us to arrive at the same conclusion [10].

Nevertheless, there is a preferred normalization choice for  $a'$ . Indeed, it turns out that the quantities  $a$  and  $a'$  have various properties in common. For example, they are both two-loop-uncorrected, while  $c$  is two-loop-corrected. The relation between  $c$  and  $a$  that is sometimes advocated in the literature [14], instead, is an artefact of the regularization technique. The radiative corrections of both  $a$  and  $a'$  begin at three loops. Actually, they coincide (see sect. 3.2).

#### 3.1 The Riegert action

The four-dimensional analogue of the Polyakov action  $S_P$  in two dimensions [27],

$$S_P = -\frac{c}{48\pi} \int d^2x \sqrt{-g_x} \int d^2y \sqrt{-g_y} R_x \square_{(x,y)}^{-1} R_y,$$

has been worked out and studied in several papers. The more ancient article containing the complete non-local action is, to our knowledge, the one by Riegert [15]. The local action for the conformal factor was found also by Fradkin and Tseytlin in ref. [28]. Buchbinder *at al.* were able to treat also the case with non-vanishing torsion. More recent studies are those by Cappelli

and Coste [30], Antoniadis and Mottola [31], and others [32, 33]. We take the expression of the action from [31], which is more symmetric than the one given by Riegert (the difference is a conformal-invariant term). One has to integrate the equation

$$\Theta = -2 \frac{1}{\sqrt{-g}} g_{\mu\nu} \frac{\delta S_R}{\delta g_{\mu\nu}}.$$

One can stay at criticality and use expression (2.5) for  $\Theta$ . The result is, in our notation,

$$\begin{aligned} S_R = & -\frac{1}{160(4\pi)^2} \frac{1}{a_*} \int d^4x \sqrt{-g_x} \int d^4x \sqrt{-g_y} \left[ c_* W^2 - \frac{1}{3} a_* \left( G - \frac{2}{3} \square R \right) \right]_x \\ & \left[ 2\square^2 + 4R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{4}{3} R \square + \frac{2}{3} (\nabla^\mu R) \nabla_\mu \right]_{(x,y)}^{-1} \left[ c_* W^2 - \frac{1}{3} a_* \left( G - \frac{2}{3} \square R \right) \right]_y \\ & + \frac{a_* - a'_*}{6480(4\pi)^2} \int d^4x \sqrt{-g} R^2. \end{aligned}$$

We stress that the Riegert action, obtained by integrating the trace anomaly, is not the complete induced action: it misses the conformal-invariant terms (local as well as non-local). Only in two dimensions is a conformally invariant term gauge-equivalent to zero.

We rederive  $S_R$  in the particular case of a metric of the form  $g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$ , which is sufficient for our purposes.

Let us start from formula (2.5). The induced action  $S_R[\phi]$  for the conformal factor  $\phi$  is the solution of the equation

$$\Theta = -e^{-4\phi} \frac{\delta S_R[\phi]}{\delta \phi}. \quad (3.26)$$

One observes that the combination  $G - \frac{2}{3} \square R$  is very simple (see [31]):

$$G - \frac{2}{3} \square R = 4e^{-4\phi} \square^2 \phi.$$

Then formula (2.5) gives, in the case of a conformally flat metric,

$$\Theta = \frac{1}{90(4\pi)^2} \left[ a_* e^{-4\phi} \square^2 \phi + \frac{1}{6} (a_* - a'_*) \square R \right].$$

It is not necessary to write  $\square R$  explicitly, since it can be integrated using

$$\square R = -\frac{1}{6\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g} R^2.$$

The solution of eq. (3.26) is then straightforward. The result is

$$S_R[\phi] = -\frac{1}{180} \frac{1}{(4\pi)^2} \int d^4x \left\{ a_* (\square \phi)^2 - (a_* - a'_*) [\square \phi + (\partial_\mu \phi)^2]^2 \right\}. \quad (3.27)$$

From this action, we immediately recover (2.7) with the correct sign (remember that (3.27) is written in the Lorentzian framework). Turning the exponential factor  $e^{iS[\phi]}$  to the Euclidean framework, one has  $e^{-S_E[\phi]}$ , in terms of the Euclidean action

$$S_E = \frac{1}{180} \frac{1}{(4\pi)^2} \int d^4x \left\{ a_* (\square \phi)^2 - (a_* - a'_*) [\square \phi + (\partial_\mu \phi)^2]^2 \right\}.$$

For  $a' = a$  this action is free, which means that the three- and four-point functions of  $\Theta$  are zero at criticality (as well as all the correlators  $\langle \Theta(x_1) \cdots \Theta(x_n) \rangle$ ,  $n > 2$ ) and that the two-point function  $\langle \Theta(x) \Theta(0) \rangle$  equals  $-\frac{1}{90(4\pi)^2} a \square^2 \delta(x)$ .

With the identification  $a' = a$ , we can suppress the primes in formula (2.13) and finally state the  $a$ -theorem.

**$a$ -theorem.**

i)  $a$  is non-negative.

ii) The total RG flow of  $a$  is non-negative and equal to the invariant area of the beta function:

$$a_{\text{UV}} - a_{\text{IR}} = - \int_{\alpha_{\text{UV}}}^{\alpha_{\text{IR}}} \frac{d\alpha}{\alpha} \beta(\alpha) f(\alpha) \geq 0. \quad (3.28)$$

This prediction will be checked in section 3.3 to the fourth-loop order in perturbation theory, for QCD in the conformal window around the asymptotic freedom point  $N_f = \frac{11}{2} N_c$ , as well as supersymmetric QCD, QED and the  $\varphi^4$ -theory.

In realistic UV free theories both the UV and IR fixed points are free theories and therefore  $a_{\text{UV}} - a_{\text{IR}}$  is a positive, integer number. In this case we see that the invariant area of the beta function is quantized, with a unit cell for each massless degree of freedom that disappears along the renormalization group flow. Furthermore, the  $a'$ -function of the previous section correctly interpolates between the critical values  $a_{\text{UV}} \geq a_{\text{IR}} \geq 0$ , so that at each intermediate energy the flow of  $a$  equals the invariant area of the graph spanned by the beta function up to that energy.

Finally, the relation  $a_* = a'_*$  has other interesting implications when the conformal factor is quantized [31, 32].

### 3.2 Discussion of the positive-definiteness of the Riegert action and its consequences

In this section I present a discussion about unitarity and the positive-definiteness of the Riegert action and give an argument for the  $a$ -theorem.

A consequence of unitarity is that

*if the classical action  $S_{cl}[\varphi]$  is positive-definite (in the Euclidean framework), then the quantum action is positive-definite.*

We refer in particular to the functional generator  $\Gamma[\varphi]$  of connected one-particle irreducible diagrams. Note that positive-definiteness does not imply the existence of a minimum. Indeed, there are examples where the quantum action does not have a minimum. Positivity of the classical action assures that the functional integral is well defined, while positivity of the functional generator  $\Gamma$  is the statement that the resulting theory makes physical sense. We assume the additive constants of  $S_{cl}$  and  $\Gamma$  are adjusted so that positive definiteness is equivalent to boundedness from below.

In stating the above property we are thinking of bosonic actions (classical and quantum). Fermions can be included in the classical action with no problem. On the other hand, we are mostly interested in induced actions for bosonic fields and sources.

The above statement would be trivial in the absence of divergences, but the regularization cuts off certain frequencies and therefore violates unitarity. The statement is therefore false in the regularized theory. Renormalization can be seen as the process of restoring positivity by compensating the undesirable effects of logarithms.

For example, the quadratic part of the induced action of fermions in an external electromagnetic field reads in momentum space

$$-\frac{\beta_1}{32\pi} \int \frac{d^4p}{(2\pi)^4} |F_{\mu\nu}(p)|^2 \ln p^2/\Lambda^2, \quad (3.29)$$

where  $\beta_1$  is the one-loop coefficient of the beta function ( $\beta = \beta_1\alpha + \mathcal{O}(\alpha^2)$ ) and  $\Lambda$  a cut-off. This expression is either positive or negative, depending on  $\beta_1$ ,  $\Lambda$  and the evaluation of the integral. If, however, the electromagnetic field is dynamical, there will be an additional contribution

$$\frac{1}{16\pi\alpha(\Lambda)} \int \frac{d^4p}{(2\pi)^4} |F_{\mu\nu}(p)|^2, \quad (3.30)$$

which removes the divergence and restores positivity. Indeed,  $\alpha(\Lambda)$  is defined in such a way that the sum

$$\frac{1}{16\pi} \int \frac{d^4p}{(2\pi)^4} |F_{\mu\nu}(p)|^2 \left( \frac{1}{\alpha(\Lambda)} - \frac{\beta_1}{2} \ln p^2/\Lambda^2 \right) \quad (3.31)$$

is independent of  $\Lambda$ . Now, the first term is always positive, while the negative contribution coming from the second term is originated by the region  $|p| < \Lambda$  if  $\beta_1 < 0$  and  $|p| > \Lambda$  if  $\beta_1 > 0$ . Either region is arbitrarily small in a suitable limit for  $\Lambda$  ( $\Lambda \rightarrow 0$  in the first case, which is the case of asymptotic freedom, and  $\Lambda \rightarrow \infty$  in the second case, which is the case of IR freedom) and therefore every negative contribution is reabsorbed. Correctly, in these limits  $\alpha(\Lambda) \rightarrow 0$  to compensate for the infinitely large negative contributions coming from the second term of (3.31). Here there is a Landau pole, so that complete positivity is not restored order-by-order in perturbation theory, but just kept far from the perturbative regime. Positivity must be fully recovered, however, in the complete theory, which has to be unitary.

Equivalently, one defines a running coupling  $\alpha(|p|)$  and writes the action (3.31) as

$$\frac{1}{16\pi} \int \frac{d^4p}{(2\pi)^4} |F_{\mu\nu}(p)|^2 \frac{1}{\alpha(|p|)}.$$

Then the statement is that the running coupling constant is everywhere positive, once it is positive at a given energy. This is clearly visible in the conformal window, where the running coupling constant is indeed positive at all energies. Positivity at a given energy is assured by the physical normalization.

We remark that it is crucial to *have* a parameter to reabsorb the violations of positivity. For example, if we just subtract the divergent part of (3.29), but we do not do it with the help of a new (running) parameter we get an expression like

$$-\frac{\beta_1}{32\pi} \int \frac{d^4p}{(2\pi)^4} |F_{\mu\nu}(p)|^2 \ln p^2/\mu^2, \quad (3.32)$$

with  $\mu$  arbitrary and finite. This expression is convergent, but has no chance to be positive. Renormalization is not just a pure subtraction of divergences, but, more deeply, it is the unique way to restore positivity.

In summary, the violations of positivity are parametrized by local terms and can be reabsorbed by appropriate local counterterms, multiplied by physical parameters that can eventually run in order to reabsorb the violations of positivity. If this goal cannot be achieved then the theory has negatively normed states and unitarity is violated.

These observations, we think, suggest an instructive way to look at renormalization, that we have not found emphasized in the existing literature<sup>3</sup>.

### Induced action for the external conformal factor

We now make a further step and consider induced actions for external sources  $\phi_{\text{ext}}$ . If we could ignore the problems associated with divergences we could state that

*if the classical action  $S_{cl}[\varphi, \phi]$  is positive definite in the full space of fields  $\varphi$  and sources  $\phi$ , then the quantum action is positive definite in the full space of fields and sources.*

In general, external sources are not such that the classical action  $S_{cl}[\varphi, \phi]$  is positive definite in the full space of fields and sources, but the (minimal) coupling to (external) gravity does satisfy this requirement.

The above statement is again spoiled by divergences or effects related to divergences (like the anomalies). Moreover, since an induced action for external fields is not equipped with the appropriate parameters under which the violations of positivity can be reabsorbed, it might have an indefinite sign.

For example, the Polyakov action  $S_P$  in two dimensions is negative definite and convergent. The Riegert action in four dimensions, as we are going to discuss, is positive and convergent. The quantities  $c$ ,  $a$ ,  $a'$  and  $h$  can be thought of as the matter contributions to the beta functions of higher-derivative quantum gravity and there is no reason why they should have a definite sign.

What we can expect nevertheless, is that if the induced action is convergent and positive at some energy scale, than it is positive at all energy scales. This statement well applies to our case.

In order to have a convergent induced effective action, one should consider sources coupled to evanescent operators [25]. The Riegert effective action, moreover, is convergent (notwithstanding the inverted  $\square^2$ -operator), up to conformal-invariant terms (we recall that the Riegert action is defined up to such terms) and total derivatives. Indeed, the divergence due to the inverted  $\square^2$ -operator is proportional to

$$\left( \int d^4x \sqrt{g} \left[ c_* W^2 - \frac{1}{3} a_* \left( G - \frac{2}{3} \square R \right) \right] \right)^2$$

and therefore harmless. Similarly in two dimensions the divergence is proportional to a total

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<sup>3</sup> We plan to elaborate further on this approach elsewhere.

derivative:

$$\left( \int d^2x \sqrt{g} R \right)^2.$$

Once we specialize to the conformal factor  $\phi$ , coupled to the evanescent operator  $\Theta$ , convergence is more apparent. This can be seen also off-criticality, where the quadratic part of the effective action is read from correlator (2.17). In momentum space we have (dropping the contribution from infinity)

$$\frac{1}{180(4\pi)^2} \int \frac{d^4p}{(2\pi)^4} (p^2)^2 |\tilde{\phi}(p)|^2 \left[ a'_{\text{IR}} + \frac{1}{2\pi^2} \int d^4x (1 - \cos(p \cdot x)) \frac{\beta^2(t) \tilde{f}(t)}{|x|^4} \right],$$

which is convergent, and positive as long as  $a'_{\text{IR}}$  is. The expression between brackets is equal to  $a'_{\text{UV}}$  in the limit  $|p| \rightarrow \infty$  and to  $a'_{\text{IR}}$  in the limit  $|p| \rightarrow 0$  (but note that this expression does not interpolate monotonically between the two values; for this purpose one should use the function  $a'(t)$  constructed in section 2.1). As we see, there are cases where it is relatively easy to get positivity at all energies and this aspect of the problem is controlled by the local terms.

Convergence is an important property in the context of our discussion, because there is no local counterterm that can cure the first term,  $\int (\square\phi)^2$ , in the Riegert action  $S_{\text{R}}$ . The second term, instead,  $\int R^2$ , should be cured by the  $a'$ -ambiguity itself. We conclude that the total action should be positive-definite throughout the RG flow if it is at some intermediate energy. In the next subsections we show that this statement is equivalent to the full  $a$ -theorem.

### Point (i) of the $a$ -theorem

The total induced gravitational action contains three types of terms. The conformally invariant terms, convergent or divergent as they might be, are not visible in the Riegert action, which contains the other two types of terms. The first one is  $\int (\square\phi)^2$ , with coefficient  $a$ , which we discuss here. The second term,  $\int \sqrt{g} R^2$ , is discussed in the next subsection.

There is no arbitrariness that can restore the eventual positivity violation in the term  $a \int (\square\phi)^2$ , as we have already remarked. Our positivity arguments, i.e. that the action should be positive definite throughout the RG flow once it is positive definite at a reference energy, imply that  $a$  be positive also in the interacting fixed point, since certainly  $a$  is positive in the free field limit. However, it is puzzling to have an induced action like  $-c \int (\partial\phi)^2$  in two dimensions, which is always negative definite. There is no contradiction with our statement, actually, since one might say that it does not apply to this case, because there is no reference energy at which the induced action is positive definite. Even better, we can observe that our statement implies also that if the induced action is negative definite, or indefinite, at some reference energy, then there cannot be any energy at which it is positive definite, which is true also in two dimensions.

### Point (ii) of the $a$ -theorem

The term  $\int \sqrt{g} R^2$  is not affected by divergences either. It is a well-known fact that there cannot be any  $R^2$ -term in the trace anomaly at criticality (the absence of divergences off-

criticality was discussed in section 2.2). Now, *there is* an arbitrary parameter associated with this term, precisely  $a'$ , and this should suffice to assure positivity, finally explaining what the  $a'$ -ambiguity is there for. Moreover, convergence implies that  $a'$  is not a running parameter, but just an additive constant, in agreement with the knowledge gained in the previous sections.

We conclude that the term  $\int \sqrt{g}R^2$  is positive-definite throughout the renormalization group flow, once its coupling constant  $a' - a$  is normalized to be positive at a given energy (we can choose one of the two fixed points). This observation is sufficient to prove point (ii) of the  $a$ -theorem, as we now show.

Now, the term proportional to  $\int \sqrt{g}R^2$  is bounded from below at criticality if

$$a'_* \geq a_*. \tag{3.33}$$

This condition has to hold throughout the renormalization group flow, in particular

$$a'_{UV} \geq a_{UV} \quad \Leftrightarrow \quad a'_{IR} \geq a_{IR}.$$

Now, we know that

$$a'_{UV} \geq a'_{IR}.$$

Let us fix  $a'$  by demanding that  $a$  and  $a'$  coincide in the UV,  $a'_{UV} = a_{UV}$ . Then we have, combining the various inequalities derived so far:

$$a_{UV} = a'_{UV} \geq a'_{IR} \geq a_{IR},$$

wherefrom the claimed inequality  $a_{UV} \geq a_{IR}$  follows.

Now, let us tentatively suppose that with the normalization  $a'_{UV} = a_{UV}$  we have the strict inequality  $a'_{IR} > a_{IR}$ . We prove that this is absurd and conclude that  $a'_{IR} = a_{IR}$ .

We can do this by changing the normalization of  $a'$  with the shift  $a' \rightarrow a'^{\text{new}} = a' - a'_{IR} + a_{IR}$ , so that  $a'_{IR}{}^{\text{new}} = a_{IR}$ . We have  $a'_{UV} \rightarrow a'_{UV}{}^{\text{new}} = a'_{UV} - a'_{IR} + a_{IR}$  and therefore  $a'_{UV}$  no longer satisfies the inequality (3.33):  $a'_{UV}{}^{\text{new}} < a_{UV}$ . This is a contradiction. We conclude that

$$a'_{UV} = a_{UV} \quad \Leftrightarrow \quad a'_{IR} = a_{IR}. \tag{3.34}$$

In conclusion, the total RG flows of  $a$  and  $a'$  are equal and given by formula (3.28). The identification  $a' = a$  is consistent and the difference  $a_{UV} - a_{IR}$  is equal to the area of the graph of the beta function. The interplay between unitarity and renormalization is able to turn a simple set of inequalities into a precise non-perturbative formula.

Another way to state our result is that the coupling constant for the term  $R^2 \sqrt{g}$ , which is  $a' - a$ , is non-renormalized. This is not surprising, in the end, since the running behaviour of a coupling constant is due to divergent contributions, but the  $a'$ -ambiguity is fully finite.

Note that the result (3.34) would follow even if positivity implied  $a'_* \leq a_*$  instead of (3.33).

### 3.3 Perturbative checks

In this section we check our predictions to the fourth loop order in perturbation theory around the free fixed point. The strategy for computing higher-loop corrections to the trace anomaly was formulated by Brown and Collins [34], applied by Hathrell [10, 11] and Freeman [35] to the third-loop order, and extended by Jack and Osborn to the fourth-loop order (and in several other directions) [36].

We begin with the third loop analysis and use mostly the results of refs. [10, 11], since, to our knowledge, the papers by Hathrell are the only ones in which the term  $\square R$  is treated explicitly. The paper by Freeman does not calculate  $a'$ , but it contains enough information to derive it, once Hathrell's formulas are used. Moreover, the Hathrell–Freeman results are easily extended to a general third-loop expression that can be directly applied, in particular, to the QCD conformal window, in the neighbourhood of the asymptotic-freedom point. Such a formula shows perfect agreement with our prediction, i.e. that the total RG flows of  $a$  and  $a'$  coincide and are equal to the invariant area of the graph of the beta function.

For the purposes of this section, there is no difference between tilded and untilded quantities of section 1. Indeed, we are just interested in comparing critical values and flows of critical values.

In massless QED we have [10]

$$\begin{aligned} \tilde{c} &= 18 + \frac{70}{3} \frac{\alpha}{4\pi} + \mathcal{O}(\alpha^2), & \tilde{a} &= 73 - 180 \left( \frac{\alpha}{4\pi} \right)^2 + \mathcal{O}(\alpha^3), \\ \tilde{a}' &= a'_* - 180 \left( \frac{\alpha}{4\pi} \right)^2 + \mathcal{O}(\alpha^3), & f_* &= 45. \end{aligned}$$

These values are independent of the subtraction scheme. While  $\tilde{c}$  required an independent calculation, the perturbative corrections of  $\tilde{a}$  and  $\tilde{a}'$  were computed from each other and turn out to be equal. This coincidence is already what we need, nevertheless the perturbative check is not exhausted by this observation, since the above flows turn out to have the wrong signs. The explanation of this fact will emerge from the final formula. For the moment, we keep this observation in mind.

Similarly, in pure Yang–Mills theory with gauge group  $G$  we have [35]

$$\begin{aligned} \tilde{c} &= 4 \dim G \left[ 3 - \frac{20}{3} \frac{\alpha}{4\pi} C(G) + \mathcal{O}(\alpha^2) \right], & \tilde{a} &= 2 \dim G \left[ 31 + 255 \left( \frac{\alpha}{4\pi} \right)^2 C(G)^2 + \mathcal{O}(\alpha^3) \right], \\ \tilde{a}' &= a'_* + 510 \dim G \left( \frac{\alpha}{4\pi} \right)^2 C(G)^2 + \mathcal{O}(\alpha^3), & f_* &= 45 \dim G. \end{aligned}$$

In [35]  $a'$  is not calculated explicitly, but the formula for the function  $h(\alpha)$  is given. We have used the techniques of Hathrell to derive  $a'$  from  $h(\alpha)$ . It is clear that the two terms  $\square R$  and  $R^2$  in  $\Theta$  are related and one can indeed work out the precise relationship as in [10]. We observe that, again, the first perturbative corrections to  $\tilde{a}$  and  $\tilde{a}'$  coincide, but have the wrong sign.

Going through Hathrell and Freeman's calculations, we have derived a very simple general formula for the flows of  $a$  and  $a'$  for asymptotically free theories with a perturbative IR fixed

point. We have

$$a_{\text{UV}} - a_{\text{IR}} = a'_{\text{UV}} - a'_{\text{IR}} = \frac{1}{2} f_{\text{UV}} \beta_2 \alpha_{\text{IR}}^2 + \mathcal{O}(\alpha_{\text{IR}}^3) = \frac{1}{2} f_{\text{UV}} \frac{\beta_1^2}{\beta_2} + \mathcal{O}\left(\frac{\beta_1^3}{\beta_2^3}\right), \quad (3.35)$$

where  $\beta_1$  and  $\beta_2$  are the first two coefficients of the beta function,  $\beta(\alpha) = \beta_1 \alpha + \beta_2 \alpha^2 + \mathcal{O}(\alpha^3)$ . If the theory is IR-free, instead, the formula has an additional minus sign. Hathrell and Freeman's results are correctly reproduced ( $\frac{1}{2} f_{\text{UV}} \beta_2 \alpha_{\text{IR}}^2 \rightarrow \frac{1}{2} f_{\text{UV}} \beta_2 \alpha^2$  for the perturbative corrections in the absence of a fixed point): in QED,  $\beta_1 = 2/(3\pi)$  and  $\beta_2 = 1/(2\pi^2)$ ; in pure Yang–Mills theory,  $\beta_1 = -11N_c/(6\pi)$ ,  $\beta_2 = -17N_c^2/(12\pi^2)$ . Hathrell does not observe the equality of the  $a$ - and  $a'$ -flows, but it is relatively simple to show that this result is, to some extent, implicit in the derivation, at least to the third-loop order. The key formula is (5.27) of [10].

At this point it is straightforward to show that our formula (3.28) gives exactly the same result as (3.35), as we wanted.

Our theorem allows us to derive the above three-loop result in a straightforward way. One just needs the one-loop value of  $f$  and the two-loop formula of the beta function. Extending the techniques of Hathrell and Freeman to all orders in perturbation theory should allow us to prove the equality of the fluxes of  $a$  and  $a'$  in a direct way. An effort in this sense is being undertaken.

The sign mismatches noted above have the following explanation: the second coefficient  $\beta_2$  is positive only when there *is* an IR fixed point, while it is negative in pure Yang–Mills theory, where there is no such point. The  $a$ -theorem does not need to hold, in general, when the IR fixed point does not exist and several cases of this kind were indeed found in ref. [7]. In particular,  $a$  is often negative off the conformal window. It is nevertheless gratifying to observe that the coefficient in question is precisely  $\beta_2$ , so that as soon as the IR fixed point exists, the theorem is satisfied and when the theorem is not satisfied, this is a signal that the IR fixed point does not exist (it might still exist, as in QCD, after the introduction of the relevant non-perturbative effects, but this would change also the formulas for the RG flows of  $a$  and  $a'$ , and in the end the  $a$ -theorem will have to be satisfied). The non-existence of an interacting fixed point in QED was established in ref. [38].

Concretely, in QCD with  $N_f$  flavours and  $N_c$  colours we have

$$a_{\text{UV}} - a_{\text{IR}} = \frac{44}{5} N_c N_f \left(1 - \frac{11}{2} \frac{N_c}{N_f}\right)^2 = a'_{\text{UV}} - a'_{\text{IR}}.$$

For a generic gauge group  $G$  and representation  $R$  we take the two-loop beta function from [39]. We have

$$a_{\text{UV}} - a_{\text{IR}} = \frac{605 C(G) \dim G}{7C(G) + 11C(R)} \left(1 - \frac{11}{4} \frac{C(G)}{T(R)}\right)^2 = a'_{\text{UV}} - a'_{\text{IR}}.$$

In supersymmetric QCD, formula (3.35) is in agreement with the exact results of [6]. For concreteness, we take  $N=1$  SQCD with group  $G = SU(N_c)$  and  $N_f$  quarks and antiquarks in

the fundamental representation. The value of  $f_{\text{UV}}$  is still  $45 \dim G$  and  $\beta_1 = -\frac{1}{2\pi}(3N_c - N_f)$ ,  $\beta_2 = N_c N_f / (4\pi^2)$ . We have for  $N_f \lesssim 3N_c$ ,

$$a_{\text{UV}} - a_{\text{IR}} = \frac{45}{2} N_c N_f \left(1 - 3 \frac{N_c}{N_f}\right)^2 = a'_{\text{UV}} - a'_{\text{IR}},$$

in agreement with the exact formula [6]

$$a_{\text{UV}} - a_{\text{IR}} = \frac{15}{2} N_c N_f \left(1 - 3 \frac{N_c}{N_f}\right)^2 \left(2 + 3 \frac{N_c}{N_f}\right)$$

(recall that there is an additional factor of 360 with respect to ref. [6] due to a change in the normalization).

As a further confirmation, we report the results for a scalar field  $\varphi$  with a  $\frac{\lambda}{4!}\varphi^4$ -interaction from [11]:

$$\begin{aligned} \tilde{c} &= 1 - \frac{5}{36} \frac{\lambda^2}{(4\pi)^4} + \mathcal{O}(\lambda^3), & \tilde{a} &= 1 + \frac{85}{288} \frac{\lambda^4}{(4\pi)^8} + \mathcal{O}(\lambda^5), \\ \tilde{a}' &= a'_* + \frac{85}{288} \frac{\lambda^4}{(4\pi)^8} + \mathcal{O}(\lambda^5) & f_* &= \frac{5}{8(4\pi)^4}. \end{aligned}$$

Ref. [11] does not give the  $a'$  correction explicitly, which was calculated using [10]. The first perturbative corrections to  $\tilde{a}$  and  $\tilde{a}'$  are still equal, although they are related in a more complicated way, as a reflection of the discussion of section 2.3. Nevertheless, to the second loop order we can neglect the renormalization mixing between the stress tensor and  $\square\varphi^2$ .

Our predictions agree with the results of Hathrell, but the formula is now slightly different from (3.35). We have  $\Theta = -\frac{\beta}{4!}\varphi^4$  and  $\beta(\lambda) = \mu \frac{d\lambda}{d\mu} = \frac{3}{(4\pi)^2} \lambda^2 - \frac{17}{3(4\pi)^4} \lambda^3 + \mathcal{O}(\lambda^4) = \beta_1 \lambda^2 + \beta_2 \lambda^3 + \mathcal{O}(\lambda^4)$ , so that

$$a_{\text{UV}} - a_{\text{IR}} = \int_0^{\lambda_{\text{UV}}} d\lambda \beta(\lambda) f(\lambda) = -\frac{1}{12} \beta_2 f_{\text{IR}} \lambda_{\text{UV}}^4 + \mathcal{O}(\lambda^5).$$

Note that in order to apply our formula correctly in the absence of an interacting fixed point, we have to treat  $\beta_1$  as an independent (“small”) parameter and pretend that an UV fixed point does exist at  $\lambda_{\text{UV}} = -\frac{\beta_1}{\beta_2}$ . Therefore we write  $\beta(\lambda) = \beta_2 \lambda^2 (\lambda - \lambda_{\text{UV}}) + \mathcal{O}(\lambda^4)$  and replace  $\beta_1$  with  $-\beta_2 \lambda_{\text{UV}}$  everywhere else. Finally, we compare the coefficient multiplying  $\lambda_{\text{UV}}^4$  in the expression of  $a_{\text{UV}} - a_{\text{IR}}$  with Hathrell’s result. The numerical factor  $\frac{1}{12}$ , instead of  $\frac{1}{2}$ , is due to the different powers of  $\lambda$  appearing in the beta function and is crucial for the test.

### Fourth-loop-order checks

We can check agreement to the fourth-loop order using the calculations done by Jack and Osborn in ref. [36]. Again, we have to merge these results with some basic formulas of Hathrell’s [10] to extract the precise expression for  $a'$ , which is not given explicitly in [36]. We do not give here the complete derivation, but provide a vocabulary that allows the reader to surf on the

various references and notations. Unfortunately, the various pieces of the puzzle are distributed in many different papers. For concreteness, we treat the case of a gauge field theory.

Our  $a'$  is

$$-720(4\pi)^2(c' + \bar{c}(\alpha) - \sigma(\alpha)) = -720(4\pi)^2(c - \sigma)$$

in Hathrell's notation (this  $c$  has nothing to do with our  $c$ ).  $c'$  denotes the arbitrary additive constant. Hathrell proves that  $c(\alpha)$  and  $\sigma(\alpha)$  are related by the formula  $\sigma = -\alpha \frac{\partial \bar{c}}{\partial \alpha}$ . The quantity  $\beta_c = -\sigma\beta$  coincides with the  $\beta_c$  of [36]. The notations for the coupling constants are as follows ([36]→[10]):  $g^i = \frac{1}{g^2} \rightarrow \frac{1}{4\pi\alpha}$ ,  $\beta^i = -\frac{2}{g^3}\beta(g) \rightarrow -\frac{1}{4\pi\alpha}\beta(\alpha)$ .

$\beta_c$  is written as  $\frac{1}{8} \left( \chi_{ij}^a \beta^i \beta^j - \beta^i \frac{\partial X}{\partial g^i} \right)$  in [36] and  $\chi_{ij}^a = \frac{g^6}{4} \chi^a(g)$ . The explicit expression of  $\chi^a(g)$  (related to our function  $f$  - see below) is given in the second line of formula (5.12) of ref. [36].

We find therefore

$$\sigma = \frac{\alpha}{8} \frac{\partial X}{\partial \alpha} - \frac{\pi}{8} \alpha \beta \chi^a.$$

Denoting the total flow  $k_{UV} - k_{IR}$  of a generic quantity  $k$  with  $\Delta k$ , we have  $\Delta\sigma = \frac{1}{8}\Delta \left( \alpha \frac{\partial X}{\partial \alpha} \right)$ . On the other hand,

$$\Delta\bar{c} = \int_{IR}^{UV} d\alpha \frac{\partial \bar{c}}{\partial \alpha} = - \int_{IR}^{UV} d\alpha \frac{\sigma}{\alpha} = -\frac{1}{8}\Delta X - \frac{\pi}{8} \int_{UV}^{IR} d\alpha \beta \chi^a.$$

Therefore we can write

$$\Delta a' = 90(4\pi)^2 \Delta \left( X + \alpha \frac{\partial X}{\partial \alpha} \right) + 90\pi(4\pi)^2 \int_{UV}^{IR} d\alpha \beta \chi^a. \quad (3.36)$$

Now, we learn from formula (29) of [37] that

$$X + \alpha \frac{\partial X}{\partial \alpha} = -2\chi_{ij}^a \beta^i g^j + \beta^i \frac{\partial X'}{\partial g^i},$$

for a certain regular function  $X'$  (called  $Y$  in [36]). This suffices to assure that the first term on the right-hand side of (3.36) vanishes. Therefore we recover our formula for  $\Delta a'$  once we identify  $f$  with  $-90\pi\chi^a(4\pi)^2\alpha$ . Using (5.12) of [36] we see that this identification agrees with our previous third-loop-order results ( $f = 45 \dim G + \mathcal{O}(\alpha)$ ).

On the other hand, we have

$$\Delta a = 360(4\pi)^2 \int_{IR}^{UV} d\alpha \frac{\partial \tilde{\beta}_b}{\partial \alpha} = -45\pi(4\pi)^2 \int_{UV}^{IR} \frac{d\alpha}{\alpha} \beta \chi^g \alpha.$$

Now  $\chi^g = -2\chi^a$  up to the fourth-loop order (see formula (5.12) of [36]). This is the analogue of Hathrell's key relation (formula (5.27) of [10]), used for the three-loop checks. We conclude that the identification  $\Delta a = \Delta a'$  is consistent to the fourth-loop order included, as we wished to show.

According to the references that we have used, the extension of the three- and four-loop agreement to all orders is not trivial. In saying that the higher order effects will conspire to

satisfy our statement we are making a strong claim. Pursuing this check to even higher orders would be desirable and is not out of reach, given that exact formulas exist in supersymmetric theories. The fifth-loop-order correction to  $a$  “just” needs the four-loop beta function [40] and the three-loop expression of  $f$ .

### 3.4 The Casimir effect

The identification  $a' = a$  allows us to give an unambiguous expression for the Casimir effect on a given manifold  $\mathcal{M}$ . The derivation, that we do not repeat here, is performed by mapping the manifold  $\mathcal{M}$  into a conformally equivalent manifold  $\mathcal{M}'$  where the effect is known. We refer to the papers of Blöte, Cardy and Nightingale [41] and Affleck [42] for details. For example, on a cylinder of radius  $r$  the formula for the vacuum energy  $E_0$  reads in our notation, using the results of Cappelli and Coste [30],

$$E_0 = \frac{1}{360} \frac{a}{r}, \quad \left( E_0 = -\frac{1}{12} \frac{c}{r} \text{ in two dimensions.} \right)$$

For  $a'$  generic the expression reads

$$E_0 = \frac{1}{1440} \frac{3a + a'}{r},$$

and the shift of  $a'$  can be seen as a shift in the vacuum energy  $E_0$ . Quantum irreversibility can be reformulated as a statement on the vacuum energy,  $E_{\text{UV}} \geq E_{\text{IR}} \geq 0$ .

### 3.5 “Proper” beta function and coupling constant

Our formula, as we have stressed already, gives a natural geometrical interpretation of quantum irreversibility, which turns out to be quantitatively measured by the invariant area of the graph of the beta function between the critical points. At intermediate scales  $\mu$ , the quantity  $a[\alpha(\mu)]$  knows about the area spanned by the part of graph up to the scale  $\mu$  (see Fig. 1). There is a universal cell of unit area for each massless degree of freedom. The number of massless degrees of freedom disappearing between two given energy scales is equal to the area of the graph of the beta function included between those scales.

Using the results of the previous sections, we can introduce a notion of covariance related to scheme dependence and define a “proper” coupling constant  $\bar{\alpha}$  and beta function  $\bar{\beta}(\bar{\alpha})$ . We observe that the function  $f(\alpha)$  is a metric in the space of couplings. Precisely, when there are more couplings and  $\Theta = \beta_i \mathcal{O}_i$  we have

$$\langle \Theta(x) \Theta(y) \rangle = \frac{1}{15\pi^4} \frac{\beta_i[\alpha(t)] f_{ij}[\alpha(t)] \beta_j[\alpha(t)]}{|x - y|^8}, \quad \text{for } x \neq y. \quad (3.37)$$

and the matrix  $f_{ij}$  is positive definite. The total  $a$ -flow is then expressed in the form

$$a_{\text{UV}} - a_{\text{IR}} = - \int_{\ln \alpha_{\text{UV}}}^{\ln \alpha_{\text{IR}}} d \ln \alpha_i \beta_j(\alpha) f^{ij}(\alpha);$$

$f^{ij}$  is a sort of “target” metric for the map  $t \rightarrow \alpha_i(t)$ . This map is the path connecting  $\alpha_{UV}$  with  $\alpha_{IR}$  in the space of couplings, and it is in general scheme-dependent, as well as the metric  $f^{ij}(\alpha)$  and  $\beta_i$ . The integral is reparametrization-invariant. In this context reparametrization invariance is precisely scheme independence.

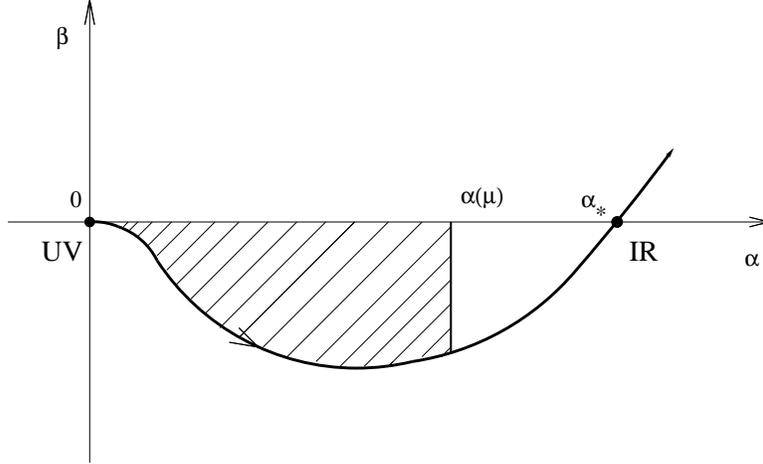


Fig. 1. Quantum irreversibility.

By definition, the proper coupling constant is the coupling constant for which the metric is identically equal to the free-field (UV) value,  $\bar{f}_{ij}(\bar{\alpha}) = (f_{UV})_{ij}$ . Both  $\beta_i$  and  $f_{ij}$  depend on the scheme and in the “proper” scheme one can measure these quantities in a universal way.

Let us focus on the case of a single coupling constant  $\alpha$ , for simplicity. By definition, we can write

$$\beta^2(\alpha)f(\alpha) = \bar{\beta}^2(\bar{\alpha})f_{UV}.$$

Moreover, we have

$$\bar{\beta}(\bar{\alpha}) = \frac{d \ln \bar{\alpha}}{d \ln \mu} = \frac{d \ln \bar{\alpha}}{d \ln \alpha} \beta(\alpha),$$

so that the formula relating the proper coupling constant to the starting one, reads

$$\bar{\alpha}(\alpha) = \bar{\alpha}(\alpha_0) \exp \left( \int_{\alpha_0}^{\alpha} \frac{d\alpha'}{\alpha'} \sqrt{\frac{f(\alpha')}{f_{UV}}} \right).$$

$\bar{\alpha}$  is a power expansion in  $\alpha$ . An arbitrary integration constant survives in  $\bar{\alpha}$  as a remnant of scheme dependence.  $\bar{\beta}$ , instead, is uniquely fixed and universal.

For example, one can fix the integration constant at the IR fixed point (which we assume to be an interacting conformal field theory), by setting  $\alpha_0 = \alpha_{IR}$ . There is no universal way to choose the overall factor  $\bar{\alpha}(\alpha_{IR})$ . If the IR fixed point is strongly coupled, then it is reasonable to set  $\bar{\alpha}_{IR} = 1$ , by definition. Independently of this value, one has  $\bar{\alpha}_{UV} = 0$ .

The loss of massless degrees of freedom along the renormalization group flow is measured by the proper area of the graph of the beta function, i.e. the area of the graph of the proper

beta function,

$$a_{\text{UV}} - a_{\text{IR}} = -f_{\text{UV}} \int_{\ln \bar{\alpha}_{\text{UV}}}^{\ln \bar{\alpha}_{\text{IR}}} d \ln \bar{\alpha} \bar{\beta}(\bar{\alpha}).$$

To the lowest order in perturbation theory we have  $\bar{\alpha} = \text{const.} \alpha + \mathcal{O}(\alpha^2)$ . If we define, instead,  $\bar{\alpha}$  as the coupling for which the metric is exactly unity throughout the RG flow, so that  $\beta^2(\alpha)f(\alpha) = \bar{\beta}^2(\bar{\alpha})$ , we have  $f(\alpha)/f_{\text{UV}} \rightarrow f(\alpha)$  in the formulas above and the transformation is no longer analytical, since around the free fixed point we have  $\bar{\alpha}(\alpha) = \text{const.} \alpha^{\sqrt{f_{\text{UV}}}}$ . For example, in QCD  $\ln \bar{\alpha} = \sqrt{45(N_c^2 - 1)} \ln \alpha + \text{const.}$

We finally remark that the definition of proper coupling constant is valid in any dimensions, even or odd, and in particular in three dimensions. In odd dimensions, the integral (3.28) is still a well-defined and interesting physical quantity (it could be considered, by extension, the effect of quantum irreversibility in odd dimensions), but there is no clear definition of  $a$  at criticality.

## 4 Conclusions

Several apparently unrelated facts suggest that the  $a'$ -ambiguity can be consistently removed by identifying  $a'$  with  $a$ . We have analysed various arguments related with unitarity, renormalization and positivity of the induced actions. In particular the statement of positive definiteness of the Riegert action throughout the RG flow is equivalent to the  $a$ -theorem.

The emerging picture clarifies several long-standing issues at the same time, among which we recall the ambiguity of the term  $\square R$ , the positivity of  $a$ , the decreasing behaviour of  $a$  along the renormalization group flow, the meaning of the Riegert action, the Casimir effect, the conceptual differences between two and four dimensional quantum field theory.

The equality of the total flows of  $a$  and  $a'$  gives an explicit formula quantifying the effect of quantum irreversibility as the invariant (i.e. scheme-independent) area of the graph of the beta function between the fixed points. This formula can be checked explicitly to the fourth-loop order in perturbation theory in all renormalizable models. There is a unity ‘‘proper’’ area associated with each massless degree of freedom disappearing along the renormalization group flow.

This establishes the intrinsic relationship between renormalization (the beta function), unitarity (absence of negatively normed states) and irreversibility (disappearance of massless degrees of freedom along the renormalization group flow), which we can schematically state as the implication

$$\mathbf{unitarity + renormalization \Rightarrow irreversibility.}$$

The interplay between unitarity and renormalization is better appreciated by observing that in a running theory the requirement that there be no negatively normed states naturally decomposes in two separate conditions: the requirement that there be no negatively normed state at some reference energy plus the requirement that no negatively normed state be generated along the renormalization group flow. This interplay turns a simple set of inequalities into the mentioned non-perturbative formula for the  $a$ -flow.

A by-product of our formula is an alternative, direct proof of the property [6] that  $a$  is invariant with respect to marginal deformations: no massless degrees of freedom can disappear along a trajectory with  $\beta \equiv 0$ . Given that also  $c$  is invariant with respect to marginal deformations [6], a formula for the non-perturbative flow of  $c$ , resembling (3.28), should exist.

A further, non-trivial, implication is that along a non-trivial RG flow the quantity  $a$  strictly decreases. Therefore two conformal field theories with the same  $a$ -values cannot be the critical points of an RG flow. This fact was conjectured in [43].

We have also traced the basic lines of an approach to the removal of divergences in quantum field theory, according to which regularization and renormalization are viewed as the violation and restoration of unitarity, respectively. Negatively normed states are introduced to regularize and then consistently removed to renormalize. In stressing the role of local terms and running parameters in this context, as well as the issues related with positive-definiteness of the induced effective actions, in particular induced effective actions for external sources, this approach seems to be more powerful than the usual one [21] and deserves further study *per se*.

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