# Quantum Conformal Algebras
# and Closed Conformal Field Theory

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## Abstract

We investigate the quantum conformal algebras of N=2 and N=1 supersymmetric gauge theories. Phenomena occurring at strong coupling are analysed using the Nachtmann theorem and very general, model-independent, arguments. The results lead us to introduce a novel class of conformal field theories, identified by a closed quantum conformal algebra. We conjecture that they are the exact solution to the strongly coupled large-$N_c$ limit of the open conformal field theories. We study the basic properties of closed conformal field theory and work out the operator product expansion of the conserved current multiplet $\mathcal{T}$. The OPE structure is uniquely determined by two central charges, $c$ and $a$. The multiplet $\mathcal{T}$ does not contain just the stress-tensor, but also $R$-currents and finite mass operators. For this reason, the ratio $c/a$ is different from 1. On the other hand, an open algebra contains an infinite tower of non-conserved currents, organized in pairs and singlets with respect to renormalization mixing. $\mathcal{T}$ mixes with a second multiplet $\mathcal{T}^*$ and the main consequence is that $c$ and $a$ have different subleading corrections. The closed algebra simplifies considerably at $c = a$, where it coincides with the N=4 one.

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Recently [1, 2] we developed techniques to study the operator product expansion of the stress-energy tensor, with the purpose of acquiring a deeper knowledge of conformal field theories in four dimensions and quantum field theories interpolating between pairs of them. These techniques are similar to those used, in the context of the deep inelastic scattering [3], to study the parton-electron scattering via the light-cone operator product expansion of the electromagnetic current. The investigation of the "graviton-graviton" scattering, i.e. the $TT$ OPE, is convenient in a more theoretical context, to study conformal windows and hopefully the low-energy limit of quantum field theories in the conformal windows.

Furthermore, an additional ingredient, supersymmetry, reveals that a nice algebraic structure [1] governs the set of operators generated by the $TT$ OPE. We called this structure the *quantum conformal algebra* of the theory, since it is the basic algebraic notion identifying a conformal field theory in more than two dimensions. We have considered in detail the maximal supersymmetric case in ref. [1] and in the present paper we extend our investigation to N=2 and N=1 theories, with special attention to the theories with vanishing beta function.

We believe that this interplay between physical techniques and more theoretical issues will be very fruitful for both.

It was observed in [1] that the relevant features of the algebra do not change with the value of the coupling constant. This was proved using a theorem due to Nachtmann [4], found 1973 in the context of the theory of deep inelastic scattering. Only at special values $g_*$ of the coupling constant can the algebra change considerably. One special point is of course the free-field limit, where infinitely many currents are conserved. Another remarkable point is the limit in which the operator product expansion closes with a finite set of conserved currents, which means only the stress-tensor in the N=4 theory, but more than the stress tensor in the N=2 algebra, as we will see.

This special situation, we believe, deserves attention *per se*. It is the simplest conformal field theory in four dimensions, simpler than free-field theory and yet non-trivial. It can be viewed as the true analogue of two-dimensional conformal field theory. Because of its simplicity, it is suitable for an algebraic/axiomatic investigation. It is expected to be relevant both physically and mathematically. For example, in [2] (sect. 4.5) we argued, using the AdS/CFT correspondence [5], in particular the results of [6], that the limit in which the $TT$ OPE closes should be the strongly coupled large-$N_c$ limit. In the present paper we argue something similar about finite N=2 theories.

The plan of the paper is as follows. In sections 1 and 2 we study the quantum conformal algebras of the N=2 vector multiplet and hypermultiplet, respectively. In sections 3 and 4 we combine the two of them into a finite N=2 theory and discuss the most important phenomena that take place when the interaction is turned on, like renormalization splitting and renormalization mixing, anomalous dimensions and so on. In the rest of the paper we argue, using the Nachtmann theorem and very general arguments, that the algebra closes in the strongly coupled large-$N_c$ limit (sect. 5). We describe various properties of closed conformal field theory (sect. 5), compare them to those of open conformal field theory (sects. 5 and 6), give the complete OPE algebra in the N=2 case (section 6) and discuss aspects of the N=1 closed quantum

conformal algebra.

For supersymmetry, we use the notation of Sohnius [7], converted to the Euclidean framework via $\delta_{\mu\nu} \to -\delta_{\mu\nu}$, $T^{V,F,S} \to -T^{V,F,S}$ (these are the vector, spinor and scalar contributions to the stress-tensor), $\varepsilon_{\mu\nu\rho\sigma} \to -i\varepsilon_{\mu\nu\rho\sigma}$ and $\gamma\mu, \gamma_5 \to -i\gamma\mu, -i\gamma_5$. Moreover, we multiply $A_i$ by a factor $\sqrt{2}$ and use ordinary Majorana spinors $\lambda_i$ instead of symplectic Majorana spinors $\lambda_s^i$ ($\lambda_s^i = \frac{1}{2}[(\delta_{ij} - i\varepsilon_{ij}) - \gamma_5(\delta_{ij} + i\varepsilon_{ij})]\lambda_j$). For the current algebra we use the notations of [1, 2].

# 1  Vector multiplet

We begin our analysis with the N=2 vector multiplet and the hypermultiplet, by repeating the steps of [1]. The current multiplets have length 2 in spin units, but the important point is that *all* of them have length 2. We recall that the stress-tensor multiplet has length 0 in the N=4 algebra [1]. Moreover, there is one multiplet for each spin, even or odd.

The vector, spinor and scalar contributions to the currents of the N=2 vector multiplet are

$$
\begin{aligned}
\mathcal{J}^V &= F_{\mu\alpha}^+ \overset{\leftrightarrow}{\Omega}_{\text{even}} F_{\alpha\nu}^- + \text{impr.}, & \mathcal{A}^V &= F_{\mu\alpha}^+ \overset{\leftrightarrow}{\Omega}_{\text{odd}} F_{\alpha\nu}^- + \text{impr.}, \\
\mathcal{J}^F &= \frac{1}{2}\bar{\lambda}_i\gamma_\mu \overset{\leftrightarrow}{\Omega}_{\text{odd}} \lambda_i + \text{impr.}, & \mathcal{A}^F &= \frac{1}{2}\bar{\lambda}_i\gamma_5\gamma_\mu \overset{\leftrightarrow}{\Omega}_{\text{even}} \lambda_i + \text{impr.}, \\
\mathcal{J}^S &= M \overset{\leftrightarrow}{\Omega}_{\text{even}} M + N \overset{\leftrightarrow}{\Omega}_{\text{even}} N + \text{impr.}, & \mathcal{A}^S &= -2iM \overset{\leftrightarrow}{\Omega}_{\text{odd}} N + \text{impr.},
\end{aligned}
$$

where $\overset{\leftrightarrow}{\Omega}_{\text{even/odd}}$ denotes an even/odd string of derivative operators $\overset{\leftrightarrow}{\partial}$ and "impr." stands for the improvement terms [2]. A simple set of basic rules suffices to determine the operation $\delta_\zeta^2$ which relates the currents of a given multiplet and is a certain combination of two supersymmetry transformations (see [1] for details). The result is

$$
\begin{aligned}
\mathcal{J}_{2s}^S &\to -2\,\mathcal{A}_{2s+1}^F + 2\,\mathcal{A}_{2s+1}^S, & \mathcal{A}_{2s-1}^S &\to -2\,\mathcal{J}_{2s}^F + 2\,\mathcal{J}_{2s}^S, \\
\mathcal{J}_{2s}^F &\to -8\,\mathcal{A}_{2s+1}^V + \mathcal{A}_{2s+1}^S, & \mathcal{A}_{2s-1}^F &\to -8\,\mathcal{J}_{2s}^V + \mathcal{J}_{2s}^S, \\
\mathcal{J}_{2s}^V &\to -2\,\mathcal{A}_{2s+1}^V + \frac{1}{4}\,\mathcal{A}_{2s+1}^F, & \mathcal{A}_{2s-1}^V &\to -2\,\mathcal{J}_{2s}^V + \frac{1}{4}\,\mathcal{J}_{2s}^F.
\end{aligned}
$$

As we see, the algebra is more symmetric than the N=4 one [1]. In particular, there is an evident even/odd-spin symmetry that was missing in [1]. We have fixed the normalization of the scalar axial current $\mathcal{A}^S$ (absent in N=4) in order to exhibit this symmetry. We recall that $T^V = -2\mathcal{J}_2^V$, $T^F = \mathcal{J}_2^F/2$ and $T^S = -\mathcal{J}_2^S/4$ are the various contributions to the stress-tensor.

The list of current multiplets generated by the $TT$ OPE is easily worked out and reads

$$
\begin{aligned}
&\mathcal{T}_0 = \tfrac{1}{2}\mathcal{J}_0^S \\
&\mathcal{T}_1 = -\mathcal{A}_1^F + \mathcal{A}_1^S && \Lambda_1 = \tfrac{1}{4}\mathcal{A}_1^F + \tfrac{1}{4}\mathcal{A}_1^S \\
&\mathcal{T}_2 = 8\mathcal{J}_2^V - 2\mathcal{J}_2^F + \mathcal{J}_2^S && \Lambda_2 = -2\mathcal{J}_2^V - \tfrac{1}{2}\mathcal{J}_2^F + \tfrac{3}{4}\mathcal{J}_2^S && \Xi_2 = \tfrac{4}{15}\mathcal{J}_2^V + \tfrac{4}{15}\mathcal{J}_2^F + \tfrac{1}{5}\mathcal{J}_2^S \\
&\Delta_3 = \tfrac{3}{7}\mathcal{A}_3^V + \tfrac{15}{56}\mathcal{A}_3^F + \tfrac{5}{28}\mathcal{A}_3^S && \Lambda_3 = 8\mathcal{A}_3^V - 2\mathcal{A}_3^F + \mathcal{A}_3^S && \Xi_3 = -\tfrac{8}{3}\mathcal{A}_3^V - \tfrac{1}{3}\mathcal{A}_3^F + \tfrac{2}{3}\mathcal{A}_3^S \\
&\Delta_4 = -3\mathcal{J}_4^V - \tfrac{1}{4}\mathcal{J}_4^F + \tfrac{5}{8}\mathcal{J}_4^S && \Upsilon_4 = \tfrac{8}{15}\mathcal{J}_4^V + \tfrac{4}{15}\mathcal{J}_4^F + \tfrac{1}{6}\mathcal{J}_4^S && \Xi_4 = 8\mathcal{J}_4^V - 2\mathcal{J}_4^F + \mathcal{J}_4^S \\
&\Delta_5 = 8\mathcal{A}_5^V - 2\mathcal{A}_5^F + \mathcal{A}_5^S && \Upsilon_5 = -\tfrac{16}{5}\mathcal{A}_5^V - \tfrac{1}{5}\mathcal{A}_5^F + \tfrac{3}{5}\mathcal{A}_5^S && \Omega_5 = \tfrac{20}{33}\mathcal{A}_5^V + \tfrac{35}{132}\mathcal{A}_5^F + \tfrac{7}{44}\mathcal{A}_5^S \\
&\cdots && \Upsilon_6 = 8\mathcal{J}_6^V - 2\mathcal{J}_6^F + \mathcal{J}_6^S && \Omega_6 = -\tfrac{10}{3}\mathcal{J}_6^V - \tfrac{1}{6}\mathcal{J}_6^F + \tfrac{7}{12}\mathcal{J}_6^S \\
& && \cdots && \Omega_7 = 8\mathcal{A}_7^V - 2\mathcal{A}_7^F + \mathcal{A}_7^S
\end{aligned}
$$
(1)

The lowest components of each current multiplet ($\mathcal{T}_2$, $\Lambda_3$, $\Xi_4$, $\Delta_5$, $\Upsilon_6$, $\Omega_7$) have the same form. The normalization is fixed in such a way that these components have also the same overall factor. In [2] we used a different convention: we fixed the normalization of each current by demanding that the coefficients of $\mathcal{A}^F$ and $\mathcal{J}^S$ be 1. Here we have to be more precise and keep track of the relative factors inside current multiplets, since we need to superpose the vector and matter quantum conformal algebras in order to obtain the most general N=2 structure (see section 3).

*Checks.* Scalar odd-spin currents appear in the algebra and their two-point functions were not computed in [2]. We can combine orthogonality checks with the indirect derivations of the amplitudes of these currents.

These currents are necessary to properly diagonalize the multiplet. For example, only the $\mathcal{A}_1^S$-independent combination $-\tfrac{1}{2}\mathcal{T}_1 + 2\Lambda_1 = \mathcal{A}_1^F$ appears in the OPE, but the scalar current $\mathcal{A}_1^S$ orthogonalizes $\mathcal{T}_1$ and $\Lambda_1$. Indeed, the two-point function of the scalar spin-1 current, easy to compute,

$$
\langle \mathcal{A}^S \mu(x)\, \mathcal{A}^S \nu(0)\rangle = \frac{4}{3}N_V \left(\frac{1}{4\pi^2}\right)^2 \pi_{\mu\nu}\left(\frac{1}{|x|^4}\right),
$$

agrees with the orthogonality relationship $\langle \mathcal{T}_1\,\Lambda_1\rangle = 0$. Similarly, $\mathcal{T}_2$ and $\Lambda_2$ are orthogonal and this can be verified with the results of [1]. $\Xi_2$ is then determined by requiring that it is orthogonal to both $\mathcal{T}_2$ and $\Lambda_2$. Note that $\Xi_2$ has the same form as $\Xi_2$ in the N=4 algebra [1], apart from a factor due to the different normalization.

Then $\Lambda_3$ and $\Xi_3$ are determined via the transformation $\delta_\zeta^2$. The two-point function of $\mathcal{A}_3^S$ is derived by the orthogonality relationship $\langle \Lambda_3\,\Xi_3\rangle = 0$. We obtain

$$
\langle \mathcal{A}_3^S(x)\, \mathcal{A}_3^S(0)\rangle = \frac{8}{35}N_V \left(\frac{1}{4\pi^2}\right)^2 \Pi^{(3)}\left(\frac{1}{|x|^4}\right),
$$

while $\langle \mathcal{A}_3^F\, \mathcal{A}_3^F\rangle$ and $\langle \mathcal{A}_3^V\, \mathcal{A}_3^V\rangle$ can be found in [2]. Then we determine $\Delta_3$ via the equations $\langle \Delta_3\,\Lambda_3\rangle = \langle \Delta_3\,\Xi_3\rangle = 0$ and $\Xi_4$, $\Delta_4$ via the $\delta_\zeta^2$ operation. The amplitudes of [2] suffice to show that $\langle \Xi_4\,\Delta_4\rangle = 0$, which is a non-trivial numerical check of the values.

Finally, once $\Upsilon_4$ is found by solving $\langle \Upsilon_4\,\Xi_4\rangle = \langle \Upsilon_4\,\Delta_4\rangle = 0$, we extract the two-point

function of $\mathcal{A}_5^S$ via the orthogonality condition of $\Delta_5$ and $\Upsilon_5$, with the result

$$\langle \mathcal{A}_5^S(x)\,\mathcal{A}_5^S(0) \rangle = \frac{2^5}{3^2 \cdot 7 \cdot 11} N_V \left( \frac{1}{4\pi^2} \right)^2 \Pi^{(5)} \left( \frac{1}{|x|^4} \right).$$

$\Omega_5$ is determined by the conditions $\langle \Omega_5\,\Upsilon_5 \rangle = 0$ and $\langle \Omega_5\,\Delta_5 \rangle = 0$, and so on.

Any current multiplet is 2-spin long and has the form

$$\Lambda_s = 4\,\frac{a_s\,\mathcal{J}_s^V + b_s\,\mathcal{J}_s^F + c_s\,\mathcal{J}_s^S}{(a_s + 8b_s + 8c_s)} \qquad \rightarrow$$

$$\Lambda_{s+1} = \frac{4}{(a_s + 8b_s + 8c_s)} \left[ -2(a_s + 4b_s)\,\mathcal{A}_{s+1}^V + \frac{1}{4}(a_s - 8c_s)\,\mathcal{A}_{s+1}^F + (b_s + 2c_s)\,\mathcal{A}_{s+1}^S \right] \qquad \rightarrow$$

$$\Lambda_{s+2} = 8\mathcal{J}_{s+2}^V - 2\,\mathcal{J}_{s+2}^F + \mathcal{J}_{s+2}^S. \tag{2}$$

for all $s$ ($\mathcal{J} \leftrightarrow \mathcal{A}$ when $s$ is odd).

We stress again the most important novelty exhibited by the N=2 algebra with respect to the N=4 one [1]: the multiplet of the stress-tensor is not shorter than the other multiplets; rather, it contains also a spin-1 current (the $R$-current) and a spin-0 partner, on which we will have more to say later on.

The theory is not finite in the absence of hypermultiplets. Nevertheless, it is meaningful to calculate the anomalous dimensions of the operators to lowest order, since at one-loop order around a free-field theory conformality is formally preserved. We give here the first few values of the anomalous dimensions for illustrative purposes. The procedure for the computation is the same as the one of ref. [1] and will be recalled in the next sections. We find $h_{\mathcal{T}} = 0$, $h_\Lambda = 2N_c\frac{\alpha}{\pi}$ and $h_\Xi = \frac{5}{2}N_c\frac{\alpha}{\pi}$. These three values obey the Nachtmann theorem [4], which states that the spectrum of anomalous dimensions is a positive, increasing and convex function of the spin. Actually, the Nachtmann theorem applies only to the lowest anomalous dimension of the even-spin levels. Nevertheless, it seems that the property is satisfied by all the spin levels in this particular case. This is not true in the presence of hypermultiplets, as we will see.

## 2  Hypermultiplet

The structure of the algebra is much simpler for the matter multiplet. The currents are

$$\mathcal{J}^F = \bar{\psi}\gamma_\mu \overset{\leftrightarrow}{\Omega}_{\text{odd}} \psi + \text{impr.}, \qquad\qquad \mathcal{A}^F = \bar{\psi}\gamma_5\gamma_\mu \overset{\leftrightarrow}{\Omega}_{\text{even}} \psi + \text{impr.},$$

$$\mathcal{J}^S = 2\bar{A}_i \overset{\leftrightarrow}{\Omega}_{\text{even}} A_i + \text{impr.},$$

The basic operation $\delta^2\zeta$ does not exhibit the even/odd spin symmetry and is more similar to the N=4 one:

$$\mathcal{J}_{2s}^S \rightarrow -4\,\mathcal{A}_{2s+1}^F,$$
$$\mathcal{J}_{2s}^F \rightarrow -2\,\mathcal{A}_{2s+1}^F, \qquad\qquad \mathcal{A}_{2s-1}^F \rightarrow -2\,\mathcal{J}_{2s}^F + \mathcal{J}_{2s}^S.$$

It gives the following list of multiplets

$$
\begin{aligned}
&\mathcal{T}_0 = -\tfrac{1}{4}\ \mathcal{J}_0^S \\
&\mathcal{T}_1 = \mathcal{A}_1^F \\
&\mathcal{T}_2 = -2\ \mathcal{J}_2^F + \mathcal{J}_2^S \qquad \Xi_2 = -\tfrac{1}{5}\ \mathcal{J}_2^F - \tfrac{3}{20}\ \mathcal{J}_2^S \\
&\qquad\qquad\qquad\qquad\qquad \Xi_3 = \mathcal{A}_3^F \\
&\qquad\qquad\qquad\qquad\qquad \Xi_4 = -2\ \mathcal{J}_4^F + \mathcal{J}_4^S \qquad\qquad \Upsilon_4 = -\tfrac{2}{9}\ \mathcal{J}_4^F - \tfrac{5}{36}\ \mathcal{J}_4^S \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Upsilon_5 = \mathcal{A}_5^F \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Upsilon_6 = -2\ \mathcal{J}_6^F + \ \mathcal{J}_6^S,
\end{aligned}
\tag{3}
$$

determined with the familiar procedure. We see that no spin-1 scalar current appears and that, again, the stress-tensor has two partners, the $R$-current and a mass term. The general form of the current hypermultiplet is particularly simple:

$$
\Lambda_{2s} = -\frac{a_s\ \mathcal{J}_{2s}^F + b_s\ \mathcal{J}_{2s}^S}{2(a_s + 2b_s)} \quad \rightarrow \quad \Lambda_{2s+1} = \mathcal{A}_{2s+1}^F \quad \rightarrow \quad \Lambda_{2(s+1)} = -2\ \mathcal{J}_{2(s+1)}^F + \ \mathcal{J}_{2(s+1)}^S.
$$

There is no anomalous dimension to compute here, since the hypermultiplet admits no renormalizable self-coupling. In the next section we combine vector multiplets and hypermultiplets to study in particular the finite N=2 theories.

## 3    Combining the two multiplets into a finite N=2 theory

In this section we work out the quantum conformal algebra of finite N=2 supersymmetric theories. We consider, as a concrete example (the structure is completely general), the theory with group $SU(N_c)$ and $N_f = 2N_c$ hypermultiplets in the fundamental representation. The beta-function is just one-loop owing to N=2 supersymmetry. Precisely, it is proportional to $N_c - \tfrac{1}{2}N_f$, so it vanishes identically for $N_f = 2N_c$ [8]. Combining the free-vector and free-hypermultiplet quantum conformal algebras is not as straightforward as it might seem. The algebra is much richer than the N=4 one and some non-trivial work is required before singling out its nice properties.

To begin with, the us write down the full multiplet $\mathcal{T} = \mathcal{T}_v + \mathcal{T}_m$ of the stress-tensor:

$$
\begin{aligned}
\mathcal{T}_0 &= \frac{1}{2}\ \mathcal{J}_{0v}^S - \frac{1}{4}\mathcal{J}_{0m}^S = \frac{1}{2}(M^2 + N^2 - \bar{A}_i A_i), \\
\mathcal{T}_1 &= -\mathcal{A}_{1v}^F + \mathcal{A}_{1m}^F + \mathcal{A}_{1v}^S = -\frac{1}{2}\lambda_i \gamma_5 \gamma\mu \lambda_i + \bar{\psi}\gamma_5\gamma\mu\psi - 2iM\overleftrightarrow{\partial}\mu N, \\
\mathcal{T}_2 &= 8\mathcal{J}_{2v}^V - 2(\mathcal{J}_{2v}^F + \mathcal{J}_{2m}^F) + \mathcal{J}_{2v}^S + \mathcal{J}_{2m}^S = -4T_{\mu\nu},
\end{aligned}
$$

where the additional subscripts $v$ and $m$ refer to the vector and matter contributions, respectively (this heavy notation is necessary, but fortunately temporary - we write down the explicit formulas in order to facilitate the reading).

In general, the full $\mathcal{T}$-multiplet appears in the quantum conformal algebra. $\mathcal{T}_1$ is the $SU(2)$-invariant $R$-current, and its anomaly vanishes because it is proportional to the beta-function.

$\mathcal{T}_0$ is one of the finite mass perturbations [9]. Our picture gives a nice argument for the finiteness of such a mass term, which follows directly from the finiteness of the stress-tensor.

The next observation is that the $\mathcal{T}$-multiplet has to be part of a pair of multiplets having the same position in the algebra. The general OPE structure of [2] shows that the singularity $1/|x|^6$ carries the sum of the squared scalar fields with coefficient 1. In our case it should be $M^2 + N^2 + 2\bar{A}_i A_i$ and not just $M^2 + N^2 - \bar{A}_i A_i$. On the other hand, the mass operator $M^2 + N^2 + 2\bar{A}_i A_i$ is not finite and cannot stay with the stress-tensor. Therefore it is split into a linear combination of two operators, precisely $\mathcal{T}_0$ and $\mathcal{T}_0^* = \frac{1}{2}\,\mathcal{J}_{0v}^S - \frac{1}{4}\,\mathcal{J}_{0m}^S = \frac{1}{2}\,(M^2 + N^2 + \bar{A}_i A_i)$. These two operators are not orthogonal: they can freely mix under renormalization, because their current multiplets $\mathcal{T}$ and $\mathcal{T}^*$ have the same position in the algebra. This means that in the N=2 quantum conformal algebra the $I$-degeneracy of [2] survives. Orthogonalization would be rather awkward, since the number of components of $M$ and $N$ is proportional to $N_c^2 - 1$, while the number of components of $A_i$ is proportional to $N_f N_c = 2N_c^2$. Coefficients of the form $\sqrt{(N_c^2 - 1)/N_c^2}$ would appear and the diagonalization would not survive once the interaction is turned on. In the presence of mixing, there is no privileged basis for the two currents, in general.

However, the $\mathcal{T}\,\mathcal{T}^*$-pair satisfies a further property, namely $\mathcal{T}$ and $\mathcal{T}^*$ do split in the large-$N_c$ limit (we will present in sects. 5 and 6 an interesting interpretation of this fact). We have fixed $\mathcal{T}_0^*$ by imposing $\langle \mathcal{T}_0 \, \mathcal{T}_0^* \rangle = 0$ in this limit. The complete $\mathcal{T}^*$ multiplet is then $\mathcal{T}^* = \mathcal{T}_v^* - \mathcal{T}_m^*$:

$$
\begin{aligned}
\mathcal{T}_0^* &= \frac{1}{2}\,\mathcal{J}_{0v}^S + \frac{1}{4}\,\mathcal{J}_{0m}^S = \frac{1}{2}(M^2 + N^2 + \bar{A}_i A_i) \\
\mathcal{T}_1^* &= -\mathcal{A}_{1v}^F - \mathcal{A}_{1m}^F + \mathcal{A}_{1v}^S = -\frac{1}{2}\lambda_i \gamma_5 \gamma \mu \lambda_i - \bar{\psi}\gamma_5 \gamma \mu \psi - 2iM \overleftrightarrow{\partial} \mu N \\
\mathcal{T}_2^* &= 8\mathcal{J}_{2v}^V - 2(\mathcal{J}_{2v}^F - \mathcal{J}_{2m}^F) + \mathcal{J}_{2v}^S - \mathcal{J}_{2m}^S = -4(T_v - T_m),
\end{aligned}
$$

where $T_v$ and $T_m$ are the vector and matter contributions to the stress-tensor.

Now we analyse the spin-1 level of the OPE. The first observation is that the scalar current contribution $\mathcal{A}_{1v}^S = -2iM \overleftrightarrow{\partial} \mu N$ appears in $\mathcal{T}_1$ and $\mathcal{T}_1^*$. We know that it does not appear in the general free-field algebra [2]. Moreover, the relative coefficient of the fermionic contributions $\mathcal{A}_{1v}^F$ and $\mathcal{A}_{1m}^F$ (coming from vector multiplets and hypermultiplets) should be 1. These two conditions cannot be satisfied by taking a linear combination of $\mathcal{T}_1$ and $\mathcal{T}_1^*$, so that a new current should appear, precisely the lowest-spin current of a new multiplet. This is the multiplet $\Lambda$ of (1), which is orthogonal to both $\mathcal{T}$ and $\mathcal{T}^*$, and therefore unaffected by the hypermultiplets (but only in the free-field limit - see below). The scalar current $-2iM \overleftrightarrow{\partial} \mu N$ is required to properly orthogonalize the multiplets, as it happens in the spin-0 case.

Some effects appear just when the interaction is turned on: the scalar current $\mathcal{A}^S$, which cancels out at the level of the free-field algebra, appears at non-vanishing $g$. The current $\Lambda$ does not depend on the hypermultiplets at the free-field level, but receives hypermultiplet contributions at non-vanishing $g$. The procedure for determining the currents at the interacting level is worked out in [1]. In particular, after covariantizing the derivatives we have to take the traces out. In the construction of $\Lambda$, such traces are proportional to the vector multiplet field

equations, which receive contributions from the hypermultiplets at $g \neq 0$.

At the spin-2 level of the OPE, the situation is similar to the spin-0 one. The basic formulas for the squares are

$$
\begin{aligned}
\langle \mathcal{J}_{2v}^V \, \mathcal{J}_{2v}^V \rangle &= 1/20 \, N_V, & \langle \mathcal{J}_{2v}^F \, \mathcal{J}_{2v}^F \rangle &= 2/5 \, N_V, & \langle \mathcal{J}_{2v}^S \, \mathcal{J}_{2v}^S \rangle &= 8/15 \, N_V, \\
\langle \mathcal{J}_{2m}^F \, \mathcal{J}_{2m}^F \rangle &= 4/5 \, N_c^2, & \langle \mathcal{J}_{2m}^S \, \mathcal{J}_{2m}^S \rangle &= 32/15 \, N_c^2,
\end{aligned}
$$

factorizing out the common factor $1/(4\pi^2)^2 \, \Pi^{(2)}(1/|x|^4)$. Three spin-2 operators come from the previous multiplets, $\mathcal{T}_2$, $\mathcal{T}_2^*$ and $\Lambda_2$, and two new operators appear, $\Xi_2$ and $\Xi_2^*$. These two mix under renormalization and do not split in the large-$N_c$ limit (see next section). They have the form

$$
\frac{4}{15} \, \mathcal{J}_{2v}^V + \frac{4}{15} \, \mathcal{J}_{2v}^F + \frac{1}{5} \, \mathcal{J}_{2v}^S + \alpha_\Xi \left( 2 \, \mathcal{J}_{2m}^F + \frac{3}{2} \, \mathcal{J}_{2m}^S \right) = \Xi_{2v} + \alpha_\Xi \Xi_{2m}.
$$

We call $\alpha_\Xi$ the coefficient for $\Xi_2$ and $\alpha_\Xi^*$ the one for $\Xi_2^*$. In order to proceed with the study of the quantum conformal algebra, it is not necessary to fix both $\alpha_\Xi$ and $\alpha_\Xi^*$, and we can treat any degenerate pair, such as $\Xi_2$ and $\Xi_2^*$, as a whole.

Summarizing, the result is that the final algebra contains the multiplets

$$
\begin{aligned}
\mathcal{T} &= \mathcal{T}_v + \alpha_{\mathcal{T}} \, \mathcal{T}_m, & \Lambda &= \Lambda_v, \\
\Xi &= \Xi_v + \alpha_\Xi \, \Xi_m, & \Delta &= \Delta_v, \\
\Upsilon &= \Upsilon_v + \alpha_\Upsilon \, \Upsilon_m, & \Omega &= \Omega_v,
\end{aligned}
$$

and so on. We have $\alpha_{\mathcal{T}} = -\alpha_{\mathcal{T}}^* = 1$, while $\alpha_\Xi$ and $\alpha_\Upsilon$ are undetermined. Fixing them by diagonalizing the matrix of two-point functions is possible to the lowest order (and in the next section we use this property to present the results of our computations), but in general it is not meaningful to all orders.

## 4 Anomalous dimensions and degenerate pairs

In this section we discuss the currents at non-vanishing $g$, compute their anomalous dimensions and study the degenerate multiplets.

We start with the spin-1 currents $\mathcal{T}_1$, $\mathcal{T}_1^*$ and $\Lambda_1$, which we call $\Sigma^i \mu$, $i = 1, 2, 3$, respectively. The currents are easily defined at non-zero coupling $g$ by covariantizing the derivative appearing in $\mathcal{A}_1^S$, i.e. $\mathcal{A}_1^S \to -2iM \overleftrightarrow{D} \mu N$.

The matrix of two-point functions has the form (see for example [11])

$$
\langle \Sigma^i \mu(x) \, \Sigma \nu^j(0) \rangle = \frac{1}{(|x|\mu)^{h_{ik}(g^2)}} \, \pi_{\mu\nu} \left( \frac{c_{kl}^{(1)}(g^2)}{|x|^4} \right) \frac{1}{(|x|\mu)^{h_{jl}(g^2)}} \left( \frac{1}{4\pi^2} \right)^2. \tag{4}
$$

To calculate the lowest order of the matrix $h_{ij}(g^2)$ of anomalous dimensions it is sufficient to take the zeroth-order $c_{ij}^{(1)}(0)$ of the matrix of central charges $c_{ij}^{(1)}(g^2)$ (see [2]). We have, at finite

$N_c$,

$$c_{ij}^{(1)}(0) = \frac{8}{3}\begin{pmatrix} 2N_c^2 - 1 & -1 & 0 \\ -1 & 2N_c^2 - 1 & 0 \\ 0 & 0 & \frac{1}{16}N_V \end{pmatrix}, \tag{5}$$

which becomes diagonal only in the large-$N_c$ limit. Now, from (4) we can compute the matrix of divergences

$$\langle \partial\Sigma^i(x)\,\partial\Sigma^j(0)\rangle = -\frac{3}{\pi^4}\frac{(ch^t + hc)_{ij}}{|x|^8}. \tag{6}$$

Calling $a$ the matrix $ch^t + hc$, the explicit computation gives $a = N_c N_V \frac{\alpha}{\pi}\,\mathrm{diag}(0, 64/3, 1)$, whence we obtain

$$h = \begin{pmatrix} 0 & \frac{1}{N_c} & 0 \\ 0 & 2N_c - \frac{1}{N_c} & 0 \\ 0 & 0 & 3N_c \end{pmatrix}\frac{\alpha}{\pi}. \tag{7}$$

This matrix is in general triangular, with entries $(i,3)$ and $(3,i)$ equal to zero, since the current multiplets $\mathcal{T}$ and $\Lambda$ are orthogonal. Moreover, the entry $h_{11}$ is zero because of the finiteness of the stress-tensor. Finally, we observe that the off-diagonal element is suppressed in the large-$N_c$ limit, as we expected, and that in this limit the anomalous dimension of $\mathcal{T}^*$ becomes $h_{\mathcal{T}^*} = 2N_c \frac{\alpha}{\pi} < h_\Lambda = 3N_c \frac{\alpha}{\pi}$.

The diagonal form of the pair $(\mathcal{T}, \mathcal{T}^*)$ is given by $(\mathcal{T}', \mathcal{T}^{*\prime}) = H\,(\mathcal{T}, \mathcal{T}^*)$ with

$$H = \begin{pmatrix} 1 & 0 \\ \frac{1}{2N_c^2} & 1 - \frac{1}{2N_c^2} \end{pmatrix}.$$

One finds $h' = \mathrm{diag}(0, h^*)$ with $h^* = \frac{\alpha}{\pi}\left(2N_c - \frac{1}{N_c}\right)$.

Now we study the spin-2 level of the OPE. A new degenerate pair $\{\Xi, \Xi^*\}$ appears and therefore we have five currents $\mathcal{J}_2^{(i)}$, $i = 1, \ldots 5$, organized into two degenerate pairs and a "singlet". The matrix $c^{(2)}$ of central charges, defined by

$$\langle \mathcal{J}_{\mu\nu}^{(i)}(x)\,\mathcal{J}_{\rho\sigma}^{(j)}(0)\rangle = \frac{1}{60}\frac{1}{(|x|\mu)^{h_{ik}}}\prod{}^{(2)}\left(\frac{c_{kl}^{(2)}}{|x|^4}\right)\frac{1}{(|x|\mu)^{h_{ik}}}\left(\frac{1}{4\pi^2}\right)^2,$$

is block-diagonal, $c^{(2)} = \mathrm{diag}(120\,c_{\mathcal{T}}^{(1)}, 36N_V, c_\Xi^{(2)})$, where the first two blocks are proportional to the spin-1 blocks, see formula (5). The third block reads

$$c_\Xi^{(2)} = \frac{16}{5}\begin{pmatrix} N_V + \frac{3}{2}\,\alpha_\Xi^2 N_c^2 & N_V + \frac{3}{2}\,\alpha_\Xi\alpha_\Xi^* N_c^2 \\ N_V + \frac{3}{2}\,\alpha_\Xi\alpha_\Xi^* N_c^2 & N_V + \frac{3}{2}\,\alpha_\Xi^{*2} N_c^2 \end{pmatrix}.$$

The matrix of divergences is

$$\langle \partial\mu\mathcal{J}_{\mu\nu}^{(i)}(x)\,\partial\rho\mathcal{J}_{\rho\sigma}^{(j)}(0)\rangle = \frac{3}{4\pi^4}(c^{(2)}h_2^t + h_2 c^{(2)})_{ij}\frac{\mathcal{I}_{\nu\sigma}(x)}{|x|^{10}}. \tag{8}$$

Again, the matrix $a^{(2)} = c^{(2)}h_2^t + h_2 c^{(2)}$ is block diagonal and the first two blocks coincide with those of the corresponding spin-1 matrix. This correctly reproduces the known anomalous dimension of $\mathcal{T}$, $\mathcal{T}^*$ and $\Lambda$. Instead, the $\Xi$-block reads

$$a_\Xi^{(2)} = \frac{16}{5}\begin{pmatrix} 7 + \frac{11}{2}\,\alpha_\Xi^2 - 2\,\alpha_\Xi & 7 + \frac{11}{2}\,\alpha_\Xi\alpha_\Xi^* - \alpha_\Xi - \alpha_\Xi^* \\ 7 + \frac{11}{2}\,\alpha_\Xi\alpha_\Xi^* - \alpha_\Xi - \alpha_\Xi^* & 7 + \frac{11}{2}\,\alpha_\Xi^{*2} - 2\,\alpha_\Xi^* \end{pmatrix}N_c N_V \frac{\alpha}{\pi}.$$

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The calculation that we have performed is not sufficient to completely determine the matrix $h$, since $h$ is not symmetric. However, at the lowest order, $a^{(2)}$ is sufficient for our purpose. In particular, let us diagonalize $c^{(2)}$ and $a^{(2)}$ in the large-$N_c$ limit. We have $\alpha, \alpha^* = \frac{1}{3}(5 \pm \sqrt{31})$ and $h = N_c \frac{\alpha}{\pi} \operatorname{diag}(1.7, 3.6)$. It appears that the entire pair acquires an anomalous dimension and moves away.

We conclude that the issue of splitting the paired currents in the large-$N_c$ limit is irrelevant to this case. What is important is that the two currents move together to infinity. The other pairs of the quantum conformal algebra ($\Xi$, $\Upsilon$, etc.) exhibit a similar behaviour and only the pair $\mathcal{T}$ is special.

The analysis of the present section could proceed to the other multiplets and multiplet pairs that appear in the algebra. However, the description that we have given so far is sufficient to understand the general properties of the algebra and proceed.

We now comment on the validity of the Nachtmann theorem [4], which states that the minimal anomalous dimension $h_{2s}$ of the currents of the even spin-$2s$ level is a positive, increasing and convex function of $s$. We have $h_2 = 0$ and $h_4 = 1.7 N_c \frac{\alpha}{\pi}$. Moreover, $h_\Lambda = 3 N_c \frac{\alpha}{\pi} > h_4$ and $h_{\mathcal{T}^*} \sim 2 N_c \frac{\alpha}{\pi} > h_4$. There is no contradiction with the Nachtmann theorem, which is restricted to the minimal even-spin values of the anomalous dimensions. Nevertheless, it is worth noting that the nice regular behaviour predicted by this theorem cannot be extended in general to the full spectrum of anomalous dimensions. In particular an odd-spin value $h_{2s+1}$ can be greater than the even-spin value $h_{2(s+1)}$.

Although the spectrum is less regular than in the N=4 case, the irregularity that we are remarking works in the sense of making certain anomalous dimensions greater than would be expected. This will still allow us to argue that all anomalous dimensions that are non-zero in perturbation theory become infinite in the strongly coupled large-$N_c$ limit. In the rest of the paper we discuss this prediction and present various consistency checks of it.

Other operators appear in the quantum conformal algebra besides those that we have discussed in detail[1]. They can be grouped into three classes:

i) symmetric operators with a non-vanishing free-field limit; these are the ones that we have discussed;

ii) non-symmetric operators with a non-vanishing free-field limit; these are not completely symmetric in their indices;

iii) operators with a vanishing free-field limit; these are turned on by the interaction.

Operators of classes ii) and iii) can often be derived from those of class i) by using supersymmetry. This is the case, for example, of the N=4 quantum conformal algebra [12]. The anomalous dimensions are of course the same as those of their class i)-partners, so that our discussion covers them and the conclusions that we derive are unaffected.

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[1] I am grateful to S. Ferrara for clarifying discussions of this point.

# 5 Closed conformal field theory

The multiplet of the stress-tensor is the most important multiplet of the algebra. Since all of its components are conserved, it will survive at arbitrary $g$ and in particular in the large-$N_c$ limit. The OPE algebra generated by this multiplet is in general not closed, but it might be closed in some special cases.

We can classify conformal field theory into two classes:

i) *open* conformal field theory, when the quantum conformal algebra contains an infinite number of (generically non-conserved) currents;

ii) *closed* conformal field theory, when the quantum conformal algebra closes within a finite set of conserved currents.

This section is devoted to a study of this classification.

We conjecture that closed conformal field theory is the boundary of the set of open conformal field theories. Roughly, one can think of a ball centred in free-field theory. The boundary sphere is the set of closed theories. The bulk is the set of open theories. As a radius $r$ one can take the value of the minimal non-vanishing anomalous dimension. In the N=4 theory, $r$ is the anomalous dimension of the Konishi multiplet, while in the N=2 finite theories $r$ is the minimal eigenvalue of the matrix of anomalous dimensions of the $(\Xi, \Xi^*)$-pair. The theory is free for $r = 0$, open for $r < 0 < \infty$ and closed for $r = \infty$. The function $r = r(g^2 N_c)$ can be taken as the true coupling constant of the theory instead of $g^2 N_c$.

The Nachtmann theorem is completely general (a consequence of unitarity and dispersion relations) and does not make any use of supersymmetry, holomorphy, chirality or whatsoever, which would restrict its range of validity. It states in particular that if $r = 0$, then $h_{2s} = 0$ $\forall s > 0$, and if $r = \infty$, then $h_{2s} = \infty$ $\forall s > 0$. The considerations of the previous section, in particular the regularity of the spectrum of critical exponents, suggest that in the former case all current multiplets are conserved and in the latter case all of them have infinite anomalous dimensions and decouple from the OPE. Very precise properties of the strongly coupled limit of the theory can be inferred from this.

It was pointed out in [2], using the AdS/CFT correspondence [5], in particular the results of [6], that the $TT$ OPE should close in the strongly coupled (which means at large 't Hooft coupling, $g^2 N_c \gg 1$) large-$N_c$ limit of N=4 supersymmetric theory. In the weakly coupled limit, the anomalous dimensions of the various non-conserved multiplets are non-zero and $r \sim g^2 N_c$. The results of [6] suggest that in the vicinity of the boundary sphere, the anomalous dimension of the Konishi operator changes to $r \sim (g^2 N_c)^{\frac{1}{4}}$, but still tends to infinity. The Nachtmann theorem then implies that all the anomalous dimensions of the non-conserved operators tend to infinity.

It is reasonable to expect a similar behaviour in the case of N=2 finite theories (to which the AdS/CFT correspondence does *not* apply, in general). We expect that in the strongly coupled large-$N_c$ limit the OPE closes just with the currents (4) [2]. This appears to be the correct generalization of the property exhibited in the N=4 case. Therefore we conjecture that

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[2] We can call $\mathcal{T}_0$ the "spin-0 current", with some abuse of language.

· *closed conformal field theory is the exact solution to the strongly coupled large-$N_c$ limit of open conformal field theory.*

The weakly coupled behaviour studied in the previous sections is consistent with this picture. We have observed that $\mathcal{T}$ and $\mathcal{T}^*$ split in the large-$N_c$ limit already at weak coupling. This suggests that $\mathcal{T}^*$ moves away from $\mathcal{T}$. Moreover, this splitting does not take place in the other multiplet-pairs ($\Xi$, $\Upsilon$ and so on) that mix under renormalization: this means that they remain paired and each pair moves to infinity, without leaving any remnant.

Secondly, we claim that

· *a closed quantum conformal algebra determines uniquely the associated closed conformal field theory*[3].

A similar property holds in two-dimensional conformal field theory and indeed we are asserting that closed conformal field theory is the proper higher-dimensional analogue of two-dimensional conformal field theory. Thirdly, we show in the next section that

· *a closed algebra is determined uniquely by two central charges: c and a.*

The two central charges, called $c$ and $a$ in ref. [13] take different values in the N=2 algebra, precisely[4]:

$$c = \frac{1}{6}(2N_c^2 - 1), \qquad\qquad a = \frac{1}{24}(7N_c^2 - 5), \qquad\qquad (9)$$

and equal values in the N=4 algebra, $c = a = \frac{1}{4}(N_c^2 - 1)$ if the gauge group is $SU(N_c)$. We recall that the values of $c$ and $a$ are independent of the coupling constant $g$ [13] if the theory is finite.

When N=2, the difference between $c$ and $a$ persists in the large-$N_c$ limit (where $c/a \sim 8/7$), both strongly and weakly coupled. The presence of more partners in the current multiplet of the stress-tensor (precisely $\mathcal{T}_0$ and $\mathcal{T}_1$) is related to the ratio $\frac{c}{a} \neq 1$ in N=2 theories, something which we will describe better in the next section. This is a remarkable difference between closed conformal field theory in four dimensions and conformal field theory in two dimensions, two types of theories that otherwise have several properties in common and can be studied in parallel.

Let us now consider N=1 (and non-supersymmetric) theories. The multiplet of the stress-tensor will not contain spin-0 partners, in general, but just the $R$-current. The above considerations stop at the spin-2 and 1 levels of the OPE, but the procedure to determine the closed algebra is the same. What is more subtle is to identify the physical situation that the closed limit should describe.

In supersymmetric QCD with $G = SU(N_c)$ and $N_f$ quarks and antiquarks in the fundamental representation, the conformal window is the interval $3/2\, N_c < N_f < 3\, N_c$. In the limit where both $N_c$ and $N_f$ are large, but the ratio $N_c/N_f$ is fixed and arbitrary in this window, the $TT$ OPE does not close [10] and $r$ is bounded by [14]

$$r \sim g^2 N_c < 8\pi^2,$$

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[3]To our present knowledge, the stronger version of this statement, i.e. its extension to open algebras, might hold also. However, this is a more difficult problem to study.

[4]We use the normalization of [13].

a relationship that assures positivity of the denominator of the NSVZ exact beta-function [15]. Therefore the closed limit $r \to \infty$ presumably does not exist in the conformal window (it is still possible, but improbable, to have $r = \infty$ for some finite value of $g^2 N_c$). The absence of a closed limit could be related to the non-integer (rational) values of $24c$ and $48a$ [5], which indeed depend on $N_c/N_f$. In the low-energy limit we have the formulas [13]

$$c = \frac{1}{16} \left( 7N_c^2 - 2 - 9\frac{N_c^4}{N_f^2} \right), \qquad a = \frac{3}{16} \left( 2N_c^2 - 1 - 3\frac{N_c^4}{N_f^2} \right). \qquad (10)$$

The $\frac{N_c^4}{N_f^2}$-contributions to $c$ and $a$ are not subleading in the large-$N_c$ limit. Presumably, an open algebra is necessary to produce non-integer values of $c$ and $a$. This problem deserves further study and is currently under investigation.

In conclusion, our picture of the moduli space of conformal field theory as a ball centred in the origin and with closed conformal field theory as a boundary works properly when supersymmetry is at least N=2 or, more generally, when the conformal field theory belongs to a one-parameter family of conformal field theories with a point at infinity, the parameter in question being the radius $r$ of the ball or, equivalently, the coupling constant $g^2 N_c$. N=1 finite families of this type are for example those studied in refs. [16].

## 6  OPE structure of closed conformal field theory.

The basic rule to determine the quantum conformal algebra of closed conformal field theory is as follows. One first studies the free-field OPE of an open conformal field theory and organizes the currents into orthogonal and mixing multiplets. Secondly, one computes the anomalous dimensions of the operators to the lowest orders in the perturbative expansion. Finally, one drops all the currents with a non-vanishing anomalous dimension.

More generically, one can postulate a set of spin-0, 1 and 2 currents, that we call $\mathcal{T}_{0,1,2}$, and study the most general OPE algebra consistent with closure and unitary.

The closed limit of the N=4 quantum conformal algebra is very simple and actually the formula of $\tilde{\mathrm{SP}}_{\mu\nu,\rho\sigma;\alpha\beta}$ that was given in [1], toghether with the value of the central charge $c = a$ encode the full closed N=4 algebra[6].

We study here the closed N=2 quantum conformal algebra for generic $c$ and $a$. The case with gauge group $SU(N_c)$ and $2N_c$ hypermultiplets in the fundamental representation is $c/a = 8/7$.

We have

$$\mathcal{T}_0(x)\,\mathcal{T}_0(y) \;=\; \frac{3}{8\pi^4}\frac{c}{|z|^4} - \frac{1}{4\pi^2}\left(3 - 4\frac{a}{c}\right)\frac{1}{|z|^2}\,\mathcal{T}_0(w) + \text{descendants} + \text{regular terms},$$

---

[5] In our normalization the free-field values of $c$ and $a$ are [13] $c = \frac{1}{24}(3N_v + N\chi)$, $a = \frac{1}{48}(9N_v + N\chi)$, where $N_v$ and $N\chi$ are the numbers of vector and chiral multiplets.

[6] This situation should be described also by the formalism of N=4 superfields [17]. Claimed by its authors to hold generically in N=4 supersymmetric Yang-Mills theory, this formally can actually be correct only in the closed limit of theories with $c = a + \mathcal{O}(1)$.

$$\mathcal{T}_\mu(x)\,\mathcal{T}_\nu(y) \;=\; \frac{c}{\pi^4}\,\pi_{\mu\nu}\left(\frac{1}{|z|^4}\right) - \frac{4}{3\pi^2}\left(1-2\frac{a}{c}\right)\mathcal{T}_0(w)\,\pi_{\mu\nu}\left(\frac{1}{|z|^2}\right)$$

$$-\frac{1}{\pi^2}\left(1-\frac{a}{c}\right)\mathcal{T}\alpha(w)\,\varepsilon_{\mu\nu\alpha\beta}\;\partial_\beta\left(\frac{1}{|z|^2}\right)$$

$$+\frac{1}{960\pi^2}\mathcal{T}_{\alpha\beta}(w){\prod}_{\mu,\nu;\alpha\beta}^{(1,1;2)}\left(|z|^2\ln|z|^2M^2\right) + \text{descendants} + \text{regular terms},$$

$$\mathcal{T}_{\mu\nu}(x)\,\mathcal{T}_{\rho\sigma}(y) \;=\; \frac{2\,c}{\pi^4}{\prod}_{\mu\nu,\rho\sigma}^{(2)}\left(\frac{1}{|z|^4}\right) - \frac{16}{3\pi^2}\left(1-\frac{a}{c}\right)\mathcal{T}_0(w){\prod}_{\mu\nu,\rho\sigma}^{(2)}\left(\frac{1}{|z|^2}\right) +$$

$$-\frac{4}{\pi^2}\left(1-\frac{a}{c}\right)\mathcal{T}\alpha(w)\;\sum_{\text{symm}}\varepsilon_{\mu\rho\alpha\beta}\;\pi_{\nu\sigma}\;\partial_\beta\left(\frac{1}{|z|^2}\right) +$$

$$-\frac{1}{\pi^2}\,\mathcal{T}_{\alpha\beta}(w)\,\tilde{\text{SP}}_{\mu\nu,\rho\sigma;\alpha\beta}\left(\frac{1}{|z|^2}\right) + \text{descendants} + \text{regular terms}, \qquad (11)$$

where $z\mu = x\mu - y\mu$, $w\mu = \frac{1}{2}(x\mu + y\mu)$ and

$$\prod_{\mu,\nu;\alpha\beta}^{(1,1;2)} \;=\; \left(4+\frac{a}{c}\right){\prod}_{\mu\nu,\alpha\beta}^{(2)} - \frac{2}{3}\left(7-5\frac{a}{c}\right)\pi_{\mu\nu}\partial\alpha\partial\beta,$$

$$\tilde{\text{SP}}_{\mu\nu,\rho\sigma;\alpha\beta}\left(\frac{1}{|z|^2}\right) \;=\; \text{SP}_{\mu\nu,\rho\sigma;\alpha\beta}\left(\frac{1}{|z|^2}\right) + \frac{1}{480}\left(102-59\frac{a}{c}\right){\prod}_{\mu\nu,\rho\sigma}^{(2)}\partial\alpha\partial\beta\left(|z|^2\ln|z|^2M^2\right)$$

$$-\frac{1}{32}\left(12-7\frac{a}{c}\right){\prod}_{\mu\nu\alpha,\rho\sigma\beta}^{(3)}\left(|z|^2\ln|z|^2M^2\right). \qquad (12)$$

The numbers appearing in these two space-time invariants are not particularly illuminating. It might be that the decomposition that we are using is not the most elegant one, but for the moment we do not have a better one to propose.

The mixed OPE's read:

$$\mathcal{T}\mu(x)\,\mathcal{T}_0(y) \;=\; \frac{1}{24\pi^2}\left(1-2\frac{a}{c}\right)\mathcal{T}\nu(w)\,\pi_{\mu\nu}\left(\ln|z|^2M^2\right) + \text{descendants} + \text{regular terms},$$

$$\mathcal{T}_{\mu\nu}(x)\,\mathcal{T}_0(y) \;=\; \frac{2}{3\pi^2}\mathcal{T}_0(w)\,\pi_{\mu\nu}\left(\frac{1}{|z|^2}\right) - \frac{1}{160\pi^2}\mathcal{T}_{\alpha\beta}(w)\left(1-\frac{a}{c}\right){\prod}_{\mu\nu,\alpha\beta}^{(2)}(|z|^2\ln|z|^2M^2)$$

$$+ \text{descendants} + \text{regular terms},$$

$$\mathcal{T}_{\mu\nu}(x)\,\mathcal{T}\rho(y) \;=\; \frac{1}{2\pi^2}\mathcal{T}\alpha(w){\prod}_{\mu\nu,\rho;\alpha}^{(2,1;1)}\left(\ln|z|^2M^2\right)$$

$$+\frac{3}{40\pi^2}\mathcal{T}_{\alpha\gamma}(w)\left(1-\frac{a}{c}\right)\sum_{\text{symm}}\varepsilon_{\mu\rho\alpha\beta}\pi_{\nu\gamma}\partial\beta(\ln|z|^2M^2)$$

$$+ \text{descendants} + \text{regular terms}, \qquad (13)$$

where

$$\prod_{\mu\nu,\rho;\alpha}^{(2,1;1)}\left(\ln|z|^2M^2\right) \;=\; (\delta_{\alpha\mu}\partial\nu\partial\rho + \delta_{\alpha\nu}\partial\mu\partial\rho + 2\delta_{\alpha\rho}\partial\mu\partial\nu - \delta_{\mu\rho}\partial\nu\partial\alpha - \delta_{\nu\rho}\partial\mu\partial\alpha)\left(\frac{1}{|z|^2}\right)$$

$$+\frac{1}{6}\left(1-2\frac{a}{c}\right)\mathcal{T}\alpha(w){\prod}_{\mu\nu,\rho\alpha}^{(2)}\left(\ln|z|^2M^2\right).$$

The $\mathcal{T}_0\,\mathcal{T}_0$ OPE closes by itself, but this fact does not appear to be particularly meaningful.

We now make some several remarks about the above algebra.

We begin by explaining how to work out (11)-(13). In a generic N=2 finite theory, we collect the hypermultiplets into a single representation $R$. The condition for finiteness is the equality of the Casimirs of $R$ and the adjoint representation: $C(G) = C(R)$. We have

$$c = \frac{1}{6}\dim G + \frac{1}{12}\dim R, \qquad a = \frac{5}{24}\dim G + \frac{1}{24}\dim R.$$

The form of the currents belonging to the $\mathcal{T}$ multiplet does not depend on the theory (i.e. on $c$ & $a$). Instead, the form of the currents $\mathcal{T}^*$ does. Let us write $\mathcal{T}_0^* = \frac{1}{2}\mathcal{J}_{0v}^S + \frac{\beta}{4}\mathcal{J}_{0m}^S$. The correlator $\langle \mathcal{T}_0 \mathcal{T}_0^* \rangle$ is proportional to $\dim G - \frac{\beta}{2}\dim R$. By definition, in the closed limit $\langle \mathcal{T}_0 \mathcal{T}_0^* \rangle = 0$, which gives

$$\beta = -2\frac{c - 2a}{5c - 4a}.$$

The scalar operator appearing in the OPE can be decomposed as

$$M^2 + N^2 + 2\bar{A}_i A_i = 2\frac{\beta - 2}{\beta + 1}\mathcal{T}_0 + \frac{6}{\beta + 1}\mathcal{T}_0^*.$$

Dropping $\mathcal{T}_0^*$, one fixes the coefficient with which $\mathcal{T}_0$ appears in the $TT$ OPE. It is proportional to $c - a$, as we see in (11). The other terms of (11) and (13) can be worked out similarly.

Let us now analyse the "critical" case $c = a$. Examples of theories with $c = a + \mathcal{O}(1)$ can easily be constructed. It is sufficient to have $\dim G = \dim R + \mathcal{O}(1)$. We do not possess the complete classification of this case.

At $c = a$ the structure simplifies enormously. The $TT$ OPE closes just with the stress-tensor, $\tilde{\text{SP}}_{\mu\nu,\rho\sigma;\alpha\beta}$ reduces to the N=4 expression [2] and the operators $\mathcal{T}_{0,1}$ behave as primary fields with respect to the stress-tensor.

The divergence of the $R$-current (which is simply $\mathcal{T}_1$ in our notation, up to a numerical factor) has an anomaly

$$\partial\mu(\sqrt{g}R\mu) = \frac{1}{24\pi^2}(c - a)R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}, \tag{14}$$

non-vanishing when $c \neq a$. The three-point function $\langle RTT \rangle$ is not zero, which means that the $TT$ OPE does contain some operator mixing with $R\mu$. This operator is just $R\mu$ in the large-$N_c$ limit, as we see in (11), but *both* $R\mu$ and $\mathcal{T}_1^*$ at generic $N_c$. On the other hand, if the theory has $c = a$ and is supersymmetric (N=1 suffices), then the above anomaly vanishes. Given that the correlator $\langle RTT \rangle$ is unique (because there is a unique space-time invariant for the spin-1 level of the OPE, see [2]), we have $\langle RTT \rangle = 0$ in such cases. In conclusion, $\mathcal{T}_1$ is kicked out of the $TT$ OPE algebra when $c = a$, even when the number of supersymmetries is less than four.

The spin-1 current $\mathcal{A}_{1v}^F + \mathcal{A}_{1m}^F$ appearing in the $TT$ OPE has to be a linear combinarion of $\Lambda_1$ and $\mathcal{T}_1^*$. A simple computation shows that this is possible only for $\mathcal{T}_1^* = -\mathcal{A}_{1v}^F - 2\mathcal{A}_{1m}^F + \mathcal{A}_{1v}^S$, which means $\beta = 2$. In turn, this implies that $\mathcal{T}_0$ is also out of the $TT$ OPE, as we have just seen. Therefore the $TT$ OPE closes just with the stress-tensor and $\tilde{\text{SP}}$ coincides precisely with the one given in [1]:

· the $c = a$ closed algebra is unique and coincides with the one of [1].

When $c \neq a$ and supersymmetry is at least N=2 the difference $c - a$ fixes the coefficients of the new terms in the OPE:

· *given c and a, there is a unique closed conformal algebra with N=2 supersymmetry.*

In the large-$N_c$ limit of the N=2 model that we have studied in section 3, $c$ and $a$ are $\mathcal{O}(N_c^2)$ and their ratio is 8/7. At finite $N_c$ (i.e. when the algebra is open) they receive different $\mathcal{O}(1)$-order corrections (see (9)). Using our algebra, we can give a very imple explanation of both effects.

The two-point function $\langle TT \rangle$ encodes only the quantity $c$. $c$ and $a$ are encoded into the three-point function $\langle TTT \rangle$, or the higher-point functions, as it is clear from the trace anomaly formula

$$\Theta = \frac{1}{16\pi^2} \left[ c(W_{\mu\nu\rho\sigma})^2 - a(\tilde{R}_{\mu\nu\rho\sigma})^2 \right],$$

and the algebra (11)-(13), in particular the space-time invariant $\tilde{\mathrm{SP}}_{\mu\nu,\rho\sigma;\alpha\beta}$ (12).

We can study the three-point function $\langle T(x)T(y)T(z) \rangle$ by taking the $x \to y$ limit and using the OPE $T(x)T(y) = \sum_n c_n(x - y)\mathcal{O}_n\left(\frac{x+y}{2}\right)$. We are lead to consider the two-point functions $\langle \mathcal{O}_n T \rangle$. Now, we know that the there are only two operators $\mathcal{O}_n$ that are not orthogonal to $T$: the stress-tensor itself *and* $\mathcal{T}^*$. $\mathcal{O}_n = T$ produces a contribution $\langle TT \rangle$, which is again $c$. This is the contribution to $a$ proportional to $c$. In the N=2 theories at finite $N_c$ there is a second contribution from $\mathcal{O}_n = \mathcal{T}^*$. Indeed, $\langle \mathcal{T}^* \mathcal{T} \rangle$ is non-vanishing and precisely $\mathcal{O}(1)$, which explains the $\mathcal{O}(1)$-difference between $c$ and $\frac{8}{7}a$. $\langle \mathcal{T}^* \mathcal{T} \rangle$ is not affected by an anomalous dimension to the second-loop order (the off-diagonal element of $a_{\mathcal{T}}^{(2)}$ vanishes), which is what we expect, since both $c$ and $a$ are radiatively uncorrected.

The reader might have noted that the operator $\mathcal{T}_1$ appearing in the $\mathcal{T}_1\mathcal{T}_1$ OPE has a coefficient proportional to $c - a$, which means that the $\langle \mathcal{T}_1\mathcal{T}_1\mathcal{T}_1 \rangle$ three-point function (which is unique by the usual arguments) is proportional to $c - a$ and not to $5a - 3c$ as in [13]. The reason in that our $R$-current is not the same as the one that is used in the N=1 context of ref. [13]: our present $R$-current is $SU(2)$-invariant, while the one of [13] is associated with a $U(1)$ subgroup of $SU(2)$.

Finally, we discuss the embedding of the N=4 open algebra of [1] into the N=2 open algebra of section 3. With the terminology "current multiplet" we have always referred to the subset of components that are generated by the $TT$ OPE. The quantum conformal algebra is embeddable into a larger set of OPE's, containing all the supersymmetric partners of the currents that we have considered so far. For example, in the N=4 theory the $R$-currents (and in particular the object called $\mathcal{T}_1$ in the N=2 frame) are not $SU(4)$ invariant and so they do not appear in the $TT$ OPE, but they appear in the superpartners of the $TT$ OPE. From the N=2 point of view, instead, the current $\mathcal{T}_1$ is $SU(2)$-invariant and indeed it does appear in the $TT$ OPE. Currents that are kicked out of the restricted algebra are of course always part of the larger web and inside that web they "move". Yet what is important is to know the *minimally* closed algebra.

Let us describe how the Konishi multiplet $\Sigma$ of [1] emerges in the N=4 case. The spin-0 current $\mathcal{T}_0^*$ coincides precisely with the spin-0 operator $\Sigma_0$ of [1]. At the spin-1 level we have, from the N=2 point of view, two operators: $\mathcal{T}_1^*$ and $\Lambda_1$ ($\mathcal{T}_1$ having been kicked out). The

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operator $\Sigma_1 = \mathcal{A}_{1v}^F + \mathcal{A}_{1m}^F$ is the linear combination of $\mathcal{T}_1^*$ and $\Lambda_1$ that does not contain the scalar current $\mathcal{A}_1^S$, forbidden in the N=4 algebra by $SU(4)$ invariance. We have $\Sigma_1 = -\frac{1}{2} \mathcal{T}_1^* + 2 \Lambda_1$.

Now, the N=2 supersymmetric algebra relates $\Sigma_1$ with $-\frac{1}{2} \mathcal{T}_2^* + 2 \Lambda_2$, which, however, is not $\Sigma_2$ and is not $SU(4)$-invariant. Actually, one has

$$\Sigma_2 = -\frac{7}{20} \mathcal{T}_2^* + \frac{7}{3} \Lambda_2 - 2 \Xi_2^*, \tag{15}$$

where $\Xi_2^* = \Xi_{2v} + \Xi_{2m}$ is orthogonal to the N=4 operator $\Xi_2 = 5 \Xi_{2v} - \frac{20}{3} \Xi_{2m}$. The N=2 supersymmetric transformation does not generate directly $\Sigma_2$. $\Sigma_2$ is recovered after a suitable $SU(4)$-invariant reordering of the currents. The meaning of this fact is that the quantum conformal algebra admits various different N=2 (and N=1) "fibrations", depending of the subgroup of $SU(4)$ that one preserves, and that all of these fibrations are meaningful at arbitrary $g$.

## 7  Conclusions

In this paper we have analysed the quantum conformal algebras of N=2 supersymmetric theories, focusing in particular on finite theories. Several novel features arise with respect to the quantum conformal algebra of [1] and each of them has a nice interpretation in the context of our approach. Known and new properties of supersymmetric theories, conformal or not, are elegantly grouped by our formal structure and descend from a general and unique mathematical notion.

Stimulated by the results of our analysis, we have introduced a novel class of conformal field theories in dimension higher than two, which have a closed quantum conformal algebra. The definition is completely general, valid in any dimension. For example, analogous considerations apply to three dimensional conformal field theory and might be relevant for several problems in condensed matter physics. Closed conformal field theory is nicely tractable from the formal/axiomatic point of view. On the other hand, it is interesting to identify the physical situations or limits of ordinary theories that it describes. Various considerations suggest that the closed algebra coincides in general with the strongly coupled large-$N_c$ limit of an open algebra. In some cases, like N=1 supersymmetric QCD as well as ordinary QCD, the identification of the closed limit, if any, is more subtle and still unclear.

In a closed conformal field theory the quantities $c$ and $a$ are always proportional to each other, but the proportionality factor depends on the theory, i.e. it feels the structure of the quantum conformal algebra. Closed conformal field theory is described uniquely by these two central charges. We have worked out the OPE algebra in detail and related it to known anomaly formulas. The value of $c$ is given by the two-point functions, while the value of $a$ is given by the three-point functions, i.e. by the structure of the algebra itself. There is a critical case, $c = a$, in which the algebra admits a closed subalgebra containing only the stress-tensor.

An open conformal field theory presents a richer set of phenomena. In particular, some operators can mix under renormalization with the stress-tensor and the main effect of this mixing is that $c$ and $a$ are not just proportional.

We think that closed conformal field theory is now ready for a purely algebraic study, i.e. mode expansion, classification of unitary representations and so on. In view of the applications, we believe that it is worth to proceed in this direction.

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