QUANTUM TOPOLOGICAL INVARIANTS, GRAVITATIONAL INSTANTONS AND THE TOPOLOGICAL EMBEDDING

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Abstract

Certain topological invariants of the moduli space of gravitational instantons are defined and studied. Several amplitudes of two and four dimensional topological gravity are computed. A notion of puncture in four dimensions, that is particularly meaningful in the class of Weyl instantons, is introduced. The *topological embedding*, a theoretical framework for constructing physical amplitudes that are well-defined order by order in perturbation theory around instantons, is explicitly applied to the computation of the correlation functions of Dirac fermions in a punctured gravitational background, as well as to the most general QED and QCD amplitude. Various alternatives are worked out, discussed and compared. The quantum background affects the propagation by generating a certain effective "quantum" metric. The topological embedding could represent a new chapter of quantum field theory.

1 Introduction and motivation

Given a manifold or, in general, a field configuration, one can define topological quantities like the Pontrjiagin number and the Euler number. In quantum field theory, one mainly deals with spaces of field configurations, rather than single field configurations. Consequently, it can be interesting to study topological invariants of such spaces. These invariants were called *quantum* in ref. [1], since they involve an integration over the chosen configuration space. The usual topological invariants were called *classical*. The quantum topological invariants are defined in a way that is originally suggested by topological field theory, if treated with the approach of ref. [2], but that actually live quite independently. No notion of functional integral is strictly necessary, so that, from the mathematical point of view, every formula is rigorously well-defined.

Among the spaces of field configurations, special interest has to be devoted to the space of instantons, or, in general, the minima of the action of a physical model. The quantum topological quantities are naturally associated with "measures" over the moduli space and such measures can be physically relevant. Following the ideas developed in ref.s [3, 1] (collected under the name of *topological embedding*), these peculiar measures can be useful to define perturbation theory around instantons, bypassing convergence problems with the moduli space integration [4]. In other words, new amplitudes in quantum field theory can be constructed and proven to be well-defined order by order in perturbation theory around instantons. They receive contributions only from a specific topological sector (the one to which the instanton belongs) and could lead to qualitatively new physical predictions.

In this paper, I investigate these issues in topological and quantum gravity. With the new method I recover known amplitudes in 2-D topological gravity and compute new ones. Then, I define and study punctures in four dimensions and compute their characteristic quantum topological invariants. Finally, I apply the topological embedding procedure to the computation of various physical (i.e. non-topological) correlation functions, illustrating many aspects of the arguments of [3, 1]. I consider first the two-point function of a fermion in a punctured background, both in two and four dimensions, and then the most general QED and QCD amplitude. The main effect of the quantum background on the propagation seems to be the generation of an effective metric, that we can call the "quantum" metric of the problem. Several different quantum backgrounds are analysed and compared.

2 Punctures in two dimensions

On a two dimensional plane, let us consider metrics of the form

$$e^a = \left(1 + \sum_{i=1}^{n} \frac{\rho_i^{2p}}{(x - x_i)^{2p}}\right)^{\frac{1}{2p}} \delta_\mu^a \mathrm{d}x^\mu \equiv D^{\frac{1}{2p}} \delta_\mu^a \mathrm{d}x^\mu \equiv \mathrm{e}^{\frac{1}{2p}\varphi} \delta_\mu^a \mathrm{d}x^\mu. \qquad (2.1)$$

The behaviour of the metric around x_i is

$$e^a \simeq \frac{\rho_i}{|x - x_i|} \mathrm{d}x^a. \qquad (2.2)$$

The points x_i are *punctures* on the plane, i.e. points that are sent to infinity. The scales ρ_i specify, in some sense, the "size" of the puncture. Here they are just external parameters (no topological quantity depends on them). Later on we shall discuss cases in which the ρ_i are treated as moduli, on the same footing as the positions x_i of the punctures. The integer number p is introduced for convenience. The topological quantities are independent of p, since they are sensitive only to the "singularities" (2.2). Unless specified, we use $p = 1$ in this section. We have

$$\omega = -\frac{1}{2D}\mathrm{d}x^\mu \varepsilon_{\mu\nu}\partial_\nu D, \qquad R = \mathrm{d}\omega, \qquad \frac{1}{2\pi}\int_{\mathbb{R}^2} R = n.$$

The singularity of the spin-connection ω is a gauge artifact. Indeed, the curvature $R = d\omega$ is everywhere regular. This means that the metrics that we are considering are physically meaningful. The *classical* topological invariant $\frac{1}{2\pi}\int_{\mathbb{R}^2} R$ counts the number of punctures.

The topological field theory of gravity with the metrics (2.1) (see [5] for the formal construction of the theory) can be solved explicitly using the method of ref. [2], combined with some remarks made in ref. [1]. There is no need to recall the procedure here, since it was extensively discussed in ref.s [2, 3, 1]. One finds that the diffeomorphism ghosts c^a are

$$c^a = -\frac{1}{D^{\frac{1}{2}}}\sum_{i=1}^n \frac{\rho_i^2 \mathrm{d}x_i^a}{(x-x_i)^2}, \qquad \hat{e}^a \equiv e^a + c^a = \frac{1}{D^{\frac{1}{2}}}\left[\mathrm{d}x^a + \sum_{i=1}^n \frac{\rho_i^2 \mathrm{d}(x-x_i)^a}{(x-x_i)^2}\right]. \qquad (2.3)$$

As explained in sect. 2 of ref. [2], everything follows from the expression of this ghost. We allow the singularity in c^a because we are allowing the same behaviour for e^a. However, the solution to the topological field theory makes sense, since the components of the BRST extended curvatures are all regular. The Lorentz ghosts c^{ab} are given by

$$c^{ab} = \mathcal{D}^{[a}c^{b]}, \qquad \hat{\omega}^{ab} \equiv \varepsilon^{ab}\hat{\omega} = \omega^{ab} + c^{ab} = \varepsilon^{ab}\frac{1}{D}\sum_{i=1}^n \frac{\rho_i^2 \mathrm{d}(x-x_i)^\mu \varepsilon_{\mu\nu}(x-x_i)^\nu}{(x-x_i)^4} \qquad (2.4)$$

One can easily check that the first descendant of the torsion, called ψ^a in ref. [5], is indeed regular:

$$\psi^a = -\mathcal{D}^{\{a}c^{b\}}e^b + \frac{1}{2}s\varphi\, e^a, \qquad s = \sum_{i=1}^n \mathrm{d}x_i^\mu \frac{\partial}{\partial x_i^\mu}, \qquad (2.5)$$

s being the exterior derivative on the moduli space. The BRST extended curvature, which is the basic ingredient in the construction of the observables, is $(\hat{\mathrm{d}} = \mathrm{d} + s)$

$$\hat{R} = \hat{\mathrm{d}}\hat{\omega} = \frac{2}{D^2}\left[\sum_{i=1}^n \frac{\rho_i^2 \mathrm{d}^2(x-x_i)}{(x-x_i)^4}\left(D - \frac{\rho_i^2}{(x-x_i)^2}\right) + \right.$$

$$\left.\sum_{i\neq j}\frac{\rho_i^2 \rho_j^2}{(x-x_i)^4(x-x_i)^4}(x-x_i)\cdot\mathrm{d}(x-x_i)\,\mathrm{d}(x-x_j)^\mu \varepsilon_{\mu\nu}(x-x_j)^\nu\right]. \qquad (2.6)$$

The observables are

$$\mathcal{O}_\gamma^{(d)} = \frac{1}{(2\pi)^d}\int_\gamma \hat{R}^d,$$

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where γ can also be a point, in which case the observable is denoted by $\mathcal{O}^{(d)}(x)$ (simply $\mathcal{O}(x)$ for $d = 1$).

The correlator $< \prod_{j=1}^{n} \mathcal{O}(y_j) >= \frac{1}{n!}\mathcal{A}_n$ with local observables placed in distinct points is equal to one. This can be proved as follows. Let us write $\mathcal{O}(y_n)$ as $\frac{1}{2}\varepsilon_{ab}sc^{ab}(y_n)$ and integrate by parts on the moduli space. The singularities of c^{ab} in x_i are responsible for the nonvanishing contributions. Each puncture contributes the same, so that we have n times a reduced amplitude $\frac{1}{n!}\mathcal{A}_{n-1}$ in which one modulus, say x_n, disappears. In $\mathcal{A}_{n-1}$ one has to set $dx_n = 0$ and take the limit $y_n \to x_n$. Before taking this limit one has to note that $\frac{1}{(n-1)!}\mathcal{A}_{n-1}$ is the same as $< \prod_{j=1}^{n-1} \mathcal{O}(y_j) >$ with the replacements $\rho_i^2 \to \frac{\rho_i^2}{1+\rho_n^2/(y_n-x_n)^2}$, $i = 1, \ldots n - 1$. This can be proven by direct inspection of (2.6). After repeating the argument $n - 1$ times, one arrives at an expression $\mathcal{A}_1$, which is the same as $< \mathcal{O}(y_1) >$ with $\rho_1^2 \to \frac{\rho_1^2}{\prod_{i=2}^{n}(1+\rho_i^2/(y_i-x_i)^2)}$. $\mathcal{A}_1$ is equal to one, whatever ρ_1 is. At this point the limits $y_i \to x_i$, $i = 2, \ldots n$ are all trivial and the final result is also one, as claimed.

When there are observables placed in coincident points, the above procedure cannot be applied and we have to do the computation in a different way. A long algebraic manipulation similar to the ones described in detail in ref. [1][1] gives

$$\hat{R}^n = \frac{2^n n!}{D^{n+1}} \prod_{i=1}^{n} \frac{\rho_i^2 \mathrm{d}^2(x - x_i)}{(x - x_i)^4}. \tag{2.7}$$

The correlation functions $< \prod_{j=1}^{k} \mathcal{O}^{(d_j)}(y_j) >$ can be computed using the cluster property. The independence of the points y_j assures that we can take the limits $|y_j - y_k| \to \infty$ $\forall j \neq k$. This shows that

$$< \prod_{j=1}^{k} \mathcal{O}(y_j)^{d_j} >= \prod_{j=1}^{k} < \mathcal{O}(y_j)^{d_j} > .$$

Formula (2.7) gives

$$< [\mathcal{O}^{(n)}(x)] >= \frac{1}{(2\pi)^n n!} \int \frac{2^n n!}{D^{n+1}} \prod_{i=1}^{n} \frac{\rho_i^2 \mathrm{d}^2 x_i}{(x - x_i)^4} = \frac{1}{n!}, \tag{2.8}$$

so that

$$< \prod_{j=1}^{k} \mathcal{O}(y_j)^{d_j} >= \prod_{j=1}^{k} \frac{1}{d_j!}. \tag{2.9}$$

These are the (known) correlation functions of 2D topological gravity [6], here computed on the plane, instead of the sphere. To recover the ones on the sphere, one has to remember that three "hidden" punctures are not visible in our approach, since they just do the job of fixing the gauge of the ghosts. Moreover, the usual normalization is to multiply the amplitude by the symmetry factor $n!$.

Before proceeding, let us make a couple of comments. The moduli space has diagonal subspaces Δ where the positions of the punctures coincide. Discarding Δ by saying that it is of

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[1] In particular, one has to use formulæ (4.24) and (4.26) of that paper.

vanishing measure could be too naive, since the measure could be singular on Δ (the measure is a descendant of the Euler characteritic, which jumps on Δ). Actually, we did not find any problem in our computation and this can be interpreted as follows. The jump of the Euler characteristic is transferred, at the quantum level, to a jump of the correlation functions, that takes place when the positions of the local observables coincide. In each correlation function Δ is of vanishing measure, but there are many different correlation functions to consider, i.e. many measures. One can obtain some topological information about Δ by comparing the values for different sets of coincident points. In particular, we have learnt that when n points come to coincide, there is a jump of $1/n!$. I argue that this is a completely general feature of punctures (and of the centers of Yang-Mills instantons, as we shall see), because the same thing happens in four dimensions (see section 3). In the case of vortices, on the other hand, there is no jump at all [1]. Exotic values of the jumps will be also found in the next section.

Now, let us study some correlation functions with nonlocal observables.

I consider the plane with one puncture and a correlation function $< \mathcal{O}_{\gamma_1} \cdot \mathcal{O}_{\gamma_2} >$, γ_1 and γ_2 being two curves. The explicit evaluation gives

$$< \mathcal{O}_{\gamma_1} \cdot \mathcal{O}_{\gamma_2} > = \frac{\rho^4}{\pi^2} \int_{\gamma_1} \int_{\gamma_2} \varepsilon_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}y^\nu \int \frac{\mathrm{d}^2 x_0}{(\rho^2 + (x - x_0)^2)^2 (\rho^2 + (y - x_0)^2)^2}. \qquad (2.10)$$

To understand what this means, it can be useful to take the limit $\rho \to 0$, where one gets

$$< \mathcal{O}_{\gamma_1} \cdot \mathcal{O}_{\gamma_2} > = \int_{\gamma_1} \int_{\gamma_2} \varepsilon_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}y^\nu \delta(x - y).$$

We see that this amplitude counts the number of intersections of γ_1 and γ_2 with signs. Roughly speaking, this is similar to the link numbers of [2, 3]. Indeed, in the multilink interpretation offered in [3], the importance of intersections among the γ's was apparent. Borrowing the graphical notation from [3], we shall denote the intersection of the two curves with $\lambdabar(\gamma_1, \gamma_2)$. One the sphere or the plane, this number is always zero, if γ_1 and γ_2 are both compact. On higher genus Riemann surfaces it is not so. Later on we shall exhibit the full set of amplitudes of this kind for any genus and and arbitrary (even) number of curves. With Yang-Mills instantons [2, 3], on the other hand, there is one more modulus (the scale) and one more element in the amplitude (the Chern-Simons form), so that it is not necessary to have a genus in order to produce a nontrivial linkage [2]. Something similar will be observed below.

With more nonlocal observables, one can consider a multiple amplitude. Straightforward manipulations allow us to show that

$$< \prod_{i=1}^{2n} \mathcal{O}_{\gamma_i} > = \sum_{\text{cyclic perm of } \{2,...2n\}} \lambdabar(\gamma_1, \gamma_2) \cdots \lambdabar(\gamma_{2n-1}, \gamma_{2n}). \qquad (2.11)$$

The fact that the multiple amplitude can be written purely in terms of the two-amplitude suggests that $\lambdabar(\gamma_1, \gamma_2)$ is in some sense the topological "propagator" between curves. Then, (2.11) is a "tree-level" amplitude. Similar phenomena take place in higher dimensions.

Finally, let us analyse what happens when treating the scales ρ_i as true moduli. (2.3) and (2.4) are unchanged, while in (2.5) the moduli space derivative s has to be replaced by

$$s = \sum_{i=1}^{n} \mathrm{d}x_i^{\mu} \frac{\partial}{\partial x_i^{\mu}} + \mathrm{d}\rho_i^2 \frac{\partial}{\partial \rho_i^2}.$$

(2.6) has to be modified consequently. It is easy to check that the quantum topological invariants are link numbers, similarly to the case worked out in ref.s [2, 3]. To be concrete, let us take $n = 1$, $p = 1$ and the observable $< \mathcal{O}_{\gamma} \cdot \mathcal{O}(x) >$. The computation can be done writing $\mathcal{O}(x)$ as $\frac{1}{2}\varepsilon_{ab}sc^{ab}$ and integrating by parts, or by direct inspection of the integral below, which I write down explicitly for later use in connection with the topological embedding. We have

$$< \mathcal{O}_{\gamma} \cdot \mathcal{O}(x) > = \frac{1}{4\pi^2} \int_{\gamma} \mathrm{d}y^{\mu} \varepsilon_{\mu\nu}(y-x)^{\nu} \int \frac{2\rho^2 \, \mathrm{d}\rho^2 \, \mathrm{d}^2 x_0}{(\rho^2 + (y-x_0)^2)^2(\rho^2 + (x-x_0)^2)^2} = \chi(\gamma, \{x\}),$$

$$(2.12)$$

where $\chi(\gamma, \{x\}) = \frac{1}{2\pi} \int_{\gamma} \mathrm{d}y^{\mu} \varepsilon_{\mu\nu}(y-x)^{\nu}/(x-y)^2$.

If one is interested in the punctured torus, one can use the metrics

$$e^1 = (\mathrm{d}\xi + \tau_1 \, \mathrm{d}\eta) \, \mathrm{e}^{\frac{1}{2p}\varphi}, \qquad e^2 = \tau_2 \, \mathrm{d}\eta \, \mathrm{e}^{\frac{1}{2p}\varphi},$$

$$\varphi = \ln \left(1 + \sum_{i=1}^{n} \rho_i^{2p} F_p(x - x_i) \right), \tag{2.13}$$

$$F_p(\lambda) = \sum_{k,l} \frac{1}{(\lambda + k\hat{1} + l\tau)^{2p}}.$$

Here $\xi, \eta \in [0, 1]$, $x = (\xi + \tau_1\eta, \tau_2\eta)$, $\hat{1} = (1, 0)$, $\tau = (\tau_1, \tau_2)$, $x_i = (\xi_i + \tau_1\eta_i, \tau_2\eta_i)$, $\xi_i, \eta_i \in [0, 1]$. τ is the modulus of the torus, (ξ_i, η_i) are the positions of the punctures. The convergence of the sum in F_p, in which k and l range over the integers, requires now $p > 1$. In the usual conventions, the above metric describes the torus with $n + 1$ punctures. The extra puncture, that fixes the translations, is not visible in our approach.

The case $n = 0$ has been discussed in sect. of [2], the local observable being the Poincarè metric

$$\mathrm{Pm} = \frac{\mathrm{d}^2 \tau}{(\tau - \bar{\tau})^2}$$

and the amplitude being the volume of the moduli space (equal to $1/24$). Pm is a good observable in presence of any number of punctures. It corresponds to the top Chern class of the Hodge bundle and can be brought into the game *via* a mechanism described in [7]. The local observable is, instead equal to the Poincaré metric plus a certain remnant.

Explicit computations are much more involved, now. A simple case is given by the amplitudes

$$\chi(\gamma_1, \ldots, \gamma_{2n}) = < \mathrm{Pm} \cdot \prod_{j=1}^{2n} \mathcal{O}_{\gamma_j} > .$$

that are equal to (2.11) times a numerical factor, now on the torus instead of the sphere. Similar considerations easily extend to higher genera.

3 Punctures in four dimensions

In four dimensions, on $\mathbb{R}^4$, we consider the conformally flat metrics

$$e^a = \left(1 + \sum_{i=1}^{n} \frac{\rho_i^{2p}}{(x - x_i)^{2p}}\right)^{\frac{1}{p}} \delta_\mu^a \mathrm{d}x^\mu \equiv \mathrm{e}^{\frac{1}{p}\varphi} \delta_\mu^a \mathrm{d}x^\mu. \tag{3.1}$$

For any p these metrics are in the same topological class (their topological invariants are the same). The Einstein action is finite (zero in the special case $p = 1$), the Weyl action is zero. From the computative point of view, $p = 2$ will be the most convenient choice. The metrics with $p = 1$ are discussed in ref. [8] from a classical point of view.

The singularity that we allow around a puncture x_i is now

$$e^a \simeq \frac{\rho_i^2}{(x - x_i)^2} \mathrm{d}x^a. \tag{3.2}$$

While the two-dimensional puncture (2.2) describes a cylinder, as one can see after the change of coordinates $|x - x_i| \to \mathrm{e}^{-|x - x_i|}$, the four-dimensional punture (3.2) corresponds to an $\mathbb{R}^4$, as it is clearly visible after the inversion $(x - x_i)_\mu \to (x - x_i)_\mu \rho_i^2 / (x - x_i)^2$.

The definition (3.1) will be justified by the results that we shall obtain. Moreover, it is such that the statement "$\mathbb{R}^4$ is S^4 with one puncture at infinity" is literally correct. Indeed, let us write the metric of S^4 as $\mathrm{d}s^2 = \mathrm{d}x^2/(1 + x^2/R^2)^2$. We perform an inversion $x^\mu \to \frac{R^2}{x^2} x^\mu$ in order to exchange the point at infinity with zero. Then we place a puncture in the origin, multiplying the metric by a conformal factor of the type (3.1) with $p = 1$, precisely $\left(1 + \frac{R^2}{x^2}\right)^2$. Finally, we undo the inversion. At the end we get the metric $\mathrm{d}s^2 = \mathrm{d}x^2$ of $\mathbb{R}^4$, as desired.

A conformal factor like the one in (3.1) "puncturizes" a Weyl instanton while remaining in the class of Weyl instantons. The process of "puncturization" encoded in (3.1) is completely general. Given a manifold M, we denote by pM the punctured version of the same. When we want to specify the number of punctures, we write pM_n. If the topological invariants of M are (χ, σ), then those of pM_n are $(\chi - 2n, \sigma)$. For example, interesting Weyl instantons with punctures are the pALE (punctured Asymptotically Locally Euclidean) manifolds, p$K3$, pT^4, pCP^2, pS^4, etc. The topological properties that will be derived for p$\mathbb{R}^4$ are quite general and hold for any punctured manifold pM.

The hyperKähler character of a gravitational instanton, instead, in *not* preserved by the puncturization process. I have not found, yet, a definition of puncture that does this job. It would be interesting to know if it exists.

Coming back to p$\mathbb{R}^4$, we have that the spin connection ω^{ab} is

$$\omega^{ab} = -\frac{1}{p}\mathrm{d}x^a \partial_b \varphi + \frac{1}{p}\mathrm{d}x^b \partial_a \varphi.$$

The curvature $R^{ab} = d\omega^{ab} - \omega^{ac}\omega^{cb}$ is regular. The Pontrijagin number vanishes, while the

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Euler number counts the number of punctures[2]:

$$-\frac{1}{64\pi^2}\int_{\mathbb{R}^4} R^{ab}\wedge R^{cd}\varepsilon_{abcd} = n, \qquad \int_{\mathbb{R}^4} R^{ab}\wedge R^{ab} = 0. \tag{3.3}$$

Now, let us solve the topological field theory of gravity [5] with the metrics (3.1), following the general recipe of ref.s [2, 1]. The ghosts c^a of diffeomorphisms are

$$c^a = -\frac{1}{D^{\frac{p-1}{p}}}\sum_{i=1}^n \frac{\rho_i^{2p}\mathrm{d}x_i^a}{(x-x_i)^{2p}}, \qquad \hat{e}^a \equiv e^a + c^a = \frac{1}{D^{\frac{p-1}{p}}}\left[\mathrm{d}x^a + \sum_{i=1}^n \frac{\rho_i^{2p}\mathrm{d}(x-x_i)^a}{(x-x_i)^{2p}}\right]. \tag{3.4}$$

The expressions of the other fields greatly simplify for $p = 2$, which is the value that we shall use from now on. The Lorentz ghosts c^{ab} are again given by the first expression of formula (2.4), so that

$$\hat{\omega}^{ab} = -\frac{2}{D}\sum_{i=1}^n \frac{\rho_i^4}{(x-x_i)^6}[(x-x_i)^a\mathrm{d}(x-x_i)^b - (x-x_i)^b\mathrm{d}(x-x_i)^a].$$

The first BRST descendant of the torsion ψ^a, again given by (2.5), is regular. Using the notation $\hat{x}_i = x - x_i$ in order to compress the formula, the BRST extended curvature $\hat{R}^{ab} = \hat{d}\hat{\omega}^{ab} - \hat{\omega}^{ac}\hat{\omega}^{cb}$ is given by

$$\hat{R}^{ab} = -\frac{4}{D^2}\left[\sum_{i=1}^n \frac{\rho_i^4}{\hat{x}_i^6}\left(D - \frac{\rho_i^4}{\hat{x}_i^4}\right)\left(\mathrm{d}\hat{x}_i^a\mathrm{d}\hat{x}_i^b - 6\frac{\hat{x}_i\cdot\mathrm{d}\hat{x}_i}{\hat{x}_i^2}\hat{x}_i^{[a}\mathrm{d}\hat{x}_i^{b]}\right)\right.$$
$$\left.+\sum_{i\neq j}\frac{\rho_i^4\rho_j^4}{\hat{x}_i^6\hat{x}_j^6}\left(4\hat{x}_i\cdot\mathrm{d}\hat{x}_i\,\hat{x}_j^{[a}\mathrm{d}\hat{x}_j^{b]} - \hat{x}_i^a\hat{x}_j^b\,\mathrm{d}\hat{x}_i\cdot\mathrm{d}\hat{x}_j - \hat{x}_i\cdot\hat{x}_j\,\mathrm{d}\hat{x}_i^a\mathrm{d}\hat{x}_j^b - 2\hat{x}_j\cdot\mathrm{d}\hat{x}_i\,\hat{x}_i^{[a}\mathrm{d}\hat{x}_j^{b]}\right)\right].$$

The observables are

$$\mathcal{O}_\gamma = -\frac{1}{64\pi^2}\int_\gamma \hat{R}^{ab}\wedge \hat{R}^{cd}\varepsilon_{abcd} \equiv \int_\gamma \hat{\mathcal{Q}}(x).$$

The correlator $< \prod_{j=1}^n \mathcal{O}(y_j) >$, $y_j \neq y_k$ for $j \neq k$, is equal to one. This can be proved exactly like in the two-dimensional case, by a sequence of partial integrations and rescalings of the ρ_i's. The detailed derivation is left to the reader. When there are coincident points, there should exist a simple formula generalizing (2.7), but it seems very difficult to work it out. In the case $n = 2$, a long work done with Mathematica gives

$$\left[\hat{\mathcal{Q}}(x)\right]^2 = \frac{384}{\pi^4}\frac{\rho_1^8\rho_2^8 d^4\hat{x}_1 d^4\hat{x}_2}{D^7\hat{x}_1^{12}\hat{x}_2^{12}}\left(D + \frac{\rho_1^4\rho_2^4}{2\hat{x}_1^4\hat{x}_2^4} - (\hat{x}_1\cdot\hat{x}_2)^2\frac{\rho_1^4\rho_2^4}{2\hat{x}_1^6\hat{x}_2^6}\right).$$

This allows us to compute the correlation function with two local observables placed in the same point, which turns out to be (putting $\rho_1 = \rho_2 = 1$)

$$< [\mathcal{O}(x)]^2 > = \frac{1}{2!}\int \frac{384}{\pi^4}\frac{\mathrm{d}^4x_1\mathrm{d}^4x_2}{D^7\hat{x}_1^{12}\hat{x}_2^{12}}\left(D + \frac{1}{2\hat{x}_1^4\hat{x}_2^4} - \frac{(\hat{x}_1\cdot\hat{x}_2)^2}{2\hat{x}_1^6\hat{x}_2^6}\right) = \frac{1}{2}. \tag{3.5}$$

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[2] In the standard normalization, $\chi = -2n$ and $\sigma = 0$. However, from the point of view of topological field theory it is convenient to normalize the invariants like in (3.3).

We see that the kind of jump that characterizes punctures in four dimensions is the same as in two, as we wanted to show. Very presumably, formula (2.9) also holds.

The Pontrijagin number of our metrics vanishes. However, this is not sufficient to say that all the related quantum topological observables

$$\tilde{\mathcal{O}}_\gamma^{(d)} = -\frac{1}{(32\pi^2)^d} \int_\gamma \text{tr}[\hat{R}^{2d}] \tag{3.6}$$

give vanishing correlation functions. $< \tilde{\mathcal{O}}^{(1)}(x) >$ trivially vanishes. One can explicitly check that $< \tilde{\mathcal{O}}^{(1)}(x)\tilde{\mathcal{O}}^{(1)}(y) >$ also vanishes. When $x \neq y$, this is straightforward. When $x = y$ similar algebraic manipulations as above give an expression proportional to

$$\int \frac{\text{d}^4 x_1 \text{d}^4 x_2}{D^7 \hat{x}_1^{12} \hat{x}_2^{12}} \left(D - \frac{2}{\hat{x}_1^4 \hat{x}_2^4} + \frac{2(\hat{x}_1 \cdot \hat{x}_2)^2}{\hat{x}_1^6 \hat{x}_2^6} \right) = 0. \tag{3.7}$$

However, due to the identity $\left[\mathcal{O}^{(1)}(x) \right]^2 = 2 \left[\tilde{\mathcal{O}}^{(1)}(x) \right]^2 + 4\tilde{\mathcal{O}}^{(2)}(x)$, $\tilde{\mathcal{O}}^{(2)}(x)$ is related to a couple of $\mathcal{O}^{(1)}(x)$ at coincident points, so that we have

$$< \tilde{\mathcal{O}}^{(2)}(x) > = \frac{1}{8}. \tag{3.8}$$

This means that Pontrjiagin and Euler numbers mix at the quantum level, although they are classically distinguished. This mixing does not violate parity conservation, since it takes place only with even powers.

It can also be checked that $\mathcal{O}(x)\tilde{\mathcal{O}}(x) \equiv 0$. In particular, decomposing the Riemann curvature R^{ab} into the self-dual and anti-self-dual components, $R^{ab} = R^{+ab} + R^{-ab}$, one can write $< \mathcal{O}(x)\tilde{\mathcal{O}}(x) > \propto < \text{tr}[(\hat{R}^+)^2]^2 - \text{tr}[(\hat{R}^-)^2]^2 > = 0$. This equality, toghether with, (3.5) and (3.7), implies that $\frac{1}{(16\pi^2)^2} < \text{tr}[(\hat{R}^\pm)^2]^2 > = \frac{1}{2}$. We point out this fact for the following reason. There is a deep relationship between punctures and the centers of Yang-Mills instantons. Let us consider (3.1) for $p = 1$. It can be easily checked that R^{+ab} turns out to coincide with the field strength of the 't Hooft $SU(2)$ Yang-Mills instantons with instanton number n [9], while R^{-ab} is the "conjugate" of the same, namely the 't Hooft solution with instanton number $-n$. Instantons and anti-instantons are placed in the same points, so that we can say that a puncture can be viewed as an instanton-anti-instanton pair[3]. We have just shown that $\frac{1}{(16\pi^2)^2} < \text{tr}[(\hat{R}^\pm)^2]^2(x) > = \frac{1}{2}$, while $\frac{1}{(16\pi^2)^2} < \text{tr}[(\hat{R}^\pm)^2](x) \, \text{tr}[(\hat{R}^\pm)^2](y) > = 1$, for $x \neq y$. This result has been obtained for $p = 2$, but it is independent of p, as we know. We conclude that the elementary jump that characterizes punctures characterizes the centers of Yang-Mills instantons also.

Similarly to the two-dimensional case, we can consider amplitudes associated with non-local observables. For example, in the case with one-puncture, we get

$$< \mathcal{O}_{\gamma_3} \cdot \mathcal{O}_{\gamma_1} > = \frac{24}{\pi^4} \int_{\gamma_3} \int_{\gamma_1} \varepsilon_{\mu\nu\rho\sigma} \text{d}x^\mu \text{d}x^\nu \text{d}x^\rho \text{d}y^\sigma \int \frac{\rho^{16}(x - x_0)^4 (y - x_0)^4 \, \text{d}^4 x_0}{(\rho^4 + (x - x_0)^4)^4 (\rho^4 + (y - x_0)^4)^4},$$

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[3] This explains also why $\chi = -2n$: a contribution $-n$ comes from instantons and another contribution $-n$ comes from anti-instantons. These two contributions cancel in the Pontrjiagin number: $\sigma = -n + n = 0$.

γ_3 and γ_1 here denoting three- and one-dimensional closed submanifolds (eventually not compact). A clearer representation of the above correlation function is obtained by taking the limit $\rho \to 0$ and using the property $\lim_{\rho \to 0} \frac{\rho^8 x^4}{(\rho^4 + x^4)^4} = \frac{\pi^2}{2} \delta^{(4)}(x)$:

$$< \mathcal{O}_{\gamma_3} \cdot \mathcal{O}_{\gamma_1} > = \frac{1}{6} \int_{\gamma_3} \int_{\gamma_1} \varepsilon_{\mu\nu\rho\sigma} \mathrm{d}x^\mu \mathrm{d}x^\nu \mathrm{d}x^\rho \mathrm{d}y^\sigma \, \delta(x-y) = \lambdabar(\gamma_3, \gamma_1).$$

So, there is a topological propagator between pairs of closed submanifolds whose dimensions sum up to four. The above correlation function is more meaningful on a topologically nontrivial manifold, like $K3$, T^4, CP^2. Using a procedure similar to the one used in section 2 for T^2, we can produce the above correlation function on any punctured manifold pM, after inserting an additional top-form of the moduli space of M. On T^4 formulæ generalizing (2.13) can be easily written down.

When the ρ_i are included in the set of the moduli, the quantum topological invariants are link numbers, again. With one puncture, one recovers exactly the multilink correlation functions of [2, 3]. For example,

$$< \mathcal{O}_{\gamma_3} \cdot \mathcal{O}(x) > = \lambdabar(\gamma_3, \{x\}).$$

With more punctures, the problem is equivalent to the same problem with 't Hooft multi-instantons [9], i.e. $SU(2)$ Yang-Mills instantons in which one keeps the gauge moduli fixed and studies only positions and scales.

Before closing this section, I would like to comment about the non uniqueness of the four dimensional puntures (3.2), and the uniqueness of the two dimentional punctures (2.1). To this purpose, let us modify (3.1) into

$$e^a = \left(1 + \sum_{i=1}^{n} \frac{\rho_i^{2p}}{(x - x_i)^{2p}}\right)^{\frac{\alpha}{2p}} \delta_\mu^a \mathrm{d}x^\mu. \tag{3.9}$$

Then the singularity (3.2) becomes $e^a \simeq \frac{\rho_i^\alpha}{|x-x_i|^\alpha} \mathrm{d}x^a$. For $\alpha = 1$ the punctures are of the kind $\mathbb{R} \times S^3$ rather than $\mathbb{R}^4$. It would seem, at first, that such punctures are the most natural candidates to generalize the two-dimensional cylindrical punctures, instead of the ones that we have used so far. This, however, is incorrect. Indeed, if we modify (2.1) like (3.9), we see that a generic α does not produce any change in the quantum topological invariants. In particular, one observes the same kind of jumps as before. Nevertheless, (3.9) do not share this universality property. Following the general recipe, one can easily solve the topological field theory with (3.9). I shall not repeat the derivation here, leaving it to the reader. The normalization (3.3) is now replaced by

$$-\frac{1}{16\pi^2 \alpha^2 (3 - \alpha)} \int_{\mathbb{R}^4} R^{ab} \wedge R^{cd} \varepsilon_{abcd} = n.$$

The technical simplifications occur for $\alpha = p$. Then, the punctures of type $\mathbb{R} \times S^3$ correspond to $p = 1$. The characteristic jump (3.5) can be easily computed to be replaced by

$$< [\mathcal{O}(x)]^2 > = \frac{(p^2 - 32p + 80)}{40 \, (3 - p)^2}. \tag{3.10}$$

The punctures of type $\mathbb{R} \times S^3$, in particular, have a jump of 49/160, a number which is very difficult to interpret. The right hand side of (3.10) has a maximum equal to 22/35 for $p = 32/13$. We conclude that the jump in not universal in four dimensions, while it is so in two dimensions. The above expression does not single out anything special about $p = 2$.

Let us consider, now, the correlation function $< [\tilde{\mathcal{O}}(x)]^2 >$, where the tilded observables are normalized by replacing the $32\pi^2$ with $8\pi^2\alpha^2(3 - \alpha)$ in (3.6). Then, after a certain nontrivial amount of work, we get

$$< [\tilde{\mathcal{O}}(x)]^2 > = \frac{(2 - p)(p + 10)}{40\,(3 - p)^2},$$

which vanishes only for $p = 2$ and equals 11/160 for $p = 1$. Thus, for $p \neq 2$ the quantum mixing between Euler and Pontrijagin numbers is much more apparent than for $p = 2$. We finally see that the $\mathbb{R}^4$ punctures are indeed the privileged ones.

We have just learned something very nontrivial from the analysis of the quantum topological invariants. This shows once again that these concepts are very useful and that the method for treating them developed in [2, 3, 1] is very powerful.

4 Physics

In this section I comment about the physical relevance of the metrics (3.1) and the notion of punctures in quantum gravity. Moreover, I study the case of a fermion placed in a given quantum topological background. For the issues of gauge invariance and renormalization in connection with the topological embedding in a general quantum field theory, the reader is referred to [3, 1].

The metrics (3.1) are instantons of Weyl gravity, but they are also important in ordinary quantum gravity, as we wish to show.

For $p = 1$ the Einstein action $\int R\sqrt{g}$ is zero, independently of the number of punctures, since the Ricci curvature vanishes identically. However, one has to take into account that the total action contains a boundary term [10], which is not zero for $p = 1$. It can be easily verified that for $p > 1$ the boundary term vanishes, while the Einstein action is negative definite.

According to ref. [10], given a metric g one has to look for the element $\bar{g}$ in the conformal class of g that satisfies $R \equiv 0$. The Positive Action Conjecture [10] says that the action for $\bar{g}$, which is entirely given by the boundary term, is positive definite. The action for g has an additional contribution, due to the conformal factor, that is positive definite once the integration over the conformal factor is taken to be parallel to the imaginary axis [10]. In our case, g is a generic metric (3.1), $\bar{g}$ is the metric with $p = 1$. So, the metrics with $p = 1$ are the ones with the least action in our class.

Since the behaviour of $\bar{g}_{\mu\nu}$ for $x \to \infty$ is $\delta_{\mu\nu}(1 + \sum_{i=1}^{n} \rho_i^2/x^2)^2$, the action for $\bar{g}$ is simply [8]

$$S = -\frac{1}{16\pi\kappa^2} \int_M \mathrm{d}^4x\sqrt{g}R + \frac{1}{8\pi\kappa^2} \int_{\partial M} [K]\sqrt{k}d^3x + \frac{i\theta}{64\pi^2} \int_M R^{ab}R^{cd}\varepsilon_{abcd} = 6\pi \sum_{i=1}^{n} \frac{\rho_i^2}{\kappa^2} - in\theta. \quad (4.1)$$

Integrating over ρ_i gives unity:

$$C\frac{1}{\kappa^n} \int \prod_{i=1}^{n} \mathrm{d}\rho_i \; \mathrm{e}^{-S} = \mathrm{e}^{in\theta},$$

where C is an appropriate numerical factor. This means that the contribution of the metrics (3.1) to the gravitational path integral is, after integrating over all possible values of the scales ρ_i, of the same order of magnitude as the contribution coming from flat space. Hence, the metrics (3.1) should be relevant for quantum gravity. Indeed, they can be considered as the analogues of the Yang-Mills instantons.

Note that the action (4.1) does not have a minimum in the topological sector specified by (3.3): tuning the values of the scales ρ_i appropriately, S can be made arbitrarily small. On the other hand, when some ρ_i's are exactly zero, the metric belongs to a different topological sector, since (3.3) jumps. A phenomenon like this one does not hold for Yang-Mills instantons, where the minimum of the action in the kth instanton sector is $8\pi^2|k|/g^2$; it holds in four dimensional quantum gravity as a consequence of the peculiarity of the gravitational action (its linearity in the curvature) and of the fact that the boundary of the moduli space of $p\mathbb{R}^4_n$ is a union of moduli spaces of $p\mathbb{R}^4_k$, $k < n$. In two dimensions, on the other hand, gravity is too simple to have an effect like this. The practical consequence of this observation is that in four dimensional quantum gravity, differently from Yang-Mills theory, the topological embedding acts on the positions x_i of the punctures only, while the scales ρ_i are taken care of by the measure $C \prod_i \mathrm{d}(\rho_i/\kappa) \; \mathrm{e}^{-S}$.

Thus, we understand that the best way to treat the topological sectors (3.3) is to integrate over all possible values of the ρ_i's. In perturbative quantum gravity around the metrics (3.1) (with $p = 1$) there will be a linear term in the graviton. It would be interesting to explore the effects of this linear term in connection with the topological embedding. This, however, is beyond the scope of the present paper and maybe will be considered elsewhere.

The integration over the scales ρ_i with the measure $C \prod_i \mathrm{d}(\rho_i/\kappa) \; \mathrm{e}^{-S}$ does not affect the topological invariants considered in the past section, since they are scale-independent.

Instead of studying the topological embedding on gravity itself, it is simpler to start from its effects on matter. Let us consider a fermion coupled to the gravitational field in d dimensions (we take $p = 1$ if $d = 2$ and $p = 2$ if $d = 4$). The action has the additional contribution

$$\sqrt{g}\,\overline{\psi}\gamma^a e_a^\mu \mathcal{D}_\mu \psi = \mathrm{e}^{\frac{d-1}{2}\varphi}\overline{\psi}\slashed{\mathcal{D}}\psi.$$

The presence of the factor e^φ, which is singular in the puctures, forces the matter field to vanish there, more or less as it happens in the case of vortices [1].

We are interested in the two-point function $< \psi(x)\overline{\psi}(y) >$ in a punctured background. This means that we have to integrate over the positions of the puncures. This integral is, in general, divergent. However, according to the general theory developed in [3] and [1], certain amplitudes are order-by-order perturbatively well-defined in the topologically nontrivial sectors. Here we want to apply those ideas to our case. One considers, instead of the ill-defined $< \psi(x)\overline{\psi}(y) >$,

amplitudes of the form

$$< \psi(x)\overline{\psi}(y) \prod_{j=1}^{n} \mathcal{O}(z_j) > \equiv < \psi(x)\overline{\psi}(y) >_{\{z\}} . \tag{4.2}$$

The insertion of the local observables $\mathcal{O}(z_j)$ (n of them, in the case of n punctures) specifies the quantum topological background on which the amplitude is defined. Intuitively, it is like studying phonon interactions on the background of the magnetic force lines that penetrate a type II superconductor [1]. An amplitude like the above one will be called a $(2,n)$-point function, the first number referring to the quantum excitations, the second number referring to the quantum background. Later on we shall also compute the fermion two-point function with a nonlocal quantum background made of curves γ.

Before beginning the computations, we have to make a couple of comments about the solutions (2.3) and (3.4) and the meaning of the topological gauge-fixing conditions on the physical side of the problem.

It has been shown in [2] that the gauge-fixing condition for the topological ghosts uniquely determine the solution to the topological field theory. In our case, $\psi^a = \psi^{ab}e^b$ should be fixed with the conditions $\psi^{ab} = \psi^{ba}$ and $\mathcal{D}^\mu \psi_\mu^a = 0$. However, (2.3) and (3.4) do not satisfy $\mathcal{D}^\mu \psi_\mu^a = 0$: (2.3) and (3.4) are only smooth deformations of the "good" solutions. The topological invariants are unaffected and, indeed, in two dimensions we were able to reproduce some known results. In the analysis of the physical side of the problem, on the other hand, the gauge-fixing condition for ψ^a is important. One can check that the physical amplitudes, like (4.2), do depend on it. As far as we know, it could be impossible to write down a gauge condition for ψ^a that rigorously produces the solutions (2.3) or (3.4) and satisfies the standard requirements of quantum field theory (locality, in particular). $\mathcal{D}^\mu \psi_\mu^a = 0$ appears to be the natural (and, to some extent, unique) choice. On the other hand, the explicit solution to this equation, that we have not found, could be so complicated to prevent us from using it for practical purposes. Therefore, we use (2.3) and (3.4) in this paragraph[4], taking full advantage of their simplicity, but being aware that in any hypothetical "comparison with experiment" our results should be mostly taken as qualitative.

In full generality, the gauge-fixing conditions $\mathcal{G}_\psi = 0$ for the topological ghosts ψ_μ^a appear to be as important as the instanton conditions $\mathcal{G}_T = 0$ themselves (for example, in Yang-Mills theory, $\mathcal{G}_T = F^{a+}$ and $\mathcal{G}_\psi = D^\mu \psi_\mu^a$). This is quite reasonable, since the topological ghosts ψ_μ^a are descendants of the curvature (F^a, R^{ab}, ...). It is for this reason, for example, that the non-covariant choice $\mathcal{G}_\psi = \partial^\mu \psi_\mu^a$ is not acceptable: indeed, in [2] it was observed that $\mathcal{G}_\psi = \partial^\mu \psi_\mu^a$ leads to an empty topological Yang-Mills theory. The instanton conditions $\mathcal{G}_T = 0$ are also a sort of gauge-fixing conditions, from the topological field theoretical viewpoint, but on the physical side they play quite a different role, because they are uniquely determined by minimizing the action.

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[4] The explicit computations will be mainly performed in two dimensions, since this is already a nontrivial and illustrative case, but simpler.

Summarizing, changing $\mathcal{G}_\psi$ does change the physical amplitudes, but, fortunately, there exists a natural choice for it. Moreover, $\mathcal{G}_\psi$ is strictly unique if one accepts that it should combine together with the ψ-field equation into a twisted Dirac equation. This "gauge-fixing dependence" is not in contraddiction with general principles. The point is that we are talking about the gauge-fixing conditions of the topological symmetry, which is *not* a symmetry of the physical theory, but rather an artifact of the perturbative expansion around instantons. The topological "gauge-symmetry" is present because in the topologically nontrivial sectors the minimum of the action is not a point, but a moduli space. We can say that the topological embedding solves the problem that arises when the very first stage of the perturbative expansion (with which I mean the zero point function) has more "gauge-symmetries" than the complete theory. In the topological embedding the quantum topological invariants play the very role of "zero point functions".

After this instructive digression, let us begin our analysis of the physical amplitudes.

In two dimensions the action of pure gravity is topological. The scales ρ_i deserve to be discussed apart. In this respect, the situation is different from the four dimensional one. The ρ_i's can or cannot be integrated over, according to the background that one wish to consider. Here, I shall examine both cases, starting from the case in which the ρ_i are fixed. The amplitudes can be ρ_i-dependent.

The covariant derivative $\mathcal{D}\psi$ is $\mathrm{d}\psi - \frac{i}{2}\omega\sigma_3\psi$. Defining $\overline{\psi} = \mathrm{e}^{-\frac{\varphi}{4}}\widetilde{\overline{\psi}}$ and $\psi = \mathrm{e}^{-\frac{1}{4}\varphi}\widetilde{\psi}$, the lagrangian can be written simply as

$$\mathrm{e}^{\frac{\varphi}{2}}\overline{\psi}\sigma^\mu\left(\partial_\mu + \frac{1}{4}\partial_\mu\varphi\right)\psi = \widetilde{\overline{\psi}}\sigma^\mu\partial_\mu\widetilde{\psi},$$

where $\mathrm{e}^{\varphi(x)} = D(x) = 1 + \sum_{i=1}^n \frac{\rho_i^2}{(x-x_i)^2}$. Thus, the desired $(2,n)$-point function is easily written down. One has

$$< \psi(x)\overline{\psi}(y) >_{\{z\}} = \frac{1}{2\pi}\frac{\not{x} - \not{y}}{(x-y)^2}G_n(x,y,\{z\}),\tag{4.3}$$

where the function $G(x,y,\{z\})$ is a sort of form factor that describes the deviation from the free propagator. In order to simplify the expression, I would like to consider the case in which the n points z_i are all distinct (coincidences appear to be quite unplausible, from the physical point of view), but concentrated in a small region (or viewed from a large distance). In this way, we can take $z_j \simeq z \,\forall j$. Were the punctures truly placed in the same point z, there would be a jump described by an additional overall factor $1/n!$, as we know.

Under these conditions, we have, from (2.8),

$$G_n(x-z, y-z) = \frac{n!}{\pi^n}\int \prod_{i=1}^n \frac{\rho_i^2\mathrm{d}^2x_i}{(z-x_i)^4}\,\frac{1}{D(z)^{n+1}D(x)^{\frac{1}{4}}D(y)^{\frac{1}{4}}}\tag{4.4}$$

Although it was not obvious from the beginning, the above formula shows that the amplitudes that we are considering are indeed well-defined: the x_i-integrations make sense. This would not be true without the insertion of the quantum background $\prod_{j=1}^n \mathcal{O}(z_j)$. Thus we have a concrete illustration and check of the general theory developed in ref.s [3, 1]. (4.3) represents

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the first perturbative contribution to the effective action in the n-puncture topological sector, obtained by expanding around the topological amplitude $< \prod_{j=1}^{n} \mathcal{O}(z_j) >= 1^5$. The result is no more topological, of course, and the factor $G(x - z, y - z)$ measures the feedback that the fermion propagation receives from the quantum background on which it is excited. G could be considered as an effective "quantum" metric. In our examples G obeys the inequality $0 \le G \le 1$. The amount of the deviation of G from the value 1 (propagator in flat space) describes the "obstacle" that the fermion finds on its way.

This phenomenon, i.e. the generation of a quantum metric by the quantum background, seems to be quite general, not related to the presence of gravity in the problem. For example, a two-point function of the same structure can be observed (see formula (2.8) of [1]) with scalars in presence of the BPST instanton [11].

In our simple theory, there is no radiative correction to the effective action and the form factor $G(x - z, y - z)$ is the entire story.

Let us focus on the singular behaviour for $x \sim y$. The function $G_n(x - z, x - z)$ has a minimum exactly on the puncture ($x = z$) and tends to 1 far from it: the deviation from the free propagator is maximal nearby the puncture and negligible elsewhere. The minimum $G_n(0,0)$ is $\frac{(2n)!!}{(2n+1)!!}$, so that

$$< \psi(x)\overline{\psi}(z) >_z \sim \frac{(2n)!!}{(2n+1)!!} \frac{1}{2\pi} \frac{\not{x} - \not{z}}{(x-z)^2}.$$

For large n this minimum goes to zero like $\frac{1}{n}$. This means that an infinite number of punctures is able to inhibit the propagation completely in the area where they are located. This is a quite reasonable physical result. But it is also interesting to note that no finite number of punctures is able to achieve this.

We can compare the effects of the quantum background to the ones of the classical background. The latter situation is achieved by saying that the metric is a fixed external field, so that no integrations over the positions x_i are performed. Then, the fermion two-point function

$$< \psi(x)\overline{\psi}(y) >= \frac{1}{2\pi} e^{-\frac{\varphi(x)}{4}} \frac{\not{x} - \not{y}}{(x-y)^2} e^{-\frac{\varphi(y)}{4}} \tag{4.5}$$

vanishes at the points x_i, whatever the total number of puctures is. We conclude that the quantum background is able to smoothen this effect.

For one puncture the form factor is

$$G(x - z, y - z) = \frac{1}{\pi} \int \frac{\rho^2 d^2 x_0}{(\rho^2 + (z - x_0)^2)^2} \frac{1}{\left[\left(1 + \frac{\rho^2}{(x-x_0)^2}\right)\left(1 + \frac{\rho^2}{(y-x_0)^2}\right)\right]^{\frac{1}{4}}} \tag{4.6}$$

and its minimum is $\frac{2}{3}$. We understand that the physical meaning of the scale ρ is that it measures the size of the region where the propagation is sensibly affected by the G-function.

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[5] The words "expanding around a topological amplitude" should not suggest that there is a smooth limit that reduces the physical theory to the topological one. Indeed, it is true that the quantum fluctuations are "small", but it is also true that, after the functional integration, their contribution is finite.

Now, we want to analyse a different possibility for defining $< \psi(x)\overline{\psi}(y) >$ in the one-puncture sector. We want to insert a couple of nonlocal observables like the ones appearing in (2.10). We have

$$< \psi(x)\overline{\psi}(y) >_{\gamma_1 \cdot \gamma_2} = \frac{1}{2\pi} \frac{\not{x} - \not{y}}{(x-y)^2} G(x, y, \gamma_1 \cdot \gamma_2).$$

where

$$G(x, y, \gamma_1 \cdot \gamma_2) = \frac{\rho^4}{\pi^2} \int_{\gamma_1} \int_{\gamma_2} \int \frac{\varepsilon_{\mu\nu} dz^\mu dw^\nu \, d^2 x_0}{(\rho^2 + (z - x_0)^2)^2 (\rho^2 + (w - x_0)^2)^2 \left[\left(1 + \frac{\rho^2}{(x-x_0)^2}\right) \left(1 + \frac{\rho^2}{(y-x_0)^2}\right) \right]^{\frac{1}{4}}}.$$

To be explicit, let us γ_1 be the x-axis and γ_2 the y-axis. The γ-integrations are easily doable. The minimum of G is in the point in which γ_1 and γ_2 intersect. A numerical integration gives, for $x \sim y \sim 0$

$$< \psi(x)\overline{\psi}(y) >_{\gamma_1 \cdot \gamma_2} \sim 0.6935 \frac{1}{2\pi} \frac{\not{x} - \not{y}}{(x-y)^2},$$

revealing that the deviation from free propagation due to two intersecting lines is slightly less than the one due to a local observable. Note, however, that the γ's distribute G in a wider area.

Finally, let us study the topological embedding when the scales ρ_i are included in the set of the moduli. We study the propagation on the quantum background (2.12). We have

$$< \psi(x)\overline{\psi}(y) >_{\gamma \cdot \{z\}} = \frac{1}{2\pi} \frac{\not{x} - \not{y}}{(x-y)^2} G(x, y, \gamma \cdot \{z\}). \tag{4.7}$$

where

$$G(x, y, \gamma \cdot \{z\}) = \frac{1}{4\pi^2} \int_\gamma \int \frac{dw^\mu \varepsilon_{\mu\nu}(w - z)^\nu \, 2\rho^2 d\rho^2 \, d^2 x_0}{(\rho^2 + (z - x_0)^2)^2 (\rho^2 + (w - x_0)^2)^2 \left[\left(1 + \frac{\rho^2}{(x-x_0)^2}\right) \left(1 + \frac{\rho^2}{(y-x_0)^2}\right) \right]^{\frac{1}{4}}}. \tag{4.8}$$

The integral in ρ and x_0 diverges as $\frac{1}{(z-w)^2}$ when $w \to z$, precisely as in the purely topological amplitude (2.12). The factor in front of the singularity, however, can be different. An instructive situation that we can easily handle is the one in which $x \sim y \sim z$. In other words, let us inspect how the propagator is modified in the neighborhood of the point z where the local observable is placed. One gets

$$< \psi(x)\overline{\psi}(y) >_{\gamma \cdot \{x\}} \sim \frac{2}{3} \chi(\gamma, \{x\}) \frac{1}{2\pi} \frac{\not{x} - \not{y}}{(x-y)^2}. \tag{4.9}$$

We see that there are both the numerical factor $2/3$ and the link number itself in front of the usual two-point function. At very large distances, on the other hand, the two point function looses the factor $2/3$ and becomes the free propagator times the link number. An intermediate situation is the one in which one point, say x, is very far and the other one is very close to z. Then the factor $2/3$ in (4.9) is replaced by $4/5$. If the curve γ and the point z are unlinked, (4.9) is exactly zero, however the two-point function is not identically zero.

Since ρ is integrated over, there is no external scale which is kept fixed. The size of the region where the propagation is sensibly affected by the quantum background is dictated by the quantum background itself: precisely it is the size of the curve γ.

We have examined two typical cases. In the first case the instantons contain a dimensionful parameter ρ that is kept fixed. Then the quantum background can be constructed purely with local observables, like in (4.2). In the second case, instead, the size of the instanton is integrated over. Then the quantum background is forced to contain at least one nonlocal observable, like in (4.7), associated with a closed submanifold γ. The brings a new scale into the game, which is the size of γ. It seems that it is not possible to have a scale-free topological embedding.

Let us recall that, in the topologically trivial sector, when no other scale is around, it is renormalization that forces the introduction of one. In section 2.1 of ref. [1] the usual renormalization scale was interpreted as the (unique) quantum background of the topologically trivial sector. Indeed, a feature that the topological embedding and renormalization have in common is that they both cure problems with divergences.

It seems that the factor G has everywhere the same sign, which we can fix conventionally to be positive. I have not found situations where this property is violated. It is straightforward to prove it for the amplitude (4.4) and for a set of simple situations in (4.8) (for example, when γ is a circle centered in z). However, I do not have a rigorous proof that this positivity condition holds in general. It would assure that our two-point function is physically meaningful without imposing restrictions on the quantum background. At the same time, it would justify the name "quantum metric" that we have used.

The analysis of the two-point function of a Dirac fermion in a four dimensional punctured background is entirely similar and will not be repeated. We just note that it is sufficient to write the Lagrangian $L = \sqrt{g}\,\overline{\psi}\gamma^a e_a^\mu \mathcal{D}_\mu \psi$ as $L = \widetilde{\overline{\psi}}\,\widetilde{\partial}\widetilde{\psi}$ for $\widetilde{\overline{\psi}} = \overline{\psi}\,\mathrm{e}^{\frac{3}{2p}\varphi}$ and $\widetilde{\psi} = \psi\,\mathrm{e}^{\frac{3}{2p}\varphi}$ ($\mathcal{D}\psi = \mathrm{d}\psi - \frac{1}{8}[\gamma_a, \gamma_b]\omega^{ab}\psi$ in our notation) and proceed as before, using the moduli-space measures of section 3. Similarly, messless QED (or QCD, with obvious modifications) in the same punctured background presents no further difficulty. The QED Lagrangian in the variables $\widetilde{\overline{\psi}}$ and $\widetilde{\psi}$ still looks like the ordinary one. Consequently, denoting by $\Gamma(x, m; y, m; z, l)$ the ordinary renormalized QED scattering amplitude of m fermions and l photons, the same amplitude in the quantum punctured background Q is given by $\Gamma_Q(x, m; y, m; z, l) = \Gamma(x, m; y, m; z, l)\, G_Q(x, m; y, m)$, where ($p = 1$)

$$G_Q(x, m; y, m) = C \int \prod_{i=1}^{n} \mathrm{d}(\rho_i/\kappa)\ \mathrm{e}^{-S_Q}\, \mathrm{d}\mu_Q \prod_{j=1}^{m} \mathrm{e}^{-\frac{3}{2}\left(\varphi(x_j)+\varphi(y_j)\right)}. \qquad (4.10)$$

Here $\mathrm{d}\mu_Q$ denotes the measure of the topological amplitude $\int \mathrm{d}\mu_Q$ associated with the quantum background Q. S_Q is the action (4.1) of the background Q. It is straightforward to check that G_Q is well-defined and that its qualitative behaviour agrees with the behaviours of the G-functions studied so far. A direct consequence of (4.10) is that the pure scattering of photons ($m = 0$) does not really feel the quantum background, in the sense that G_Q is just the topological invariant in that case. (4.10) could have some physical applications, for example in solid state physics, if one finds some material that is able to simulate the effects of the punctures. In

this hypothetical situation, (4.10) could describe the QED scattering inside such a material. Finally, we can see explicitly that there is no conflict between the topological embedding and renormalization.

In conclusion, we have tested the general theory of the topological embedding in a set of simple models in which we can write down explicit formulæ. The amplitudes are well-defined and give quite plausible physical predictions. We have seen that the known properties of quantum field theory are generalized in a reasonable way. Up to now, the analysis of the topological embedding has not revealed the need of physical restrictions on the quantum background one is expanding around. The topological embedding could be a new chapter of quantum field theory.

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