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ON FIELD THEORY QUANTIZATION AROUND INSTANTONS

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Abstract

With the perspective of looking for experimentally detectable physical applications of the so-called topological embedding, a procedure recently proposed by the author for quantizing a field theory around a non-discrete space of classical minima (instantons, for example), the physical implications are discussed in a "theoretical" framework, the ideas are collected in a simple logical scheme and the topological version of the Ginzburg-Landau theory of superconductivity is solved in the intermediate situation between type I and type II superconductors.

1 Introduction and motivation

So far, the nonperturbative aspects of quantum field theory have been mainly investigated in the context of supersymmetry. This research began with the computation of certain exact "topological" correlation functions that are independent of the positions of the local observables (condensates) [1]. In many cases, supersymmetry greatly reduces the computational effort and various quantities of a topological nature can be identified. In non-supersymmetric theories, for example, primed determinants do not simplify between bosons and fermions and have to be calculated explicitly. Although technically involved, this is not impossible, as shown by t' Hooft in ref. [2]. At that time, however, no well-defined exact amplitude was identified and the convergence of the integration over the moduli space remained an open issue [2, 3, 4].

It is natural to ask ourselves, nowadays, whether it can be fruitful to reconsider non-supersymmetric gauge field theory and explore the viewpoints that are offered by the new theoretical devices at our disposal.

In ref. [5], a proposal was made to expand perturbatively a quantum field theory in the topologically nontrivial sectors, or, more generally, when the minima of the classical action are not a discrete set, but a *moduli* space $\mathcal{M}$. One needs to separate conveniently the integration over $\mathcal{M}$, which remains a nonperturbative issue, from the integration over the quantum fluctuations "perpendicular" to $\mathcal{M}$, that, instead, can be treated perturbatively. The topological field theory associated with $\mathcal{M}$ is suitably embedded into the physical theory, so that the physical theory is viewed as a perturbative expansion around the topological one. For example, QCD is formulated as an expansion around topological Yang-Mills theory. Nothing changes in the topologically trivial sector with respect to the usual approach, but many problems encountered so far in the topologically nontrivial sectors can be easily bypassed and nonperturbative quantities can be defined and computed exactly. The insertion of suitable topological observables $\mathcal{O}_\gamma$ guarantees the convergence of the integration over $\mathcal{M}$.

The key idea behind the topological embedding is very simple and natural. One can say that the space $\mathcal{M}$, around which one would like to define perturbation theory, enjoys an enhanced gauge symmetry, which is precisely of a topological nature. Indeed, in the problem under consideration any two $\mathcal{M}$-configurations that are continuously deformable into each other have to be considered equivalent. So, the troubles found in the past with the $\mathcal{M}$-integral are here viewed as due to an extra gauge symmetry that was not gauge-fixed. The $\mathcal{O}_\gamma$-insertions gauge-fix the extra symmetry. Effectively, they act as projectors onto some special point $m \in \mathcal{M}$, the Poincarè dual of the differential form $\prod_i \mathcal{O}_{\gamma_i}$. Thus, the topological embedding is a sophysticated way to reduce perturbation theory around a moduli space $\mathcal{M}$ to perturbation theory around a point $m \in \mathcal{M}$, which is the quotient of $\mathcal{M}$ by the topological symmetry. The topological amplitudes classify the inequivalent ways of doing this consistently.

The first nontrivial consequence is that the topological quantities have a role in the physical theory. This should have deep implications, even at the qualitative level, that hopefully will be compared with experiment in a non distant future. The purpose of this paper is precisely to begin a deeper exploration of this fascinating subject in order to look for the physical meaning of the topological aspects of quantum field theory.

Some physical implications of the topological embedding can be clarified immediately, in a "theoretical" framework. In ref. [5], the concrete example of pure non-abelian Yang-Mills theory was considered. The first difficulty is to imagine what the topologically nontrivial sectors of QCD are in nature, since our physical intuition is limited, at least at present. Soon or later, some experiment should reveal them. For now, it is very convenient to stimulate our imagination with an analogy. There is another interesting physical theory, closer to experiments, in which the nontrivial topological sectors could play a crucial role: it is the Ginzburg-Landau theory of superconductivity, that is a $U(1)$ gauge theory with a Higgs charged scalar. At a very special value ($\lambda = 1$) of the parameter λ that distinguishes type I ($\lambda < 1$) form type II ($\lambda > 1$) superconductors, the theory admits "instantons", whose proper name is in fact "vortices". The instanton number is the vorticity. The various topological sectors are thus labelled by the number of magnetic flux units penetrating the superconductor, a number that can be measured: this is a very clear picture of the physical meaning of the topologically nontrivial sectors. It suggests that any event (amplitude) is naturally placed in a *specific* topological sector, and does not receive, as it is commonly believed, contributions from all the topological sectors: either one flux unit penetrates the superconductor, or two, or zero $\ldots$, but not all contemporarily. The same argument should apply to QCD and to any other similar case. This is a basic feature of the topological embedding. Sometimes, to avoid confusion, I shall call QCD* the theory treated with the new approach.

Another important implication, that is stressed very much in the present paper, is that it is not sufficient, at the quantum field theoretical level, to specify the vorticity or the instanton number, i.e. what we can call the *classical* topological sector, but even when this is specified, there are discrete inequivalent possibilities, represented by the topological observables $\mathcal{O}_\gamma$ that are inserted in order to make the $\mathcal{M}$-integration convergent. We say that such possibilities characterize different *quantum* topological sectors. Correspondingly, the topological amplitudes are called the quantum topological properties of the instantons. Of course, the (classical and quantum) topological sectors represent metastable configurations, the transition from one sector to another one requiring a finite perturbation.

In the example of pure non-abelian Yang-Mills theory the Pontrjiagin number k is the classical topological property of the instantons, while the generalized multilink intersection theory uncovered in ref. [5] classifies the quantum topological properties of the BPST instanton [6]. In the case of superconductivity, the classical property is the vortex number, while the quantum properties are classifed in the second part of the present paper, where the topological version of the theory is solved.

So, to compare the experimental results with the predictions of the theory it is necessary to specify the quantum topological sector where the experiment takes place. Being a global (because topological) property (i.e. it is sensible to the *boundary* of the spacetime manifold), the quantum topological sector describes the interaction between the quantum fluctuations and the experimental apparatus: it is meaningless, in the topologically nontrivial sectors, to speak about *free* asymptotic states and the fluctuations over the instanton background interact with the instanton background itself.

Recently, another relation between topological field theory and quantum field theory has been proposed. In ref. [7] Cattaneo, Cotta-Ramusino, Gamba and Martellini formulated QCD as an expansion around a B-F theory. The idea is as follows. In the first order formalism, the action of pure Yang-Mills theory is written as

$$\int \mathrm{Tr}[B \wedge F_A] - g^2 \int \mathrm{Tr}[B^2]. \tag{1.1}$$

Then, a perturbation around $g = 0$ is indeed a perturbation around a B-F theory. Notice that this trick does not work after the rescaling $A \to gA$. In other words, this formulation treats instantons on the same footing as any configuration with zero Pontrjiagin number, a feature also shared by the topological embedding.

An interesting problem is to understand the correspondence between the two formulations of QCD that I have just recalled [9]. Abstractly, one is naturally lead to conjecture that there exists a deep relation (that I call *the topological map*) between topological Yang-Mills theory and the B-F theory (see fig. 1). This relation may seem a bit unplausible, at first sight, and, indeed, has never been conjectured so far. However, we already have a strong evidence in favour of it. It is well-known that link numbers are natural topological invariants of the B-F theories. On the other hand, the results collected in ref. [5] (firstly discovered in [8]) show that the amplitudes of topological Yang-Mills theory with the BPST instanton are also link invariants. This was quite unexpected, indeed. Presumably, the topological map passes very nontrivially through the physical theory (i.e. QCD itself), so that understanding abstractly the topological map could give insight to uncover other nonperturbative aspects of QCD [9].

$$\begin{tikzcd} TYMT \arrow[r, "{[5]}"] \arrow[rr, leftrightarrow, bend right, "\text{topological map}"'] & QCD & B\text{-}F \arrow[l, "{[7]}"'] \end{tikzcd}$$

Fig. 1: relation between QCD and topological field theory.

As we see, there are many interesting open problems, both of a mathematical and a physical nature. On the physical side, surely the more attractive one, other questions are:

i) what are the quantum topological sectors of ref. [5] in nature?

ii) how to detect the non-abelian analogue of the Aharonov-Bohm effect described in ref. [5]?

iii) can we find a second quantization or a statistical distribution collecting the quantum topological sectors into a general picture[1]?

iv) can topological field theory describe properties of zero temperature phenomena?

v) can interactions with the boundary be adequately described by the topological observables and the topological embedding?

vi) is the quantum Hall effect related to all of this?

[1]This interesting question was raised to me by P. Frè.

I shall not answer all these questions in the present paper, but many interesting remarks will be made along with the discussion. Moreover, it is very useful to describe the various aspects of the new approach from different viewpoints. The first part of the paper is devoted to this. The long-range aim of the investigation is to compare the consequences of the topological embedding with experiment. Therefore, it is necessary to identify the most convenient theories, situations and physical quantities. It would not be upsetting if topological field theories turned out to give good descriptions of phenomena where the quantum fluctuations are naturally frozen, like at zero temperature. Having this in mind, I consider, in the second part of the paper, the Ginzburg-Landau theory of superconductivity and solve its topological version in the intermediate case between type I and type II superconductors. This investigation could stimulate the experimental study of superconductors in this very special situation.

Point vi) is suggested by the observation that the quantum Hall effect is a zero temperature phenomenon characterized by a quantity, the filling factor, that takes integer and fractional values. Such values have all the features of the topological quantities and could be a direct or indirect trace of the quantum topological sectors.

As far as point iii) is concerned, one might ask whether there is something missing in the theory, and why, otherwise, the topological embedding is unable to describe transitions among different quantum topological sectors. To clarify this point, I stress that the topological embedding is not an exact treatment of the theory, but a way of defining perturbation theory around non-discrete spaces of minima. As such, the topological embedding is perfectly satisfactory to describe phenomena involving the fluctuations over the (classical and quantum) background, but it fails to describe nonperturbative transitions. The complete theory, nevertheless, is expected to contain the answer to this. Yet, it could be very interesting to try to supplement the topological embedding with something like a second quantization or a generalized statistical distribution, in order to get closer to the complete theory.

The paper is organized as follows. In section 2 the basic facts about the topological embedding are recalled, the experimental normalization of coupling constants is analysed in the new approach and the concrete meaning of the notion of quantum topological sector $\mathcal{Q}$ is developped. In section 3 the quantum topological properties of instantons computed so far are summarized. In section 4 the topological version of the Ginzburg-Landau theory of superconductivity is solved.

2 Quantum field theory around a non-discrete space of minima

In this section, the proposal of ref. [5] for quantizing a theory when the action has not a unique minimum, but a non-zero dimensional space $\mathcal{M}$ of minima, is briefly recalled. In subsection 2.1 the normalization of physical constants in the quantum topological sectors is elucidated. This gives a concrete interpretation of the physical meaning of the $\mathcal{O}_\gamma$-insertions (see question i) of the introduction).

Since there is no way to privilege *one* solution to the field equations, the classical theory is not satisfactory. In quantum field theory, on the other hand, the functional integral should bypass the problem. However, in general, the $\mathcal{M}$-integral is not guaranteed to converge [2, 3, 4]. The question that one has to answer in order to correctly define the quantum theory is: *how* to do the integral over $\mathcal{M}$? How many inequivalent possibilities are there? What do they mean physically?

The *topological embedding* proposed in ref. [5] in order to overcome the problem correctly is a way of expanding the true theory around the topological version of the same theory. Notice that there is a topological version of *any* quantum field theory. Moreover, the topological theory can be defined for *any* given finite dimensional subspace $\mathcal{M}$ of the total functional space. In this respect, there is nothing special with the objects that we usually call "instantons". Topological field theory is a useful device to compute the quantum topological properties of any given $\mathcal{M}$. It is the physical theory that selects the interesting $\mathcal{M}$'s (the minima of the classical action). Keeping this in mind, I shall nevertheless refer to $\mathcal{M}$ as to the space of "instantons".

Here are the basic features of the topological embedding. To be concrete, I take Yang-Mills theory. The gauge field A is written as $A = A_0 + gA_q$, A_0 parametrizing the space $\mathcal{M}$ and A_q denoting the quantum fluctuation. In the topologically trivial sector $A_0 = 0$, so that $A = gA_q$ corresponds to the replacement $A \rightarrow gA$ that defines the usual perturbative approach. The Yang-Mills BRST algebra for A is written as the semidirect product of the topological BRST algebra for A_0 and the consequent remnant for A_q. This formulation, that otherwise would be nothing else but a refined version of the Faddeev-Popov procedure of "introducing 1" [10, 11], has the advantage of providing a set of topological observables $\mathcal{O}_\gamma$ (constructed with A_0 and promoted now to physical observables of the complete theory), that can be used to make the integration over the moduli space well-defined. The physical amplitudes are thus defined as

$$\ll A_q(x_1) \cdots A_q(x_n) \gg_{\mathcal{Q}} \equiv < \prod_i \mathcal{O}_{\gamma_i} \cdot A_q(x_1) \cdots A_q(x_n) > . \tag{2.1}$$

$\mathcal{Q}$ is a label identifying completely the topological sector where the experiment takes place. One is lead to define a notion of *classical* topological sector and a notion of *quantum* topological sector, as follows.

i) The classical topological sector is identified by the value of classical topological invariant, in our case the instanton number k, contributing to the amplitude. For (2.1) it is the total ghost number of the $\mathcal{O}_\gamma$-insertions:

$$|k| = \sum_i \mathrm{gh}\#\, [\mathcal{O}_{\gamma_i}]. \tag{2.2}$$

ii) The quantum topological sector $\mathcal{Q}$ is identified by the specific set of $\mathcal{O}_\gamma$'s that have been inserted.

The classical topological invariant is a common property of *any* single instanton in the moduli space $\mathcal{M}$, not a property of the *space* $\mathcal{M}$ of instantons. The amplitudes of the topological theory, on the other hand, arise as integrals over $\mathcal{M}$. It is proper of a *quantum* theory to deal, via the functional integral, with the space of configurations and not with single configurations. This is the reason why the instanton number is here called the *classical* topological invariant, while the amplitudes of the topological field theory are called the *quantum* topological invariants of the instanton.

When $n = 0$ in (2.1) (i.e. when no functional derivative is taken with respect to the source J_q associated to the quantum fluctuations A_q and J_q is set to zero), the amplitude $\ll 1 \gg_{\mathcal{Q}}$ is proportional to the pure quantum topological invariant $< \prod_i \mathcal{O}_{\gamma_i} >$ (see subsection 2.1 for the detailed justification of this). It plays the role, in the topologically nontrivial sectors, that is played by the partition function in the topologically trivial sector (which is simply equal to 1, at $J_q = 0$, by normalization). As such, $\ll 1 \gg_{\mathcal{Q}}$ is not detectable in a direct way. Indeed, a "particle" is an excitation *above* a background, not a property of the background. What one can concretely do, instead, is to study gluon scattering *over* the given quantum background $\mathcal{Q}$ and compare the predictions with the results found in the topologically trivial sector. The effect of the quantum background should be enough for an eventual (perhaps only hypothetical, for now) experimental test.

$\ll A_q A_q A_q A_q \gg_{\mathcal{Q}} =$

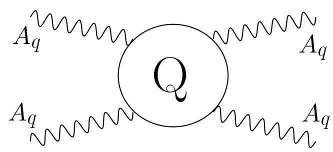

Fig. 2: scattering over a quantum background.

I stress that the topological embedding (and the approach of ref. [7], on the side of B-F theories) is the answer to the question about the physical relevance of topological field theory, a subject that, surprisingly, had not been considered seriously in the previous literature.

2.1 Normalization of physical constants in QCD*

Let us write the generic amplitude of the $\mathcal{Q}$-quantum topological sector of QCD* in the form

$$G^{(n)}_{\mathcal{Q}}(p, q, g_{\mathcal{Q}}) = \int_{\mathcal{M}} d\rho\, \mathcal{Q}(\rho, q)\, \gamma_{(n)}(p, \rho, \Lambda, g_R, Z, Z_g). \tag{2.3}$$

Here, most terms are symbolic. ρ often denotes the entire set of moduli, but it is only on the scale that we need to focus on. $\mathcal{Q}(\rho, q)$ represents the (explicitly known [8, 5]) insertion of topological observables $\prod_i \mathcal{O}_{\gamma_i}$, that depend on the moduli and on the γ_i. The γ_i-dependence is here symbolically denoted by the momentum q, because we are working in momentum space. In general, this dependence affects any dynamical amplitude. Only the topological amplitudes are q-independent. $\gamma_{(n)}$ is the perturbative part of the amplitude, regularized with a cut-off Λ. p are the external A_q-momenta and n is the number of external A_q-legs. $g_{\mathcal{Q}}$ is the measured coupling constant, at a certain reference scale that will be specified. g_R, instead, has no direct physical meaning. It represents the correct expansion parameter for the perturbative part $\gamma_{(n)}$. The relation between $g_{\mathcal{Q}}$ and g_R will come out of the argument. Z and Z_g are the wave function and coupling renormalization constants. They depend on g_R, Λ and a certain reference scale s.

Now, we have to understand the meaning of q, s, $g_{\mathcal{Q}}$ and g_R and say what is measured and how. In particular, $\mathcal{Q}$ fixes the quantum topological sector, but what is the meaning of q?

I have recalled, in the introduction, that in QCD* each event takes place in a single topological sector and does not receive contributions from any sector. The first consequence of this fact is that the physical normalization of the coupling constant is not the same in any sector. The second consequence is that it is meaningless to say that instanton contributions are suppressed: in the topologically nontrivial sectors they are the entire story.

Z and Z_g should be such that $\gamma_{(n)}$ is convergent in Λ. If we do not require this *before* the $\mathcal{M}$-integration, we can loose the powerful theorems about the classification of divergences. On the other hand, since g_R has no direct physical meaning, its reference scale s does not need to have a physical meaning. Moreover, in $\gamma_{(n)}$ there is already a scale (that is an "external" scale, from the point of view of $\gamma_{(n)}$), namely ρ. Consequently, the natural choice is $s=\rho$.

After the $\Lambda\rightarrow\infty$ limit, we have

$$G^{(n)}_{\mathcal{Q}}(p,q,g_{\mathcal{Q}})=\int_{\mathcal{M}}d\rho\,\mathcal{Q}(\rho,q)\,\gamma_{(n)}(p,\rho,g_R). \tag{2.4}$$

Now, let us consider $n=3$, namely the vertex function that fixes the coupling constant. It is reasonable to look at q as the *generalized* reference scale at which the physical parameters are normalized and this is consistent with the interpretation according to which $\mathcal{Q}$ specifies some kind of interaction with the experimental apparatus. Thus, for $p=q$ we write

$$G^{(3)}_{\mathcal{Q}}(q,q,g_{\mathcal{Q}})=\int_{\mathcal{M}}d\rho\,\mathcal{Q}(\rho,q)\,\gamma_{(3)}(q,\rho,g_R)\equiv g_{\mathcal{Q}}(q). \tag{2.5}$$

q and $g_{\mathcal{Q}}(q)$ have physical meaning: q as a chosen (generalized) scale, $g_{\mathcal{Q}}(q)$ as a measured number at the scale q. On the other hand, $g_{\mathcal{Q}}(q)$ is a function of g_R. So, indirectly we have $g_R=g_R(q)$, which fixes g_R. The running coupling constant is

$$g_{\mathcal{Q}}(p)\equiv G^{(3)}_{\mathcal{Q}}(p,q,g_{\mathcal{Q}})=\int_{\mathcal{M}}d\rho\,\mathcal{Q}(\rho,q)\,\gamma_{(3)}(p,\rho,g_R). \tag{2.6}$$

Its behaviour could be different from the running of the coupling constant in the topologically trivial sector.

Now, let us take $n=0$, so that no external momentum p is present. Since $\gamma_{(0)}$ is dimensionless, it cannot depend on ρ. This means that

$$\ll 1 \gg_{\mathcal{Q}}=G^{(0)}(q,g_{\mathcal{Q}}(q))=\gamma_{(0)}(g_{\mathcal{Q}}(q))\int_{\mathcal{M}}d\rho\,\mathcal{Q}(\rho,q)=\gamma_{(0)}(g_{\mathcal{Q}}(q))<\prod_i\mathcal{O}_{\gamma_i}>, \tag{2.7}$$

as desired. In the last term, $<\ldots>$ refers to the pure topological theory. $\ll 1\gg_{\mathcal{Q}}$ is the partition function in the $\mathcal{Q}$-sector.

The above normalization prescription is not in contraddiction with the usual one for the topologically trivial sector. In that case a fictitious intermediate scale ρ can be also introduced. Then, $\mathcal{Q}(\rho,q)$ is $\delta(\rho-1/q)$: this illustrates the operation with which the observer fixes a definite reference scale q. The "topological invariant" is simply $\int d\rho\,\delta(\rho-1/q)=1$.

The analysis of this subsection offers a clear interpretation of the quantum topological sectors $\mathcal{Q}$: they classify the ways in which the observer can fix the reference momenta in order to normalize the fundamental parameters of the theory, which is the notion of "generalized reference scale".

The simplest example of non-topological amplitude is perhaps the propagator of a scalar field in the fundamental representation of $SU(2)$ above the BPST instanton background. In this case, the perturbative Green function is (see appendix D of ref. [1] for details)

$$\gamma_{(2)}(x,x',\rho,x_0)=\frac{i}{4\pi^2}\frac{1}{(x-x')^2}\frac{\rho^2+(x-x_0)^\mu(x'-x_0)^\nu e_\mu\bar{e}_\nu}{[\rho^2+(x-x_0)^2]^{1/2}[\rho^2+(x'-x_0)^2]^{1/2}}\gamma_{(0)}(g_{\mathcal{Q}}). \tag{2.8}$$

As before, the factor $\gamma_{(0)}(g_{\mathcal{Q}})$ is due to the integral over the gauge field A_q, with no A_q-external legs. The behaviors of (2.8) for $\rho\to 0,\infty$, and for $x_0\to x,x',\infty$ show that the convergence of the $\mathcal{M}$-integral is preserved. A complete amplitude is, for example,

$$G^{(2)}_{\mathcal{Q}}(x,x',y,z,g_{\mathcal{Q}})=\frac{1}{(x-x')^2}\int_{\mathcal{M}}\omega^{(4)}_y\,\omega^{(1)}_z(x-x')^2\,\gamma_{(2)}(x,x',\rho,x_0), \tag{2.9}$$

where $\omega^{(4)}_y$ is given in formula (4.13) of ref. [8] and corresponds to the local observable, while $\omega^{(1)}_z$ is given in formula (2.18) of ref. [5] and is a nonlocal observable integrated over a 3-sphere.

3 The quantum topological properties of the instantons

In this section I briefly recall how link numbers appear in topological Yang-Mills theory with the BPST instanton and what happens in the other topological field theories that have been solved explicitly so far. The purpose of this section is to collect the essential features of the matter, while, for the detailed proofs and calculations, the reader should check ref.s [8, 5].

The topological observables $\mathcal{O}_{\gamma_i}$ correspond to closed differential forms ω_{γ_i} on the moduli space $\mathcal{M}$. In the interior of $\mathcal{M}$ such forms are also exact and we can define Ω_{γ_i}'s such that $\omega_{\gamma_i}=d\Omega_{\gamma_i}$. We can thus write

$$\mathcal{A}=<\prod_i\mathcal{O}_{\gamma_i}>=\int_{\mathcal{M}}\prod_i\omega_i=\int_{\partial\mathcal{M}}\Omega_1\prod_{i\neq 1}\omega_i. \tag{3.1}$$

Now, ω_{γ_i} are generated by $\frac{1}{16\pi^2}\hat{F}^a\hat{F}^a$ [8], which can also be written as $\hat{d}\hat{C}$, $\hat{C}$ being the BRST extended Chern-Simons form, while Ω_{γ_j} are generated by $\hat{C}$. On $\partial\mathcal{M}$, i.e. when $\rho\to 0$ and $d\rho$ is set to zero the explicit solution elaborated in ref.s [8, 5] gives, not surprisingly,

$$\frac{1}{16\pi^2}\hat{F}^a\hat{F}^a(x)\to\frac{1}{4!}\delta(x-x_0)\,dV(x-x_0)=-\frac{1}{4!\pi^2}\hat{d}\,\partial_\mu\frac{1}{(x-x_0)^2}\,d\sigma_\mu(x-x_0), \tag{3.2}$$

so that

$$\omega_{\gamma_i}\to\sim\int_{\gamma_i}\delta(x_i-x_0),\qquad \Omega_{\gamma_j}\to\sim\int_{\gamma_j}\partial\frac{1}{(x_j-x_0)^2}. \tag{3.3}$$

Consequently,

$$<\prod_i\mathcal{O}_{\gamma_i}>\sim\int_{\mathbb{R}^4}dx_0\int_{\gamma_1}\partial\frac{1}{(x_1-x_0)^2}\prod_{i\neq 1}\int_{\gamma_i}\delta(x_i-x_0)\sim \chi(\gamma_1,\ldots\gamma_n). \tag{3.4}$$

This facts show that there is a very simple generalization of the concept of linking; it is the so-called multilink invariant $\chi(\gamma_1, \ldots \gamma_n)$ that can be reduced to the usual link number between one chosen submanifold, say γ_1, and the intersection among the other ones, the result being independent of the chosen γ_1:

$$\chi(\gamma_1, \ldots \gamma_n) = \chi(\gamma_1, \cap_{i\neq 1}\gamma_i). \tag{3.5}$$

In the formulas sketched above, the symbol $\sim$ means that numerical factors have been neglected. One can check [8, 5], nevertheless, that the link numbers turn out to be normalized correctly.

Now, due to the topological embedding, the link invariants $\ll 1 \gg_{\mathcal{Q}}$ just recalled are also exact QCD amplitudes. Therefore, since we expect that QCD exhibits confinement at the nonpreturbative level, it is natural to ask ourselves whether the above results are compatible with it or not. Indeed they are, if we interpret them as a non-abelian analogue of the Aharonov-Bohm effect [5]. In the investigation of the physical relevance of the topological quantities, the Aharonov-Bohm effect is very important, because it is a noticeable example of an experimental phenomenon in which a purely topological quantity (precisely a link number) plays a central role. The link number reveals that the magnetic field is trapped inside the (closed, circular) solenoid Γ, like in Fig. 3. Γ can be conveniently idealized to a closed circle, corresponding to a single magnetic force line. Since $\ll 1 \gg_{\mathcal{Q}}$ are the only exact nonperturbative QCD amplitudes that we possess today, they can be viewed as a trace that non-abelian Yang-Mills theory confines, in the sense that they are consistent with this fact and no exact result available at present is in contraddiction with it. The same cannot be said of the link numbers in QED, of course: it is sufficient to open the solenoid to deconfine the field. In the realm of the amplitudes $\ll 1 \gg_{\mathcal{Q}} \propto < \prod_i \mathcal{O}_{\gamma_i} >$, the observables $\mathcal{O}_\gamma$ are necessarily associated to *closed* $\mathbb{R}^4$-submanifolds γ_i: opening them would deconfine the field, but this is forbidden by the gauge invariance. The intuitive picture that we have just worked out also suggests that the closed submanifolds γ can be viewed as effective color force *lines, surfaces, 3-spheres, etc.*

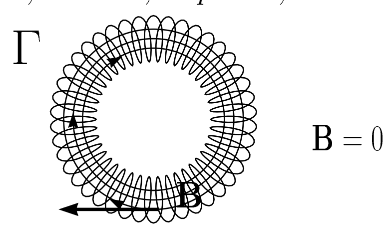

Fig. 3: trapping the magnetic field in a compact region.

Another important consequence of the investigation of ref. [5] is that an observable $\mathcal{O}$ can be 'sensitive' to the positions of the other ones appearing in the same amplitude $\mathcal{A} = < \prod_i \mathcal{O}_{\gamma_i} >$. This is also a novelty with respect to the previous idea about this kind of topological amplitudes, that were expected to be only sensitive to the spacetime manifold (like the so-called Donaldson invariants) or that were explicitly shown to be constants (like the gaugino condensates [1]) and not step functions. Consequently, the results of ref.s [8, 5] drastically change our vision of four dimensional topological field theories. The topological map mentioned in the introduction can push much further in this new direction.

Topological gravity, in the formulation proposed by Frè and the author in ref. [12], was solved by the author in ref. [8] with the Eguchi-Hanson gravitational instanton (and coupled to topological abelian Yang-Mills theory). This theory exhibits nonvanishing amplitudes associated to 3-dimensional closed submanifolds. Now, there is no nontrivial 3-cycle on the Eguchi-Hanson manifold, apart from one special case. At the boundary of the moduli space, the Eguchi-Hanson manifold degenerates to $\mathbb{R}^4/\mathbb{Z}_2$. If the 3-cycle is *linked* to the singular point of $\mathbb{R}^4/\mathbb{Z}_2$, then the result is finite and nonzero, otherwise the result is zero. The Eguchi-Hanson manifold has a noncontractible 2-sphere, to which a topological observable is associated that also gives nonvanishing amplitudes [8]. Anyway, it is interesting to notice that the appearance of some concept of *linking* seems to be quite a general feature of four dimensional topological field theory.

The method proposed in section 2 of ref. [8] for solving topological field theories explicitly when the explicit expression of the instantons is known turns out to be more powerful than expected. Indeed, in certain cases, it is not necessary to know the explicit expressions of the instantons in order to compute their quantum topological properties. This fact will be exemplified by the theory solved in the next section. It is enough to have the explicit parametrization of a continuous deformation $\mathcal{M}'$ of the true moduli space $\mathcal{M}$.

Collecting the results of [8, 5] and the present paper, the method of ref. [8] has been successful, up to now, for solving:

i) topological Yang-Mills theory with the BPST instanton; this theory was also coupled to hyperinstantons [13, 14] in section 3 of ref. [5];

ii) four dimensional topological gravity [12] with the Eguchi-Hanson instanton; this theory was also coupled to topological abelian Yang-Mills theory;

iii) the topological version of the two dimensional Ginzburg-Landau theory of superconductivity, both on the plane and on the torus, with and without external magnetic field. There are no link invariants and the computation of the topological amplitudes reduces to a straightforward, but tedious matter of counting. It is somehow similar to what happens in two dimensional topological gravity [15].

4 The topological version of the Ginzburg-Landau theory of superconductivity

In this section I solve the topological version of the Ginzburg-Landau theory of superconductivity with $\lambda = 1$. Due to a theorem proved in ref. [16], the result collects the quantum topological properties of the full set of solutions to the classical field equations of the theory. First, the theory is solved on $\mathbb{R}^2$; in subsection 4.3 the results are generalized to the torus.

The Ginzburg-Landau theory of superconductivity is described by the free-energy

$$\mathcal{L} = F_{\mu\nu}^2 + \frac{1}{2}|D_\mu q^i|^2 + \frac{\lambda}{8}\mathcal{P}^2. \tag{4.1}$$

where $F_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)$, $Dq^i = dq^i + \varepsilon^{ij} A q^j$ and $\mathcal{P} = 1 - q^2$. λ is the unique parameter that survives trivial redefinitions and distinguishes type I superconductors ($\lambda < 1$) from type II superconductors ($\lambda > 1$). The value of λ depends on the material, on the amount of impurity and, very slightly, on the temperature. q is an order parameter that describes the distribution of Cooper pairs. For other details and the relation with the BCS microscopic theory the reader is referred to ref. [17].

The above "free-energy" will be called "lagrangian". I take it as the lagrangian of a two dimensional quantum field theory in the Euclidean framework and I consider the associated functional integral. In the intermediate situation $\lambda = 1$, on which I focus from now on and that corresponds to the bosonic lagrangian obtained by topologically twisting an N=2 supersymmetric theory [18], $\mathcal{L}$ can be written as the sum of the squares of two instantonic conditions plus a topological invariant

$$\mathcal{L} = \left(F_{\mu\nu} + \frac{1}{4}\varepsilon_{\mu\nu}\mathcal{P}\right)^2 + \frac{1}{4}\left(D_\mu q^i - \varepsilon_{\mu\nu}D_\nu q^j \varepsilon_{ji}\right)^2 - \frac{1}{2}\varepsilon_{\mu\nu}\Omega_{\mu\nu}. \tag{4.2}$$

The differential form $\Omega = \varepsilon_{ij}Dq^iDq^j + F\mathcal{P} = \Omega_{\mu\nu}dx^\mu dx^\nu$ is closed, $d\Omega = 0$, so that the last term of (4.2) is indeed a topological invariant, related to the so-called *fluxoid*. One is interested in studying the *vortex* equations

$$F_{\mu\nu} + \frac{1}{4}\varepsilon_{\mu\nu}\mathcal{P} = 0, \qquad D_\mu q^i - \varepsilon_{\mu\nu}D_\nu q^j\varepsilon_{ji} = 0, \tag{4.3}$$

the second ones being the covariantized version of the Cauchy-Riemann equations.

The quantization of the *fluxoid* corresponds to the Bohr-Sommerfeld quantization rule [17]. Indeed, Ω is locally exact, $\Omega = d\omega$. One finds $\omega = \varepsilon_{ij}q^idq^j + A\mathcal{P} = J + A$, $J = \rho v$ being the electromagnetic current, $\rho = q^2$ being the density of Cooper pairs and v being the velocity. Now, the requirement of finite action implies, in particular, that $\rho \to 1$ at infinity, so that

$$\int_{\mathbb{R}^2}\Omega = -\oint_{C_\infty}\omega = -\oint_{C_\infty} v + A = -\oint p\cdot dq = -2\pi n, \tag{4.4}$$

having written $\partial\mathbb{R}^2 = -C_\infty$. It is clear that the integer n, that I call the *vorticity*, can only take values with a definite sign. The conventions have been chosen so that $n \geq 0$. We have $\mathcal{L} = \pi n$, so that $n = 0$ only with $q^2 = 1$, $A =$pure gauge.

From the mathematical point of view, the integer number n is the *degree* of the map $q : \mathbb{R}^2 \to \mathbb{R}^2$ and is generalized to the case $q : \Sigma_g \to M$ and gauge group G according to $\omega = i(\bar\partial\mathcal{K} - \partial\mathcal{K}) + A^a\mathcal{P}_a$, Σ_g denoting a genus g Riemann surface and M denoting a Kähler target manifold with Kähler potential $\mathcal{K}$ and Kähler form $K = 2i\partial\bar\partial\mathcal{K} \equiv K_{ij}dq^idq^j$. The vector potential A^a gauges M-isometries associated with Killing vectors $\mathbf{k}_a = k_a^i(q)\frac{\partial}{\partial q^i}$, $[\mathbf{k}_a, \mathbf{k}_b] = -f^c{}_{ab}\mathbf{k}_c$. $\mathcal{P}_a$ is defined by $i_{\mathbf{k}_a}K = -d\mathcal{P}_a$, the arbitrary constants being fixed by imposing the equivariance condition $0 = K_{ij}k_a^ik_b^j + \frac{1}{2}f^c{}_{ab}\mathcal{P}_c$.

The vortex equations admit two generalizations to four dimensions. One is simply their straightforward reinterpretation on four dimensional Kähler manifolds and is related to N=1 supersymmetry, in the same way as in two dimensions vortices are related to N=2 supersymmetry [18]. The second type of equations, instead, were introduced by Fré and the author in ref. [13]. Their solutions are called *hyperinstantons* and are related to N=2 supersymmetry. They generalize the Cauchy-Riemann equations to maps (called *tri-holomorphic maps*) between hyperKähler, quaternionic Kähler, or simply almost quaternionic manifolds. They can be naturally gauged and coupled to gravity. The fluxoid becomes the so-called *hyperinstanton number*.

Coming back to our problem, we note that the second equations of (4.3) can be solved explicitly for the gauge field A in terms of the scalar $q^i = |q|(\cos\theta, \sin\theta)$ as follows

$$A = d\theta - dx^\mu \varepsilon_{\mu\nu}\partial_\nu \ln|q|. \tag{4.5}$$

The angle θ is uniquely fixed only up to continuous deformations, that indeed correspond to the $U(1)$ gauge transformations. In general, there are points a_j, $j = 1, \ldots n$ such that $d^2\theta = -2\pi\sum_{k=1}^n \delta(x-a_k)d^2x$ and one can choose

$$\theta = \sum_{k=1}^n \mathrm{arctg}\frac{(x-a_k)_1}{(x-a_k)_2} \equiv \sum_{k=1}^n \theta_k. \tag{4.6}$$

θ and $d\theta$ are singular in the points a_k. However, A and q^i must be regular everywhere. The only possibility for this to happen is that $|q| \to 0$ for $x \to a_k$. Form (4.5) we see that in order for the singularity of $d\theta$ to be cancelled by the one of $d\ln|q|$, $|q|$ has to behave like $|x-a_k|^{n_k}$ for $x \to a_k$, $n_k > 0$, so that $d^2\theta = -2\pi\sum_k n_k\delta(x-a_k)d^2x$. $n_k > 0$ $\forall k$ means that there are only vortices and not antivortices. For the moment, we assume $n_k = 1$ $\forall k$, the other cases taking place when some a_j's coincide.

Inserting (4.5) into the first of (4.3) produces the following Liouville equation for $\phi = \ln q^2$:

$$\Box\phi - 4\pi\sum_{k=1}^n \delta(x-a_k) = \mathrm{e}^\phi - 1. \tag{4.7}$$

Although I am not going to prove it rigorously, this equation suggests that there is one and only one solution with $\phi \to 0$ at infinity for any given set of a_j's. The proof can be found in ref. [19]. We conclude that the a_j's are the moduli and that the moduli space $\mathcal{M}_n$ is the symmetric product $S^n\mathbb{R}^2$ of n copies of the plane. In practice, we can integrate each a_j over $\mathbb{R}^2$ and divide the result by $n!$. When some a_j's coincide, there should be a different symmetry factor, but this does not concern us, at least for now, because it only affects an $\mathcal{M}_n$-subspace of vanishing measure. I shall return later to this point.

Thus, when $\lambda = 1$ the vortices can have arbitrary positions, without interacting with one another. Instead, when $\lambda < 1$ they attract, while when $\lambda > 1$ they repel, this being, qualitatively, the difference between type I and type II superconductors [17].

The BRST algebra of the theory is

$$sA_\mu = \partial_\mu C, \quad sC = 0, \quad sq^i = -C\varepsilon^{ij}q^j. \tag{4.8}$$

The BRST algebra of the topological version of the same theory is obtained by introducing additional ghosts ψ_μ, ϕ and ξ^i so as to kill any local degree of freedom, while preserving $s^2 = 0$:

$$\begin{aligned} sA_\mu &= \psi_\mu + \partial_\mu C, \quad sC = \phi, \quad sq^i = \xi^i - C\varepsilon^{ij}q^j, \\ s\psi_\mu &= -\partial_\mu\phi, \qquad s\phi = 0, \quad s\xi^i = \phi\varepsilon^{ij}q^j - C\varepsilon^{ij}\xi^j. \end{aligned} \tag{4.9}$$

I shall solve the topological theory by using the method explained in section 2 of ref. [8] and by combining this method with the theorems proved by Taubes in ref.s [19, 16] about the solutions of (4.3).

First of all, let us recall a very important fact proved in ref. [16]: the set of solutions to the vortex equations (4.3) is the complete set of solutions to the field equations of the theory. In other words, not only it is true that the solutions to the instanton equations are solutions to the field equations, which is obvious, but the converse is also true (with the boundary conditions that follow from the requirement of finite action). This implies that we shall find the quantum topological properties of the full set of solutions to the field equations of the theory.

The second noticeable fact, already recalled in section 3, is the following: the method of section 2 of ref. [8] is more powerful than expected, in the sense that it is not necessary to have the explicit parametrization of the space $\mathcal{M}$ of minima in order to find the quantum topological properties of this space itself: it is enough to have the explicit parametrization of a subset $\mathcal{M}'$ of the functional space that is a continuous deformation of the true moduli space $\mathcal{M}$. In our case, I take

$$q^i = \prod_{k=1}^{n} \frac{|x-a_k|}{\sqrt{\zeta+(x-a_k)^2}}(\cos\theta, \sin\theta), \tag{4.10}$$

ζ being a useful extra parameter (but not a modulus). The gauge field A, determined from (4.5), and its field strength are

$$A = -\sum_{k=1}^{n} \frac{\varepsilon_{\mu\nu}(x-a_k)^\mu dx_\nu}{\zeta+(x-a_k)^2}, \qquad F = dA = -\zeta \sum_{k=1}^{n} \frac{\varepsilon_{\mu\nu} dx^\mu dx^\nu}{(\zeta+(x-a_k)^2)^2}. \tag{4.11}$$

In this way, the covariantized Cauchy-Riemann equations are satisfied. What is not satisfied is the first equation of (4.3) or, alternatively, the Liouville equation (4.7). Nevertheless, the above configurations are, for any $\zeta > 0$, in one-to-one correspondence with the true solutions [19] and belong to the same topological sector:

$$-\frac{1}{2\pi}\int_{\mathbb{R}^2} \Omega = -\frac{1}{2\pi}\int_{\mathbb{R}^2} F = n. \tag{4.12}$$

Consequently, I argue that the quantum topological properties are also the same and I solve the theory with (4.10) and (4.11).

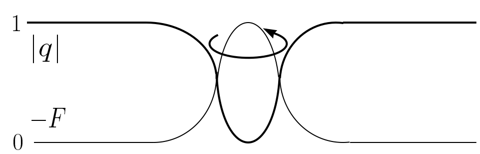

Fig. 4: typical aspect of a vortex.

Notice that the "wrong" configurations that we use in replacement of the true solutions, nevertheless have the same basic physical features of the true solutions themselves. See Fig. 4.

For example, q vanishes in the centers a_k of the vortices and only there, while the field strength is maximal in those points.

The key step in order to solve explicitly the topological theory is, according to section 2 of ref. [8], the calculation of the ghost C, from which everything else follows automatically. C is fixed by a gauge-fixing the "second" gauge symmetry (I use the terminology of [8]), i.e. the symmetry generated by ϕ. Here, we do not possess a natural choice of this gauge-fixing condition. Moreover, we have used very simple explicit configurations (that, nevertheless, could be the solutions of very complicated equations) in replacement of possibly complicated (and not explicitly known) solutions of very simple instanton equations. Very presumably the gauge-fixing that we need is also very complicated. In brief, it is better to guess C directly, without referring to any explicit gauge condition. As before, we argue that the topological properties are independent of all such details, as long as we deal with well-defined quantities.

Noticing that every modulus is a translational one and taking inspiration from the solution of topological Yang-Mills theory with the BPST instanton [8], we can guess

$$C=\sum_{k=1}^{n}\frac{\varepsilon_{\mu\nu}(x-a_k)^\mu da_k^\nu}{\zeta+(x-a_k)^2}, \tag{4.13}$$

so that the BRST extensions of $\hat{A}=A+C$ and $\hat{F}=\hat{d}\hat{A}=F+\psi+\phi$ of A and F (where $\hat{d}=d+s$ and $s=\sum_{k=1}^{n}da_k^\mu\frac{\partial}{\partial a_k^\mu}$ is the $\mathcal{M}_n$-exterior derivative) take the very simple forms

$$\hat{A}=-\sum_{k=1}^{n}\frac{\varepsilon_{\mu\nu}(x-a_k)^\mu d(x-a_k)^\nu}{\zeta+(x-a_k)^2},\quad \hat{F}=-\zeta\sum_{k=1}^{n}\frac{\varepsilon_{\mu\nu}d(x-a_k)^\mu d(x-a_k)^\nu}{(\zeta+(x-a_k)^2)^2}. \tag{4.14}$$

The choice of C is not arbitrary, as it may seem at first sight ($C=0$, for example, is not good). Indeed, while A_μ, q and sA_μ, sq are not normalizable, in general (and they do not need to be, since they are not physical fields), ψ_μ and ξ^i *have* be normalizable, because they are strict relatives of F and Ω and so they appear in the physical observables, that are generated by the BRST extensions $\hat{F}$ and $\hat{\Omega}$. This requirement guarantees that the physical amplitudes are well-defined and topological.

The above choice of C was studied in order to produce a normalizable ψ_μ. A consistency-check is that it also produces a regular and normalizable ξ^i. Indeed, (4.9) gives

$$\xi=(\xi^1,\xi^2)=-\zeta\sum_{k=1}^{n}\frac{|q|}{|x-a_k|(\zeta+(x-a_k)^2)}\Big(\sin(\theta_k-\theta)da_k^1+\cos(\theta_k-\theta)da_k^2,$$
$$\cos(\theta-\theta_k)da_k^1+\sin(\theta-\theta_k)da_k^2\Big). \tag{4.15}$$

This expression goes like $\frac{1}{|x|^3}$ for $|x|\to\infty$, so that we only have to check that it is regular for $x\to a_k$. Consider the k-th term in (4.15): when $x\to a_l$ with $l\neq k$, such term tends to zero, due to the factor $|q|$; instead when $x\to a_k$, $\frac{|q|}{|x-a_k|}$ is regular and $\theta-\theta_k$ has a well-defined limit (note that θ_k has no well-defined limit for $x\to a_k$).

The observables of the theory are

$$\mathcal{O}^m(x)=\frac{(-1)^m}{(2\pi)^m}\phi^m(x),\quad \mathcal{O}^{(d)}=\frac{(-1)^d}{(2\pi)^d}\int_{\mathbb{R}^2}\hat{F}^d, \tag{4.16}$$

plus the ones generated in a completely similar way by the BRST extension $\hat{\Omega}$ of Ω. Here we focus on the observables (4.16), because (4.12) suggests that $\hat{\Omega} \sim \hat{F}$ inside the amplitudes. This fact has been explicitly checked in the simplest cases. Moreover, the calculations with $\hat{F}$ are much simpler than the ones with $\hat{\Omega}$. Indeed, recalling that (4.10) and (4.11) are a continuous deformation of the true solutions for *any* value of ζ, we can choose the most convenient ζ, that is $\zeta = 0$. In this limit we get only delta functions

$$-\frac{1}{2\pi}\hat{F} \rightarrow \sum_{k=1}^{n} \delta(x-a_k)\, d^2(x-a_k), \tag{4.17}$$

so that the calculation of the topological amplitudes reduces to a pure combinatorial counting, while any integration is trivial. This is more or less what happens in two dimensional topological gravity, where Strebel's theorem [20] allows one to reduce the problem to a simple combinatorial counting, that can also be encoded into a matrix model [15]. There, the limit $\zeta \rightarrow 0$ corresponds to deforming the Riemann surface so as to take the punctures at infinity and reduce basically to the case in which the Riemann curvature has a delta support on the punctures.

The situation is different, instead, in the case of topological Yang-Mills theory with the BPST instanton, as I recalled in section 3, where not every observable inside an amplitude can be reduced to a delta function, but one, and only one (the result being independent of *which* one), has to be replaced by the Chern-Simons form, this being the simple reason why link numbers appear instead of constant amplitudes.

The generic amplitude that we consider is

$$\mathcal{A}_{\{d\}m} =< \mathcal{O}^m(x) \cdot \prod_{i=1}^{k} \mathcal{O}^{(d_i)} > . \tag{4.18}$$

The vorticity contributing to this amplitude is

$$n = \sum_{i=1}^{k}(d_i - 1) + m. \tag{4.19}$$

(4.17) shows immediately that $\mathcal{A}$ is independent of the positions of the local observables. This is why in (4.18) I have put the local observables in the same point x. The computation of the above amplitudes is now a simple, but tedious matter of counting. One has to decompose

$$d^2(x-a_i) = d^2x + d^2a_i - \varepsilon_{\mu\nu}dx^\mu da_i^\nu, \tag{4.20}$$

Were it not for the last term, to which we shall refer as the *double product* in the sequel, it would be immediate to extract the relevant components to the various observables. Let us write

$$\mathcal{A}_{\{d\}m} = \frac{1}{n!}\sum_{j=0}^{k} C^{(j)}_{\{d\}m}, \tag{4.21}$$

$C^{(0)}_{\{d\}m}$ denoting the contribution in which the double products are completely neglected and $C^{(j)}_{\{d\}m}$ being the contribution where j and only j pairs of double products are taken into account.

I show in the appendix that

$$C^{(0)}_{\{d\}m} = n! \prod_{i=1}^{k} d_i(n-d_i+1), \qquad C^{(j)}_{\{d\}m} = -(j-1)\, C^{(0)}_{\{d\}m} \sum_{\sigma_j} \prod_{p\in\sigma_j} \frac{(d_p-1)}{(n-d_p+1)}, \tag{4.22}$$

σ_j denoting a j-uple of elements of the set $\{1,\dots k\}$. The final sum gives

$$\mathcal{A}_{\{d\}m} = n^{k-1}\, m \prod_{i=1}^{k} d_i. \tag{4.23}$$

Indeed, it is easy to show that $\forall n$, given k integer numbers $d_1,\dots d_k$, the following identity holds

$$f_k \equiv \sum_{j=0}^{k} \sum_{\sigma_j} \prod_{p\in\sigma_j} (d_p-1) \prod_{q\notin\sigma_j} (n-d_q+1) = n^k. \tag{4.24}$$

This is straightforward, by induction. For $k=0$ we have $1=1$. For generic k, distinguishing those σ_j that contain k from those that do not contain it, we can write

$$\begin{aligned} f_k = (d_k-1) &\sum_{j=1}^{k} \sum_{\sigma'_{j-1}} \prod_{p\in\sigma'_{j-1}} (d_p-1) \prod_{q\notin\sigma'_{j-1}} (n-d_q+1) \\ &+(n-d_k+1) \sum_{j=0}^{k-1} \sum_{\sigma'_j} \prod_{p\in\sigma'_j} (d_p-1) \prod_{q\notin\sigma'_j} (n-d_q+1) = n\, f_{k-1}, \end{aligned} \tag{4.25}$$

where σ'_j are j-uples of elements of the set $\{1,\dots k-1\}$. In a similar way one proves that

$$\begin{aligned} g_k \equiv \sum_{j=0}^{k} j \sum_{\sigma_j} \prod_{p\in\sigma_j} (d_p-1) \prod_{q\notin\sigma_j} (n-d_q+1) = (d_k-1)n^{k-1} \\ +n \sum_{j=0}^{k-1} j \sum_{\sigma'_j} \prod_{p\in\sigma'_j} (d_p-1) \prod_{q\notin\sigma'_j} (n-d_q+1) = n g_{k-1} + n^{k-1}(d_k-1). \end{aligned} \tag{4.26}$$

We have $\mathcal{A}_{\{d\}m} = h_k \prod_{i=1}^{k} d_i$, with $h_k = f_k - g_k = n^k - g_k = n h_{k-1} - n^{k-1}(d_k-1)$. With $h_0=1$ it is easy to prove that the recursion relation is solved by $h_k = n^{k-1}\left[n - \sum_{i=1}^{k}(d_i-1)\right] = n^{k-1} m$, from which the result (4.23) follows.

A check that formula (4.23) is correct is that it turns out to be proportional to m, although no $C^{(j)}_{\{d\}m}$ is. Indeed, $\mathcal{A}_{\{d\}0}$ has to be zero, for the following simple reason. Expression (4.17) shows that the arguments of the delta functions are *differences* of points. When $m=0$, the local observable is absent and there are n delta functions depending on $n-1$ *differences*. Consequently, there would be a $\delta(0)$, which, however, is multiplied by a zero coefficient. Instead, when $m\neq 0$, there is one additional point around, namely the point x where the local observable $\mathcal{O}^m(x)$ is placed, so that there are n delta functions for n differences.

Moreover, $\mathcal{A}_{\{d\}m}$ is proportional to any d_i, consistently with the fact that $\mathcal{O}^{(d)}=0$ for $d=0$.

4.1 The τ-function

With the result (4.23) one can compute the τ-function

$$\tau[t] = \int d\mu \, \exp\left(\sum_{i=0}^{\infty} t_i \mathcal{O}^{(i)}\right), \tag{4.27}$$

$\mathcal{O}^{(0)}$ denoting the local observable $\mathcal{O}(x)$. One finds

$$\tau[t] = \sum_{n=0}^{\infty} \mathrm{e}^{nt_1} \tau_n[t], \quad \tau_0[t] = 1, \quad \tau_n[t] = \frac{1}{n} \sum_{\mathcal{K}_n} \frac{t_0^{k_0}}{(k_0-1)!} \prod_{i=2}^{\infty} \frac{(int_i)^{k_i}}{k_i!} \text{ for } n > 0, \tag{4.28}$$

$\mathcal{K}_n$ denoting a string of natural numbers $\{k_0, k_2, k_3, \ldots\}$ such that $k_0 + \sum_{i=2}^{\infty}(i-1)k_i = n$, $k_0 > 0$ and $k_i \geq 0 \; \forall i \geq 1$. $\tau_n[t]$ is a finite polynomial $\forall n$.

Putting $t_i = 0$ for $i > 1$ in (4.27) one has

$$\tau[t_0, t_1] = \exp\left(t_0 \, \mathrm{e}^{t_1}\right). \tag{4.29}$$

Putting $t_i = 0$ for $i > 2$, instead, the τ-function is convergent for $|2\mathrm{e}t_2\mathrm{e}^{t_1}| < 1$:

$$\tau[t_0, t_1, t_2] = t_0 \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{e}^{nt_1} (t_0 + 2nt_2)^{n-1}, \tag{4.30}$$

The first $\tau_n[t]$'s are

$$\begin{aligned}
\tau_1[t] &= t_0, \qquad \tau_2[t] = \frac{1}{2} t_0^2 + 2t_0 t_2, \qquad \tau_3[t] = \frac{1}{6} t_0^3 + 2t_0^2 t_2 + 6t_0 t_2^2 + 3t_0 t_3, \\
\tau_4[t] &= \frac{1}{24} t_0^4 + t_0^3 t_2 + 8t_0^2 t_2^2 + 3t_0^2 t_3 + \frac{64}{3} t_0 t_2^3 + 24 t_0 t_2 t_3 + 4t_0 t_4, \\
\tau_n[t] &= \frac{1}{n!} t_0^n + \frac{2}{(n-2)!} t_0^{n-1} t_2 + \frac{1}{(n-3)!} t_0^{n-2} (2nt_2^2 + 3t_3) \\
&\quad + \frac{1}{(n-4)!} t_0^{n-3} \left(\frac{4}{3} n^2 t_2^3 + 6nt_2 t_3 + 4t_4\right) + \cdots.
\end{aligned} \tag{4.31}$$

Summing term by term, we get expressions like

$$\begin{aligned}
\tau[t] &= (1 + 2t_0 t_2 + 3t_0 t_3 + 4t_0 t_4 + \cdots + 6t_0 t_2^2 + 24 t_0 t_2 t_3 + \cdots) \exp\left(t_0 \, \mathrm{e}^{t_1}\right) \\
&\quad + (2t_0^2 t_2^2 + 6t_0^2 t_2 t_3 + \cdots) \exp\left(t_1 + t_0 \, \mathrm{e}^{t_1}\right) + \cdots.
\end{aligned} \tag{4.32}$$

4.2 Impurities

Let us now suppose that, for some reason[2], the integral over the moduli space $\mathcal{M}$ is repalced by the integral over some proper $\mathcal{M}$-subspace $\mathcal{M}^{\{n\}}$, for example a diagonal subspace, i.e. a subset where the positions of certain vortices coincide, so that

$$-\frac{1}{2\pi} \hat{F} = \sum_{j=1}^{\infty} j \sum_{i=1}^{n_j} \delta(x - a_j^i) \, d^2(x - a_j^i). \tag{4.33}$$

[2] In the realm of topological field theory, such situation can perhaps be obtained *via* a constraining mechanism like the one studied in ref. [21] (constrained topological field theory).

This means that there are n_j vortices with vorticity j. The total vorticity is $v = \sum_{j=1}^{\infty} j n_j$. The (constrained) moduli space is $\mathcal{M}_v^{\{n\}} = \bigotimes_{j=1}^{\infty} S^{n_j}\mathbb{R}^2$, $n = \dim \mathcal{M}_v^{\{n\}} = \sum_{j=1}^{\infty} n_j$. The symmetry factor is $\prod_{j=1}^{\infty} n_j!$. Because of the reduced symmetry, we speak about "impurity", when referring to the projection onto $\mathcal{M}_v^{\{n\}}$, the kind of impurity being specified by the set of integer numbers $\{n\} = \{n_1, \ldots n_j, \ldots\}$. The amplitudes are formally the same as in (4.18) and I denote them by $\mathcal{A}_{\{d\}m}^{\{n\}}$.

In the general case, the counting is much more involved than before. One has to go through the detailed derivation of the appendix, where the case without impurities is analysed in full detail, and improve the arguments when necessary.

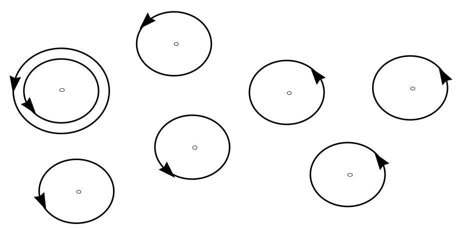

Fig. 5: an impurity.

For example, in the case with one impurity, say when the vortex placed in a_1 has vorticity I, one finds

$$\begin{aligned}\mathcal{A}_{\{d\}m}^{(I)} &= I \prod_{i=1}^{k} d_i (n - d_i + I) \sum_{j=0}^{k} (1-j) \sum_{\sigma_j} \prod_{p \in \sigma_j} \frac{d_p - 1}{n - d_p + I} \left[n + (I-1) \left(j - \sum_{q \notin \sigma_j} \frac{d_q - 1}{n - d_q + I} \right) \right] \\ &= I v^k m \prod_{i=1}^{k} d_i, \end{aligned} \tag{4.34}$$

where $n = m + \sum_{i=1}^{k} (d_i - 1)$, as before, while $v = n + I - 1$. As we see, the various contributions still sum up nicely as before. Assuming that this also holds in the general case, a general formula that agrees with the results found so far is

$$\mathcal{A}_{\{d\}m}^{\{n\}} = < \mathcal{O}^m(x) \cdot \prod_{i=1}^{k} \mathcal{O}^{(d_i)} >= v^k (n-1)!\, m \prod_{i=1}^{k} d_i \prod_{j=1}^{\infty} \frac{j^{n_j}}{n_j!}. \tag{4.35}$$

The statistical impurity distribution is the Bose-Einstein one, where the role of the energy is played by v. More precisely, I denote the energy of a vortex unity by ε, so that the total energy is $E = \varepsilon v$. A good parameter that quantifies the amount of impurity is

$$d \equiv \frac{\bar{v} - \bar{n}}{\bar{n}} \geq 0. \tag{4.36}$$

A small d corresponds to a small amount of dirtiness. It is easy to check that lowering the temperature reduces the number of impurities: $d = \frac{1}{\bar{n}+1} \mathrm{e}^{-\beta\varepsilon} + \mathcal{O}(\mathrm{e}^{-2\beta\varepsilon})$ for $\beta \to \infty$. The standard Bose statistical distribution partially answers the question raised in point iii) of the introduction. It is restricted, however, to the classical topological invariant v only. It should be possible, using (4.35), to define a generalized statistical distribution for the quantum topological sectors $\mathcal{Q}$.

To conclude this subsection, let us note that a straightforward four dimensional generalization of the counting problem that we had to deal with is obtained by considering observables like

$$\mathcal{O}^{(d)} = \int_{\mathbb{R}^4} \hat{\mathcal{Q}}^d, \quad \mathcal{O}^m(x) = \left[\sum_{k=1}^{n} n_k \delta(x - a_k)\, d^4 a_k\right]^m, \quad \hat{\mathcal{Q}} = \sum_{k=1}^{n} n_k \delta(x - a_k)\, d^4(x - a_k). \tag{4.37}$$

The amplitudes are formally the same as in (4.18), although their evaluation appears to be less straightforward. This problem could be related to some kind of hyperinstantons [13, 14].

4.3 Case of the torus and presence of an external magnetic field

With a simple recursion relation, the results of the previous section can be extended to the case of the torus and to the presence of an external magnetic field. The existence theorems for solutions to the vortex equations on compact Rieman surfaces can be found in ref. [22].

Let V denote the volume of the torus. Despite the fact that the volume is not a topological quantity, it is relevant to our problem. Indeed, integrating the first equation of (4.3) on a compact Riemann surface Σ, one gets [22] in our units

$$v \leq \left[\frac{V}{4\pi}\right] \equiv N. \tag{4.38}$$

This is an interesting example in which a non-topological quantity enters a topological field theory: still, any correlation function is topological, but the *range* of the classical topological invariant depends on the size of Σ. The τ-function becomes a finite sum.

On the torus, besides the pointlike vortices that we have on the plane, there can be a vorticity around the handle, corresponding to the constant two-form

$$-\frac{1}{2\pi} F = \frac{p}{V}\, d^2x, \qquad p \in \mathbb{Z}. \tag{4.39}$$

This extra contribution does not change the moduli space. So, we are lead to consider observables generated by

$$-\frac{1}{2\pi} \hat{F}' = \sum_{j=1}^{\infty} j \sum_{i=1}^{n_j} \delta(x - a_j^i)\, d^2(x - a_j^i) + \frac{p}{V}\, d^2x. \tag{4.40}$$

The prime is used to denote genus one quantities. Now, the total vorticity is $p+v = p+\sum_{j=1}^{\infty} j n_j$, while the dimension of the moduli space $\otimes_{j=1}^{\infty} S^{n_j}\Sigma$ is $n = \sum_{j=1}^{\infty} n_j$, as before. The symmetry factor is also unchanged.

A glance at the correlation functions shows that they are independent of V, as it must be, although V enters (4.40) explicitly. In practice, one can replace (4.40) with a more convenient expression obtained with the substitution $\frac{1}{V} \to \delta(x - y)$, y being an arbitrary point of Σ. One can write

$$\mathcal{O}'^m(x) = \mathcal{O}^m(x), \qquad \mathcal{O}'^{(d)} = p\, d\, \mathcal{O}^{d-1}(y) + \mathcal{O}^{(d)}. \tag{4.41}$$

Thus, provided (4.38) holds, we have, using (4.35),

$$\mathcal{A}^{\{n\}p}_{\{d\}m} = < \mathcal{O}'^{m}(x) \cdot \prod_{i=1}^{k} \mathcal{O}'^{(d_i)} >= (n-1)! \prod_{i=1}^{k} d_i \prod_{j=1}^{\infty} \frac{j^{n_j}}{n_j!} \sum_{j=0}^{k} p^{k-j} v^j \sum_{\sigma_j} \left(n - \sum_{i \in \sigma_j} (d_i - 1) \right)$$
$$= (np + mv)\,(p+v)^{k-1} (n-1)! \prod_{i=1}^{k} d_i \prod_{j=1}^{\infty} \frac{j^{n_j}}{n_j!}. \tag{4.42}$$

The inclusion of the Abelian differentials of the torus can be achieved by replacing A with $A + u_\mu dx^\mu$. The vector u_μ is a modulus and belongs to the fundamental cell of the reciprocal lattice of the torus. F is unchanged, but $\hat{F}$ gets the extra contribution $du_\mu dx^\mu$. In this case, one can also construct observables by integrating over the cycles of the torus. However, the correlation functions reduce to the known ones.

Taking $V \to \infty$, the bound (4.38) drops out and p is no longer required to be an integer. Then one recovers the case of $\mathbb{R}^2$ in presence of an external magnetic field p. For large p, the amplitude behaves like p^k. So, for k even there is a minimum when the magnetic field has certain fractional values, precisely when

$$p = -\frac{v\,(n + m\,(k-1))}{n\,k}. \tag{4.43}$$

It is curious to note that some fractional values of the external magnetic field play a special role (also note the condition k=even), although they do not seem to be related to the values observed in the fractional quantum Hall effect.

4.4 Perspectives

Assuming that (4.1) is the complete free-energy (as a matter of fact, it only contains the first few terms of an expansion in q and its derivatives [17]), the general remarks made in the first part of this paper and the results found in the present section should stimulate the curiosity of seeing what happens experimentally in the special case $\lambda = 1$, where the classical theory fails and the functional integral is required. Such situations have to be treated *via* the topological embedding, so that the quantum topological invariants computed in this section should have a relevant counterpart in the experimental results. When including the higher order corrections to the free energy, it can happen that the minima become discrete, nevertheless the consequences of the topological embedding should still be visible, since (4.1) is a good approximation. Experimentally, one should manage to tune λ across the critical value and examine what sort of transition occurs. The general theoretical set-up developed here suggests that one should find a qualitatively new kind of transition.

5 Conclusion

The best feature of the topological embedding is that it is *testable* in nature and this possibility does not seem so distant. For example, one should estabilish whether QCD or QCD* better describe the real world. A key qualitative feature of the topological embedding is that any event is placed in a *single* topological sector. However, the new idea is quite general and can be fruitfully applied to many other problems, for example superconductivity in a very special case. This fact makes the new approach more powerful and more easily testable. In the present paper the fundamental guidelines in this research area were estabilished. Surely it is worth insisting in this direction.

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6 Appendix: proof of formulæ (4.22)

In this appendix, I prove formulæ (4.22). The proof is done in several steps. It is one of those cases in which it can be easier to work out the proof independently than reading it. Anyway, for completeness, I have to write down the details.

Step 1: $C^{(0)}_{\{d\}m}$.

Let us begin by computing $C^{(0)}_{\{d\}m}$. Each observable involves the polynomial $\hat{F}$, that is a sum of monomials of the form $\delta(x-a)\,d^2(x-a)$, raised to the power m (local observable) or d_i (nonlocal). When expanding the polynomial, each monomial can only be raised to the powers 0 or 1, because the square of $d^2(x-a)$ vanishes. So, there is a combinatorial factor

$$m!\prod_{i=1}^{k} d_i! \tag{6.1}$$

simply coming from the expansion of the polynomials. Now, one has distribute the powers 0 and 1 and count the number of possibilities. The local observable $\mathcal{O}^m(x)$, for which $d^2(x-a)$ reduces to d^2a, has to saturate the integrations over m moduli. There are $\begin{pmatrix} n \\ m \end{pmatrix}$ ways of achieving this. So, we can multiply by $\begin{pmatrix} n \\ m \end{pmatrix}$ and assume that, say, the first m moduli-integrations are saturated. Up to now, we have

$$n!\prod_{i=1}^{k} d_i!\frac{1}{(n-m)!}. \tag{6.2}$$

At this point, we have to distribute the integrations coming from the first nonlocal observable $\mathcal{O}^{(d_1)}$. There are d_1-1 moduli-integrations and one spacetime-integration (such integration being part of the observable and not of the amplitude). The spacetime-integration is naturally associated to a certain modulus a, since one only has differentials like $d^2(x-a)$. Two situations can happen:

i) if the spacetime-integration is associated to one of the first m moduli $a_1, \dots a_m$ (m possibilities), then the d_1-1 moduli integrations can be chosen in $\begin{pmatrix} n-m \\ d_1-1 \end{pmatrix}$ ways;

ii) if the spacetime-integration is associated to any other modulus $\bar{a}=a_i$, $i>m$ ($n-m$ possibilities), then the d_1-1 moduli integrations can be chosen in $\begin{pmatrix} n-m-1 \\ d_1-1 \end{pmatrix}$ ways (the $\bar{a}$-integration cannot be included in this case).

The factor due to $\mathcal{O}^{(d_1)}$ is thus

$$m\begin{pmatrix} n-m \\ d_1-1 \end{pmatrix}+(n-m)\begin{pmatrix} n-m-1 \\ d_1-1 \end{pmatrix}=(n-d_1+1)\begin{pmatrix} n-m \\ d_1-1 \end{pmatrix}, \tag{6.3}$$

so that, together with (6.2), we have, so far,

$$n!\, d_1(n-d_1+1)\frac{1}{(n-m-d_1+1)!}\prod_{i=2}^{k} d_i!. \tag{6.4}$$

and we can assume that the first $m+d_1-1$ moduli integrations are saturated. Now, for $\mathcal{O}^{(d_2)}$, one can proceed exactly as before, with $m\rightarrow m+d_1-1$, and so on. The final result is the claimed one.

Step 2: $C^{(1)}_{\{d\}m}$.

$C^{(1)}_{\{d\}m}$ is trivially zero. Indeed, there is no possibility with only one pair of double products. We can assume that the nonlocal observable interested in this pair of double products is $\mathcal{O}^{(d_1)}$. We call it the *special observable*. Each double product is associated with a moduli integration. Let us say that the $\mathcal{O}^{(d_1)}$-pair of double products is associated to $\bar{a}$ and $\bar{b}$, with $\bar{a}\neq\bar{b}$. That means that the $\bar{a}$ and $\bar{b}$ integrations cannot be completely saturated, unless some other double products, coming from other nonlocal observables, join the game. This cannot happen for $C^{(1)}_{\{d\}m}$ by assumption, but happens in the other cases.

Step 3: $C^{(2)}_{\{d\}m}$.

In this case, instead, the $\bar{a}$ and $\bar{b}$ integrations can be completely saturated, because there are two nonlocal special observables. There will be a sum $\sum_{\sigma_2}$ over the set of couples σ_2 of the special observables. So, we can restrict to $\sigma_2=\{1,2\}$. We can assume that the pairs of double products in $\mathcal{O}^{(d_1)}$ and $\mathcal{O}^{(d_2)}$ correspond to a_{m+1} and a_{m+2}, this producing a factor $\begin{pmatrix} n-m \\ 2 \end{pmatrix}$. The remaining d_1-2 integrations of $\mathcal{O}^{(d_1)}$ can be chosen to saturate $a_{m+3}, \dots a_{m+d_1}$, this producing a factor $\begin{pmatrix} n-m-2 \\ d_1-2 \end{pmatrix}$. Finally, the remaining d_2-2 integrations of $\mathcal{O}^{(d_2)}$ can be chosen to saturate $a_{m+d_1+1}, \dots a_{m+d_1+d_2-2}$, this producing a factor $\begin{pmatrix} n-m-d_1 \\ d_2-2 \end{pmatrix}$. Taking into account that

$$(-\varepsilon_{\mu\nu}dx_1^\mu d\bar{a}^\nu)(-\varepsilon_{\rho\sigma}dx_1^\rho d\bar{b}^\sigma)(-\varepsilon_{\alpha\beta}dx_2^\alpha d\bar{a}^\beta)(-\varepsilon_{\gamma\delta}dx_2^\gamma d\bar{b}^\delta)=-2\,d^2x_1\,d^2x_2\,d^2\bar{a}\,d^2\bar{b}, \tag{6.5}$$

we see that there is a further factor -2 with respect to before. Collecting the factors computed so far, we have

$$(-2)n!\prod_{i=1}^{k} d_i!\frac{1}{(n-m)!}\sum_{\sigma_2=\{i,j\}}\binom{n-m}{2}\binom{n-m-2}{d_i-2}\binom{n-m-d_i}{d_j-2}(\cdots), \tag{6.6}$$

the dots standing for the factors that remain to be computed. From this point onwards one can proceed, for any i and j, as in Step 1, assuming that the first $m+d_i+d_j-2$ moduli integrations are saturated. So, one gets

$$-n!\prod_{i=1}^{k} d_i!\sum_{\sigma_2=\{i,j\}}\frac{1}{(d_i-2)!(d_j-2)!}\prod_{k\neq i,j}\frac{n-d_k+1}{(d_k-1)!}, \tag{6.7}$$

as desired.

Step 4: $C^{(j)}_{\{d\}m}$, $j>2$.

The case $j=2$ is sufficient to illustrate, with some straightforward adaptations, what happens in general. When $j>2$, one has many more possible ways of rearranging the pairs of double products. Of course there is a sum $\sum_{\sigma_j}$ and we focus on $\sigma_j=\{1,\dots j\}$, which means that the special observables are $\mathcal{O}^{(d_1)},\dots\mathcal{O}^{(d_j)}$. One can distinguish some closed paths, in the following way. We have to consider a $j\times j$ matrix. In each row, two and only two entries are 1, all the other ones being 0. The same for each column. The 1's correspond to the positions of the double products. The rows represent different moduli, while the columns represent different special observables. One can always exchange the rows and the columns in such a way that a certain number of closed paths is obtained. For example, for $j=6$ one can have situations like

$$\begin{bmatrix}1&1&&&&\\1&&1&&&\\&1&&1&&\\&&1&&1&\\&&&1&&1\\&&&&1&1\end{bmatrix}\begin{bmatrix}1&1&&&&\\1&1&&&&\\&&1&1&&\\&&1&&1&\\&&&1&&1\\&&&&1&1\end{bmatrix}\begin{bmatrix}1&1&&&&\\1&&1&&&\\&1&1&&&\\&&&1&1&\\&&&1&&1\\&&&&1&1\end{bmatrix}\begin{bmatrix}1&1&&&&\\1&1&&&&\\&&1&1&&\\&&1&1&&\\&&&&1&1\\&&&&1&1\end{bmatrix} \tag{6.8}$$

In the first case there is only one closed path, in the second and third cases there are two, in the last case there are three. In general, the number of such closed paths is between 1 and the integral part of $j/2$. A concrete example of a closed path is given in eq. (6.5), which represents $\begin{pmatrix}1&1\\1&1\end{pmatrix}$. One has to count the number of possibilities like the ones illustrated above, with the appropriate weight. Any closed path has a factor -2, like in (6.5). Of course, there is the overall factor (6.2).

For convenience, let us define

$$a_j=n!\prod_{i=1}^{k} d_i!\sum_{\sigma_j}\prod_{p\in\sigma_j}\frac{1}{(d_i-2)!}\prod_{q\notin\sigma_j}\frac{(n-d_q+1)}{(d_q-1)!}. \tag{6.9}$$

a) The case with one and only one closed path has a weight $-a_j(j-1)!$. Refer to the first example of (6.8). There is a factor -2 for the closed path. The 1's in the first column can be fixed in $\binom{n-m}{2}$ ways. Consider the first row. I can fix the position of the second 1 in $j-1$ ways. With a factor $j-1$, I can put it in position $(1,2)$. Now, consider the second column. I cannot put a 1 in position $(2,2)$, since it would close a path. This is what happens in the second case of (6.8) and will be discussed below. With a factor $n-m-2$, I put the 1 in position $(3,2)$. Then, consider the second row. I can fix the position of the second 1 in $(2,3)$, gaining a factor $j-2$. Proceeding in this way, one obtains the weight $-(j-1)!\frac{(n-m)!}{(n-m-j)!}$ after distributing the double products. Next, the d_i-2 remaining moduli integrations, $i=1,\ldots j$, give factors

$$\binom{n-m-j}{d_1-2}\binom{n-m-j-d_1+2}{d_2-2}\cdots\binom{n-m-j-\sum_{i=1}^{j-1}(d_i-2)}{d_j-2}. \tag{6.10}$$

For the other observables $\mathcal{O}^{(l)}$, $l=j+1,\ldots k$, one can proceed as in the second part of Step 1, with $m\rightarrow m+\sum_{i=1}^{j}(d_i-1)$. Collecting everything and the sum $\sum_{\sigma_j}$, on gets the desired result.

b) Two paths have a weight $a_j(j-1)!\sum_{k=2}^{j-2}\frac{1}{k}$. First of all, there is a factor $(-2)^2$, due to the closed paths. The first column gives $\binom{n-m}{2}$, as before. In the first row, I fix the second 1 in position $(1,2)$, gaining a factor $j-1$. Now, consider the second column: this time I *can* put a 1 in $(2,2)$, as in the second case of (6.8). This will close the first path, and only one path will remain to be closed. Now, consider the third column: with a factor $\binom{n-m-2}{2}$, I can always fix the two 1's in $(3,3)$ and $(4,3)$ and proceed as in point *a)*. I shall have, in total, a weight $a_j(j-1)(j-3)!=a_j\frac{(j-1)!}{(j-2)}$. If, instead, I do not close the first path by putting a 1 in position $(2,2)$, I can always arrange the second column by putting a 1 in $(3,2)$, this giving a factor $n-m-2$. Then, I can close the first path on the third column, for example, which is what happens in the third case of (6.8). This situation is characterized by a weight $a_j\frac{(j-1)!}{(j-3)}$, as it can be easily checked. In other words I can close the first path in any column k such that $2\leq k\leq j-2$. This will produce a weight $a_j\frac{(j-1)!}{(j-k)}$. Summing over the various possibilities, one gets the claimed weight for the two paths.

c) Reasoning in a completely similar way, one realizes that three paths have a weight equal to $-a_j(j-1)!\sum_{\sigma_2^{(j)}}\prod_{p\in\sigma_2^{(j)}}\frac{1}{p}$. $\sigma_2^{(j)}$ stands for the couples $\{k,l\}$ such that $|k-l|>1$ (this is due to the fact that any closed path occupies at least two columns) and $2\leq k,l\leq j-2$. At this point, one easily learns the general rule and immediately proves that k paths have a weight equal to $(-1)^k a_j(j-1)!\sum_{\sigma_{k-1}^{(j)}}\prod_{p\in\sigma_{k-1}^{(j)}}\frac{1}{p}$.

The total coefficient is thus simply $C^{(j)}_{\{d\}m}=c_j a_j$ with

$$c_j\equiv(j-1)!\sum_{k=1}^{[j/2]}(-1)^k\sum_{\sigma_{k-1}^{(j)}}\prod_{p\in\sigma_{k-1}^{(j)}}\frac{1}{p}=1-j. \tag{6.11}$$

The last equality is easily proven by induction. Distinguishing those $\sigma_{k-1}^{(j)}$'s that contain $j-2$

from those that do not, one finds the following recursion relation:

$$c_j = (j-1)! \left(\sum_{k=1}^{\left[\frac{j-1}{2}\right]} (-1)^k \sum_{\sigma_{k-1}^{(j-1)}} \prod_{p \in \sigma_{k-1}^{(j-1)}} \frac{1}{p} + \frac{1}{j-2} \sum_{k=2}^{\left[\frac{j}{2}\right]} (-1)^k \sum_{\sigma_{k-2}^{(j-2)}} \prod_{p \in \sigma_{k-2}^{(j-2)}} \frac{1}{p} \right) = (j-1)(c_{j-1} - c_{j-2}). \tag{6.12}$$

The first values are

$$c_2 = -1, \quad c_3 = -2, \quad c_4 = 3!\left(-1 + \frac{1}{2}\right) = -3, \quad c_5 = 4!\left(-1 + \frac{1}{2} + \frac{1}{3}\right) = -4. \tag{6.13}$$

This concludes the proof.

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