HUTP-94/A040
hepth@xxx/9411049
November, 1994

ANOMALIES IN INSTANTON CALCULUS

Damiano Anselmi

Lyman Laboratory, Harvard University, Cambridge MA 02138, U.S.A.

Abstract

I develop a formalism for solving topological field theories explicitly, in the case when the explicit expression of the instantons is known. I solve topological Yang-Mills theory with the $k = 1$ Belavin *et al.* instanton and topological gravity with the Eguchi-Hanson instanton. It turns out that naively empty theories are indeed nontrivial. Many unexpected interesting hidden quantities (punctures, contact terms, nonperturbative anomalies with or without gravity) are revealed. Topological Yang-Mills theory with $G = SU(2)$ is not just Donaldson theory, but contains a certain *link* theory. Indeed, local and nonlocal observables have the property of *marking* cycles. Moreover, from topological gravity one learns that an object can be considered BRST exact only if it is so all over the moduli space $\mathcal{M}$, boundary included. Being BRST exact in any interior point of $\mathcal{M}$ is not sufficient to make an amplitude vanish. Presumably, recursion relations and hierarchies can be found to solve topological field theories in four dimensions, in particular topological Yang-Mills theory with $G = SU(2)$ on $\mathbf{R}^4$ and topological gravity with the full set of asymptotically locally Euclidean manifolds.

1 Introduction

Not many explicit computations have been done so far in four dimensional topological field theory, in particular when the ghost number anomaly is greater than zero and one needs to insert nontrivial observables to construct nonvanishing amplitudes. So, it is still not enough clear how to compute quantities and what is the relevance of topolgical field theory for physics. The purpose of this paper is to look for the basic rules to make explicit computations in the cases in which the explicit expressions of the instantons are known.

In quantum field theory, a theory is entirely encoded in the functional integral and the set of rules for manipulating it. Consequently, it is very important to make explicit direct computations in the very special class of quantum field theory represented by topological field theory, in order to test whether the functional integral is well-defined or not. Even if it were not so, something very interesting could be learned.

All the examples considered in this paper suggest that the functional integral is well-defined in topological field theory. Nevertheless, one learns that sometimes it is crucial to "follow the instructions" very carefully. For example, one can appreciate the fact that certain problems with apparent divergences disappear only when the equivariant cohomology requirement is correctly understood and fulfilled.

Some topological field theories are believed to coincide with certain previously defined mathematical problems. For example, this is the case of topological Yang-Mills theory in four dimensions, which is believed to be equivalent to Donaldson theory [1]. Note, however, that in physics, once the functional integral is specified, i.e. the fields, their boundary conditions, the gauge symmetry and the gauge-fixings, then the content of the theory is also fixed. So, explicit computations can be a useful tool for testing the expectations. Moreover, some theories produced by the so-called topological twist [1] suggest new problems not considered before in the mathematical literature [2].

One can reasonably say that in theoretical physics a problem is satisfactorily understood and solved, once it has been reduced to an algorithmic procedure. This is certainly one of the reasons for the enormous success of perturbative quantum field theory. The situation is far from being like this today in topological field theory, or, more generically, in instanton calculus. In this paper, I try to reduce explicit computations in topological field theory to an algorithmic procedure. In section 2 I develop the formal set-up, which is the guide-light for the rest of the paper, devoted to applications of the general procedure, from which all the results follow, sometimes unexpectedly.

To begin with, an illustrative example is presented in section 3, namely 2D topological gravity on the torus T^2 with one puncture.

In section 4 I consider 4D topological Yang-Mills theory with $G = SU(2)$ on $M = \mathbf{R}^4$. In particular I solve the theory with the $k = 1$ Belavin *et al.* instanton [3]. Explicit computations show that the observables have the property of *marking* cycles. In other words, amplitudes like $< \mathcal{O}_{\gamma_1} \cdot \mathcal{O}_{\gamma_2} >$ are not necessarily trivial when γ_1 and γ_2 are trivial in the sense of the M-homology. $< \mathcal{O}_{\gamma_1} \cdot \mathcal{O}_{\gamma_2} >$ can be different from zero if

γ_1 is nontrivial in the $M\backslash\gamma_2$-homology and γ_2 is nontrivial in the $M\backslash\gamma_1$-homology. So, topological Yang-Mills theory is not just Donaldson theory, but contains a certain *link* theory [4].

In section 5 I consider 4D topological gravity with the Eguchi-Hanson instanton. I solve the theory and compute some *anomalous* amplitudes, that turn out to be nonzero even if the observables are integrated over representatives of trivial cycles. Although there is no evidence, yet, of a link theory contained in topological gravity, these nonzero results are due to a subtlety related to the boundary of the moduli space, that is able to turn a naively BRST exact object into a nontrivial one. In this sense, such anomalies look like the holomorphic anomaly of ref. [5]. Moreover, I show that these anomalies are *necessarily* present. They reveal that an object can only be considered BRST exact if it is so all over the moduli space, boundary included. If it is BRST exact in any interior point of the moduli space, but not at the boundary, then it is not BRST exact. In section 6 I couple topological gravity to abelian topological Yang-Mills theory and compute some other anomalous amplitudes.

2 Formalism

In this section I develop a formalism for solving a topological field theory explicitly, when the explicit expression of the instanton is known. Just to fix notation, I manipulate the BRST algebra of topological Yang-Mills theory with gauge group G on a manifold M with instantons defined by the self-duality of the field strength. Nevertheless, the method is completely general, as I show in the applications. In the last part of the section, I briefly discuss some straightforward modifications required by topological gravity.

First of all, what do we mean by *solving* a topological field theory explicitly? A topological field theory can be seen as a map $\pi : H(M) \times \mathcal{A} \longrightarrow \mathcal{H}(\mathcal{M})$ acting from the homology $H(M)$ of the manifold M times the algebra $\mathcal{A}$ of observables $\mathcal{O}$, to the cohomology $\mathcal{H}(\mathcal{M})$ of the moduli space $\mathcal{M}$ of instantons[1]. Thus, solving a topological field theory amounts to find the map π, i.e. writing down the observables $\mathcal{O}_\gamma$, $\gamma \in H(M)$ as forms $\omega_\gamma(\mathcal{O}) \in \mathcal{H}(\mathcal{M})$ on the moduli space $\mathcal{M}$. The physical amplitudes are then the integrals over $\mathcal{M}$ of top-forms constructed with $\omega_\gamma(\mathcal{O})$.

If A^a_μ is the instanton, i.e. a *coordinate* on $\mathcal{M}$, let δA^a_μ denote the *differential* on $\mathcal{M}$. Then, one would like to find the explicit expressions for

$$\omega_\gamma(\mathcal{O}) = \mathcal{F}^{(\mathcal{O}_\gamma)}_{i_1,\ldots i_n}(A)\delta A^{i_1} \wedge \cdots \wedge \delta A^{i_n}, \qquad (2.1)$$

for any $\gamma \in H(M)$ and any observable $\mathcal{O} \in \mathcal{A}$.

[1]Actually, it is Donaldson theory that is a map $H_k(M) \to \mathcal{H}^{4-k}(\mathcal{M})$. Commonly, Donaldson theory and topological Yang-Mills theory are believed to be equivalent [1]. However, one of the byproducts of the investigation carried on in the present paper is that it is not so, rather topological Yang-Mills theory contains more, in particular a certain kind of *link* theory. Then, the definition of the map π will have to be suitably amended.

2.1 Topological Yang-Mills theory

2.1.1 BRST algebra and observables

Before seeing how this can be achieved, let us write down the BRST algebra of topological Yang-Mills theory in standard notation

$$\begin{aligned}
sA^a_\mu &= \psi^a_\mu + D_\mu C^a \equiv \psi'^a_\mu, \\
s\psi^a_\mu &= -D_\mu\phi^a - f^a{}_{bc}\psi^b_\mu C^c, \\
s\phi^a &= f^a{}_{bc}\phi^b C^c, \\
sC^a &= \phi^a - \frac{1}{2}f^a{}_{bc}C^b C^c,
\end{aligned} \tag{2.2}$$

where $D^{ab}_\mu = \delta_{ab}\partial_\mu + f^a{}_c{}^b A^c_\mu$ and $f^a{}_{bc}$ are the structure constants of G.

The observables are generated by the BRST extensions of identities like $d\,\mathrm{tr}\,[F\wedge F] = 0$. For future use, I write down here the simplest ones, namely

$$\begin{aligned}
&\mathcal{O}^{(0)}_x = \mathrm{tr}[\phi\phi](x), \qquad \mathcal{O}^{(1)}_{\gamma_1} = 2\int_{\gamma_1}\mathrm{tr}[\psi\phi], \\
&\mathcal{O}^{(2)}_{\gamma_2} = \int_{\gamma_2}\mathrm{tr}[\psi\psi + 2F\phi], \qquad \mathcal{O}^{(3)}_{\gamma_3} = 2\int_{\gamma_3}\mathrm{tr}[F\psi],
\end{aligned} \tag{2.3}$$

γ_i being representatives of $i-$cycles on M. Note that the integrands are differential forms on M and have a ghost number. Let us call such objects *ghost-forms*. When dealing with ghost-forms, the relevant grading is the *ghost-form number*, i.e. the sum of the ghost number and the form degree. The integrands in (2.3) have ghost-form number 4, form degree i and ghost number $4-i$. The integrals, instead, have only a ghost number. Consequently, for consistency, we have to assign a negative form degree to the integral symbols, so that $\int_{\gamma_i}$ has form degree $-i$. This has to be kept into account when commuting the BRST operator s with integrals.

I am going to show that all the task for solving the theory amounts to find the explicit expression of the ghost C^a. The equation that determines it will be written down in a moment. The concrete examples that I will discuss in the next section suggest that whenever the explicit expression of the instanton is known, the equation for C^a can also be solved explicitly.

2.1.2 Gauge-fixings

In general, the instanton A^a_μ will satisfy a certain gauge-fixing condition, $\partial^\mu A^a_\mu = 0$ for example. The explicit form of this gauge-fixing condition is totally immaterial to our purpose, as it will be clear in the sequel. One only needs to know that the instanton satisfies a *certain* gauge-fixing condition. Instead, what is crucial is the gauge-fixing condition for the topological ghost ψ^a_μ. It is this condition that determines C^a and solves the problem. So, I must discuss this gauge condition in detail.

The role of the gauge-fixing condition for ψ^a_μ is to fix the so-called *gauge of the gauge*. Indeed, in topological field theory one frequently has to do with a hierarchy of gauge-symmetries. In the case we are dealing with, it is convenient to distinguish three such symmetries: the first one is the topological symmetry (ghost ψ^a_μ) and is the most important; the second one is the gauge of the gauge, i.e. the symmetry that acts on ψ^a_μ like an ordinary gauge-symmetry acts on A^a_μ (the ghost is ϕ^a, which, to be precise, is called ghost for the ghosts); the third one is the ordinary gauge-symmetry (ghost C^a). These three symmetries are combined together so as to produce a nilpotent ($s^2=0$) BRST operator s.

Let us now describe the gauge-fixings. The instantonic condition ($F^{+a}_{\mu\nu}=0$, for example) is the gauge-fixing of the first symmetry and has to preserve the other two. The gauge condition we are looking for, instead, has to break the second symmetry (and eventually also the first), while preserving the third one. Finally, the usual condition $\partial^\mu A^a_\mu=0$ breaks the third symmetry (it can also break the other two). These are the requirements of the so-called equivariant BRST-cohomology. If one does not satisfy them, then one gets wrong or meaningless results.

Instead, breaking the second symmetry while preserving the third one is crucial: to be more explicit, a condition like $\partial^\mu\psi^a_\mu=0$ is wrong, while a condition like $D^\mu\psi^a_\mu=0$ is correct. Indeed, the condition $\partial^\mu\psi^a_\mu=0$ would kill all the local observables. It is obvious that $\psi^{\prime a}_\mu=\delta A^a_\mu$ satisfies $\partial^\mu\psi^{\prime a}_\mu=0$: this is proved simply by taking the δ-variation of $\partial^\mu A^a_\mu=0$ (s and δ are essentially the same, as far as our calculations are concerned). Then one can take $\psi^{\prime a}_\mu=\psi^a_\mu$ and C^a=0. This implies $\phi^a=0$: all the local observables (like $\mathrm{tr}[\phi^2]$) vanish. So, our gauge-fixing for the gauge of the gauge will be $D^\mu\psi^a_\mu=0$.

2.1.3 Solution

Now, we are ready to describe the procedure for solving a topological field theory. A^a_μ is explicitly known by assumption. $\psi^{\prime a}_\mu=\delta A^a_\mu$ is found by a simple differentiation with respect to the moduli. Writing $\psi^a_\mu=\psi^{\prime a}_\mu-D_\mu C^a$, the condition $D^\mu\psi^a_\mu=0$ then becomes an equation for C^a:

$$D_\mu D^\mu C^a=D^\mu\psi^{\prime a}_\mu. \tag{2.4}$$

Once this equation is solved, C^a and ψ^a_μ are found. It is clear that ψ^a_μ satisfies its field equation, whatever C^a is: the ψ^a_μ field equation (in our case $D_{[\mu}\psi^a_{\nu]^+}=0$) is the δ-variation of the instantonic condition ($F^{+a}_{\mu\nu}=0$) and does not depend on C^a because the instantonic condition has to preserve the third symmetry.

Finally, ϕ^a is found as

$$\phi^a=\delta C^a+\frac{1}{2}f^a{}_{bc}C^bC^c. \tag{2.5}$$

Again, ϕ^a automatically satisfies its field equation.

Let $G^{ab}(x,y)$ denote the Green function for equation (2.4),

$$[D_\mu D^\mu(x)]^{ac}G^{cb}(x,y)=\delta^{ab}\delta(x-y). \tag{2.6}$$

In many cases $G^{ab}(x,y)$ has been worked out explicitly [6]. We shall not need the explicit form of $G^{ab}(x,y)$ in our calculations. One can formally write

$$\begin{aligned} C^a(x) &= \int_M G^{ab}(x,y) D^\mu \delta A^b_\mu(y)\, dy, \\ \psi^a_\mu(x) &= \int_M \{\delta^{ab}\delta(x-y)\delta^\nu_\mu - [D_\mu(x)]^{ac} G^{cd}(x,y)[D^\nu(y)]^{db}]\}\, \delta A^b_\nu(y)\, dy. \end{aligned} \tag{2.7}$$

C^a and ψ^a_μ have been written as one-forms on $\mathcal{M}$. Writing down ϕ^a according to the above prescription, one finds

$$\phi^a(x) = \int_M G^{ab}(x,y) f^a{}_{bc} \psi^b_\mu(y) \psi^{\mu c}(y)\, dy. \tag{2.8}$$

Indeed, taking the BRST variation of (2.4) one checks that ϕ^a satisfies

$$D_\mu D^\mu \phi^a = f^a{}_{bc} \psi^b_\mu \psi^c_\mu. \tag{2.9}$$

2.1.4 Functional integral

One may wonder when the functional integral enters the game. Since the topological field theory action is a set of gauge-fixings, one can simply deal with the gauge-fixing conditions directly, as we have done. To justify this, let us choose the gauge-fermion [7]

$$\Psi = \bar{\chi}^{\mu\nu}_a F^{+a}_{\mu\nu} + \bar{\phi}_a D^\mu \psi^a_\mu + \bar{C}_a \partial^\mu A^a_\mu. \tag{2.10}$$

Then, the Lagrangian will be the BRST variation of this gauge-fermion (plus eventually the topological invariant, that I do not include, since I shall work with a fixed instanton number):

$$\begin{aligned} \mathcal{L} = s\Psi = & b^{\mu\nu}_a F^{+a}_{\mu\nu} - \bar{\chi}^{\mu\nu}_a D_{[\mu}\psi^a_{\nu]+} + \bar{\eta}_a D^\mu \psi^a_\mu - \bar{\phi}_a (D_\mu D^\mu \phi^a + f^a{}_{bc} \psi^b_\mu \psi^{c\mu}) \\ & + b_a \partial^\mu A^a_\mu - \bar{C}_a(\partial_\mu D^\mu C^a + \partial^\mu \psi^a_\mu). \end{aligned} \tag{2.11}$$

Integrating the Lagrange multipliers $b^{\mu\nu}_a$, $\bar{\eta}_a$ and b_a and the antighosts $\bar{\chi}^{\mu\nu}_a$, $\bar{\phi}_a$ and $\bar{C}_a$ away (this is allowed, since the observables do not depend on these fields), one gets a set of delta functions that agree with the equations that we wrote and solved before. In this way, the functional integral is performed exactly and there is no perturbative correction. Instead, the Lagrangian that Witten wrote in [1], obtained by twisting N=2 super Yang-Mills theory, contains extra BRST exact [7] terms, which spoil the linearity of $\mathcal{L}$ in antighosts. Indeed, there is no need to introduce the full set of renormalizable interactions in the gauge-fixing sector [8, 9]: the theory does not depend on (the continuous deformations of) the gauge-fixing. The role of the gauge-fixing sector is that of permitting to define the propagators and a minimal choice is quite sufficient. On the other hand, one could even choose a power-counting non-renormalizable gauge-fixing, without affecting the results [8].

2.1.5 Functional measure

The functional measure $d\mu$ is not a problem in topological field theory, since there exists a canonical one [1], which reads

$$d\mu = dm\, d\hat{m}, \tag{2.12}$$

m denoting the moduli and $\hat{m}$ their ghost partners ($\hat{m} = sm$). To pass from the original functional measure to the above one, one has to deal with Jacobian determinants, that, however, mutually simplify between bosons and fermions. Then, the net effect of (2.12) is that of replacing, *via* the $\hat{m}$-integration, $\hat{m}$ with dm wherever $\hat{m}$ appears.

2.1.6 Zero modes

I shall only consider cases in which C^a and ϕ^a possess no zero modes or their zero modes can be simply dealt with. In general, one must include them in the most general solution to (2.4). Getting rid of them in the physical amplitudes requires the introduction of suitable *puncture operators*. It is better to stop a moment and discuss this point in general, because it will be useful in the applications.

So, let us assume that C^a possesses zero modes:

$$C^a = \tilde{C}^a + C^a_0, \qquad C^a_0(x) = C^a_i(x)\theta^i, \tag{2.13}$$

$\tilde{C}^a$ denoting a particular solution to the inhomogeneous equation (2.4), and C^a_0 denoting the most general solution to the homogeneous equation $D_\mu D^\mu C^a_0 = 0$, expanded in a basis $C^a_i(x)$. Eq. (2.9) shows that ϕ^a has the same zero modes. We write $\phi^a = \tilde{\phi}^a + \phi^a_0$, $\phi^a_0(x) = \phi^a_i(x)\gamma^i$. θ_i have ghost number 1, while γ_i have ghost number 2. So, they correspond to 1-forms and 2-forms on $\mathcal{M}$, respectively, and we have to work out the explicit expressions of these forms.

Zero modes are to be regarded as a further symmetry of the Lagrangian $\mathcal{L}$. I call it the *zero mode symmetry*. Since $\mathcal{L}$ does not depend on zero modes, the functional integral is still ill-defined: fermionic zero modes integrate to give zero, while bosonic zero modes integrate to give infinity. There is a very well-established machinery to treat such problems, which is the BRST technique. So, the first thing to do is to choose a gauge-fixing for the zero mode symmetry. This can be the requirement that $C^a(x)$ vanishes in a certain set of points $\{x_i\}$. Let us introduce antighosts $\bar{\gamma}_i$ and Lagrange multipliers $\bar{\theta}_i$ of ghost numbers -2 and -1, respectively ($s\bar{\gamma}_i = \bar{\theta}_i$, $s\bar{\theta}_i = 0$). The gauge-fermion is

$$\Psi_0 = \sum_i \bar{\gamma}_i C^a(x_i). \tag{2.14}$$

The total Lagrangian is $\mathcal{L}_{\rm tot} = \mathcal{L} + \mathcal{L}_0$, where

$$\mathcal{L}_0 = s\Psi_0 = \sum_i \bar{\theta}_i C^a(x_i) + \sum_i \bar{\gamma}_i \left(\phi^a(x_i) + \frac{1}{2} f^a{}_{bc} C^b(x_i) C^c(x_i)\right). \tag{2.15}$$

Integrating $\bar{\theta}_i$ and $\bar{\gamma}_i$ away, one ends up with the insertion of the following puncture operator:

$$P = \prod_{i,a} C^a(x_i)\delta[\phi^a(x_i)]. \tag{2.16}$$

Now, the conditions $C^a(x_i) = 0$ and $\phi^a(x_i) = 0$, imposed by this insertion, can be easily recognized as equations of θ_i and γ_i, that permit to find their explicit expressions as forms on the moduli space $\mathcal{M}$.

Collecting the information that we have found so far, we can conclude that the BRST algebra is solved consistently, so that the map π can be written down and the amplitudes can be calculated. I shall do this explicitly in section 4, for $G = SU(2)$, $k = 1$ and $M = \mathbf{R}^4$.

2.2 Topological gravity

Here I briefly describe how to deal with topological gravity. This will be used in sections 5 and 6.

Several formalisms appeared in the literature for writing down the BRST algebra and the observables of topological gravity (in arbitrary dimension) [10]. The computations of sections 5 and 6 will show that the most convenient formalism is the most similar to the one used for topological Yang-Mills theory. In particular, it is important to express the observables in a simple form. As it was shown in sect. 3 of [11] (see there for further details), one can write the BRST algebra as

$$\begin{aligned}
se^a &= \psi^a - \mathcal{D}\varepsilon^a - \varepsilon^{ab}e^b = \psi'^a = \psi'^{ab}e^b,\\
s\omega^{ab} &= \chi^{ab} - \mathcal{D}\varepsilon^{ab} \equiv \chi'^{ab},\\
s\psi^a &= -\mathcal{D}\phi^a - \varepsilon^{ab}\psi^b + \chi^{ab}\varepsilon^b + \eta^{ab}e^b,\\
s\chi^{ab} &= \chi^{ac}\varepsilon^{cb} - \varepsilon^{ac}\chi^{cb} - \mathcal{D}\eta^{ab},\\
s\phi^a &= \eta^{ab}\varepsilon^b - \varepsilon^{ab}\phi^b,\\
s\eta^{ab} &= \eta^{ac}\varepsilon^{cb} - \varepsilon^{ac}\eta^{cb},\\
s\varepsilon^a &= \phi^a - \varepsilon^{ab}\varepsilon^b,\\
s\varepsilon^{ab} &= \eta^{ab} - \varepsilon^{ac}\varepsilon^{cb}.
\end{aligned} \tag{2.17}$$

Note that there is a change in notation with respect to [11]: the spin connection and curvature are defined so that $R^{ab} = d\omega^{ab} + \omega^{ac}\omega^{cb}$. This is just to fit with the notation in which the Eguchi-Hanson metric is commonly written down.

I now discuss how the procedure for solving a topological theory has to be applied to gravity. We have learned that it is crucial to choose correct gauge-fixing conditions for $\psi^a = \psi^{ab}e^b$. Convenient ones are

$$\mathcal{D}_b\psi^{ab} = 0, \qquad \psi^{ab} = \psi^{ba}, \tag{2.18}$$

where $\mathcal{D}_c\psi^{ab}=e^\mu_c\partial_\mu\psi^{ab}-\psi^{ae}\omega^{eb}_c+\omega^{ae}_c\psi^{eb}$. Being ψ^{ab} and ε^{ab} symmetric and antisymmetric, respectively, the relation $\psi'^{ab}=\psi^{ab}+\mathcal{D}^b\varepsilon^a-\varepsilon^{ab}$ allows us to write

$$\psi^{ab}=\psi'^{\{ab\}}-\mathcal{D}^{\{a}\varepsilon^{b\}},\qquad \varepsilon^{ab}=-\psi'^{[ab]}-\mathcal{D}^{[a}\varepsilon^{b]}. \tag{2.19}$$

This solves the second equation of (2.18). Instead, the condition $\mathcal{D}_b\psi^{ab}=0$ becomes a differential equation for ε^a. Once it is solved, ψ^{ab} and ε^{ab} are determined. All the rest follows by a straightforward application of the formulæ (2.17). ε^a plays the role that was played by C^a in topological Yang-Mills theory.

The observables we shall deal with are derived from the BRST extension of $d\,\mathrm{tr}[R\wedge R]=0$ ($d\,\mathrm{tr}[R\wedge\tilde{R}]=0$ will give the same thing), in particular

$$\mathcal{O}^{(1)}_{\gamma_1}=2\int_{\gamma_1}\mathrm{tr}[\chi\eta],\quad \mathcal{O}^{(2)}_{\gamma_2}=\int_{\gamma_2}\mathrm{tr}[\chi\chi+2R\eta],\quad \mathcal{O}^{(3)}_{\gamma_3}=2\int_{\gamma_3}\mathrm{tr}[R\chi], \tag{2.20}$$

while the local observable $\mathrm{tr}[\eta^2]$ is not interesting to our problem, since we shall focus on metrics with less than four moduli. In section 6 I also consider, in abelian topological Yang-Mills theory coupled to topological gravity, observables derived from identities like $d\left[(\mathrm{tr}[RR])^m F^n\right]=0$.

2.2.1 Changes of variables

Before concluding this section, I discuss what happens when changing coordinates. It is useful to work on the bundle X that has $\mathcal{M}$ as base manifold and $M(m)$ as fiber on $m\in\mathcal{M}$ (this is sometimes called the *tautological bundle*), because the BRST extended objects (like $\hat{e}^a=e^a+\varepsilon^a$, $\hat{\omega}^{ab}=\omega^{ab}+\varepsilon^{ab}$, etc.; let me call them *hatted forms*) are differential forms on X and $\hat{d}=d+s$ is the exterior derivative on X. The points of the manifold X are denoted by (x,m) and we are interested in generic moduli-dependent changes of variables on M, which have the form $x=x(x',m)$. These changes of variables can be useful in many applications, because sometimes it is convenient to parametrize cycles in a very peculiar coordinate system. One could simply re-start from the beginning, working out the new fields and the new BRST transformations in the new reference frame. However, the two solutions can be simply related to each other, overcoming the problem that a BRST variation (i.e. the derivative with respect to m) at fixed x is essentially different from the BRST variation at fixed x'. The answer is the following. The hatted forms are unaffected, but the decompositions according to ghost number and form degree have to be rederived. Indeed,

$$dx=dx'\left.\frac{\partial x}{\partial x'}\right|_m+dm\left.\frac{\partial x}{\partial m}\right|_{x'}, \tag{2.21}$$

so that forms on M acquire ghost terms. For example, the vielbein becomes the sum of an $(1,0)$ piece (the new vielbein) plus a $(0,1)$ piece. The latter has to be added to ε^a, so that the hatted form $\hat{e}^a=e^a+\varepsilon^a$ is unaffected.

In particular, the BRST extension of the torsion is $\hat{R}^a = R^a + \psi^a + \phi^a$. ψ^a cannot acquire ghost terms from R^a, because we work at vanishing torsion. So, as far as the new ψ^a is concerned, one can forget about the m dependence in $x = x(x', m)$. That means that the new ψ^{ab} is related to the old one by a diffeomorphism and a Lorentz rotation (i.e. by a transformation of the *third* gauge symmetry). We know that the gauge conditions that break the second gauge symmetry have to respect the third one. So, we are guaranteed that the new ψ^{ab} satisfies them, if the old one did. Concretely, in our case (2.18) are preserved by changes of variables. This is another example of the importance of the equivariant cohomology prescription and at the same time shows that notions like differential forms on M and ghost fields have not an *invariant* meaning. Only hatted forms have an invariant meaning.

2.3 Conclusion

The procedure elaborated so far works for many cases and I will not go further in this paper. Thus, I shall also be able to compute well-defined amplitudes in power-counting non-renormalizable quantum field theories (like quantum gravity), by studying their topological versions (which are perturbatively finite: there is no beta function, because there is no coupling constant). As discussed in [8], the infinitely many types of counterterms that may appear perturbatively do not affect the topological results (in absence of anomalies). If, instead, there are anomalies, then the theory could be even more interesting, since the breaking of the topological symmetry could generate quantum gravity. In perturbation theory, an anomaly could turn some gauge-fixing parameter into a physical one. Then, one should count the number of such anomalies: if they are a finite number, they generate a predictive theory. This motivation is quite sufficient to justify such a kind of investigation.

The following sections are devoted to applications of the formalism established in the present one. I begin with an illustrative very simple case and then I turn to topological Yang-Mills theory and topological gravity.

3 A trivial example: T^2

As a first example, I consider a very simple case that leads to a non-vanishing amplitude: the torus T^2 in two dimensional topological gravity. It will be described by the cube $(\xi, \eta) \in [0, 1] \times [0, 1]$ with flat metric

$$ds^2 = d\xi^2 + |\tau|^2 d\eta^2 + 2\,\mathrm{Re}\,\tau\; d\xi d\eta. \tag{3.22}$$

Coordinates like $z = \xi + \tau\eta$ are not good for the computational procedure that is described in this paper. In a field theoretical approach, the coordinates must parametrize the topological manifold and the moduli are fields.

I choose the zweibein

$$e^0 = d\xi + \mathrm{Re}\,\tau\ d\eta, \qquad e^1 = \mathrm{Im}\,\tau\ d\eta. \tag{3.23}$$

The variation of the zweibein is

$$se^a = \psi^a - \mathcal{D}\varepsilon^a - \varepsilon^{ab}e_b = \psi'^{ab}e_b, \tag{3.24}$$

ε^a and ε^{ab} being the diffeomorphism and Lorentz rotation ghosts, respectively. One finds

$$\psi'^{ab} = \begin{pmatrix} 0 & i\frac{d\tau+d\bar{\tau}}{\tau-\bar{\tau}} \\ 0 & \frac{d\tau-d\bar{\tau}}{\tau-\bar{\tau}} \end{pmatrix}. \tag{3.25}$$

Imposing the gauge-conditions (2.18) on $\psi^a = \psi^{ab}e_b$, it turns out that ε^a is constant, while ψ^{ab} and ε^{ab} are the symmetric and antisymmetric parts of ψ'^{ab}, respectively. In particular,

$$\varepsilon^{ab} = -\frac{i}{2}\frac{d\tau + d\bar{\tau}}{\tau - \bar{\tau}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv c_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.26}$$

The local observable γ_0 of the theory is the BRST variation of c_0, i.e.

$$\gamma_0 = i\frac{d\tau \wedge d\bar{\tau}}{(\tau - \bar{\tau})^2}, \tag{3.27}$$

which is the Poincarè metric, the Poincarè dual of a point. Thus the amplitude $< \gamma_0 >$ is the volume of the moduli space of the torus, which is certainly a well-defined amplitude. In the usual description [12], it corresponds to the amplitude $< \sigma_1(x) >$. The presence of one puncture is revealed by the fact that ε^a has two real zero modes (the constants) and so one has to introduce a puncture operator to get rid of them.

Knowing the explicit expression for the metric of the torus with more punctures would allow to recover the remaining amplitudes. It could be interesting to recover the full set of amplitudes of the punctured sphere, first. Perhaps, combining the formalism developed in the previous section with Strebel's theory of quadratic differentials on Riemann surfaces [13] one can recover Kontsevich's result [14].

4 Topological Yang-Mills theory

In this section, I consider topological Yang-Mills theory on $M = \mathbf{R}^4$ with gauge group $G = SU(2)$. Let k denote the instanton number. Then the moduli space $\mathcal{M}$ has dimension $8k-3$. $H(M)$ does not contain nontrivial cycles other than the point and M itself. So, in eq. (2.3) only the observable $\mathrm{tr}[\phi^2]$ is nontrivial and the selection rule can never be fulfilled.

I shall answer the question: is this theory empty? There is surely a non-empty intersection theory on the instanton moduli space $\mathcal{M}$ and the topological field theory should contain at least a part of it. Actually, we shall see that the theory contains many unexpected things, more related to the manifold M than to intersection theory on the moduli space $\mathcal{M}$.

I focus on the well-known $k=1$ instanton [3]

$$A^a_\mu(x)=\frac{2}{D}\eta^a_{\mu\nu}(x-x_0)^\nu, \qquad D=(x-x_0)^2+\rho^2. \tag{4.1}$$

Here, $x_0\in\mathbf{R}^4$ and $\rho\in(0,\infty)$ are the moduli, so that $\mathcal{M}=(0,\infty)\otimes\mathbf{R}^4$. In the sequel, I strictly follow the notation of [15]. I do not introduce any compactification of M or $\mathcal{M}$: it is not necessary in my approach, because, since everything follows automatically from the physical formalism (functional integral manipulated as explained in section 2), that will produce well-defined finite results, one can say that, in some sense, a privileged kind of compactification is already encoded in it. Anyway, one can consider $M=\mathbf{R}^4$ as a chart of S^4. The point at infinity will be treated appropriately in a moment.

According to (2.2), one has

$$\psi'^a_\mu=\delta A^a_\mu=-\frac{2}{D^2}\eta^a_{\mu\nu}[Ddx_0^\nu+2(x-x_0)^\nu(\rho d\rho-(x-x_0)\cdot dx_0)], \tag{4.2}$$

so that equation (2.4) becomes

$$D_\mu D^\mu C^a=-\frac{8}{D^2}\eta^a_{\mu\nu}(x-x_0)^\mu dx_0^\nu. \tag{4.3}$$

Note that the right hand side does not contain $d\rho$. Thus, C^a has the form $C^a=g^a_\mu dx_0^\mu$. A natural ansatz is

$$C^a=f(D)\eta^a_{\mu\nu}(x-x_0)^\mu dx_0^\nu. \tag{4.4}$$

(4.3) reduces to the following equation for f:

$$f''(D-\rho^2)+3f'+2\frac{\rho^2}{D^2}f+\frac{2}{D^2}=0, \tag{4.5}$$

the primes denoting derivatives with respect to D. This equation is solved by $f=\frac{2}{D}$, so that

$$C^a=\frac{2}{D}\eta^a_{\mu\nu}(x-x_0)^\mu dx_0^\nu. \tag{4.6}$$

Note the $\sim\frac{1}{x}$ behavior of A^a_μ and C^a for $x\to\infty$: it will produce well-defined topological amplitudes. In general, the "unphysical" quantities like A^a_μ and C^a should be regular everywhere and bounded at infinity. In the case of topological gravity the diffeomorphism ghosts ε^a will possess a similar $\sim\frac{1}{x}$ behavior, while the Lorentz ghosts ε^{ab} will tend to constants (global Lorentz rotation at infinity).

If there are zero modes, then (4.6) is not the most general solution. As a matter of fact, in [15] it is shown that there are the following three zero modes, corresponding to the isospin infinitesimal rotations,

$$C^a_b\theta^b=\frac{1}{D}\eta^a_{\mu\nu}\bar\eta^b_{\mu\rho}(x-x_0)^\nu(x-x_0)^\rho\theta^b. \tag{4.7}$$

We see that these expressions have not a well-defined limit at infinity, rather (so to speak) they have a three-parameter limit at infinity, while (4.6) tends to zero. One can get rid of (4.7) following the procedure explained in section 2. We have to choose three points x_1, x_2 and x_3 where C^a should vanish. The most convenient choice seems to be $x_1 = x_2 = x_3 = \infty$ (more precisely, one should choose an arbitrary triplet of points and then let these points tend to infinity). The fact that (4.7) have a three-parameter limit at infinity says that precisely three conditions are necessary to make the ghost vanish there. The expressions for θ^a as forms on the moduli space are then very simple: $\theta^a = 0\ \forall a$. Consequently, (4.6) is the correct expression we have to deal with. The same argument, when applied to ϕ^a, gives $\gamma^a = 0\ \forall a$.

With the solution (4.6), one can write down the explicit expression of any quantity. Note the very simple expression of the BRST extension $\hat{A}^a$:

$$\hat{A}^a = A^a + C^a = \frac{2}{D}\, d(x-x_0)^\mu \eta^a_{\mu\nu}(x-x_0)^\nu. \tag{4.8}$$

For ϕ^a one gets

$$\phi^a = -\frac{2\rho}{D^2}\, \eta^a_{\mu\nu}[\rho dx_0^\mu + 2(x-x_0)^\mu d\rho)]dx_0^\nu, \tag{4.9}$$

while ψ^a_μ turns out to be

$$\psi^a_\mu = -4\frac{\rho}{D^2}\eta^a_{\mu\nu}[\rho dx_0^\nu + (x-x_0)^\nu d\rho]. \tag{4.10}$$

The BRST extension $\hat{F}^a$ of the field strength $F^a = dA^a + \frac{1}{2}\varepsilon^a{}_{bc}A^bA^c$ also has a very simple expression:

$$\hat{F}^a = F^a + \psi^a + \phi^a = -2\frac{\rho}{D^2}d(x-x_0)^\mu\eta^a_{\mu\nu}[\rho d(x-x_0)^\nu - 2(x-x_0)^\nu\, d\rho]. \tag{4.11}$$

$\hat{F}^a$ tends to zero on $\partial\mathcal{M}$ sufficiently rapidly to assure that the amplitudes are topological.

Other useful formulæ are

$$\begin{aligned}\hat{F}^a\hat{F}^a &= 4\frac{\rho^3}{D^4}[\rho dV(x-x_0) - 4d\rho\wedge(x-x_0)^\mu d\sigma_\mu(x-x_0)] = \hat{d}\hat{C},\\ \hat{C} &= \hat{A}^a\hat{F}^a - \frac{1}{6}\varepsilon_{abc}\hat{A}^a\hat{A}^b\hat{A}^c = \frac{4}{3}\frac{1}{D^3}[3\rho^2 + (x-x_0)^2](x-x_0)^\mu d\sigma_\mu(x-x_0),\end{aligned} \tag{4.12}$$

where $\hat{d} = d + s$.

Any expression involving $\hat{A}$ and not just $\hat{F}$ holds only locally. A form on $\mathcal{M}$ is exact only if it can be written as $s\Lambda$, Λ being constructed with the components of $\hat{F}$, but independent of $\hat{A}$. This is the analogue of the fact that $\mathrm{tr}[F\wedge F]$ is not globally exact over M, because it cannot be written as dC with a gauge-invariant C.

4.1 Amplitudes

Let us now come to the observables. The full set (2.3) is generated by the identity $\hat{d}\,\mathrm{tr}[\hat{F}\hat{F}] = 0$. The local observable $\mathcal{O}_x^{(0)}$ can be promptly written down as a 4-form on $\mathcal{M}$:

$$\omega_x^{(4)} = \phi^a\phi^a(x) = 4\frac{\rho^3}{D^4}[\rho dV(x_0) + 4d\rho \wedge (x-x_0)_\mu d\sigma^\mu(x_0)], \tag{4.13}$$

where

$$dV(x_0) = \varepsilon_{\mu\nu\rho\sigma}dx_0^\mu dx_0^\nu dx_0^\rho dx_0^\sigma, \qquad d\sigma^\mu(x_0) = \varepsilon_{\mu\nu\rho\sigma}dx_0^\nu dx_0^\rho dx_0^\sigma. \tag{4.14}$$

One can explicitly check that $\omega_x^{(4)}$ is closed. By integrating it on a 4-cycle in $\mathcal{M}$, like $\{\rho_0\} \otimes \mathbf{R}^4 = \gamma_4$ for example, one can show that it is not exact:

$$\int_{\gamma_4} \omega_x^{(4)} = 16\pi^2. \tag{4.15}$$

Let us consider another 4-cycle, $\gamma_4' = (0,\infty) \otimes S_r^3$, r being the radius of a 3-sphere $S^3 \subset \mathbf{R}^4$ centered in the origin. Then

$$\int_{\gamma_4'} \omega_x^{(4)} = -384\pi \int_0^\infty d\rho \int_0^\pi d\theta \frac{\rho^3 r^3 (r - x\cos\theta)\sin^2\theta}{(\rho^2 + r^2 + x^2 - 2rx\cos\theta)^4}. \tag{4.16}$$

Before computing the above integral, let us see what is the expected result. First of all, it should not depend on the representative of the 4-cycle. Second, it should not depend on the point x that defines the 4-form $\omega_x^{(4)}$. These are nothing but the requirements that the amplitude be topological. If these expectations are correct, we can choose the most convenient values of r and x. So, let us choose $r = 0$, i.e. let us shrink the 3-sphere to a point. Since there is no singularity at $r = 0$, the result is zero. Independence of r suggests that the above integral is identically zero. However, this is not true. Let us perform the ρ-integration:

$$\int_{\gamma_4'} \omega_x^{(4)} = -32\pi \int_0^\pi d\theta \frac{r^3(r - x\cos\theta)\sin^2\theta}{(r^2 + x^2 - 2rx\cos\theta)^2}. \tag{4.17}$$

Now, for $r \to 0$ the above expression tends to zero, in agreement with the above argument, but for $r \to \infty$ it tends to $-16\pi^2$ and for $r = x$ it is equal to $-8\pi^2$. So, the amplitude is surely r and x dependent and does not seem to be topological. To solve the puzzle, let us perform the complete integration:

$$\int_{\gamma_4'} \omega_x^{(4)} = -8\pi^2 \left(1 + \frac{\sqrt{(r^2 - x^2)^2}}{r^2 - x^2}\right) = -8\pi^2(1 + H(r - x)), \tag{4.18}$$

$H(x)$ denoting the step function, $H(x) = 1$ for $x > 0$ and $H(x) = -1$ for $x < 0$. Thus, a very simple interpretation of the result comes to one's mind: the amplitude "feels" the location x of the observable or, vice versa, the local observable modifies the geometry.

When $r > x$ one cannot shrink the 3-sphere to zero safely: it is necessary to cross the observable. On the other hand, when $r < 0$ no problem is encountered and the result is zero. The major consequence is the following: with the insertion of one local observable, there are noncontractible 3-cycles even in $\mathbf{R}^4$. Keeping this in mind, in a moment we shall go back to our original question ("is the theory empty?") and we shall be able to give a negative answer.

What I have described is the first evidence of the main subject of the present paper: anomalies in instanton calculus. As a matter of fact, I did not discover them in this example, rather in the case of topological gravity with the Eguchi-Hanson metric (see the next section). Nevertheless, I preferred to start with an example that everybody is more familiar with.

3-cycles play a role in the definition of the observable $\mathcal{O}^{(3)}_{\gamma_3}$ of (2.3). So, let us compute $\omega^{(1)}_r = \mathcal{O}^{(3)}_{S^3_r}$. It is convenient to use (4.12) in order to write $\mathcal{O}^{(3)}_{S^3_r}$ as

$$\mathcal{O}^{(3)}_{S^3_r} = -s \int_{S^3_r} \varepsilon_{\mu\nu\rho\sigma} C^\mu dx^\nu dx^\rho dx^\sigma, \tag{4.19}$$

where C^μ is the Chern-Simons form

$$C^\mu = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\left(A^a_\nu \partial_\rho A^a_\sigma + \frac{1}{3}\varepsilon_{abc} A^a_\nu A^b_\rho A^c_\sigma\right) = \frac{4}{3D^3}(x-x_0)^\mu[3\rho^2 + (x-x_0)^2]. \tag{4.20}$$

In (4.19), the minus sign is due to having commuted s with $\int_{S^3_r}$. Actually, (4.12) permits to write $\mathcal{O}^{(3)}_{S^3_r}$ as the above expression plus something of the form $\int_{S^3_r} d[\cdots]$. This extra term vanishes, since although the $[\cdots]$ is not globally defined on M, certainly it is on S^3_r.

It is thus simple to prove that

$$\omega^{(1)}_r = df, \qquad f(\rho, x_0) = 32\pi r^3 \int_0^\pi d\theta \frac{(r - x_0\cos\theta)(3\rho^2 + r^2 + x_0^2 - 2rx_0\cos\theta)\sin^2\theta}{(\rho^2 + r^2 + x_0^2 - 2rx_0\cos\theta)^3}. \tag{4.21}$$

So, the interesting physical amplitude is

$$\mathcal{A} = < \mathcal{O}^{(3)}_{S^3_r} \cdot \mathcal{O}^{(0)}_x > = \int_{\mathcal{M}} \omega^{(1)}_r \wedge \omega^{(4)}_x. \tag{4.22}$$

Using (4.21) and $d\omega^{(4)}_x = 0$, we get

$$\mathcal{A} = \int_{\partial\mathcal{M}} f \omega^{(4)}_x. \tag{4.23}$$

Now, the boundary of the moduli space is $\partial\mathcal{M} = \partial_1\mathcal{M} \cup \partial_2\mathcal{M} \cup \partial_3\mathcal{M}$, where $\partial_1\mathcal{M} = \{0\} \otimes \mathbf{R}^4$, $\partial_2\mathcal{M} = \{\infty\} \otimes \mathbf{R}^4$ and $\partial_3\mathcal{M} = (0,\infty) \otimes S^3_\infty$. We have

$$\lim_{\rho\rightarrow\infty} f = 0, \quad \lim_{x_0\rightarrow\infty} f = 0, \quad \lim_{\rho\rightarrow 0} f = 8\pi^2(1 + H(r - x_0)), \tag{4.24}$$

the last limit being easily evaluated, since the integral in (4.21) reduces to the one in (4.18) when $\rho \to 0$. One can easily check that $\int_{\partial_2 \mathcal{M}} \omega_x^{(4)}$ and $\int_{\partial_3 \mathcal{M}} \omega_x^{(4)}$ are finite[2], so that $\mathcal{A}$ receives contribution only from $\partial_1 \mathcal{M}$. We thus have

$$\mathcal{A} = 64 \cdot 96\pi^3 \lim_{\rho \to 0} \rho^4 \int_0^r x_0^3 dx_0 \int_0^\pi \frac{\sin^2\theta \; d\theta}{(x^2 + x_0^2 + \rho^2 - 2xx_0 \cos\theta)^4}. \tag{4.25}$$

Now, let us discuss this formula. The ρ^4 factor kills the full expression at least when the remaining integral is regular. This surely happens for $r < x$. Thus

$$\mathcal{A} = 0, \qquad \text{for}\, r < x. \tag{4.26}$$

However, one must be careful when $r > x$, because the integral is singular at $x = x_0$ and $\theta = 0$: it is better to keep ρ different from zero and take the limit at the very end. Performing the θ-integration, one gets

$$\mathcal{A} = 32 \cdot 48\pi^4 \lim_{\rho \to 0} \int_{\rho - \frac{x^2}{\rho}}^{\rho + \frac{r^2 - x^2}{\rho}} \frac{(v\rho + 2x^2)(v\rho - \rho^2 + x^2) dv}{(v^2 + 4x^2)^{5/2}}, \tag{4.27}$$

where $v = \rho + \frac{x_0^2 - x^2}{\rho}$. Amazingly, the v-integration gives back the by now familiar step function:

$$\mathcal{A} = 128\pi^4 (1 + H(r - x)), \tag{4.28}$$

confirming once for all the presence of an anomaly. Thus, there is no doubt that one should accept the fact that the presence of local observables alters the theory in a visible way.

This can can be thought as something similar to the contact terms that appear in 2D topological gravity [12] and to the holomorphic anomaly of [5]. Indeed, $\mathcal{O}^{(3)}_{S^3_r}$ is naively BRST-exact, since S^3_r is naively a boundary. Thus $< \mathcal{O}^{(3)}_{S^3_r} \cdot \mathcal{O}^{(0)}_x >$ is expected to be zero, since it is the average value of a BRST exact object. However, the naive expectation is affected by the boundary of the moduli space: this is revealed by formula (4.24), which clearly stresses that the whole result is due to the "instanton of zero size ρ".

Summarizing, the above result can be understood as follows. When the local observable $\mathcal{O}^{(0)}_x$ is placed *inside* S^3_r, then S^3_r is not homotopically trivial: it cannot be contracted to a point without crossing either the observable placed at x or the puncture (2.16) placed at infinity. In this case, the amplitude is different from zero. Instead, if $\mathcal{O}^{(0)}_x$ is placed *outside* S^3_r, then S^3_r is homotopically trivial and the amplitude vanishes.

From what we have discovered so far (that is not the full story), we can say that, instead of classifying the homology of M, one should classify the homology of $M \backslash \{x_1, \ldots x_n\}$, $x_1, \ldots x_n$ being the positions of the local observables and of the punctures (2.16). Thus, we learn that concepts like puncture, contact term and nonperturbative BRST anomaly[3]

[2]Such integrals are nothing but peculiar cases of (4.15) and (4.18).

[3]It is worth stressing that such BRST anomalies do not affect the unitarity of the theory. As a matter of fact, if things are correctly understood, they are not anomalies at all. More details on this will be given in the next section.

are meaningful even in four dimensions and without gravity. The existence of a correspondence (topological twist) relating topological Yang-Mills theory to N=2 supersymmetric Yang-Mills theory [1] suggests that similar things should be present in ordinary supersymmetric theories. This should be true also for N=1 theories, since they also possess topological amplitudes [16].

We can thus conclude that the theory is all but empty: anomalies are responsible for giving sense to a naively empty theory.

The next question that comes naturally to one's mind is the following: if a local observable is able to mark a point, can a non-local observable mark a higher dimensional cycle? It happens that the answer is positive, so we are going to discover that topological Yang-Mills theory contains a certain *link*-theory .

Consider the observables $\mathcal{O}^{(1)}_{\gamma_1}$ and $\mathcal{O}^{(2)}_{\gamma_2}$ of formula (2.3). They correspond to a 3-form $\omega^{(3)}$ and a 2-form $\omega^{(2)}$ on $\mathcal{M}$, respectively. So, their product can be integrated over $\mathcal{M}$ to give an amplitude. To define these observables and the corresponding $\mathcal{M}$-forms, one needs a 1-cycle and a 2-cycle on $M=\mathbf{R}^4$. There are no nontrivial such cycles on $\mathbf{R}^4$, however, enlightened by the previous results, we can easily figure out what happens. Let us choose γ_1 and γ_2 in such a way that γ_2 is a nontrivial 2-cycle of $M\backslash\gamma_1$ and viceversa: this should produce a finite non-vanishing amplitude $\mathcal{A}$. On the other hand, if we choose γ_1 to be a trivial loop of $M\backslash\gamma_2$, we expect $\mathcal{A}$ to be zero. We are now going to show that it is precisely so.

We choose γ_1 and γ_2 in the following way. Let us write $M=\mathbf{R}^4=\mathbf{R}\otimes\mathbf{R}^3$, $\mathbf{R}$ being "time" and $\mathbf{R}^3$ being three-space. Then, let

$$\gamma_1=(-\infty,\infty)\otimes\{\mathbf{x}_1\},\qquad \gamma_2=\{t_2\}\otimes S^2_r. \tag{4.29}$$

One finds

$$\mathcal{O}^{(1)}_{\gamma_1}=\omega^{(3)}_{\mathbf{x}_1}=d\Omega^{(2)}_{\mathbf{x}_1},\qquad \Omega^{(2)}_{\mathbf{x}_1}=\pi\frac{(x_1-x_0)^i\varepsilon_{ijk}dx_0^j dx_0^k}{[\rho^2+(\mathbf{x}_1-\mathbf{x}_0)^2]^{5/2}}[5\rho^2+2(\mathbf{x}_1-\mathbf{x}_0)^2]. \tag{4.30}$$

One has

$$\mathcal{A}=<\mathcal{O}^{(1)}_{\gamma_1}\cdot\mathcal{O}^{(2)}_{\gamma_2}>=\int_{\mathcal{M}}\omega^{(3)}_{\mathbf{x}_1}\wedge\omega^{(2)}_r=\int_{\partial\mathcal{M}}\Omega^{(2)}_{\mathbf{x}_1}\wedge\omega^{(2)}_r. \tag{4.31}$$

It is convenient to write x_0 as $(t_0,\mathbf{x}_0)$. Since, $\omega^{(3)}_{\mathbf{x}_1}$ does not contain dt_0, one can focus on the terms of $\omega^{(2)}_r$ that contain it. One finds

$$\omega^{(2)}_r=48\rho^3dt_0\int_{S^2_r}\frac{\rho dx_0^i+d\rho\,(x-x_0)^i}{(t_0^2+\rho^2+(\mathbf{x}-\mathbf{x}_0)^2)^4}\varepsilon_{ijk}dx^j dx^k. \tag{4.32}$$

Due to the fact that

$$\begin{aligned}
\Omega^{(2)}_{\mathbf{x}_1}&\sim\frac{1}{\rho^3}(x_1-x_0)^i\varepsilon_{ijk}dx_0^j dx_0^k, &&\text{for } \rho\rightarrow\infty,\\
\Omega^{(2)}_{\mathbf{x}_1}&\sim\frac{1}{x_0^3}(x_1-x_0)^i\varepsilon_{ijk}dx_0^j dx_0^k, &&\text{for } x_0\rightarrow\infty,\\
\lim_{\rho\rightarrow 0}\Omega^{(2)}_{\mathbf{x}_1}&=2\pi\frac{(x_1-x_0)^i\varepsilon_{ijk}dx_0^j dx_0^k}{|\mathbf{x}_1-\mathbf{x}_0|^3}, &&
\end{aligned} \tag{4.33}$$

one can check that $\partial_2\mathcal{M}$ and $\partial_3\mathcal{M}$ do not contribute. After a straightforward manipulation, one can write

$$\mathcal{A}=32\pi\lim_{\rho\to 0}\rho^4\int_{-\infty}^{\infty}dt_0\int_{\mathbf{R}^3}\frac{\varepsilon_{ijk}dx_0^i dx_0^j dx_0^k}{x_0^3}\int_{S_r^2}\frac{x_0^m\varepsilon_{mnp}dx^n dx^p}{(t_0^2+\rho^2+(\mathbf{x}-\mathbf{x}_1-\mathbf{x}_0)^2)^4}. \tag{4.34}$$

Let us now study this amplitude in two limiting cases, representing the two possible situations described above.

1) Let $x_1=0$. Then, whatever r is, S_r^2 is a nontrivial cycle of $M\backslash\gamma_1$ and we expect a nonzero amplitude. Indeed, one finds, after rescaling all quantities by ρ,

$$\mathcal{A}=128\pi^2\lim_{r\to\infty}r^2\int_{-\infty}^{\infty}dt_0\int_{\mathbf{R}^3}\frac{\varepsilon_{ijk}dx_0^i dx_0^j dx_0^k}{x_0^3}\int_{-1}^{1}\frac{udu}{(1+t_0^2+r^2+x_0^2-2rx_0u)^4}. \tag{4.35}$$

Integrating over u and setting $s_0=\frac{1}{2r}(x_0^2+1+t_0^2-r^2)$, one arrives at

$$\mathcal{A}=512\pi^3\lim_{r\to\infty}\frac{1}{r}\int_{-\infty}^{\infty}dt_0\int_{\frac{1+t_0^2}{2r}-\frac{r}{2}}^{\infty}\frac{(s_0+r)ds_0}{(s_0^2+t_0^2+1)^3}. \tag{4.36}$$

Now we can take the limit $r\to\infty$ and find

$$\mathcal{A}=256\pi^4. \tag{4.37}$$

2) Let $r\to 0$, $x_1\neq 0$. Then, γ_1 is a contractible loop of $M\backslash\gamma_2$ and we predict $\mathcal{A}=0$. Indeed,

$$\mathcal{A}\sim\lim_{\rho\to 0}\rho^4\int_{-\infty}^{\infty}dt_0\int_{\mathbf{R}^3}\frac{\varepsilon_{ijk}dx_0^i dx_0^j dx_0^k}{x_0^3}\frac{x_0^m}{(t_0^2+\rho^2+(\mathbf{x}_1+\mathbf{x}_0)^2)^4}\lim_{r\to 0}\int_{S_r^2}\varepsilon_{mnp}dx^n dx^p. \tag{4.38}$$

Now, the factor with the limit $r\to 0$ tends to zero and the remaining integral is easily shown to be convergent, with a manipulation similar to the one of point 1). So, $\mathcal{A}=0$.

The final expression of the amplitude is thus

$$\mathcal{A}=128\pi^4(1+H(r-x_1)). \tag{4.39}$$

We now check the above result by parametrizing the cycles in a different way. Let us write $M=\mathbf{R}^2\otimes\mathbf{R}^{2\prime}$, $x=(\mathbf{x},\mathbf{x}')$, $x_0=(\mathbf{x}_0,\mathbf{x}_0')$ and

$$\gamma_1=C_r\otimes\{\mathbf{x}_1'\},\qquad\gamma_2=\{\mathbf{x}_2\}\otimes\mathbf{R}^{2\prime}, \tag{4.40}$$

C_r denoting a circle of radius r centered in the origin. One finds

$$\omega_{\mathbf{x}_2}^{(2)}=d\Omega_{\mathbf{x}_2}^{(1)},\quad\Omega_{\mathbf{x}_2}^{(1)}=-8\pi\frac{2\rho^2+(\mathbf{x}_2-\mathbf{x}_0)^2}{(\rho^2+(\mathbf{x}_2-\mathbf{x}_0)^2)^2}[(x_2^0-x_0^0)dx_0^1-(x_1^1-x_0^1)dx_0^0]. \tag{4.41}$$

we have

$$\mathcal{A}=\int_{\mathcal{M}}\omega_{\mathbf{x}_2}^{(2)}\wedge\omega_r^{(3)}=\int_{\partial\mathcal{M}}\Omega_{\mathbf{x}_2}^{(1)}\wedge\omega_r^{(3)}=-\lim_{\rho\to 0}\int_{\mathbf{R}^4}\Omega_{\mathbf{x}_2}^{(1)}\wedge\omega_r^{(3)}. \tag{4.42}$$

As before, one can check that $\partial_2\mathcal{M}$ and $\partial_3\mathcal{M}$ do not contribute. One finds

$$\mathcal{A} = 256\pi^2 \lim_{\rho\to 0}\rho^4 \int_{\mathbf{R}^2} \frac{dx_0^0 dx_0^1}{x_0^2} \int_{\gamma_1} \frac{x_0^0 dx_1^1 - x_0^1 dx_1^0}{[\rho^2 + (\mathbf{x}_0 + \mathbf{x}_2 - \mathbf{x}_1)^2]^3}. \tag{4.43}$$

In the limit $r = |\mathbf{x}_1| \to 0$, $\mathbf{x}_2 \neq 0$, one finds 0, while for $\mathbf{x}_2 = 0$ one can go on as in the previous calculation and find

$$\mathcal{A} = 256\pi^4, \tag{4.44}$$

in agreement with (4.37). It is clear that choosing the cycles as

$$\gamma_1 = \cup_i (-1)^{\sigma_i}[(-\infty,\infty)\otimes\{\mathbf{x}_i\}], \qquad \gamma_2 = \cup_j(-1)^{\pi_j}[\{t_2\}\otimes S^2_{r_j}], \tag{4.45}$$

$(-1)^{\sigma_i}$ and $(-1)^{\pi_j}$ denoting the orientations of the various components of γ_1 and γ_2, one gets

$$\mathcal{A} = 128\pi^4 \sum_{ij}(-1)^{\sigma_i+\pi_j}[1 + H(r_j - x_i)], \tag{4.46}$$

i.e. $\mathcal{A}$ counts the link number.

Note that the numerical coefficients in (4.28) and (4.39) are the same (the eventual sign depends on the orientations of the cycles). This means that the observables (2.3) have a correct relative normalization. As a matter of fact, due to $256\pi^4 = (16\pi^2)^2$, we find that the normalized expression generating the full set of observables is

$$\frac{1}{16\pi^2}\hat{F}^a\hat{F}^a. \tag{4.47}$$

This agrees with the normalization of the instanton number: $\frac{1}{16\pi^2}\int_M F^aF^a = 1$. Amplitude (4.46) becomes $1/2\sum_{ij}(-1)^{\sigma_i+\pi_j}[1+H(r_j-x_i)]$. The fact that we have noticed has not to be underestimated: it seems that the moduli space is made exactly to adjust the factors, so that, once the instanton number is normalized correctly, there is no more freedom in topological Yang-Mills theory.

One can think of computing links in much more complicated situations, where, for example, γ_2 is a genus g Riemann surface.

4.2 Conclusion

The computations made up to now should suffice to show that "step amplitudes" are part of life when dealing with instanton calculus in non-abelian Yang-Mills theory.

The richness of the homology of $M\backslash(\gamma_{i_1}\cup\ldots\gamma_{i_n})$ suggests that there is a fully open problem, in contrast with the naive expectations of an empty theory, suggested by the well-known claims. Thus, topological Yang-Mills theory is not just Donaldson theory, rather it also contains a *link* theory. The definition of the map π has to be corrected by taking into account that the important homology is that of $M\backslash(\gamma_{i_1}\cup\ldots\gamma_{i_n})$ (and so it depends on the amplitude) and not simply that of M.

For higher instanton number k, there are many more observables. One can construct them from identities like $d\,(\mathrm{tr}[F \wedge \cdots \wedge F])^n = 0$. In the $k = 1$ case, it is easily proved, by using (4.12), that $(\hat{F}^a\hat{F}^a)^n = 0\ \forall n > 1$. Instead, for $k = 2$ there are 13 moduli and, for example, one has amplitudes like

$$< \phi^a\phi^a(x_1) \cdot \phi^b\phi^b(x_2) \cdot \phi^c\phi^c(x_3) \cdot \mathcal{O}^{(3)}_{\gamma_3} > . \tag{4.48}$$

Here there are three marked points, plus the one at infinity, so there are three independent noncontractible 3-spheres γ_3. If one inserts two $\mathrm{tr}[\phi^2]$ operators, one marks two points. Then there are two nontrivial three-cycles $S^{(3)}$ and $S^{(3)\prime}$. Descent equation from $d(\hat{F}^a\hat{F}^a)^2 = 0$ produce, in particular, a five dimensional forms $\omega^{(5)}$ obtained by integrating over one of these 3-cycles γ_3:

$$\omega^{(5)}_{\gamma_3} = 4\int_{\gamma_3} F^a\psi^a\phi^b\phi^b + 2F^a\phi^a\psi^b\phi^b + \psi^a\psi^a\psi^b\phi^b. \tag{4.49}$$

So, good amplitudes are

$$< \phi^a\phi^a(x_1) \cdot \phi^b\phi^b(x_2) \cdot \omega^{(5)}_{\gamma_3} > . \tag{4.50}$$

The selection rule can also be saturated with one local observable $\mathrm{tr}[\phi^2](x)$ and a non-local observables $\omega^{(9)}_{\gamma_3}$ coming from the descent equations generated by $(\hat{F}^a\hat{F}^a)^3$, again integrated over a nontrivial 3-cycle γ_3. Lots of other amplitudes, that I do not list here, come from the couples (γ_1, γ_2) of 1-cycles γ_1 and 2-cycles γ_2.

The solution to the problem, that would seem very difficult at first, could instead be simple, to some extent . Presumably, one can find a set of recursion relations relating the amplitudes with different values of k and some kind of hierarchy collecting the full set of them. In particular, what is the role of k in connection with link theory? It would also be interesting to know what happens with other gauge-groups G, in particular $SU(3)$ and to find out the general characterization of topological Yang-Mills theory.

This concludes the discussion on topological Yang-Mills theory. In the next section, I turn to topological gravity.

5 Topological gravity

In this section, I explore topological gravity with the Eguchi-Hanson instanton, $M = T^*(P_1(\mathbf{C}))$ [17, 18]. I parametrize the vierbein with three moduli as follows

$$e^0 = \sqrt{\frac{\rho^2 + a^2}{\rho^2 + 2a^2}}d\rho, \quad e^1 = \sqrt{\rho^2 + a^2}\sigma'_x, \quad e^2 = \sqrt{\rho^2 + a^2}\sigma'_y, \quad e^3 = \rho\sqrt{\frac{\rho^2 + 2a^2}{\rho^2 + a^2}}\sigma'_z. \tag{5.1}$$

a is one modulus, namely the size of the instanton. $\rho \in (0, \infty)$ is related to the usual radial coordinate r by $r^2 = \rho^2 + a^2$. It is better to avoid using $r \in (a, \infty)$, because its range is a-dependent and this affects the differentiation with respect to a and the parametrizations of cycles (for example, the 3-sphere $r =$constant would depend on a).

σ'_x, σ'_y and σ'_z are one-forms satisfying the $SU(2)$ Maurer-Cartan equations

$$d\sigma'_i = \varepsilon_{ijk}\sigma'_j\sigma'_k \tag{5.2}$$

and can be expressed as

$$\begin{aligned}
\sigma'_x &= \cos\beta\,\sigma_z - \sin\alpha\,\sin\beta\,\sigma_x - \cos\alpha\,\sin\beta\,\sigma_y = \frac{1}{2}(\sin\psi'\,d\theta' - \sin\theta'\,\cos\psi'\,d\phi'),\\
\sigma'_y &= \cos\alpha\,\sigma_x - \sin\alpha\,\sigma_y = -\frac{1}{2}(\cos\psi'\,d\theta' + \sin\theta'\,\sin\psi'\,d\phi'),\\
\sigma'_z &= \sin\beta\,\sigma_z + \sin\alpha\cos\beta\,\sigma_x + \cos\alpha\,\cos\beta\,\sigma_y = \frac{1}{2}(d\psi' + \cos\theta'\,d\phi').
\end{aligned} \tag{5.3}$$

σ'_i is nothing but a rotated version of the usual basis σ_i [17]:

$$\begin{aligned}
\sigma_x &= \frac{1}{2}(\sin\psi\,d\theta - \sin\theta\,\cos\psi\,d\phi),\\
\sigma_y &= -\frac{1}{2}(\cos\psi\,d\theta + \sin\theta\,\sin\psi\,d\phi),\\
\sigma_z &= \frac{1}{2}(d\psi + \cos\theta\,d\phi).
\end{aligned} \tag{5.4}$$

One can write θ', ψ' and ϕ' as functions of θ, ψ, ϕ and the moduli α and β. Nevertheless, the ranges are the same for unprimed as for primed angles:

$$0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi, \qquad 0 \le \psi \le 2\pi. \tag{5.5}$$

The angles α and β have been introduced to deal explicitly with a three dimensional moduli space. The metric is self-dual for any α and β, because in proving self-duality, one only needs to use (5.2). The relations (5.2) are preserved by rotations in the space of the forms σ_i. However, σ_x and σ_y appear symmetrically in the metric $ds^2 = e^a e^a$, so that one of the three rotations (the one around the z axis) is a Lorentz rotation and not a true modulus. So, one remains precisely with two meaningful angles. This matches with the counting of the number of moduli in the Gibbons-Hawking [19] description of multicenter metrics.

To some extent, the number of moduli can be a convention. One can say that α and β are trivial parameters and not moduli, because they can be eliminated with a change of coordinates. However, such a change of coordinates is a rotation and so is not bounded at infinity. A ghost field tending to a constant at infinity (i.e. to a global gauge transformation) should be allowed. Nevertheless, from the examples of the previous and the present section, it seems reasonable to restrict the ghost fields to be bounded at infinity.

Moreover, the theory of topological gravity that we are considering, is related *via* topological twist to N=2 supergravity [11] and it seems that in supergravity the full set of moduli should be treated [20]. So, in order to develop a background for a future comparison with N=2 supergravity computations [21], I keep the full set of three moduli.

The moduli space we are dealing with is $\mathcal{M} = (\mathbf{R}^3 \backslash \{0\})/\mathbf{Z}_2$. This can be easily seen in the Gibbons-Hawking description [19], where the metric is identified by the positions of two non-coincident points in $\mathbf{R}^3$: one point can always be placed in the origin (with a translation) and the quotient by $\mathbf{Z}_2$ means that *which* one is placed there is immaterial. $\mathcal{M}$ can also be written as $(0,\infty) \otimes S^2/\mathbf{Z}_2$, the radial coordinate being $a \in (0,\infty)$ and the polar angles of S^2 being α and β, which have ranges

$$0 \leq \alpha \leq 2\pi, \qquad -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}. \tag{5.6}$$

Indeed, for $\beta = \pm\frac{\pi}{2}$ the metric is α-independent, so that $\beta = \pm\frac{\pi}{2}$ are the two poles of the sphere S^2. The quotient by $\mathbf{Z}_2$ will be taken into account by dividing the results by a factor 2.

For future use, let me write down the spin connection and the curvature explicitly

$$\begin{array}{ll} \omega^{01} = -\omega^{23} = -\rho \frac{\sqrt{\rho^2+2a^2}}{\rho^2+a^2}\sigma'_x, & R^{01} = -R^{23} = \frac{2a^4}{(\rho^2+a^2)^3}(-e^0e^1 + e^2e^3), \\ \omega^{02} = -\omega^{31} = -\rho \frac{\sqrt{\rho^2+2a^2}}{\rho^2+a^2}\sigma'_y, & R^{02} = -R^{31} = \frac{2a^4}{(\rho^2+a^2)^3}(-e^0e^2 + e^3e^1), \\ \omega^{03} = -\omega^{12} = -\left[1 + \frac{a^4}{(\rho^2+a^2)^2}\right]\sigma'_z, & R^{03} = -R^{12} = -\frac{4a^4}{(\rho^2+a^2)^3}(-e^0e^3 + e^1e^2). \end{array} \tag{5.7}$$

Now, we are ready to compute observables and amplitudes.

5.1 The simplest amplitude

We begin with the calculation of the easiest observable, namely $\mathcal{O}^{(3)}_{\gamma_3}$ of (2.20). The simplest choice of γ_3 is a 3-sphere S^3_ρ of radius ρ, even if naively it is not the representative of a 3-cycle and so it is expected to produce a trivial result, i.e. an exact form $\omega^{(1)}_\rho = \mathcal{O}^{(3)}_{S^3_\rho}$. One can write

$$\mathcal{O}^{(3)}_{S^3_\rho} = 2\int_{S^3_\rho} R^{ab}\chi^{ab} = 2\int_{S^3_\rho} R^{ab}\chi'^{ab}. \tag{5.8}$$

The last equality follows from the fact that $R^{ab}(\chi^{ab} - \chi'^{ab}) = R^{ab}\mathcal{D}\varepsilon^{ab}$ is a total derivative, due to the Bianchi identity $\mathcal{D}R^{ab} = 0$. Now, χ'^{ab} is easily computed, since (2.17) shows that it is the exterior derivative of ω^{ab} with respect to the moduli:

$$\begin{aligned} \chi'^{01} = -\chi'^{23} &= \frac{2\rho a^3 da\,\sigma'_x}{(\rho^2+a^2)^2\sqrt{\rho^2+2a^2}} + \rho\frac{\sqrt{\rho^2+2a^2}}{\rho^2+a^2}(d\beta\,\sigma'_z + \sin\beta\,d\alpha\,\sigma'_y), \\ \chi'^{02} = -\chi'^{31} &= \frac{2\rho a^3 da\,\sigma'_y}{(\rho^2+a^2)^2\sqrt{\rho^2+2a^2}} + \rho\frac{\sqrt{\rho^2+2a^2}}{\rho^2+a^2}d\alpha\,(\cos\beta\,\sigma'_z - \sin\beta\,\sigma'_x), \\ \chi'^{03} = -\chi'^{12} &= -\frac{4\rho^2 a^3 da\,\sigma'_z}{(\rho^2+a^2)^3} - \left[1 + \frac{a^4}{(\rho^2+a^2)^2}\right](d\beta\,\sigma'_x + \cos\beta\,d\alpha\,\sigma'_y), \end{aligned} \tag{5.9}$$

The straightforward computation leads to

$$\omega^{(1)}_\rho = -192\frac{a^7\rho^2 da}{(\rho^2+a^2)^5}\int_{S^3_\rho}\sigma'_x\sigma'_y\sigma'_z = 192\pi^2\frac{a^7\rho^2 da}{(\rho^2+a^2)^5}. \tag{5.10}$$

We can also write

$$\omega_{\rho}^{(1)} = df, \qquad f(\rho, a) = -24\pi^2\rho^2 \frac{4a^6 + 6a^4\rho^2 + 4a^2\rho^4 + \rho^6}{(a^2 + \rho^2)^4}. \tag{5.11}$$

In df, the exterior derivative d acts on a. We see that $\omega_{\rho}^{(1)}$ is a well behaving 1-form on the moduli space $\mathcal{M}$. Integrating it over the 1-cycle $\gamma_1 \subset \mathcal{M}$, $\gamma_1 = (0, \infty) \times \{\hat{n}\}$, $\hat{n}$ denoting a certain point of $S^2/\mathbf{Z}_2$, one gets a finite ρ- and $\hat{n}$-independent result:

$$\int_{\gamma_1} \omega_r^{(1)} = 24\pi^2, \tag{5.12}$$

which can also be considered as the amplitude $< \mathcal{O}^{(3)}_{S^3_\rho} >$, when one restricts the moduli space to be only the range $(0, \infty)$ of the scale a.

The above result means once again that an amplitude naively expected to be zero is instead nonvanishing. So, it is an anomaly. Nevertheless, we should find a more satisfactory explanation, since there is no other observable around that can mark a cycle and change the topology of the Eguchi-Hanson manifold. Indeed, (5.12) is always different from zero: there is no step function. So, what is the reason why the 3-sphere S^3_ρ should be considered indeed a 3-cycle? Once again, the whole story is originated by the boundary of the moduli space. Consider the limiting case $a = 0$. There, the Eguchi-Hanson manifold degenerates to $\mathbf{C}^2/\mathbf{Z}_2$, which is singular in the origin. The consequence is that if one wants to shrink S^3_ρ, one has necessarily to cross the singular point. Nothing can be said a priori about this procedure: it could be safe, or produce infinities, or, in the most interesting case, produce finite nonvanishing anomalous values, as it happens in fact.

As it was partially anticipated in the previous section, one can say that the above anomalies are not, strictly speaking, "anomalies". Indeed, if an object is not BRST exact on the entire moduli space (boundary included), then it should not be considered as BRST exact. So, one should simply change the naive definition of BRST cohomology accordingly.

The check of the independence of result (5.12) from the coordinate system is left to the appendix. There, I use the Gibbons-Hawking coordinates. This check is interesting for preparing the future computations in multi-center metrics, because it reveals some technical subtleties that arise due to the presence of Dirac strings.

The reason for the above result to be necessarily nonvanishing is quite simple. Indeed, the observable $\mathcal{O}^{(3)}$ can be written, like in formula (4.20), in terms of the Chern-Simons form. In other words, the function $f(\rho, a)$ is precisely the integral of the Chern-Simons form over $S^{(3)}_\rho$. Since in our example only two dimensionful parameters are around (a and ρ), one can write $f(\rho, a) = f(\rho/a)$ and conclude

$$< \mathcal{O}^{(3)}_{S^3_\rho} > = \int_0^\infty \frac{\partial f}{\partial a} da = f(\rho/a)\Big|_{a=0}^{a=\infty} = -f(\rho/a)\Big|_{\rho=0}^{\rho=\infty} = -\int_0^\infty \frac{\partial f}{\partial \rho} d\rho. \tag{5.13}$$

The last expression is proportional to the topological invariant $\int_M \mathrm{tr}[R \wedge R]$ on the Eguchi-Hanson manifold M. So, it is surely nonzero. This is an aspect of the fact that many topological invariants of moduli space $\mathcal{M}$ computed by the topological field theory are related to topological invariants of the manifold M (see for example [22]). One can turn this argument backwards and say: since the Eguchi-Hanson manifold has no 3-cycles (for generic a), but the above amplitude *must* be nonzero (because the Pontrjiagin number is nonvanishing), then *necessarily* there must be some singularity in the boundary of the moduli-space. In other words, it would have been impossible for the Eguchi-Hanson instanton to have a regular limit for $a \to 0$. This seems a general property of instantons and, perhaps, some similar argument could give general information about the singularities that the moduli space of instantons *necessarily* possesses.

When considering the full three dimensional moduli space, one has to deal with a more complicated amplitude, as we are going to see.

5.2 Solution

In order to compute the remaining observables, we have to work out the explicit expressions of some ghosts. We have

$$\psi'^{ab} = \begin{pmatrix} -\frac{a\rho^2\, da}{(\rho^2+a^2)(\rho^2+2a^2)} & 0 & 0 & 0 \\ 0 & \frac{a\, da}{\rho^2+a^2} & -\sin\beta\, d\alpha & -K d\beta \\ 0 & \sin\beta\, d\alpha & \frac{a\, da}{\rho^2+a^2} & -K\cos\beta\, d\alpha \\ 0 & d\beta/K & \cos\beta\, d\alpha/K & \frac{a\rho^2\, da}{(\rho^2+a^2)(\rho^2+2a^2)} \end{pmatrix} = \psi^{ab} + \mathcal{D}^b\varepsilon^a - \varepsilon^{ab}, \tag{5.14}$$

where $K = \frac{\rho^2+a^2}{\rho\sqrt{\rho^2+2a^2}}$. Note the bad divergence at $\rho = 0$, that, however, is not dangerous, since it will disappear at the end. Indeed, the ghost ε^a, although being regular, subtracts the divergence in question from ψ'^{ab}, so that ψ^{ab} turns out to be also regular. It is worth to stress again that this is due to the correct choice of the gauge-fixing conditions that break the gauge of the gauge.

The equation for ε^a will be solved in two steps. In general, ε^a is a linear combination of da, $d\alpha$ and $d\beta$. The coefficient of da and those of $d\alpha$, $d\beta$ satisfy independent equations and they will be discussed independently. So, in the first step, we put $d\alpha = d\beta = 0$ and choose the following ansatz

$$\varepsilon^0 = f(\rho, a)\frac{\rho}{a}\sqrt{\frac{\rho^2+a^2}{\rho^2+2a^2}}da, \qquad \varepsilon^1 = \varepsilon^2 = \varepsilon^3 = 0. \tag{5.15}$$

The calculation goes on straightforwardly. One has to combine the first of (2.18) with the first of (2.19). This produces the following equation for $F = 1 + f$:

$$F'' + \frac{5\rho^2 + 3a^2}{\rho(\rho^2+a^2)}F' + \frac{4a^2}{(\rho^2+a^2)^2}F = 0. \tag{5.16}$$

The ansatz

$$F = \rho^m(\rho^2 + a^2)^n, \tag{5.17}$$

gives $m = -2$, $n = \pm 1$. The coefficients of the linear combination of the corresponding two solutions have to be chosen in order to have a regular ε^0 everywhere and with a regular limit at infinity. This produces the answer

$$f(\rho, a) = \frac{a^2}{\rho^2 + a^2}. \tag{5.18}$$

Note the $\frac{1}{\rho}$ behavior of ε^0 at $\rho \to \infty$ that we have already encountered when discussing C^a.

In the second step of the calculation, we put $da = 0$ and choose the ansatz

$$\varepsilon^0 = 0, \quad \varepsilon^1 = -f_1\sqrt{\rho^2 + a^2}\cos\beta\, d\alpha, \quad \varepsilon^2 = f_2\sqrt{\rho^2 + a^2} d\beta, \quad \varepsilon^3 = 0. \tag{5.19}$$

It is then easy to see that f_1 and f_2 satisfy independent equivalent equations, so that the choice $f_1 = f_2 = f$ is consistent. One finds the equation

$$(f'L)' = \frac{2a^8}{L}(1 + 2f), \tag{5.20}$$

where $L = \rho(\rho^2 + a^2)(\rho^2 + 2a^2)$. Setting $F = 1 + 2f$, $G = F'L$, one has

$$GG' = 4a^8 F F', \tag{5.21}$$

which is solved by $F = \frac{1}{2a^4}G$. Integrating this last equation and choosing the constant so as to have a bounded ε^a for $\rho \to \infty$, one finally finds

$$f = \frac{1}{2}\left(\rho\frac{\sqrt{\rho^2 + 2a^2}}{\rho^2 + a^2} - 1\right). \tag{5.22}$$

Collecting the whole information that we have accumulated so far, the final result is

$$\begin{aligned}
\varepsilon^0 &= \frac{\rho a da}{\sqrt{(\rho^2 + a^2)(\rho^2 + 2a^2)}}, \qquad \varepsilon^1 = -\frac{1}{2}\cos\beta\, d\alpha\left(\rho\sqrt{\frac{\rho^2 + 2a^2}{\rho^2 + a^2}} - \sqrt{\rho^2 + a^2}\right),\\
\varepsilon^2 &= \frac{1}{2}d\beta\left(\rho\sqrt{\frac{\rho^2 + 2a^2}{\rho^2 + a^2}} - \sqrt{\rho^2 + a^2}\right), \qquad \varepsilon^3 = 0.
\end{aligned} \tag{5.23}$$

We have thus arrived at the explicit expression of ε^{ab} which is all what we need:

$$\begin{aligned}
\varepsilon^{01} &= -\frac{1}{2}\cos\beta\, d\alpha\left(\rho\frac{\sqrt{\rho^2 + 2a^2}}{\rho^2 + a^2} - 1\right), \quad \varepsilon^{02} = \frac{1}{2}d\beta\left(\rho\frac{\sqrt{\rho^2 + 2a^2}}{\rho^2 + a^2} - 1\right), \quad \varepsilon^{03} = 0,\\
\varepsilon^{12} &= \sin\beta\, d\alpha, \quad \varepsilon^{13} = \frac{1}{2}d\beta\left(\rho\frac{\sqrt{\rho^2 + 2a^2}}{\rho^2 + a^2} + 1\right), \quad \varepsilon^{23} = \frac{1}{2}\cos\beta\, d\alpha\left(\rho\frac{\sqrt{\rho^2 + 2a^2}}{\rho^2 + a^2} + 1\right).
\end{aligned} \tag{5.24}$$

Note that there is no da in the expression of ε^{ab}. I do not write down the explicit expression of ψ^{ab}, nevertheless it can be easily checked that it is not singular at $\rho \to 0$, as promised.

5.3 Other amplitudes

We are now ready to discuss observables and amplitudes. Let us start from the observable $\mathcal{O}^{(2)}_{\gamma_2}$. It is well-known that multicenter manifolds possess nontrivial 2-cycles [23]. In particular, the Eguchi-Hanson manifold possesses one noncontractible 2-sphere S^2. In the usual Gibbons-Hawking description [19], multicenter metrics are parametrized by the positions of n points (centers) in $\mathbf{R}^3$ and by a cyclic coordinate $\tau \in [0, 4\pi]$. The nontrivial 2-spheres S^2 are represented by lines l in $\mathbf{R}^3$ joining couples of centers, "multiplied" by the full range of τ. The line l can be thought as a meridian on S^2, while the cyclic coordinate τ is the longitudinal angle. There are also noncompact 2-cycles: for example, one can choose a line l that goes from one of the centers to infinity.

Before going on, we have to pay attention to the following fact: the line l that joins the two centers of the Eguchi-Hanson manifold depends on the positions on the centers, so it seems difficult to parametrize γ_2 in a moduli-independent way. If a cycle γ is parametrized in a moduli-dependent way, one cannot interchange the BRST operator with the integral over the cycle and prove that (2.3) or (2.20) are BRST closed. In the Gibbons-Hawking coordinates, there is at least one noncompact 2-cycle that can be parametrized in a moduli-independent way: it is described by a line l going from the center placed in the origin to infinity. Instead, in our coordinates it is easy to see that the two centers of the Eguchi-Hanson metric correspond to $\rho = 0$, $\theta' = 0, \pi$. So, we can parametrize a class of representatives of the 2-cycle γ_2 by

$$S^2_g: \qquad \rho = g(\theta'), \quad \theta' \in [0, \pi], \quad \phi' \in [0, 2\pi], \quad \psi' = \text{const}, \tag{5.25}$$

$g(\theta')$ being an arbitrary function such that $g(0) = g(\pi) = 0$. This is a moduli-dependent parametrization, because the primed angles depend on α and β. So, we are not guaranteed to compute meaningful quantities. Eventually, one should change variables from θ, ϕ, ψ to θ', ϕ', ψ', according to the rules explained at the end of section 2[4].

Instead of doing this lengthy work, we can disentangle the situation as follows (it should be kept in mind the what follows is an *ad hoc* argument, differently from the other arguments contained in the present paper).

Our purpose is to compute the amplitude

$$\mathcal{A} = < \mathcal{O}^{(3)}_{S^3_\rho} \cdot \mathcal{O}^{(2)}_{S^2_g} > = \int_{\mathcal{M}} \omega^{(1)}_\rho \wedge \omega^{(2)}_g, \tag{5.26}$$

$\omega^{(1)}_\rho$ being given by (5.10) and $\omega^{(2)}_g$ being $\mathcal{O}^{(2)}_{S^2_g}$. Since $\omega^{(1)}_\rho$ is proportional to da, we can work out $\omega^{(2)}_g$ at fixed a. The amplitude can be rewritten as

$$\mathcal{A} = \int_{\mathcal{M}} df \wedge \omega^{(2)}_g = \int_{\partial\mathcal{M}} f \omega^{(2)}_g = \frac{1}{2} 24\pi^2 \lim_{a \to 0} \int_{S^2} \omega^{(2)}_g. \tag{5.27}$$

The factor $1/2$ is due to the quotient by $\mathbf{Z}_2$.

[4]This change of variables makes the new metric α, β-independent. Then, $d\alpha$ and $d\beta$ are the coefficients of two unbounded zero modes of ε^a. This is due to the peculiarity of the two moduli α and β.

We shall use the parametrization (5.25), although it is moduli-dependent. At fixed a, $\omega_g^{(2)}$ is surely closed, because it is a top form of $S^2 \in \mathcal{M}$. Moreover, $\mathcal{A}$ should not depend on the representative of γ_2, since, although it is necessary to use the moduli for parametrizing the line l joining the two centers, this is no longer true for the *difference* $\delta\gamma_2$ of two representatives of γ_2: indeed, $\delta\gamma_2$ is represented by a loop δl in $\mathbf{R}^3$. This permits to repeat safely the proof that adding a boundary to γ_2 amounts to adding an exact form to $\omega^{(2)}$. Summarizing, we have reasons to expect that the above amplitude is well-defined and topological, notwithstanding the unconventional parametrization of γ_2.

We have to be careful before taking the limit $a \to 0$, since $\mathcal{O}^{(2)}_{S^2_g}$ contains expressions like

$$I = \int_0^\pi \frac{a^8 g(\theta')dg(\theta')\,\cos\theta'}{(g(\theta')^2 + a^2)^5} \tag{5.28}$$

(the cyclic coordinate ϕ having already been integrated away), coming from terms containing e^0. For $a \to 0$, one finds $0 \times \int \frac{dg}{g^9}$, which is surely problematic, since $g(0) = g(\pi) = 0$. We can overcome the difficulty with an integration by parts:

$$I = -\frac{1}{8}\left[\frac{a^8\cos\theta'}{(g(\theta')^2 + a^2)^8}\right]_0^\pi + \frac{1}{8}\int_0^\pi \frac{a^8\sin\theta'\,d\theta'}{(g(\theta')^2 + a^2)^4} = \frac{1}{4} + \frac{1}{8}\int_0^\pi \frac{a^8\sin\theta'\,d\theta'}{(g(\theta')^2 + a^2)^4}. \tag{5.29}$$

The second term tends to zero, thus the net result is a finite contribution. As a matter of fact, one can show that the only nonzero contributions to $\mathcal{A}$ are originated in this way. Collecting all the terms together, one ends up with

$$\mathcal{A} = 24\pi^2 \cdot 16\pi^2. \tag{5.30}$$

We have thus made a very powerful test of topological field theory: we have found a topological amplitude that is not only independent of some parameters, but of a full function $g(\theta')$.

The above correlation function has a simple geometrical interpretation. Apart from the quotient with respect to $\mathbf{Z}_2$, the moduli space is $\mathbf{R}^3 \backslash \{0\}$. $\omega^{(1)}$ is the Poincarè dual of a noncontractible 2-sphere H_2 around the origin, while $\omega^{(2)}$ can be thought as the Poincarè dual of a noncompact 1-cycle H_1 joining the point 0 with infinity. It is clear that the intersection of these two cycles on $\mathcal{M}$ is a point:

$$\#(H_1 \cap H_2) = 1. \tag{5.31}$$

Our amplitude can be interpreted as this (the normalization is immaterial, at this level). In other words, thanks to the anomaly that we have discovered, the topological field theory is able to reproduce some intersection theory on the moduli space.

Note the following funny fact: $\omega^{(2)}$ was calculated on a compact cycle of the manifold M, but it gives a noncompact cycle on the moduli space $\mathcal{M}$. Instead, $\omega^{(1)}$ relates a compact cycle of M with a compact cycle of $\mathcal{M}$. Thus, the map π mixes compact and non-compact homologies in a nontrivial way.

One can also consider noncompact 2-cycles like

$$\gamma_g^{(2)}: \qquad \rho = g(\theta'), \quad \theta' \in [0, \theta'_{max}], \quad \phi' \in [0, 2\pi], \quad \psi' = \text{const}, \tag{5.32}$$

with $g(0) = 0$, $g(\theta'_{max}) = \infty$. Then, since the full result is encoded in expressions like (5.29), one finds a half than before:

$$\mathcal{A} = 24\pi^2 \cdot 8\pi^2. \tag{5.33}$$

The correct normalization seems to be obtained by dividing by $24\pi^2 \cdot 8\pi^2$, so that the two amplitudes (5.30) and (5.33) become 2 and 1, respectively. Such numbers are not so surprising, since (modulo signs) they are the entries of the Cartan matrix of the simply laced groups [24]. The phenomenon that we noticed in topological Yang-Mills theory, i.e. the strict relation among the normalizations of the observables and the meaningful topological invariant, does not seem to occur in topological gravity. Notice, however, that in topological gravity the observables are related to the Pontrjiagin number, while the meaningful topological invariant is the Hierzebruch signature, that differs from the Pontrjiagin number by boundary corrections.

Actually, there could be another nontrivial amplitude, since there is another nonvanishing intersection number: this is the intersection between the full manifold $\mathcal{M}$ (3-cycle) and a point x (zero-cycle),

$$\#(\{x\} \cap \mathcal{M}) = 1. \tag{5.34}$$

The observable that is Poincarè dual to $\mathcal{M}$ is the identity operator, while the observable that is Poincaré dual to a point should be a 3-form on $\mathcal{M}$. Thus, the natural candidate for the latter is $\mathcal{O}^{(1)}_{\gamma_1}$. One natural 1-cycle γ_1 (a circle) that comes to one's mind is the following: in the Gibbons-Hawking description, fix a point $\mathbf{x}$ in $\mathbf{R}^3$ and take the parameter of γ_1 to be the cyclic coordinate $\tau \in [0, 4\pi]$. Such a circle will be denoted by $\gamma_{\mathbf{x}}$. This is surely a contractible cycle, as far as a in nonzero. Indeed, there exists a line l joining the two centers and passing through $\mathbf{x}$. Such line, "multiplied" by the range of τ, represents a non-contractible 2-sphere S^2_l, as already recalled. τ is the longitudinal angle on it. Then, one can deform $\gamma_{\mathbf{x}}$ on S^2_l and make it collapse on the northern or the southern poles (the centers). This corresponds to move $\mathbf{x}$ to one of the two centers. However, when a is 0 the two centers coincide, i.e. northern and southern poles of S^2_l coincide $\forall l$. What we learned in the computations that we have done so far tell us that we should be cautious before drawing conclusions. We simply trust in the physical formalism, that should already contain the correct answer.

Since we are not using the Gibbons-Hawing coordinates, let us choose a convenient 1-cycle γ_1 in our parametrization. Now, τ corresponds to $\phi/2$ (indeed, ϕ is the cyclic coordinate of out metric). So, let us take

$$\gamma_1: \qquad \rho = \rho_0, \quad \theta = \theta_0, \quad \psi = \psi_0, \quad \phi \in [0, 2\pi]. \tag{5.35}$$

One then finds that the α and β-integrations kill every term, and the result is zero. Actually, it must be so, since $d\phi$ always appears multiplied by $\cos\theta$ or by $\sin\theta$: a nonzero result would be θ-dependent. As a nontrivial check of this and of the consistency of the computational methods that we have used so far, let us take a moduli-dependent parametrization

$$\gamma_1: \qquad \rho=\rho_0, \quad \theta'=\theta'_0, \quad \psi'=\psi'_0, \quad \phi'\in[0,2\pi], \tag{5.36}$$

and write the observable as

$$\mathcal{O}^{(1)}_{\gamma_1}=s\int_{\gamma_1} 2\varepsilon^{ab}\chi'^{ab}+\varepsilon^{ab}\mathcal{D}\varepsilon^{ab}, \tag{5.37}$$

(notice that the BRST operator s is *outside* the integral, to avoid problems with the moduli-dependent parametrization). Now, the α and β-integrations do not kill anything, and it is a nontrivial cancellation that makes the total to be zero. So, apparently the theory is not able to reproduce the intersection number (5.34). I do not have any better explanation of this fact. Something more on this will be said in the next section.

Summarizing, in topological gravity, we have found consistent amplitudes. Another question remains without answer: does topological gravity contain a kind of link theory? Can we find step amplitudes?

6 Coupled theories

In this section, I consider topological gravity on the Eguchi-Hanson manifold coupled to abelian topological Yang-Mills theory. Indeed, there is exactly one self-dual $U(1)$ gauge-connection on the Eguchi-Hanson manifold, namely

$$A=\frac{2a^2}{\rho^2+a^2}\sigma'_z, \tag{6.38}$$

the normalization being such that $\int_M F\wedge F=8\pi^2$. Moreover, $A=A_a e^a$ is gauge-fixed with the condition $\mathcal{D}_a A^a=0$. Applying the method of section 2, one finds

$$\psi=\frac{4a\rho^2 da}{(\rho^2+a^2)^2}\sigma'_z+\frac{2a^2}{\rho^2+a^2}(d\beta\,\sigma'_x+\cos\beta\,d\alpha\,\sigma'_y), \quad C=\text{const}, \quad \phi=\text{const}, \tag{6.39}$$

$\psi=\psi_a e^a$ satisfying the condition $\mathcal{D}_a\psi^a=0$. The C and ϕ zero modes are eliminated by inserting the puncture operator

$$C(x)\delta[\phi(x)], \tag{6.40}$$

x being any point of M. Thus, $\hat{F}=F+\psi+\phi=F+\psi$. F can be thought as the Poincarè dual of the compact noncontractible 2-sphere of the Eguchi-Hanson manifold. In the abelian case, there are observables generated by $\hat{d}\,[\hat{F}^n]=0$. We can thus compute the amplitude

$$\mathcal{A}=<\int_{S^3_\rho}\psi^3>=\int_{\mathcal{M}}96\pi^2\frac{a^5\rho^2 da}{(\rho^2+a^2)^4}\cos\beta\,d\alpha\,d\beta=32\pi^3. \tag{6.41}$$

There is another amplitude, indeed, showing that the coupling between topological gravity and topological Yang-Mills theory is nontrivial .

It corresponds to the form

$$\omega^{(3)}=\int_{S^3_\rho}2\eta^{ab}\chi^{ab}F+\chi^{ab}\chi^{ab}\psi+2R^{ab}\eta^{ab}\psi, \tag{6.42}$$

generated by $\hat{d}[\hat{R}^{ab}\hat{R}^{ab}\hat{F}]=0$. The calculation is a bit long. The effort can be a bit reduced by writing $\hat{R}^{ab}\hat{R}^{ab}\hat{F}$ as $\hat{d}[\hat{R}^{ab}\hat{R}^{ab}\hat{A}]$ and

$$\omega^{(3)}=d\Omega^{(2)},\qquad \Omega^{(2)}=-\int_{S^3_\rho}(\chi^{ab}\chi^{ab}+2R^{ab}\eta^{ab})\,A. \tag{6.43}$$

One has to work out η^{ab} and χ^{ab} from (5.24) according to (2.17). At the end one finds

$$\Omega^{(2)}=32\pi^2\frac{a^{10}}{(\rho^2+a^2)^5}\cos\beta\,d\alpha\,d\beta,\qquad \mathcal{A}=\int_{\mathcal{M}}\omega^{(3)}=64\pi^3. \tag{6.44}$$

One could introduce a parameter ξ in front of A, so that

$$A=\xi\frac{2a^2}{\rho^2+a^2}\sigma'_z,\qquad \int_M F\wedge F=8\pi^2\xi^2. \tag{6.45}$$

However, ξ cannot be considered as a modulus, since it changes the Chern class. In other words, $s\xi=0$, otherwise the Chern class in not a good observable. The integration over ξ is made convergent by the Chern class itself in the action:

$$\int d\xi\,\mathrm{e}^{-8\pi^2\xi^2}. \tag{6.46}$$

Such an integral replaces, in the abelian case, the usual sum over topological numbers. It gives an overall constant factor, that we can normalize to one. This justifies the previous calculations, in which ξ was 1.

The amplitudes (6.41) and (6.44) could be perhaps related to the intersection form (5.34), that we were unable to reproduce in pure topological gravity.

It would also be interesting to work out the relation between the calculations made in the previous and in the present sections and the theory obtained by twisting N=2 supergravity [11], in which there is a $U(1)$ gauge connection (graviphoton), that, however, is related to the ghosts for the ghosts.

Acknowledgements

I would like to thank M. Bianchi, F. Fucito and G.C. Rossi for interesting discussions on instanton calculus in supersymmetric theories. This research was supported in part by the Packard Foundation and by NSF grant PHY-92-18167.

7 Appendix: Checks

In the Gibbons-Hawking multicenter coordinates [19], one describes the multicenter manifold M as an $\mathbf{R}^3$ space, where the centers are placed, plus a cyclic coordinate τ ranging from 0 to 4π. One center can always be placed in the origin, while the positions of the other centers are the moduli. The general form of the metric is

$$ds^2 = U^{-1}(d\tau + \omega)^2 + U d\mathbf{x} \cdot d\mathbf{x}. \tag{7.47}$$

In the Eguchi-Hanson case (two centers), placing one center in the origin and the other one on the z-axis and choosing natural vertical positions for the Dirac strings, one has

$$U = \frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x} - \mathbf{a}|}, \qquad \omega = \left(\frac{z}{|\mathbf{x}|} + \frac{z-a}{|\mathbf{x} - \mathbf{a}|} - 2 \right) \frac{ydx - xdy}{x^2 + y^2}. \tag{7.48}$$

We write the observable $\mathcal{O}^{(3)}_{\gamma_3}$ as

$$\mathcal{O}^{(3)}_{\gamma_3} = -s \int_{\gamma_3} C, \tag{7.49}$$

C being the Chern-Simons form, that one can easily calculate in general (i.e. for any multicenter metric), finding the expression

$$C = U^{-9/2} \partial_j U \left(U \partial_i \partial_j U - \frac{3}{2} \partial_i U \partial_j U \right) \varepsilon_{imn} e^m e^n e^0. \tag{7.50}$$

We choose the cycle γ_3 as the full range $[0, 4\pi]$ of the cyclic coordinate τ, multiplied by a 2-sphere S^2_r of radius r contained in the space $\mathbf{R}^3$. In particular,

$$\int_{\gamma_3} C = 4\pi \int_{S^2_r} U^{-4} \partial_j U \left(U \partial_i \partial_j U - \frac{3}{2} \partial_i U \partial_j U \right) \varepsilon_{imn} dx^m dx^n \equiv f(a, r), \tag{7.51}$$

and our amplitude is

$$\mathcal{A} = < \mathcal{O}^{(3)}_{\gamma_3} > = - \int_0^\infty df(a, r). \tag{7.52}$$

The above integration looks trivial, at first sight, so that one is lead to conclude

$$\mathcal{A} = - \lim_{a \to \infty} f(a, r) + \lim_{a \to 0} f(a, r). \tag{7.53}$$

In these two limits the integral (7.51) is easily evaluated. It gives $-16\pi^2$ and $-8\pi^2$, respectively, so that $\mathcal{A} = 8\pi^2$. This result is clearly wrong, however, because we found $24\pi^2$ in the Eguchi-Hanson coordinates, see (5.12). The point is that the function $f(a, r)$ is not continuous in the entire range of values of a: there is a jump at $a = r$, which is due to the presence of the Dirac string in the Gibbons-Hawking coordinates. Indeed, one can check that

$$\lim_{a \to r^+} f(a, r) = \frac{112}{27} \pi^2, \qquad \lim_{a \to r^-} f(a, r) = -\frac{320}{27} \pi^2, \tag{7.54}$$

so that

$$\mathcal{A} = 16\pi^2 + \frac{112}{27}\pi^2 + \frac{320}{27}\pi^2 - 8\pi^2 = 24\pi^2, \tag{7.55}$$

correctly. This calculation illustrates the technical complications, that one has to deal with when using in the Gibbons-Hawking coordinates. This is useful in view of the future calculations with multicenter metrics, where the Gibbons-Hawking coordinates are the only ones available.

To conclude, let me present some computation of the kind of (5.12).

In the case of the Taub-Nut metric with

$$e^a = \left\{\frac{1}{2}\sqrt{\frac{x+2m}{x}}dx, \sqrt{x(x+2m)}\sigma_x, \sqrt{x(x+2m)}\sigma_y, 2m\sqrt{\frac{x}{x+2m}}\sigma_z\right\} \tag{7.56}$$

($x > 0$) the calculation gives

$$\omega_x^{(1)} = -96 \cdot 16\pi^2 \frac{x^2 m^2\, dm}{(x+2m)^5}, \qquad \int_0^\infty \omega_r^{(1)} = -16\pi^2. \tag{7.57}$$

For the $\mathbf{CP}^2$ Fubini-Study metric with

$$e^a = \left\{\frac{dr}{1+\Lambda r^2}, \frac{r\sigma_x}{\sqrt{1+\Lambda r^2}}, \frac{r\sigma_y}{\sqrt{1+\Lambda r^2}}, \frac{r\sigma_z}{1+\Lambda r^2}\right\}, \tag{7.58}$$

one gets

$$\omega_r^{(1)} = -48\pi^2 \frac{r^4\Lambda\, d\Lambda}{(1+\Lambda r^2)^3}, \qquad \int_0^\infty \omega_r^{(1)} = -24\pi^2. \tag{7.59}$$

References

[1] E. Witten, Commun. Math. Phys. 117 (1988) 353.

[2] D. Anselmi and P. Frè, Nucl. Phys. B416 (1994) 255;

D. Anselmi, P. Frè, L. Girardello and P. Soriani, Nucl. Phys. B427 (1994) 351; Phys. Lett. B 335 (1994) 416.

[3] A.A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. 59B (1975) 85.

[4] G.T. Horowitz and M. Srednicki, Commun. Math. Phys. 130 (1990) 83.

[5] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Nucl. Phys. B405 (1993) 279.

[6] E.F. Corrigan, D.B. Fairlie, S. Templeton and P. Goddard, Nucl. Phys. B140 (1978) 31;

E.F. Corrigan, P. Goddard and S. Templeton, Nucl. Phys. B151 (1979) 93.

[7] L. Baulieu and I.M. Singer, Nucl. Phys. B (Proc. Suppl.) 5B (1988) 12.

[8] D. Anselmi, Class. and Quantum Grav. 11 (1994) 2181.

[9] D. Anselmi, "More on the subtraction algorithm", preprint SISSA/ISAS 90/94/EP and hepth 9407023, to appear in Class. and Quantum Grav.

[10] E. Witten, Phys. Lett. 206B (1988) 601;

J.M.F. Labastida and M. Pernici, Phys. Lett. 212B (1988) 56; 213B (1988) 319.

L. Baulieu, A. Bilal and M. Picco, Nucl. Phys. B346 (1990) 507;

I. Oda and A. Sugamoto, Phys. Lett. B266 (1991) 280;

A. Nanamiki, I. Oda and A. Sugamoto, Phys. Rev. D44 (1991) 3835;

H. Kunitomo, Mod. Phys. Lett. A6 (1991) 2389;

F. Nagamori, I. Oda and A. Sugamoto, Progr. Theor. Phys. 88 (1992) 797;

I. Oda, Progr. Theor. Phys. Suppl. 110 (1992) 41;

D. Birmingham and M. Rakowski, Phys. Lett. B289 (1992) 271;

M.J. Perry and E. Teo, Nucl. Phys. B401 (1993) 206;

H.Y. Lee, A. Nanamiki and T. Ueno, Phys. Rev. D47 (1993) 1563.

[11] D. Anselmi and P. Fré, Nucl. Phys. B392 (1993) 401.

[12] E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457.

[13] K. Strebel, *Quadratic Differentials*, Springer-Verlag, Berlin, 1984.

[14] M. Kontsevich, Comm. Math. Phys. 147 (1992) 1.

[15] G. 't Hooft, Phys. Rev. D 14 (1076) 3432.

[16] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. 162 (1988) 169.

[17] T. Eguchi and A. Hanson, Ann. Phys. 120 (1979) 82.

[18] T. Eguchi, P.B. Gilkey and A. Hanson, Phys. Rep. 66 (1980) 213.

[19] S. Hawking, Phys. Lett. A 60 (1977) 81;

G.W. Gibbons and S. Hawking, Phys. Rev. D 15 (1977) 2752; Phys. Lett. B 78 (1978) 430.

[20] K. Konishi, N. Magnoli and H. Panagopoulos, Nucl. Phys. B309 (1988) 201; B323 (1989) 441.

[21] M. Bianchi, F. Fucito, G.C. Rossi and M. Martellini, "ALE instantons in string effective theory", preprint ROM2F-94-17 .

[22] E. Witten, "Supersymmetric Yang-Mills theory on a four manifold", preprint IASSNS-HEP-94-5 and hepth/9403195.

[23] A.L. Yuille, Class. and Quantum Grav. 4 (1987) 1409.

[24] D. Anselmi, M. Billó, P. Frè, L. Girardello and A. Zaffaroni, Int. Jou. Mod. Phys. A9 (1994) 3007.