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REMOVAL OF DIVERGENCES WITH THE BATALIN-VILKOVISKY FORMALISM [1]

Damiano Anselmi

International School for Advanced Studies (ISAS), via Beirut 2-4, I-34100 Trieste, Italy
and Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Trieste, Trieste, Italy

Abstract

We consider the problem of removing the divergences in an arbitrary gauge-field theory (possibly nonrenormalizable). We show that this can be achieved by performing, order by order in the loop expansion, a redefinition of some parameters (possibly infinitely many) and a canonical transformation (in the sense of Batalin and Vilkovisky) of fields and BRS sources. Gauge-invariance is turned into a suitable quantum generalization of BRS-invariance. We define quantum observables and study their properties. We apply the result to renormalizable gauge-field theories that are gauge-fixed with a nonrenormalizable gauge-fixing and prove that their predictivity is retained. A corollary is that topological field theories are predictive. Analogies and differences with the formalisms of classical and quantum mechanics are pointed out.

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1 Introduction

Suppose one is studying an ordinary renormalizable gauge field theory and that, for some unspecified reason, one wants to choose a nonrenormalizable gauge-fixing, namely a gauge-fixing that gives rise to nonrenormalizable vertices in the BRS action. In the present paper, we want to study the problem of predictivity of these theories. We determine in full generality the algorithm that permits to remove the divergences of a gauge-field theory, order by order in the perturbative loop expansion. We show that the independence of the physical amplitudes from the continuous deformations of the gauge-fixing survives the correction algorithm (apart from eventual BRS anomalies).

The gauge-fixing is thus supposed to contain constants of negative dimensions in mass units. We denote them collectively by κ throughout this paper. The classical renormalizable action $\mathcal{L}_{class}$ is κ-independent. One expects the quantum field theory to retain its predictivity, although infinitely many types of κ-dependent counterterms are needed. What guarantees the preservation of predictivity is not *a priori* obvious. The theorems about renormalizability of gauge-field theories that appear in the literature [1, 2, 3] are adapted to renormalizable BRS actions, i.e. renormalizable field theories that are gauge-fixed with an ordinary renormalizable gauge-fixing. Nevertheless, the investigation of the problem that we have in mind turns out to be very instructive.

Let Σ_0 be the starting BRS action. We find it convenient to use the formalism of Batalin and Vilkovisky [4, 5, 6, 7], although we never distinguish between irreducible and reducible field theories, to retain generality. Instead of speaking of antifields Φ^*, we shall speak of BRS sources K. Σ_0 has to satisfy the master equation [4]. We make the sufficiently general assumptions that Σ_0 is linear in K and that the functional measure is BRS invariant: $(\Sigma_0, \Sigma_0) = 0$ and $\Delta\Sigma_0 = 0$. Precisely, Σ_0 is the sum of the classical Lagrangian $\mathcal{L}_{class}(\phi, \lambda)$ plus the BRS variation $s\Psi$ of the gauge-fermion $\Psi(\Phi)$ plus the terms $K_A s\Phi^A$ that couple the BRS sources K_A to the BRS variations $s\Phi^A$ of the fields Φ^A. λ denote the constants that multiply the gauge-invariant terms of $\mathcal{L}_{class}$. Batalin and Vilkovisky prove [4] that if the action satisfies the master equation, then the functional integral Z is independent from the continuous deformations of the gauge-fermion Ψ. Since the nonrenormalizability that we plan to study only comes from Ψ, at first sight one could think that the argument by Batalin and Vilkovisky is enough to assure predictivity. Indeed, the renormalized action Σ also satisfies the master equation. However, the structure of Σ is not so simple as the structure of Σ_0: one is not even sure to be able to identify a *renormalized* gauge-fermion inside Σ. In conclusion, the problem that we have in mind is to prove that the subtraction procedure can be performed while preserving the independence of the physical amplitudes from the continuous deformations of the gauge-fermion Ψ that gauge-fixes the *starting* action Σ_0.

The first step is to find the subtraction algorithm in full generality (i.e. for any gauge-field theory, eventually a nonrenormalizable one). The algorithm that we find extends the well-known ones[2].

[2]See for example [1, 2] in the case of ordinary Yang-Mills theories and [3] in the case of higher

We have to construct, order by order in the loop expansion, the renormalized action Σ that is able to guarantee the removal of all the divergences in the effective action Γ while retaining gauge-invariance (which is in fact converted into invariance defined by new nilpotent operators), although infinitely many kinds of counterterms are needed. The correction of divergences is achieved by performing, order by order, a redefinition of the constants λ that multiply the gauge-invariant terms of the classical Lagrangian $\mathcal{L}_{class}$, together with a redefinition of fields and BRS sources. In particular, the redefinition of fields and BRS sources is a canonical transformation, in the sense of Batalin and Vilkovisky [4, 5, 7], i.e. a transformation that preserves the antibrackets $(. , .)$. This is the key observation of the paper.

In other words, one starts from a *classical* BRS action Σ_0 that is the most simple solution to the master equation, i.e. it is linear in the BRS sources (notice that Σ_0 is only *one* particular solution to the master equation). Then one implements the correction algorithm, which automatically yields a *quantum* action Σ, such that the effective action Γ is convergent. Σ is related to Σ_0 by a set of canonical transformations and redefinitions of suitable parameters λ. Consequently, Σ is another particular solution to the master equation. As a matter of fact, Σ is the *good* solution. The only problem is that one does not know the good solution Σ to the master equation from the beginning. The subtraction algorithm can thus be considered as the *principle of correspondence* that produces the right *quantum* action Σ starting from the known *classical* action Σ_0. Such a correspondence principle is nothing but the search for the *correct* variables $\{\Phi, K\}$ and the correct definitions of the parameters λ of the classical Lagrangian $\mathcal{L}_{class}$. One has nonrenormalizability when the correspondence principle produces infinitely many good solutions Σ and there is no way of privileging a finite subset of them. Indeed, when a theory is nonrenormalizable, $\mathcal{L}_{class}$ must depend on infinitely many λ's, otherwise one cannot remove the divergences by *redefining* them.

The second step is the application of the general subtraction algorithm to renormalizable field theories that are treated with a nonrenormalizable gauge-fixing. The fact that divergences can be made disappear by a set of canonical transformations and redefinitions of parameters makes the algorithm nicely tractable and all the properties that we need can be proved without too much difficulty.

The idea is the following: one would like to prove that the infinitely many κ-dependent counterterms are all BRS-exact, so that the normalization of the coupling constants associated to them is immaterial[3]. *A priori* it is not clear what "BRS-exact counterterm"

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derivative quantum gravity. The latter is an example of algorithm in which the field redefinitions are not simply field renormalizations; they are nevertheless independent of the derivatives of the fields and linearity in the BRS sources is preserved; in general, instead, the field redefinitions contain derivative terms and linearity in the BRS sources is lost, as we shall see in an example towards the end of the paper.

[3] As a matter of fact, we are tacitly assuming that there are not BRS anomalies. If possible, it is convenient to move the eventual BRS anomalies from gauge-symmetries to non-gauge ones and completely neglect the non-gauge symmetries, otherwise the master equation cannot be solved. BRS anomalies can cause a dependence of the physical amplitudes on a coupling constant that multiplies a BRS-exact term

means, since the BRS-operator s gets renormalized in a highly nontrivial way (if it is compared, for example, with the simplicity of the renormalization of s when the gauge-fixing is an ordinary renormalizable one). As a matter of fact, a "quantum BRS operator" Ω can be introduced [8]: Ω generalizes s to the space of fields and BRS sources. It is nilpotent ($\Omega^2 = 0$) and is fundamental for the definition of the observables.

We show that, if the κ-derivative $\frac{\partial\Sigma_0}{\partial\kappa}$ of the zeroth loop order BRS action Σ_0 is s-exact (as it happens when only the gauge-fermion Ψ depends on κ), then the derivative $\frac{\partial\Sigma}{\partial\kappa}$ of the renormalized action Σ is Ω-exact and the derivative $\frac{\partial\Gamma}{\partial\kappa}$ of the effective action Γ is $(\,.\,,\Gamma)$-exact. These properties permit to derive the conclusion that any physical amplitude is κ-independent.

Regularization is always understood. The dimensional technique and the minimal subtraction scheme can be convenient for most purposes [1, 2, 3]. Notice, however, that the operator Δ introduced by Batalin and Vilkovisky is proportional to $\delta(0)$ and dimensional regularization looses any trace of these terms (see [9] for the study of the Pauli-Villars regularization of Δ and [10] for a discussion of $\delta(0)$ divergent terms related to the measure in quantum gravity). It is not necessary to use a regularization scheme that preserves explicit gauge-invariance, since gauge-invariance can always be retrieved with local counterterms. The formal arguments of the present paper can suggest what quantum generalization of gauge-invariance has to be retrieved with suitable local counterterms when the regularization scheme explicitly breaks it.

A remark is required in order to specify what we shall mean by "locality" (local functional, local canonical transformation, and so on). Indeed, when a nonrenormalizable Lagrangian is allowed, an arbitrary number of derivatives can appear. This is not, strictly speaking, a problem intrinsic in nonrenormalizable theories. It is also present in renormalizable theories, when one wants to study amplitudes that involve insertions of composite local operators $\mathcal{O}(x)$ of sufficiently high dimensionality. In that case, some constant κ of negative dimension has to be introduced and infinitely many kinds of counterterms can appear. What one can do is to truncate each step of the derivation and each formula up to some order κ^D. In this way, all amplitudes with D or less $\mathcal{O}(x)$-insertions are made finite. Similarly, we can always assume to neglect the powers greater than D in the negatively dimensioned parameters and, contemporarily, to neglect all the parameters of dimension less than $-D$. Then, the effective action is only made finite up to a given order in the negatively dimensioned parameters (but to all orders in $\hbar$). A "local" functional is a functional such that its "D-truncation" is local. Since the trick can be used for arbitrarily large D, none of the results gets affected.

For simplicity, we can assume that the propagators are of the form $\frac{1}{k^2+m^2}$. In other words, the propagator is defined from the Lagrangian term $\mathcal{Q}$ that is quadratic both in the fields and in derivatives plus the mass terms. All the other terms will be considered as interactions. This means that in our perturbation series all the parameters but the

in the Lagrangian and this can even happen in a renormalizable theory. Moreover, BRS anomalies can spoil unitarity. The detailed analysis of these aspects of the problem are beyond the scope of the present paper and will be eventually treated elsewhere.

constant λ_0 that multiplies $\mathcal{Q}$ are considered "small". λ_0, instead, is considered finite and not small. This remark is required, since the renormalizablity or nonrenormalizability of the theory depends on what parameters are considered small and what are considered finite. In higher derivative quantum gravity [3], for example, if the constants that multiply the higher derivative terms $\sqrt{g}R^2$ and $\sqrt{g}R_{\mu\nu}R^{\mu\nu}$ in the classical Lagrangian are "small", then the theory is nonrenormalizable; if, on the other hand, they are considered as finite (on the same footing as the parameter that multiplies the Hilbert-Einstein term $\sqrt{g}R$), then the theory is renormalizable (but not unitary). As a matter of fact, our arguments are also valid if the propagators behave like $\frac{1}{(k^2)^n}$, for $k^2 \to \infty$, as long as $n < \infty$. In other words, only a finite number of terms that are quadratic in the fields can be multiplied by finite (not "small") parameters, otherwise the previous observations about locality are nonsense (the theory is truly nonlocal).

Let us now make some comments of a general character. A quantum field theory *can* depend on infinitely many arbitrary parameters, without loosing predictivity: it happens when the on-shell physical amplitudes depend only on finitely many parameters. If this is the case, the remaining infinitely many parameters can be fixed at any energy at whatever value one prefers.

This fact suggests that the following classification of quantum field theories deserves attention. Let us divide quantum field theories into predictive and nonpredictive theories. Predictive field theories are those theories, whose on-shell physical amplitudes depend on finitely many parameters. Nonpredictive field theories are those theories, whose on-shell amplitudes depend on infinitely many parameters.

Renormalizable field theories are *a fortiori* predictive. Treated with the usual gauge-fixings, they necessarily depend on finitely many parameters. However, predictive field theories do not necessarily depend on finitely many parameters, nevertheless the physical amplitudes can still depend on finitely many parameters. This is for example the case of renormalizable theories that are treated with a nonrenormalizable gauge-fixing, on which we focus in the present paper. Correspondingly, nonrenormalizable field theories are not necessarily nonpredictive. For example, finite theories can be nonrenormalizable and predictive.

What are the possible applications of our results? First of all, notice that a subtraction algorithm that is applyable to nonrenormalizable field theories can also be used to study the renormalization of highly-dimensioned composite operators in ordinary renormalizable field theories. Concerning nonrenormalizable gauge-fixings, on the other hand, it is hard to think such gauge-fixings will ever be used in Yang-Mills theories or in the Standard Model, since the ordinary renormalizable ones are quite satisfactory. However, there are theories in which an exotic gauge-fixing can be convenient for some peculiar reasons. This is the case, for example, of four dimensional topological fields theories of Witten type [11, 12, 13, 14, 15, 16], that are now attracting a lot of interest. Among these, an interesting one is surely the topological sigma-model formulated in [16], which describes the triholomorphic embeddings of four dimensional Riemannian manifolds into almost quaternionic manifolds[4]. It is an example of an irreducible gauge-field theory, which is nonrenormalizable, but, since it is also topological, the nonrenormalizability is entirely due to the gauge-fixing (the classical action is either zero or a topological invariant). It is well-known that the mathematical interpretation of topological field theories shows that there is a dependence on the gauge-fixing: two gauge-fixings that are not continuously deformable one into the other give rise to inequivalent field theories[5]. Thus one cannot turn the gauge-fixing (which in the present case is the triholomorphicity condition on the map) to a renormalizable one without either spoiling general covariance or changing completely the theory. Nevertheless, one expects the nonrenormalizable theory to be perturbatively well defined and predictive. This is a corollary of our result, namely topological field theories (of Witten type) are predictive.

Notice that topological field theories are intrinsically nonperturbative. On the other hand, in this paper we are only concerned with the perturbative behaviour of quantum field theories. Our results can thus by described by saying that the nonrenormalizability of topological models produces no perturbative obstruction to a good definition of them (apart from the eventual BRS anomalies).

An example of first stage reducible nonrenormalizable topological gauge-field theory is topological gravity (there are various versions of it: topological conformal gravity, see [12]; topological gravity with the self-duality condition on the Riemann tensor [13]; topological gravity derived from the twist of N=2 supergravity [14]). The topological Yang-Mills theory originally formulated by Witten [11] is, instead, renormalizable, so that its perturbative definition is straightforward.

An exotic gauge-fixing can also be used in order to make computations easier. For example, in ref. [17] a nonlinear gauge-fixing plays an essential role in simplifying the perturbative two-loop computation in quantum gravity[6].

The paper is organized as follows. In section 2 we fix notation and conventions and give the fundamental properties that will be useful in successive derivations. In section 3 we define the nilpotent quantum operators Ω and $\mathrm{ad}\,\Gamma$ that generalize the BRS operator s. We define the observables and study their change under canonical transformations. We point out analogies and differences with the formalisms of classical and quantum mechanics. In section 4 we derive the algorithm for removing the divergences of a generic quantum field theory, while preserving gauge-invariance. In section 5 we show that the redefinitions of the parameters λ of $\mathcal{L}_{class}$ do not depend on the constants that are only introduced *via* the gauge-fixing. This result applies to the case of renormalizable gauge-field theories treated with a nonrenormalizable gauge-fixing showing that predictivity is retained. We also define the convergent physical amplitudes. In section 6 we give some examples: Q.E.D. with an exotic gauge-fixing and the topological σ-model of ref.

[4] I would like to thank E. Witten for a couple of inspiring discussions on the nonrenormalizability of this model, that gave rise to my interest in the problem investigated in the present paper.

[5] A similar behaviour, of course, is expected to happen in *any* gauge-field theory.

[6] That computation confirmed the Goroff-Sagnotti result [18] that quantum gravity is not two-loop finite.

[16]. We also show how the usual coupling constant renormalizations and wave function renormalizations are retrieved within our approach. Section 7 contains the conclusions, while the appendix is devoted to the lengthy, but straightforward proof of a formula that is needed in the paper.

2 Preliminars

In this section we introduce the notation and the basic definitions. For self-consistence, we also report some simple arguments by Batalin and Vilkovisky [4, 5, 7] that will be useful in the following.

The partition function is

$$Z[J_A, K_A] = \int \mathrm{d}\Phi \, \mathrm{e}^{\frac{i}{\hbar}\Sigma(\Phi^A, K_A) + \frac{i}{\hbar} J_A \Phi^A}, \tag{1}$$

where Φ^A denote the fields, while J_A are the corresponding sources. Notice that Φ^A is not the *minimal* set of fields [4] that is usually necessary to solve the master equation. Rather, it is enlarged to contain the classical fields ϕ, the ghosts, the antighosts, the Lagrange multipliers, the eventual extraghosts and so on. K_A is the source associated to the BRS transformation $s\Phi^A$ of the field Φ^A (it will be called the *BRS source*). K_A differs from the antifields Φ^*_A introduced by Batalin and Vilkovisky by a derivative of the gauge-fermion Ψ:

$$K_A = \Phi^*_A - \frac{\partial \Psi}{\partial \Phi^A}. \tag{2}$$

The transition from $\{\Phi^A, \Phi^*_A\}$ to $\{\Phi^A, K_A\}$ is a canonical transformation [4, 5]. The operators that we use are defined in terms of $\{\Phi^A, K_A\}$ rather than $\{\Phi^A, \Phi^*_A\}$. The antibrackets are

$$(X, Y) = \frac{\partial_r X}{\partial \Phi^A} \frac{\partial_l Y}{\partial K_A} - \frac{\partial_r X}{\partial K_A} \frac{\partial_l Y}{\partial \Phi^A}. \tag{3}$$

The subscripts r and l denote right and left derivatives, respectively. When there is no subscript, that means that left and right derivatives are equivalent. The statistics of the field Φ^A is denoted by ε_A, which is an integer modulo two (zero for bosons, one for fermions). The statistics of K_A is $\varepsilon_A + 1$. We report some simple properties of the antibrackets [7] that will be very useful in the calculations, namely

$$\begin{array}{c} \varepsilon[(X, Y)] = \varepsilon(X) + \varepsilon(Y) + 1, \\ (X, Y) = -(-1)^{(\varepsilon(X)+1)(\varepsilon(Y)+1)}(Y, X), \\ (-1)^{(\varepsilon(X)+1)(\varepsilon(W)+1)}(X, (Y, W)) + \text{cyclic permutations} = 0. \end{array} \tag{4}$$

We also introduce the operator Δ, the definition of which differs from the one given by Batalin and Vilkovisky because of the replacements of antifields with BRS sources, namely

$$\Delta = \frac{\partial_r}{\partial \Phi^A} \frac{\partial_r}{\partial K_A} (-1)^{\varepsilon_A + 1}. \tag{5}$$

The properties of Δ that will be useful in the calculations are $\varepsilon(\Delta)=1$ and

$$\Delta^2=0,\quad \Delta(X,Y)=(X,\Delta Y)-(-1)^{\varepsilon(Y)}(\Delta X,Y). \tag{6}$$

The action Σ is supposed to satisfy the master equation

$$(\Sigma,\Sigma)=2i\hbar\Delta\Sigma. \tag{7}$$

A canonical transformation is a transformation of fields Φ^A and BRS sources K_A into new fields Φ'^A and new BRS sources K'_A that preserves the antibrackets. As in classical mechanics, a generating functional $F(\Phi^A,K'_A)$ can be introduced, such that

$$\Phi'^A=\tfrac{\partial F}{\partial K'_A},\quad K_A=\tfrac{\partial F}{\partial \Phi^A}. \tag{8}$$

F is a fermionic functional. The rule for the change of the action under a canonical transformation $\mathcal{C}$ is determined by the requirement that the new action Σ' satisfies the master equation. Our convention is that the arguments of Σ' are still called $\{\Phi^A,K_A\}$. The expression for $\Sigma'(\Phi^A,K_A)$, that will be proved in a moment, then turns out to be

$$\Sigma'(\Phi^A,K_A)=\mathcal{C}\Sigma(\Phi^A,K_A)=\Sigma(\Phi'^A(\Phi,K),K'_A(\Phi,K))+\frac{1}{2}i\hbar\ln\,J, \tag{9}$$

where J is the Berezinian determinant associated to the change of fields and sources (which is not a change of variables in the functional integral)

$$J=\det\frac{\partial(\Phi,K)}{\partial(\Phi',K')}. \tag{10}$$

A useful property of J is the following [5]

$$\Delta\,\ln\,J=\frac{1}{4}(\ln\,J,\ln\,J). \tag{11}$$

We adopt the following convention: any $\{\Phi^A,K_A\}$-dependent functional or operator will be marked with a tilde when we mean that the fields Φ^A and the BRS sources K_A have to be replaced by $\Phi'^A(\Phi,K)$ and $K'_A(\Phi,K)$, respectively. Thus, equation (9) will be briefly written as

$$\Sigma'=\tilde{\Sigma}+\frac{1}{2}i\hbar\ln\,J. \tag{12}$$

Similarly, $\tilde{\Delta}$ is the same as the operator Δ, in which the derivatives with respect to unprimed fields and BRS sources are replaced by derivatives with respect to the primed ones. The fact that the transformation that we are considering is canonical can be simply expressed by the following equation

$$(\,.\,,\,.\,)\tilde{}=(\,.\,,\,.\,). \tag{13}$$

The proof that Σ' still satisfies the master equation if Σ does is immediate consequence of the following identity [5]

$$\tilde{\Delta} X = \Delta X - \frac{1}{2}(X, \ln\, J), \tag{14}$$

for any X. Indeed, let us consider the master equation (7) satisfied by Σ. Since the name that one gives to fields and BRS sources is immaterial, it is clear that the identity

$$(\tilde{\Sigma}, \tilde{\Sigma})\tilde{} = 2i\hbar \tilde{\Delta}\tilde{\Sigma}, \tag{15}$$

also holds. Using (13) and (14) one has

$$(\tilde{\Sigma}, \tilde{\Sigma}) = 2i\hbar \Delta\tilde{\Sigma} - i\hbar(\tilde{\Sigma}, \ln\, J). \tag{16}$$

At this point, using (11), it is easy to see that definition (12) is chosen precisely to have

$$(\Sigma', \Sigma') = 2i\hbar \Delta \Sigma', \tag{17}$$

as desired.

Now, consider the functional integral (1). Let us perform the following infinitesimal change of variables

$$\Phi^A \rightarrow \Phi^A + \frac{\partial_l \Sigma}{\partial K_A}\Lambda = \Phi^A + (\Phi^A, \Sigma)\Lambda, \tag{18}$$

where Λ is some constant, infinitesimal fermionic parameter. Due to the eventual non-linearity of Σ in K_A, (18) is in general a source-dependent change of variables. The variation δZ of Z is zero and can be written in the form

$$0 = \delta Z = \int \mathrm{d}\Phi\, \mathrm{e}^{\frac{i}{\hbar}\Sigma + \frac{i}{\hbar} J_A \Phi^A} \left\{ -\frac{i}{2\hbar}(2i\hbar\Delta\Sigma - (\Sigma, \Sigma)) + \frac{i}{\hbar} J_A (\Phi^A, \Sigma) \right\} \Lambda. \tag{19}$$

The same result can be derived from the identity

$$0 = \int \mathrm{d}\Phi\, \frac{\partial_r}{\partial \Phi^A} \left\{ \mathrm{e}^{\frac{i}{\hbar}\Sigma + \frac{i}{\hbar} J_A \Phi^A} \frac{\partial_l \Sigma}{\partial K_A} \right\}. \tag{20}$$

Since Σ satisfies the master equation (7), formula (19) reduces to the following, fundamental Ward identity:

$$< J_A(\Phi^A, \Sigma) >_J = 0. \tag{21}$$

The subscript J is to mean that the sources J are not set to zero.

Let us introduce, as usual, the generating functional $W[J_A, K_A]$ of connected Green functions, as follows:

$$Z[J_A, K_A] = \mathrm{e}^{\frac{i}{\hbar} W[J_A, K_A]}. \tag{22}$$

The Ward identity (21) can be rewritten as

$$J_A \frac{\partial_l W}{\partial K_A} = 0. \tag{23}$$

Let us also introduce the generating functional $\Gamma[\Phi^A, K_A]$ of one particle irreducible Green functions, defined as the Legendre transform of $W[J_A, K_A]$ with respect to J_A and with K_A inert:

$$\Gamma[\Phi^A, K_A] = W[J_A(\Phi, K), K_A] - J_A(\Phi, K)\Phi^A, \tag{24}$$

where the function $J_A(\Phi, K)$ is defined as the inverse of

$$\Phi^A(J, K) = \frac{\partial_l W}{\partial J_A}. \tag{25}$$

The properties of Legendre transforms guarantee that

$$J_A = -\tfrac{\partial_r \Gamma}{\partial \Phi^A}, \quad \tfrac{\partial_l W}{\partial K_A} = \tfrac{\partial_l \Gamma}{\partial K_A}, \tag{26}$$

since the BRS sources K are simple spectators. Thus the Ward identity (23) can be rewritten as [7]

$$(\Gamma, \Gamma) = 0. \tag{27}$$

So, whenever the action Σ satisfies the master equation (7), then the effective action Γ satisfies the Ward identity (27).

3 Observables

In this section, we identify the observables and study their properties. Let us first make a brief digression on classical and quantum mechanics.

In classical mechanics, the Hamiltonian $H(p, q)$ (let us assume it is time independent) satisfies the "master equation"

$$\{H, H\} = 0, \tag{28}$$

$\{\,.\,,\,.\,\}$ denoting the Poisson brackets. A time independent integral of motion $\mathcal{O}$ satisfies

$$\{\mathcal{O}, H\} \equiv (\mathrm{ad}\, H)\, \mathcal{O} = 0. \tag{29}$$

Notice that the operator $\mathrm{ad}\, H$ is nilpotent, $(\mathrm{ad}\, H)^2 = 0$. Suppose H depends on some parameter g: $H = H(p, q, g)$. Consider a time independent (but possibly g-dependent) canonical transformation generated by $f(p', q, g)$, namely

$$p = \tfrac{\partial f}{\partial q}, \quad q' = \tfrac{\partial f}{\partial p'}. \tag{30}$$

The Hamiltonian H transforms into

$$H'(p, q, g) = H(p'(p, q, g), q'(p, q, g), g) = \tilde{H}. \tag{31}$$

Similarly, an integral of motion $\mathcal{O}$ transforms into $\mathcal{O}' = \tilde{\mathcal{O}}$. Since canonical transformations preserve the Poisson brackets, $\mathrm{ad}\, H$-closure and $\mathrm{ad}\, H$-exactness are converted into ad H'-closure and ad H'-exactness, respectively. We are interested in the transformation of $\frac{\partial H}{\partial g}$, i.e. $\frac{\partial H'}{\partial g}$. The explicit computation gives

$$\frac{\partial H'}{\partial g}=\widetilde{\frac{\partial H}{\partial g}}-\left\{\frac{\partial f}{\partial g},H'\right\}. \tag{32}$$

In particular, this assures that if $\frac{\partial H}{\partial g}$ is ad H-exact, then $\frac{\partial H'}{\partial g}$ is ad H'-exact.

Let us now turn to quantum mechanics. The "master equation" is simply

$$[H,H]=0, \tag{33}$$

where the square brackets denote the commutator. A (time independent) observable is an operator $\mathcal{O}$ that commutes with H. A canonical transformation is performed by a unitary operator U and

$$H'=UHU^{-1}=\tilde{H}. \tag{34}$$

Similarly, $\mathcal{O}$ transforms into $\mathcal{O}'=\tilde{\mathcal{O}}$. Again, ad H-closure and ad H-exactness are converted into ad H'-closure and ad H'-exactness. Moreover,

$$\frac{\partial H'}{\partial g}=U\frac{\partial H}{\partial g}U^{-1}+\left[\frac{\partial U}{\partial g}U^{-1},UHU^{-1}\right]=\widetilde{\frac{\partial H}{\partial g}}+\left[\frac{\partial U}{\partial g}U^{-1},H'\right]. \tag{35}$$

Once more it is true that if $\frac{\partial H}{\partial g}$ is ad H-exact, then $\frac{\partial H'}{\partial g}$ is ad H'-exact.

Inspired by this digression, it is simple to work out the definition of the nilpotent operator Ω that extends the BRS operator s to the space of fields and BRS sources [8]. This will also be useful to understand the limits of the analogy among antibrackets, commutators and Poisson brackets. Suppose Σ depends on some parameter g. The change of $\frac{\partial \Sigma}{\partial g}$ under a canonical transformation (8) generated by $F(\Phi^A,K'_A,g)$ turns out to be very similar to (32) and (35), namely

$$\frac{\partial \Sigma'}{\partial g}=\widetilde{\frac{\partial \Sigma}{\partial g}}-\Omega'\frac{\partial F}{\partial g} \tag{36}$$

(the proof can be found in the appendix), where we have introduced the fundamental operator

$$\Omega'=\mathrm{ad}\,\Sigma'-i\hbar\Delta, \tag{37}$$

ad Σ being now defined by $(\mathrm{ad}\,\Sigma)\,X=(X,\Sigma)$. We are thus lead to interpret the operator Ω' as the canonically transformed version of the operator Ω, defined by

$$\Omega X\equiv (X,\Sigma)-i\hbar\Delta X. \tag{38}$$

Notice that Ω acts on the space of fields and BRS sources. Ω is the candidate for the definition of the physical quantities. The key check that we are on the right way is the nilpotence of Ω. Indeed, let us suppose that $X=\Omega\chi$ for some χ. Then,

$$\Omega X=\Omega^2\chi=((\chi,\Sigma)-i\hbar\Delta\chi,\Sigma)-i\hbar\Delta((\chi,\Sigma)-i\hbar\Delta\chi). \tag{39}$$

Using the Jacobi identity of (4) and the properties (6) of the operator Δ, we get

$$\Omega^2\chi = \frac{1}{2}(\chi, (\Sigma, \Sigma)) - i\hbar(\chi, \Delta\Sigma), \tag{40}$$

which vanishes due to the master equation (7).

Differentiation of the master equation (7) gives

$$\left(\frac{\partial\Sigma}{\partial g}, \Sigma\right) = i\hbar\Delta\frac{\partial\Sigma}{\partial g}. \tag{41}$$

This means that $\frac{\partial\Sigma}{\partial g}$ is Ω-closed. We then expect $\frac{\partial\Sigma'}{\partial g}$ to be Ω'-closed. In particular, as we shall prove in a moment, the first term on the right hand side of equation (36) is Ω'-closed, while the second term is trivially Ω'-exact. In classical or quantum mechanics, we cannot say that $\frac{\partial H}{\partial g}$ is $\mathrm{ad}\, H$-closed: indeed, a g-differentiation of equations (28) and (33) gives no new information. This is because (28) and (33) are trivial identities, i.e. they would be satisfied by any H. The master equation (7), however, has a nontrivial content: the definition of antibrackets does not imply that it is identically true[7]. This suggests that the analogy with classical and quantum mechanics has to be taken *cum grano salis*. This is a luck, rather that a handicap, since we shall be able to prove useful properties that have no counterpart in classical or quantum mechanics. In particular, the nontriviality of (27) will be extremely important in the following.

The proof of formula (36) is a little involved, although conceptually rather simple [it is only a matter of change of variables and it can be obtained by mimicking the analogous steps that permit to prove (32) in classical mechanics]. That is why we postpone it to the appendix. For the moment, the reader should be satisfied with the plausibility that is suggested by the similarities with equations (32) and (35) and the fact that Ω' is the only reasonable nilpotent generalization of the BRS operator s to the space of fields and BRS sources. The minus sign in (36) can be promptly checked by considering a canonical transformation infinitesimally close to the identity, i.e. a transformation with $F(\Phi^A, K'_A) = \Phi^A K'_A + \varepsilon\mathcal{R}(\Phi^A, K'_A)$, ε being a constant infinitesimal parameter.

Let us prove in detail that if $\mathcal{O}$ is Ω-closed, then $\tilde{\mathcal{O}}$ is Ω'-closed. In particular, this assures that $\widetilde{\frac{\partial\Sigma}{\partial g}}$ is Ω'-closed. Indeed, we have

$$\Omega\mathcal{O} = (\mathcal{O}, \Sigma) - i\hbar\Delta\mathcal{O} = 0. \tag{42}$$

Changing name to fields and BRS sources wherever their appear, we also get

$$(\tilde{\mathcal{O}}, \tilde{\Sigma})^{\sim} - i\hbar\tilde{\Delta}\tilde{\mathcal{O}} = 0. \tag{43}$$

Using (12), (13) and (14), we get

$$(\tilde{\mathcal{O}}, \Sigma') - i\hbar\Delta\tilde{\mathcal{O}} = \Omega'\tilde{\mathcal{O}} = 0. \tag{44}$$

[7] As a matter of fact, the definition of antibrackets only implies that $(F, F) = 0$ for any fermionic F.

Thus, if $\mathcal{O}$ is Ω-closed, it is natural to define its variation under a canonical transformation according to

$$\mathcal{O}' = \tilde{\mathcal{O}} + \Omega' \Lambda. \tag{45}$$

In general, Λ can be chosen to be zero. Notice, however, that for $\mathcal{O} = \frac{\partial \Sigma}{\partial g}$ one has $\Lambda = -\frac{\partial F}{\partial g} \neq 0$, according to (36). With $\Lambda = 0$, (45) is reminiscent of the usual quantum mechanical rule $\mathcal{O}' = U \mathcal{O} U^{-1} = \tilde{\mathcal{O}}$.

We now prove that if $\mathcal{O}$ is Ω-exact, then $\tilde{\mathcal{O}}$ is Ω'-exact. Let χ be such that $\mathcal{O} = \Omega \chi$. We have

$$\mathcal{O} = (\chi, \Sigma) - i\hbar \Delta \chi. \tag{46}$$

Changing name to fields and BRS sources wherever their appear, we also get

$$\tilde{\mathcal{O}} = (\tilde{\chi}, \tilde{\Sigma})\tilde{} - i\hbar \tilde{\Delta} \tilde{\chi}. \tag{47}$$

Using (12), (13) and (14), we finally obtain

$$\tilde{\mathcal{O}} = (\tilde{\chi}, \Sigma') - i\hbar \Delta \tilde{\chi} = \Omega' \tilde{\chi}, \tag{48}$$

as desired. Consequently, $\mathcal{O}'$ is also Ω'-exact. A corollary is that if $\frac{\partial \Sigma}{\partial g}$ is Ω-exact, then $\frac{\partial \Sigma'}{\partial g}$ is Ω'-exact. This fact will be very useful in the following.

The formulæ for the change of the functional Γ under a canonical transformation can be also conjectured by analogy with the classical and quantum mechanical formulæ. In particular, (31) and (34) suggest

$$\Gamma' = \tilde{\Gamma}. \tag{49}$$

Moreover, (32) and (35) suggest

$$\frac{\partial \Gamma'}{\partial g} = \widetilde{\frac{\partial \Gamma}{\partial g}} - \left(\frac{\partial F}{\partial g}, \Gamma' \right) \equiv \widetilde{\frac{\partial \Gamma}{\partial g}} - \mathrm{ad}\, \Gamma' \, \frac{\partial F}{\partial g}. \tag{50}$$

Although formula (49) is very natural and can be easily checked in the case of canonical transformations infinitesimally close to the identity, we give it only as a conjecture. We shall be able to prove our results independently of (49). Equation (50), instead, can be simply proved by taking the g-derivative of (49) and mimicking the standard arguments that are contained in the appendix. However, it must be kept in mind that it depends on the validity of (49), that we leave without proof. A direct consequence of (50) is that if $\frac{\partial \Gamma}{\partial g}$ is $\mathrm{ad}\, \Gamma$-exact, then $\frac{\partial \Gamma'}{\partial g}$ is $\mathrm{ad}\, \Gamma'$-exact.

Moreover, a g-differentiation of (27) shows that $\frac{\partial \Gamma}{\partial g}$ is $\mathrm{ad}\, \Gamma$-closed.

Let us now prove that if $\mathcal{O}$ is Ω-closed, then $< \mathcal{O} >_J$ is $\mathrm{ad}\, \Gamma$-closed (the subscript J means that the sources J are not set to zero). We start from

$$0 =< \Omega \mathcal{O} >_J =< \frac{\partial_r \mathcal{O}}{\partial \Phi^A} \frac{\partial_l \Sigma}{\partial K_A} - \frac{\partial_r \mathcal{O}}{\partial K_A} \frac{\partial_l \Sigma}{\partial \Phi^A} - i\hbar (-1)^{\varepsilon_A + 1} \frac{\partial_r}{\partial \Phi^A} \frac{\partial_r \mathcal{O}}{\partial K_A} >_J . \tag{51}$$

By performing the change of variables

$$\delta\Phi^A = \frac{\partial_l \Sigma}{\partial K_A}\Lambda, \tag{52}$$

(Λ being a fermionic infinitesimal parameter) in the functional integral $<\mathcal{O}>_J$, one finds

$$< \frac{\partial_r \mathcal{O}}{\partial \Phi^A}\frac{\partial_l \Sigma}{\partial K_A} >_J = -\frac{i}{\hbar} < \mathcal{O}\frac{\partial_l \Sigma}{\partial K_A} >_J \; J_A. \tag{53}$$

Moreover, with an integration by parts one can show that

$$< \Delta\mathcal{O} >_J = \frac{i}{\hbar} < \frac{\partial_r \mathcal{O}}{\partial K_A}\frac{\partial_l \Sigma}{\partial \Phi^A} >_J + \frac{i}{\hbar} < \frac{\partial_r \mathcal{O}}{\partial K_A} >_J \; J_A(-1)^{\varepsilon_A}. \tag{54}$$

It is thus possible to write (51) in the form

$$\begin{aligned} 0 &= \frac{i}{\hbar} < \mathcal{O}\frac{\partial_l \Sigma}{\partial K_A} >_J \; J_A - (-1)^{\varepsilon_A} < \frac{\partial_r \mathcal{O}}{\partial K_A} >_J \; J_A \\ &= -(-1)^{\varepsilon_A}\frac{\partial_r < \mathcal{O} >_J}{\partial K_A}J_A + \frac{i}{\hbar} < \mathcal{O} >_J \; \frac{\partial_l W}{\partial K_A}J_A = -(-1)^{\varepsilon_A}\frac{\partial_r < \mathcal{O} >_J}{\partial K_A}J_A, \end{aligned} \tag{55}$$

where we have used the Ward identity (23) for W. Using

$$\left.\frac{\partial_r < \mathcal{O} >_J}{\partial K_A}\right|_J = \left.\frac{\partial_r < \mathcal{O} >_J}{\partial K_A}\right|_\Phi - \left.\frac{\partial_r < \mathcal{O} >_J}{\partial J_B}\right|_K \left.\frac{\partial_r J_B}{\partial K_A}\right|_\Phi, \tag{56}$$

and some standard manipulation, one gets

$$0 = (< \mathcal{O} >_J, \Gamma) - \left.\frac{\partial_r < \mathcal{O} >_J}{\partial J_B}\right|_K (J_B, \Gamma) = (< \mathcal{O} >_J, \Gamma), \tag{57}$$

which is the claimed result. We have used the fact that

$$(J_B, \Gamma) = 0, \tag{58}$$

as one can prove by differentiating (27) with respect to Φ^B. Notice that a nontrivial formula like (58) is once more consequence of the nontrivial content of (27). No similar formula can be derived in classical or quantum mechanics.

With a similar argument, one can prove that if $\mathcal{O}$ is Ω-exact, then $<\mathcal{O}>_J$ is $\mathrm{ad}\,\Gamma$-exact.

We are thus allowed to define an observable as an $\mathrm{ad}\,\Gamma_\infty$-closed finite functional $\mathcal{O}$, where Γ_∞ denotes the finite effective action (corrected at any loop order in $\hbar$). We shall say more about this definition of observable in the following. It is the analogue of the classical concept of invariant of motion or the quantum mechanical concept of observable. Notice that $\mathcal{O}$ is defined up to $\mathrm{ad}\,\Gamma_\infty$-exact terms. The nice property is that an $\mathrm{ad}\,\Gamma_\infty$-exact functional Λ is zero on shell (i.e. at $J_A = 0$ and $K_A = 0$ or, equivalently, at $\Phi^A = 0$ and $K_A = 0$). This is because if Λ is $\mathrm{ad}\,\Gamma_\infty$-exact, then there exists a $\mathcal{Q}$ such that $\Lambda = \left.\frac{\partial_r \mathcal{Q}}{\partial K_A}\right|_J J_A(-1)^{\varepsilon_A}$.

4 Removal of divergences in a generic gauge field theory

Let us now consider a generic nonrenormalizable gauge field theory. The classical Lagrangian $\mathcal{L}_{class}$ will be supposed to be the most general gauge-invariant one. The constants that multiply the possible gauge-invariant terms of the classical Lagrangian (infinite in number) will be denoted by λ.

As we anticipated in the introduction, we suppose that the zero-th order action Σ_0 is linear in the BRS sources K,

$$\Sigma_0 = \mathcal{L}_{class}(\phi) + s\Psi(\Phi) + K_A s\Phi^A, \tag{59}$$

and moreover that the functional measure is BRS invariant, i.e. $\Delta\Sigma_0 = 0$.

The algorithm that removes the divergences is derived by induction. Quantities with a subscript k refer to the theory in which the divergences have been removed up to the k^{th}-loop order (for example Z_k, W_k, Γ_k, Σ_k, $\ldots$).

The inductive hypothesis is the following. We assume that the operator $\mathcal{L}_{n-1}$ that removes the $(n-1)^{th}$-order divergences (when the lower order ones have already been removed) is a transformation acting on λ and $\{\Phi, K\}$ in the following way

$$\begin{array}{c} \mathcal{L}_{n-1}\lambda = \lambda + \delta_{n-1}\lambda, \\ \mathcal{L}_{n-1}\{\Phi^A, K_A\} = \mathcal{C}_{n-1}\{\Phi^A, K_A\}, \end{array} \tag{60}$$

where $\mathcal{C}_{n-1}$ denotes a local canonical transformation ($\mathcal{C}_{n-1} = 1 + \mathcal{O}(\hbar^{n-1})$), while $\delta_{n-1}\lambda = \mathcal{O}(\hbar^{n-1})$ denote the $(n-1)^{th}$-loop order corrections of the constants λ. The action of $\mathcal{L}_{n-1}$ on Σ_{n-2} is

$$\mathcal{L}_{n-1}\Sigma_{n-2}(\Phi, K, \lambda) = \mathcal{C}_{n-1}\Sigma_{n-2}(\Phi, K, \lambda + \delta_{n-1}\lambda). \tag{61}$$

The actions on W_{n-2} and Γ_{n-2} to give W_{n-1} and Γ_{n-1}, respectively, are direct consequences of (61).

Let us define

$$\mathcal{R}_{n-1} = \mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_1 = \mathcal{L}_{n-1} \circ \mathcal{R}_{n-2}. \tag{62}$$

$\mathcal{R}_{n-1}$ acts on Σ_0, W_0 and Γ_0 and gives Σ_{n-1}, W_{n-1} and Γ_{n-1}. It removes the divergences up to the $(n-1)^{th}$ loop order. In general, $\mathcal{R}_{n-1}$ is not directly defined on λ and $\{\Phi, K\}$. It is a composition of redefinitions of λ and local canonical transformations on $\{\Phi, K\}$. Notice that $\mathcal{C}_k$, $\mathcal{L}_k$ and $\mathcal{R}_k$ preserve locality $\forall k$. So, Σ_k is local.

We are thus assuming that the divergences up to the $(n-1)^{th}$-loop order have been corrected by a set of redefinitions of the constants λ and local canonical transformations of fields and BRS sources. This guarantees that Σ_{n-1} satisfies the master equation, since Σ_0 does. To show this, let us go back to the proof that a canonical transformation preserves the master equation [see formulæ (15), (16) and (17)]. We have to improve it with a redefinition of the constants λ. This is extremely simple. Consider (15). It was derived from (7) by changing names to fields and BRS sources (from unprimed to primed objects). A similar identity can be derived by changing names to fields, BRS sources and the constants λ (i.e. replacing λ with $\lambda + \delta\lambda$). None of the remaining steps of the proof gets affected. With this simple improvement, all the properties that hold for canonical transformations can be extended to $\mathcal{L}_k$-transformations, apart from those concerning the derivatives with respect to g (indeed, the parameters g can also enter in the redefinitions of λ). So, for example, $\mathcal{L}_k$-transformations preserve Ω-closure and Ω-exactness, as well as the Ward identity (27) and so on. Consequently, the same holds for any $\mathcal{R}_k$ and, in particular, for $\mathcal{R}_{n-1}$. We are going to prove that the n^{th}-order correction is of the same type. In other words, we have to define suitable $\mathcal{L}_n$ and $\mathcal{R}_n$ operators.

When we speak about *redefinitions* of the parameters λ, we are assuming that the classical Lagrangian $\mathcal{L}_{class}$ is the most general one, i.e. that it contains any possible λ. Indeed, if we miss one, we cannot redefine it. So, when we want to consider a classical Lagrangian with some missing terms, we have to set the corresponding constants λ to zero at the very end. The present point is a key one: when the nonrenormalizability is only due to the gauge-fixing, $\mathcal{L}_{class}$ is not the most general classical Lagrangian and one has to be sure that the missing terms do not appear when applying the correction algorithm.

Now, we have a functional Γ_{n-1} that is convergent up to the $(n-1)^{th}$-loop order. Let us call $\Gamma^{(n)}_{div}$ its n^{th}-order divergent part. $\Gamma^{(n)}_{div}$ is a local functional. We want to show that we are able to remove $\Gamma^{(n)}_{div}$ with a canonical transformation $\mathcal{C}_n = 1 + \mathcal{O}(\hbar^n)$ and a redefinition $\lambda \to \lambda + \delta_n\lambda = \lambda + \mathcal{O}(\hbar^n)$ of the constants λ. This reproduces the inductive hypothesis up to the n^{th}-loop order. As a consequence of the inductive proof, there will exist a $\mathcal{R}$-transformation ($\mathcal{R}_\infty$, to be precise) that is able to remove all the divergences of the theory.

Since Σ_{n-1} satisfies the master equation (7), then Γ_{n-1} satisfies a Ward identity analogous to (27), namely

$$(\Gamma_{n-1}, \Gamma_{n-1}) = 0. \tag{63}$$

Taking the divergent part of the n^{th}-loop order of this expression, we get

$$(\Gamma^{(n)}_{div}, \Sigma_0) \equiv \sigma\Gamma^{(n)}_{div} = 0. \tag{64}$$

Notice that σ is nilpotent: $\sigma^2 = 0$, due to $(\Sigma_0, \Sigma_0) = 0$. (64) is a very useful characterization of $\Gamma^{(n)}_{div}$. As a matter of fact, the general solution of this equation on local functionals of zero ghost number [1, 2, 3, 19, 20] is

$$\Gamma^{(n)}_{div} = \mathcal{G}^{(n)}(\phi) + (R^{(n)}, \Sigma_0), \tag{65}$$

for some local gauge-invariant functional $\mathcal{G}^{(n)}(\phi)$ depending only on the classical fields ϕ and a suitable local functional $R^{(n)}(\Phi^A, K_A)$. For Yang-Mills theories, this statement was conjectured in ref. [19] and proved in detail in ref. [20]. The analogous statement for higher derivative quantum gravity was assumed without proof in ref. [3]. In this paper we assume the same property without proof for our gauge field theory. If formula (65) were violated, for example if $\mathcal{G}^{(n)}$ depends on more fields than the classical ones, this could simply mean that one has to enlarge the definition of "classical fields" and "classical Lagrangian" in order to include $\mathcal{G}^{(n)}$-type functionals. Serious troubles, instead, can appear if $\mathcal{G}^{(n)}$ cannot be made independent of the BRS sources K: some arguments of our derivations should be reconsidered. See also [21].

$\mathcal{G}^{(n)}(\phi)$ has the same form as $\mathcal{L}_{class}$, of course, since we are assuming that the classical Lagrangian is the most general one. So, $\mathcal{G}^{(n)}$ is responsible for the corrections $\delta_n\lambda$ of the constants λ. In other words, there exist suitable $\delta_n\lambda$ such that the corrected action $\Sigma_{n-1}(\Phi, K, \lambda+\delta_n\lambda)$ gives a Γ-functional whose n^{th}-loop divergent part is σ-exact: $\mathcal{G}^{(n)}$ is removed from $\Gamma^{(n)}_{div}$ and only $(R^{(n)}, \Sigma_0)$ remains. This last piece can in fact be removed by a local canonical transformation $\mathcal{C}_n$ with a generating functional equal to

$$F^{(n)}(\Phi, K') = \Phi^A K'_A + R^{(n)}(\Phi, K'). \tag{66}$$

Notice that in the argument of $R^{(n)}$ the BRS sources are K' and not K. In this way, it is easy to check that

$$\Sigma_n = \mathcal{C}_n \Sigma_{n-1}(\Phi, K, \lambda+\delta_n\lambda) = \tilde{\Sigma}_{n-1} + \frac{i\hbar}{2}\ln J_n = \Sigma_{n-1} - \Gamma^{(n)}_{div} + \mathcal{O}(\hbar^{n+1}), \tag{67}$$

where now the tilde on Σ_{n-1} means not only that the fields and BRS sources are substituted by their primed versions, but also that the constants λ are substituted by $\lambda+\delta_n\lambda$. In deriving (67), we have used the fact that, since $R^{(n)}$ is of order $\hbar^n$, (8) give

$$\Phi'^A = \Phi^A + \frac{\partial R^{(n)}}{\partial K_A} + \mathcal{O}(\hbar^{n+1}), \quad K'_A = K_A - \frac{\partial R^{(n)}}{\partial \Phi^A} + \mathcal{O}(\hbar^{n+1}). \tag{68}$$

Moreover, $\ln J_n = \mathcal{O}(\hbar^n)$. We conclude that Γ_n is convergent up to the n^{th}-loop order.

The composition of the n^{th}-loop order correction $\mathcal{L}_n$ with the operation $\mathcal{R}_{n-1}$ of removal of the lowest order divergences defines an operator $\mathcal{R}_n = \mathcal{L}_n \circ \mathcal{R}_{n-1}$, that acts on the zero-th order theory and removes all the divergences up to the n^{th}-loop order. Moreover, $\mathcal{L}_n$ has the same structure as $\mathcal{L}_{n-1}$, namely it is a redefinition of the constants λ and a local canonical transformation $\mathcal{C}_n$.

According to the results of the previous section, we have shown that there exists an operator $\mathcal{R}_\infty$ that is able to remove all the divergences, while preserving suitable extensions of gauge-invariance, namely Ω_∞-invariance on local functionals and $\mathrm{ad}\,\Gamma_\infty$-invariance on their average values (which are also generalizations of BRS-invariance).

Finally, notice that the canonical transformation (66) is not uniquely fixed. Any higher order correction can be introduced without affecting the results. Moreover, due to (65), $R^{(n)}$ is defined up to σ-closed local functionals $T^{(n)}$, that can also be of order $\hbar^n$.

5 Predictivity with a nonrenormalizable gauge-fixing

The purpose of the present section is to improve the argument of the previous one in order to show that when a renormalizable theory is treated with a nonrenormalizable gauge-fixing, predictivity is retained. Now, the classical Lagrangian $\mathcal{L}_{class}$ is not the most general one, since it has to be renormalizable. The parameters λ are finite in number. The κ-dependence is entirely due to the gauge-fermion Ψ. So, the previous argument can be adapted to the present case, only if we are able to prove that $\mathcal{G}^{(n)}(\phi)$ is κ-independent, so that it is only made of renormalizable terms, i.e. terms contained in the starting renormalizable classical Lagrangian $\mathcal{L}_{class}(\phi,\lambda)$. Indeed, only in that case $\mathcal{G}^{(n)}$ can be absorbed with redefinitions $\delta_n\lambda$ of the parameters λ of $\mathcal{L}_{class}$.

Again, we proceed by induction. Let us suppose that the algorithm works well up to the $(n-1)^{th}$-loop order, i.e. that $\forall k=1,\dots n-1$, $\mathcal{L}_k$ is made by κ-independent redefinitions $\delta_k\lambda$ of the parameters λ and a canonical transformation $\mathcal{C}_k$. Formula (36) for the change of $\frac{\partial\Sigma}{\partial g}$ under a canonical transformation is extendable to any $\mathcal{L}_k$-transformation, $k=1,\dots n-1$, if we take $g=\kappa$, since the correction of the parameters λ is κ-independent by inductive hypothesis. So, if $\frac{\partial\Sigma_{k-1}}{\partial\kappa}$ is Ω_{k-1}-exact, then $\frac{\partial\Sigma_k}{\partial\kappa}$ is Ω_k-exact. A similar property extends to $\mathcal{R}_{n-1}$: if $\frac{\partial\Sigma_0}{\partial\kappa}$ is Ω_0-exact, then $\frac{\partial\Sigma_{n-1}}{\partial\kappa}$ is Ω_{n-1}-exact.

Notice that in a renormalizable gauge-field theory that is gauge-fixed with a non-renormalizable gauge-fixing, $\frac{\partial\Sigma_0}{\partial\kappa}$ is BRS-exact. As a matter of fact,

$$\Sigma_0=\mathcal{L}_{class}(\phi,\lambda)+s\Psi(\Phi,\kappa)+K_As\Phi^A, \tag{69}$$

Ψ being the gauge-fermion, and so,

$$\frac{\partial\Sigma_0}{\partial\kappa}=s\frac{\partial\Psi}{\partial\kappa}. \tag{70}$$

Since Ψ is independent of the BRS sources K, we can also write

$$\frac{\partial\Sigma_0}{\partial\kappa}=\left(\frac{\partial\Psi}{\partial\kappa},\Sigma_0\right)=\Omega_0\frac{\partial\Psi}{\partial\kappa}, \tag{71}$$

Ω_0 denoting the zeroth order Ω-operator. Since $\frac{\partial\Sigma_0}{\partial\kappa}$ is local and Ω_0-exact, then (36) and the above remarks assure that $\frac{\partial\Sigma_{n-1}}{\partial\kappa}$ is local and Ω_{n-1}-exact. Let χ_{n-1} be such that

$$\frac{\partial\Sigma_{n-1}}{\partial\kappa}=\Omega_{n-1}\chi_{n-1}. \tag{72}$$

Since $\chi_{n-1}=\mathcal{R}_{n-1}\frac{\partial\Psi}{\partial\kappa}$, we see that χ_{n-1} is a local functional. (72) is sufficient to prove that $\frac{\partial\Gamma_{n-1}}{\partial\kappa}$ is $\mathrm{ad}\,\Gamma_{n-1}$-exact, which can be obtained following the same steps of the proof that if $\mathcal{O}$ is Ω-closed, then $<\mathcal{O}>_J$ is $\mathrm{ad}\,\Gamma$-closed.

Indeed, let us start from

$$\frac{\partial W_{n-1}}{\partial\kappa}=<\frac{\partial\Sigma_{n-1}}{\partial\kappa}>_J=<\Omega_{n-1}\chi_{n-1}>_J\ . \tag{73}$$

By performing the change of variables

$$\delta\Phi^A=\frac{\partial_l\Sigma_{n-1}}{\partial K_A}\Lambda, \tag{74}$$

in the functional integral $<\chi_{n-1}>_J$, one finds a formula analogous to (53), with χ_{n-1} in replacement of $\mathcal{O}$. Moreover, with an integration by parts, one can prove the analogue of (54). It is thus possible to write (73) in the form

$$\frac{\partial W_{n-1}}{\partial \kappa} = \frac{\partial_r <\chi_{n-1}>_J}{\partial K_A} J_A (-1)^{\varepsilon_A}. \tag{75}$$

Now, the definition of Γ_{n-1} [see (24)], permits to write

$$\left.\frac{\partial \Gamma_{n-1}}{\partial \kappa}\right|_{\Phi,K} = \left.\frac{\partial W_{n-1}}{\partial \kappa}\right|_{J,K} = \left.\frac{\partial_r <\chi_{n-1}>_J}{\partial K_A}\right|_J J_A(-1)^{\varepsilon_A}. \tag{76}$$

Using the analogue of (56) and (58), one gets

$$\frac{\partial \Gamma_{n-1}}{\partial \kappa} = (<\chi_{n-1}>_J, \Gamma_{n-1}). \tag{77}$$

We conclude that $\frac{\partial \Gamma_{n-1}}{\partial \kappa}$ is $\mathrm{ad}\,\Gamma_{n-1}$-exact.

Let us call $S_{n-1} =<\chi_{n-1}>_J$. S_{n-1} is the average value (at nonzero sources J) of a local functional. Notice that S_{n-1} is determined up to additions of $\mathrm{ad}\,\Gamma_{n-1}$-closed functionals. We introduce the additional inductive hypothesis that S_{n-1} is finite up to the $(n-1)^{th}$-loop order. Of course, this hypothesis is satisfied at lowest order: $S_0 =<\chi_0>_J =<\frac{\partial \Psi}{\partial \kappa}>_J =$ finite $+\,\mathcal{O}(\hbar)$. We shall have to prove that the additional hypothesis is reproduced to the n^{th} loop order, i.e. that $S_n =<\chi_n>_J$ can be chosen finite up to order $\hbar^n$.

Let $S^{(n)}_{div}$ denote the n^{th}-loop order divergent part of S_{n-1}. $S^{(n)}_{div}$ is local, since we are assuming that all the subdivergences have been removed. Let us focus on the n^{th}-loop order divergent part of equation (77), namely

$$\frac{\partial \Gamma^{(n)}_{div}}{\partial \kappa} = (\chi_0, \Gamma^{(n)}_{div}) + (S^{(n)}_{div}, \Sigma_0). \tag{78}$$

On the other hand, (65) gives

$$\frac{\partial \Gamma^{(n)}_{div}}{\partial \kappa} = \frac{\partial \mathcal{G}^{(n)}}{\partial \kappa} + \left(R^{(n)}, \frac{\partial \Sigma_0}{\partial \kappa}\right) + \left(\frac{\partial R^{(n)}}{\partial \kappa}, \Sigma_0\right), \tag{79}$$

so that we can write

$$\frac{\partial \mathcal{G}^{(n)}}{\partial \kappa} = (\chi_0, \Gamma^{(n)}_{div}) - \left(R^{(n)}, \frac{\partial \Sigma_0}{\partial \kappa}\right) + (\ldots, \Sigma_0). \tag{80}$$

Using (71), we get

$$\frac{\partial \mathcal{G}^{(n)}}{\partial \kappa} = (\chi_0, \mathcal{G}^{(n)}) + (\ldots, \Sigma_0). \tag{81}$$

Now, since χ_0 and $\mathcal{G}^{(n)}(\phi)$ do not depend on the BRS sources K, we have

$$(\chi_0, \mathcal{G}^{(n)}) = 0, \tag{82}$$

and we can conclude that $\frac{\partial \mathcal{G}^{(n)}}{\partial \kappa}$ is σ-exact and that it is equal to the action of σ on a local functional. This implies that it can only be zero[8]. This proves that the redefinitions $\delta_n \lambda$ of the constants λ are κ-independent.

The κ-independence of $\delta_n \lambda$ generalizes the well-known gauge-independence of the coupling constant renormalization. See also section 6 for a comment about this fact within Yang-Mills theory treated in the usual way.

As a consequence of $\frac{\partial \delta_n \lambda}{\partial \kappa} = 0$, we can write

$$\frac{\partial \Sigma_n}{\partial \kappa} = \Omega_n \chi_n, \tag{83}$$

and

$$\frac{\partial \Gamma_n}{\partial \kappa} = (S_n, \Gamma_n), \tag{84}$$

being $S_n =< \chi_n >_J$. Formula (36) gives

$$\chi_n = \tilde{\chi}_{n-1} - \frac{\partial F^{(n)}}{\partial \kappa} \tag{85}$$

(the tilde, as usual now, means that the parameters λ have to be substituted with the new ones and the fields and BRS sources have to be substituted with the canonically transformed ones). Moreover, $S_n =< \chi_n >_J = S_{n-1} + \mathcal{O}(\hbar^n)$. Consequently, S_n is surely finite up to order $\hbar^{n-1}$. In order to fully reproduce the inductive hypothesis, we have to prove that S_n can be chosen finite up to order $\hbar^n$.

Let $\mathcal{S}_{div}^{(n)}$ denote the n^{th} loop order divergent part of S_n. $\mathcal{S}_{div}^{(n)}$ is local, since the subdivergences vanish by inductive assumption. Taking the n^{th} loop order divergent part of (84), one has

$$\sigma \mathcal{S}_{div}^{(n)} = 0. \tag{86}$$

We shall give two different arguments for removing $\mathcal{S}_{div}^{(n)}$. The first method is based on the fact that S_n is defined up to $\mathrm{ad}\,\Gamma_n$-closed functionals, that are averages of Ω_n-closed local functionals, while the second method is based on the fact that the generating functional $F^{(n)}$ of (66) is defined up to σ-closed functionals $T^{(n)} = \mathcal{O}(\hbar^n)$.

Let us begin with the first method. It is fundamental to be able to extend $\mathcal{S}_{div}^{(n)}$ to an Ω_0-closed local functional $\mathcal{S}_{div}^{(n)\prime}$ by adding higher order terms. Were $\mathcal{S}_{div}^{(n)}$ of ghost number zero, this would be very easy: $\mathcal{S}_{div}^{(n)}$ would be of the form

$$\mathcal{S}_{div}^{(n)} = f(\phi) + (h, \Sigma_0) = f + \sigma h, \tag{87}$$

[8]See [20]. As a matter of fact, one can always assume that, in formula (65), $\mathcal{G}^{(n)}$ does not contain any gauge-invariant term of the form (local functional, Σ_0). This simply amounts to a redefinition of $R^{(n)}$. If this is the case, then $\frac{\partial \mathcal{G}^{(n)}}{\partial \kappa}$ is also a sum of gauge-invariant terms, none of which can be written as (local functional, Σ_0).

where f is a gauge-invariant local functional of the classical fields ϕ and h is local. Of course, f and h are of order $\hbar^n$, as $\mathcal{S}^{(n)}_{div}$. Then, the Ω_0-closed extension of $\mathcal{S}^{(n)}_{div}$ is

$$\mathcal{S}^{(n)\prime}_{div} \equiv f + \Omega_0 h = \mathcal{S}^{(n)}_{div} + \mathcal{O}(\hbar^{n+1}). \tag{88}$$

However, $\mathcal{S}^{(n)}_{div}$ has ghost number -1 and it is not simple to estabilish the cohomology content of σ on ghost number -1 local functionals. Nevertheless, due to $\Omega_0 = \sigma - i\hbar\Delta$, one can notice that at least in a dimensional regularization framework, $\mathcal{S}^{(n)\prime}_{div} = \mathcal{S}^{(n)}_{div}$ is trivially Ω_0-closed. Then, we know from section 3 and the remarks of section 4, that the operator $\mathcal{R}_n$ permits to find a local Ω_n-closed extension $\mathcal{R}_n\mathcal{S}^{(n)\prime}_{div} = \mathcal{S}^{(n)}_{div} + \mathcal{O}(\hbar^{n+1})$ of $\mathcal{S}^{(n)\prime}_{div}$. Finally, we know from section 3 that

$$\mathcal{S}_n \equiv < \mathcal{R}_n\mathcal{S}^{(n)\prime}_{div} >_J \tag{89}$$

is ad$\,\Gamma_n$-closed. Moreover, $\mathcal{S}_n = \mathcal{S}^{(n)}_{div} + \mathcal{O}(\hbar^{n+1})$. Consequently, $\mathcal{S}_n$ can be safely subtracted from S_n: this cancels the divergent part $\mathcal{S}^{(n)}_{div}$ and preserves (84). The subtraction of $\mathcal{S}_n$ from S_n corresponds to a subtraction of $\mathcal{R}_n\mathcal{S}^{(n)\prime}_{div}$ from χ_n: (83) is also preserved.

The second method does not require any restriction on the regularization technique. We know that $F^{(n)}$ of formula (66) is defined up to σ-colsed functionals $T^{(n)}$. The addition of such $T^{(n)}$'s to $F^{(n)}$ only changes Σ_n and Γ_n to order $\hbar^{n+1}$ and neither affects (83) nor (84). However, due to (85), it affects χ_n and also $S_n =< \chi_n >_J$. Moreover, $T^{(n)}$ has ghost number -1: it is thus a good candidate for our purposes. Let us choose $T^{(n)}$ to be $\mathcal{O}(\hbar^n)$ and divergent. (85) assures that when introducing a $T^{(n)}$ in $F^{(n)}$, χ_n goes into $\chi_n - \frac{\partial T^{(n)}}{\partial\kappa} - (T^{(n)}, \chi_0) + \mathcal{O}(\hbar^{n+1})$ and that $\mathcal{S}^{(n)}_{div}$ changes according to

$$\mathcal{S}^{(n)}_{div} \to \mathcal{S}^{(n)}_{div} - \frac{\partial T^{(n)}}{\partial\kappa} - (T^{(n)}, \chi_0). \tag{90}$$

$\sigma T^{(n)} = 0$ also implies

$$0 = \frac{\partial(\sigma T^{(n)})}{\partial\kappa} = \sigma\left(\frac{\partial T^{(n)}}{\partial\kappa} + (T^{(n)}, \chi_0)\right), \tag{91}$$

as it must be, due to (86). One would like to find a $T^{(n)}$ such that the right hand side of (90) is zero. This can be done perturbatively in κ. Let us start with $T^{(n)} = \kappa\mathcal{S}^{(n)}_{div}$: $\sigma(\kappa\mathcal{S}^{(n)}_{div}) = 0$. Then

$$\mathcal{S}^{(n)}_{div} \to -\kappa\frac{\partial\mathcal{S}^{(n)}_{div}}{\partial\kappa} - \kappa(\mathcal{S}^{(n)}_{div}, \chi_0) \equiv \kappa\mathcal{T}^{(1)}. \tag{92}$$

$\mathcal{S}^{(n)}_{div}$ has not been set to zero, however its power expansion in κ starts now with order one. Clearly, $\sigma\mathcal{T}^{(1)} = 0$, due to (86). Let us now add $\frac{\kappa^2}{2}\mathcal{T}^{(1)}$ to $T^{(n)}$. We have

$$\mathcal{S}^{(n)}_{div} \to -\frac{\kappa^2}{2}\left(\frac{\partial\mathcal{T}^{(1)}}{\partial\kappa} + (\mathcal{T}^{(1)}, \chi_0)\right) \equiv \kappa^2\mathcal{T}^{(2)}. \tag{93}$$

Again, $\sigma\mathcal{T}^{(2)}=0$. Moreover, $\mathcal{S}_{div}^{(n)}$ has become quadratic in κ. Then we can add $\frac{\kappa^3}{3}\mathcal{T}^{(2)}$ to $T^{(n)}$ and go on: $\mathcal{S}_{div}^{(n)}$ can be made of arbitrarily high order in κ (always of order n in $\hbar$) and so we can conclude that it can be made to vanish, as desired.

A key remark, now, is the following. The proof that we have made can be extended to any nonrenormalizable gauge field theory, to show that if some parameter g (not necessary of negative dimension in mass units) only appears in the gauge-fermion Ψ, but neither in the classical Lagrangian $\mathcal{L}_{class}(\phi,\lambda)$, nor in the BRS variations $s\Phi^A$ of the fields Φ^A, then the order by order redefinitions of the parameters λ of $\mathcal{L}_{class}$ (now infinitely many) is g-independent and the derivative of Γ_∞ with respect to g is $\mathrm{ad}\,\Gamma_\infty$-exact.

This is useful for the definition of the observables of a *renormalizable* field theory. Let $\{\mathcal{O}_i(\phi)\}$ be a basis of local gauge-invariant operators, constructed only with the classical fields ϕ. Let us substitute the classical Lagrangian $\mathcal{L}_{class}(\phi,\lambda)$ with

$$\mathcal{L}_{class}^{(\beta)}(\phi,\lambda,\beta)\equiv\mathcal{L}_{class}(\phi,\lambda)+\beta_i\mathcal{O}_i(\phi). \tag{94}$$

The β_i can be point-dependent. For simplicity, such a dependence will be undestrood. The Lagrangian (94) is the same as a nonrenormalizable Lagrangian, so that the correction algorithm permits to define a finite functional

$$\Gamma_\infty^{(\beta)}(\Phi,K,\lambda,\kappa,\beta) \tag{95}$$

such that its derivative with respect to κ is $\mathrm{ad}\,\Gamma_\infty$-exact and the order by order redefinitions of λ and β are κ-independent. The only (fundamental) difference with a true nonrenormalizable field theory is that β_i are not true coupling constants, but artificial parameters that permit to define the quantum extensions $\mathcal{O}_i^{(q)}$ of the observables $\mathcal{O}_i$, that are

$$\mathcal{O}_i^{(q)}(\Phi,K)=\left.\frac{\partial\Gamma_\infty^{(\beta)}}{\partial\beta_i}\right|_{\beta=0}. \tag{96}$$

The quantum generalization of the amplitude $<\mathcal{O}_{i_1}\cdots\mathcal{O}_{i_n}>$ is given by

$$\mathcal{A}_{i_1\cdots i_n}\equiv\left.\frac{\partial^n\Gamma_\infty^{(\beta)}}{\partial\beta_{i_1}\cdots\partial\beta_{i_n}}\right|_{\beta=0,\,\Phi=0,\,K=0}. \tag{97}$$

We know that, on shell,

$$\frac{\partial\Gamma_\infty^{(\beta)}}{\partial\kappa}=0 \tag{98}$$

and so we conclude that the on shell amplitudes are κ-independent, namely

$$\frac{\partial\mathcal{A}_{i_1\cdots i_n}}{\partial\kappa}=0. \tag{99}$$

Differentiating $(\Gamma_\infty^{(\beta)},\Gamma_\infty^{(\beta)})=0$ with respect to β_i and setting $\beta=0$, we get

$$\mathrm{ad}\,\Gamma_\infty\,\mathcal{O}_i^{(q)}=(\mathcal{O}_i^{(q)},\Gamma_\infty)=0, \tag{100}$$

where $\Gamma_\infty = \Gamma^{(\beta)}_\infty|_{\beta=0}$. In other words, $\mathcal{O}^{(q)}_i$ are quantum observables in the sense defined in section 3. Moreover,

$$\mathcal{O}^{(q)}_i = \left(\left. \frac{\partial \Gamma^{(\beta)}_\infty}{\partial \beta_i} \right|_{\Phi,K} \right) \Bigg|_{\beta=0} = \left(\left. \frac{\partial W^{(\beta)}_\infty}{\partial \beta_i} \right|_{J,K} \right) \Bigg|_{\beta=0} = \left. < \frac{\partial \Sigma^{(\beta)}_\infty}{\partial \beta_i} >_J \right|_{\beta=0}. \tag{101}$$

The differentiation of the master equation for $\Sigma^{(\beta)}_\infty$ with respect to β_i implies

$$\Omega_\infty \left. \frac{\partial \Sigma^{(\beta)}_\infty}{\partial \beta_i} \right|_{\beta=0} = 0, \tag{102}$$

so that the quantum observables $\mathcal{O}^{(q)}_i$ are average values of Ω_∞-closed local functionals, again in agreement with the definitions developed in section 3.

Concluding, when the nonrenormalizability is only due to the gauge-fixing, the physical amplitudes depend on finitely many parameters and so a finite number of measurements is necessary to determine uniquely the theory.

Again, the previous conclusions can also be extended to a true nonrenormalizable field theory: the physical amplitudes can depend on infinitely many parameters, nevertheless they cannot depend on the parameters that were introduced only through the gauge-fixing, i.e. the independence from the gauge-fermion Ψ survives the subtraction algorithm.

6 Examples

In this section we examine some examples of renormalizable field theories that are treated with a nonrenormalizable gauge-fixing. The first example is pure Q.E.D. (to further simplify things, we consider the case of space-time dimension two): it is a free theory and so there is no nontrivial physical amplitude. Let us regularize with the dimensional technique. We choose the gauge-fixing

$$\partial_\mu A_\mu + \kappa (\partial_\mu A_\nu)(\partial_\mu A_\nu) = 0. \tag{103}$$

This is a continuous deformation of the usual gauge-fixing $\partial_\mu A_\mu = 0$, so that we expect the same physical results. The BRS algebra is

$$\begin{array}{ll} sA_\mu = \partial_\mu c, & sc = 0, \\ s\bar{c} = b, & sb = 0, \end{array} \tag{104}$$

with obvious notation. We choose the following gauge fermion Ψ

$$\Psi = \frac{1}{2} \bar{c} \, (b + 2\partial \cdot A + 2\kappa (\partial_\mu A_\nu)^2), \tag{105}$$

so that the BRS action is

$$\Sigma_0 = -\frac{1}{4} F^2_{\mu\nu} + s\Psi + K_A sA + K_c sc + K_{\bar{c}} s\bar{c} + K_b sb. \tag{106}$$

Notice that the Lagrange multiplier b is not integrated away. A_μ and b must be treated as a whole to define a nonsingular propagator. One then checks that there is propagation from A_μ to A_ν and from A_μ to b (and *viceversa*), but no propagation from b to b. There is no dependence on K_c and K_b, since $sc = sb = 0$. Moreover, there is no radiative correction to the other BRS transformations, since they are linear (correspondingly $K_A sA + K_{\bar{c}} s$ is quadratic). This implies that the linear dependence of the action on the BRS sources is radiatively preserved[9]. Let us rewrite Σ_0 in a more explicit form, namely

$$\begin{aligned}\Sigma_0 = &\frac{1}{2}A_\mu \Box A_\mu + \frac{1}{2}(\partial\cdot A)^2 + \frac{1}{2}b^2 + b(\partial\cdot A + \kappa(\partial_\mu A_\nu)^2)\\ &- \bar{c}(\Box c + 2\kappa\partial_\mu\partial_\nu c\,\partial_\mu A_\nu) + K_A^\mu \partial_\mu c + K_{\bar{c}} b.\end{aligned} \tag{107}$$

Let us consider the one loop amplitudes with external photonic legs. They are the sum of two diagrams: in one of them A_μ and b circulate in the loop, while in the other one, $\bar{c}$ and c circulate. It is simple to check that these two diagrams exactly cancel, so that the amplitude is identically zero, independently of the number of external photonic legs.

Next, consider the one loop diagram with two external b-legs (of momentum p and $-p$): there is a divergence of the kind

$$\frac{1}{\varepsilon}\kappa^2 p^2. \tag{108}$$

This means that it is necessary to introduce a counterterm of the form

$$\frac{1}{2}\kappa^2 b\Box b. \tag{109}$$

We conclude that the Lagrange multiplier becomes "propagating". Counterterms of the form $\kappa^3 b^2\Box b$ and $\kappa^3 b(\partial_\mu b)^2$ are also required, so that the corrected action is not even quadratic in b. That is why, for the simplicity of the computation, it is preferable to avoid integrating b away.

Instead, starting from a Lagrangian Σ_0 in which b has been integrated away, namely

$$\begin{aligned}\Sigma_0 = &\frac{1}{2}A_\mu\Box A_\mu - \kappa\partial\cdot A(\partial_\mu A_\nu)^2 - \frac{1}{2}\kappa^2((\partial_\mu A_\nu)^2)^2\\ &- \bar{c}(\Box c + 2\kappa\partial_\mu\partial_\nu c\,\partial_\mu A_\nu) + K_A^\mu\partial_\mu c + K_{\bar{c}} b,\end{aligned} \tag{110}$$

it is no more true that the one loop amplitudes with only photonic external legs

$$< A_{\mu_1}(p_1)\cdots A_{\mu_n}(p_n) > \Big|_{\text{one loop}} \tag{111}$$

[9] Strictly speaking, one should say that the condition of linearity in the BRS sources *can* be radiatively preserved, i.e. that this condition is compatible with the subtraction algorithm. Indeed, if this restriction is not specified, one could introduce arbitrary finite nonlinear terms.

are zero. Nevertheless, it is true that the physical projections

$$\left(\delta_{\mu_1\nu_1}-\frac{p_{1\mu_1}p_{1\nu_1}}{p_1^2}\right)\cdots\left(\delta_{\mu_n\nu_n}-\frac{p_{n\mu_n}p_{n\nu_n}}{kp_n^2}\right)< A_{\nu_1}(p_1)\cdots A_{\nu_n}(p_n)>\Big|_{\text{one loop}} \tag{112}$$

vanish. Check, for example, the case $n=2$.

Let us now couple fermions to the electromagnetic field, by adding to Σ_0 the terms

$$\bar{\psi}(\partial\!\!\!/+A\!\!\!/)\psi+\bar{D}s\psi+s\bar{\psi}D. \tag{113}$$

The BRS algebra is

$$s\psi=-c\psi,\quad s\bar{\psi}=c\bar{\psi}. \tag{114}$$

The nonlinearity of these transformations gives rise to nontrivial radiative corrections to them together with the lost of linearity in the BRS sources. As a matter of fact, although it is simple to check that there is no divergent one loop diagram with both $\bar{D}$- and D-external legs, nevertheless two loop divergent diagrams with both $\bar{D}$- and D-external legs do exist.

Analogous considerations apply for non-abelian Yang-Mills theory.

Let us now describe where, in our description, the usual coupling constant and wave function renormalization come from, when a renormalizable gauge-field theory is treated with one of the usual renormalizable gauge-fixings. For simplicity, we consider pure non-abelian Yang-Mills theory. In the usual approach, three independent renormalization constants Z are needed: one for the coupling constant g, one for the wave-function renormalization of the vector A_μ and one for the ghosts-antighost wave-function renormalization. On the other hand, in our approach the removal of divergences is performed by a redefinition of the parameters that multiply the gauge-invariant terms of the starting classical Lagrangian (in the present case there is only one such term, namely $F^a_{\mu\nu}F^a_{\mu\nu}$; let us call λ the constant in front of it) and a canonical transformation of fields and BRS sources. If the gauge-fixing is renormalizable, the canonical transformation is linear, i.e. the fields Φ^A and the BRS sources K_A are simply multiplied by some constants, which we call Z_{Φ^A} and Z_{K_A}, respectively. The requirement of preservation of antibrackets implies $Z_{\Phi^A}Z_{K_A}=1\ \forall A$. This reduces the number of independent Z-factors for fields and BRS sources to a half, namely 4 (those for A_μ, c, $\bar{c}$ and b, for instance). However, since $K_{\bar{c}}b$ and $b\partial\cdot A$ are not radiatively corrected, we have $Z_b=Z_{\bar{c}}=\frac{1}{Z_{A_\mu}}$. Thus, we remain with three independent Z-factors: those for A_μ and c and that for λ. The correct counting is thus retrieved and one can also check that these three Z-factors are indeed sufficient to produce the usual coupling constant renormalization and wave function renormalizations. Notice that there is no redefinition of the coupling constant g in our description; indeed the usual redefinition of g is recovered from the redefinitions of λ and A_μ. Precisely, the usual renormalization factor Z_g for the gauge-coupling constant g results to be equal to $Z_\lambda^{-1/2}$. Thus, the gauge-independence of Z_λ, proved in section 5, is nothing but the familiar gauge-independence of Z_g.

Let us now consider the topological σ-model formulated in [16]. It is an irreducible gauge-field theory and its BRS algebra [formula (13) of [16]] can be written, after a natural redefinition of the Lagrange multiplier b^i_μ, in the form

$$\begin{array}{ll} sq^i = \xi^i, & s\xi^i = 0, \\ s\zeta^i_\mu = b^i_\mu, & sb^i_\mu = 0. \end{array} \tag{115}$$

The simplicity of the BRS algebra assures that linearity in the BRS sources is preserved and that there is no radiative correction to the BRS transformations. Thus, the observables that were listed in [16] are directly promoted to quantum observables. The expression $\mathcal{G}^{(n)}$ appearing in formula (65) is zero, since any gauge-invariant functional (i.e. a topological invariant) is perturbatively trivial. This implies that there is no redefinition of the parameters λ. The expression $R^{(n)}$ appearing in the same formula (65) is independent of the BRS sources. This implies that the canonical transformation that absorbs the divergent terms leaves the fields invariant and only changes the BRS sources according to

$$\delta K_A = -\frac{\partial R^{(n)}}{\partial \Phi^A}. \tag{116}$$

Formula (2) shows that this is precisely a redefinion of the gauge-fermion Ψ:

$$\Psi_{n-1} \to \Psi_n = \Psi_{n-1} - R^{(n)}. \tag{117}$$

We conclude that the removal of divergences simply reduces to a redefinition of the gauge-fermion and thus has no physical consequence.

The nonrenormalizability of topological fields theories coming from the twist of some N=2 nonrenormalizable quantum field theory is thus turned into a positive feature: it shows that a suitable subset of the physical amplitudes of a nonrenormalizable N=2 quantum field theory is in any case predictive and physically well-defined.

7 Conclusions

Apart from eventual applications, the investigation about the removal of divergences in a nonrenormalizable gauge-field theory turns out to be enlightening rather than extravagant. The usual theorem of renormalizability of Yang-Mills theories is not, strictly speaking, a *renormalizability* theorem, since our improved version works for any (eventually nonrenormalizable) gauge-field theory. Rather, it is a theorem on the compatibility (up to BRS anomalies) of gauge-invariance with the subtraction algorithm, a fact that is fundamental for unitarity. When combining that theorem with power counting, one can determine the renormalizability or nonrenormalizability of the theory. This is not the whole story about predictivity. As a matter of fact, when the counterterms are infinitely many, one has to determine how many of them are nontrivial (i.e. non "BRS exact"): if the nontrivial counterterms are infinitely many, then the theory is not predictive. If the nontrivial counterterms are finitely many, then the theory is predictive. We have shown in detail that when nonrenormalizability is only due to the gauge-fixing, predictivity is preserved, a fact that is naturally expected. One is lead to wonder whether there are more general cases of predictive nonrenormalizability. This could require a revision of our idea of physically acceptable field theories.

Finally, the simple description of the removal of divergences as a redefinition of some parameters together with a canonical transformation of fields and BRS sources opens the possibility that in some simple models such a set of redefinitions, or, in other words, the identification of the *correct* variables and the correct parameters, is derivable from first principles with a synthetic argument, i.e. without any analytic computation.

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Appendix: Proof of formula (36)

We want to prove formula (36). We find it convenient to use the notation of ref. [22] and some of the formulæ proven there. Let us define

$$\begin{array}{ll} M^A{}_B = \frac{\partial_l \partial F}{\partial K'_A \partial \Phi^B}, & N_B{}^A = \frac{\partial_l \partial F}{\partial \Phi^B \partial K'_A}, \\ F^{AB} = \frac{\partial_l \partial F}{\partial K'_B \partial K'_A}, & F_{AB} = \frac{\partial_l \partial F}{\partial \Phi^A \partial \Phi^B}. \end{array} \tag{118}$$

The statistics are as follows

$$\varepsilon(M^A{}_B) = \varepsilon(N_B{}^A) = \varepsilon_A + \varepsilon_B, \quad \varepsilon(F^{AB}) = \varepsilon(F_{AB}) = \varepsilon_A + \varepsilon_B + 1. \tag{119}$$

The following rules for transposition hold:

$$N_A{}^B = M^B{}_A (-1)^{\varepsilon_A(\varepsilon_B+1)}, \quad (N^{-1})_A{}^B = (M^{-1})^B{}_A (-1)^{\varepsilon_A(\varepsilon_B+1)}. \tag{120}$$

The differentials of Φ' and K are

$$\begin{aligned} d\Phi'^A &= M^A{}_B d\Phi^B + F^{AB} dK'_B + \frac{\partial \Phi'^A}{\partial g} dg, \\ dK_A &= d\Phi^B F_{BA} + dK'_B M^B{}_A + \frac{\partial K_A}{\partial g} dg. \end{aligned} \tag{121}$$

As a convention, when there cannot be any misunderstanding, we do not specify the variables that are taken to be constant in a partial derivative. It is understood that the differentiated function is considered as a function of its natural variables: F, Φ' and K

are functions of $\{\Phi, K', g\}$ [see (8)], Σ and Σ' are functions of $\{\Phi, K, g\}$, $\tilde{\Sigma}$ is a function of $\{\Phi', K', g\}$, and so on.

Let us differentiate (12) with respect to g at constant Φ and K.

$$\frac{\partial \Sigma'}{\partial g} = \frac{\partial \tilde{\Sigma}}{\partial g}(\Phi'(\Phi, K), K'(\Phi, K)) + \left.\frac{\partial \Phi'^A}{\partial g}\right|_{\Phi,K} \frac{\partial_l \tilde{\Sigma}}{\partial \Phi'^A} + \left.\frac{\partial K'_A}{\partial g}\right|_{\Phi,K} \frac{\partial_l \tilde{\Sigma}}{\partial K'_A} + i\hbar \left.\frac{\partial \ln J^{\frac{1}{2}}}{\partial g}\right|_{\Phi,K}. \tag{122}$$

Now, $\frac{\partial \tilde{\Sigma}}{\partial g}(\Phi'(\Phi, K), K'(\Phi, K))$ is the same as $\widetilde{\frac{\partial \Sigma}{\partial g}}$. Let us define

$$S(\Phi, K', g) = \frac{\partial F(\Phi, K', g)}{\partial g}. \tag{123}$$

(121) gives

$$\left.\frac{\partial S}{\partial \Phi'^A}\right|_{K'} = \left.\frac{\partial_r \Phi^B}{\partial \Phi'^A}\right|_{K'} \frac{\partial S}{\partial \Phi^B} = \frac{\partial S}{\partial \Phi^B}(M^{-1})^B{}_A. \tag{124}$$

Moreover, (8) gives

$$\left.\frac{\partial S}{\partial \Phi'^A}\right|_{K'} = \frac{\partial K_B}{\partial g}(M^{-1})^B{}_A. \tag{125}$$

On the other hand, the second of (121) permits to write

$$\left.\frac{\partial K'_B}{\partial g}\right|_{\Phi,K} M^B{}_A = -\frac{\partial K_A}{\partial g}, \tag{126}$$

so that we conclude

$$\left.\frac{\partial S}{\partial \Phi'^A}\right|_{K'} = -\left.\frac{\partial K'_A}{\partial g}\right|_{\Phi,K}. \tag{127}$$

Following similar steps, one can prove

$$\left.\frac{\partial S}{\partial K'_A}\right|_{\Phi'} = \frac{\partial \Phi'^A}{\partial g} - \frac{\partial K_B}{\partial g}(M^{-1})^B{}_C F^{CA} = \left.\frac{\partial \Phi'^A}{\partial g}\right|_{\Phi,K}. \tag{128}$$

This formula, together with (127) and (13), permits to rewrite (122) in the form

$$\frac{\partial \Sigma'}{\partial g} = \widetilde{\frac{\partial \Sigma}{\partial g}} - (S, \tilde{\Sigma}) + i\hbar \left.\frac{\partial \ln J^{\frac{1}{2}}}{\partial g}\right|_{\Phi,K}. \tag{129}$$

Using (12) and (14) we also get

$$\frac{\partial \Sigma'}{\partial g} = \widetilde{\frac{\partial \Sigma}{\partial g}} - (S, \Sigma') + i\hbar\Delta S - i\hbar \left\{ \tilde{\Delta} S - \left.\frac{\partial \ln J^{\frac{1}{2}}}{\partial g}\right|_{\Phi,K} \right\}. \tag{130}$$

Our thesis will be proved if we are able to show that

$$\tilde{\Delta} S = \left.\frac{\partial \ln J^{\frac{1}{2}}}{\partial g}\right|_{\Phi,K} . \tag{131}$$

Now, the chain rule gives

$$\begin{aligned}
\tilde{\Delta} S &= (-1)^{\varepsilon_A+1}\, \frac{\partial_r}{\partial K'_A}\left(\left.\frac{\partial S}{\partial \Phi'^A}\right|_{K'}\right)\Bigg|_{\Phi'} = (-1)^{\varepsilon_A}\, \frac{\partial_r}{\partial K'_A}\left(\left.\frac{\partial K'_A}{\partial g}\right|_{\Phi,K}\right)\Bigg|_{\Phi'} \\
&= (-1)^{\varepsilon_A}\, \frac{\partial_r}{\partial \Phi^B}\left(\left.\frac{\partial K'_A}{\partial g}\right|_{\Phi,K}\right)\Bigg|_{K} \left.\frac{\partial_r \Phi^B}{\partial K'_A}\right|_{\Phi'} + (-1)^{\varepsilon_A}\, \frac{\partial_r}{\partial K_B}\left(\left.\frac{\partial K'_A}{\partial g}\right|_{\Phi,K}\right)\Bigg|_{\Phi} \left.\frac{\partial_r K_B}{\partial K'_A}\right|_{\Phi'} \\
&= (-1)^{\varepsilon_A}\, \frac{\partial}{\partial g}\left(\left.\frac{\partial_r K'_A}{\partial \Phi^B}\right|_{K}\right)\Bigg|_{\Phi,K} \left.\frac{\partial_r \Phi^B}{\partial K'_A}\right|_{\Phi'} + (-1)^{\varepsilon_A}\, \frac{\partial}{\partial g}\left(\left.\frac{\partial_r K'_A}{\partial K_B}\right|_{\Phi}\right)\Bigg|_{\Phi,K} \left.\frac{\partial_r K_B}{\partial K'_A}\right|_{\Phi'} .
\end{aligned} \tag{132}$$

With the help of (121), one can prove that

$$\begin{aligned}
\left.\tfrac{\partial_r K'_A}{\partial K_B}\right|_{\Phi} &= (N^{-1})_A{}^B, & \left.\tfrac{\partial_r K_B}{\partial K'_A}\right|_{\Phi'} &= N_B{}^A - F_{BC}(M^{-1})^C{}_D F^{DA}, \\
\left.\tfrac{\partial_r K'_A}{\partial \Phi^B}\right|_{K} &= -(N^{-1})_A{}^C F_{CB}, & \left.\tfrac{\partial_r \Phi^B}{\partial K'_A}\right|_{\Phi'} &= -(M^{-1})^B{}_C F^{CA}.
\end{aligned} \tag{133}$$

Moreover, noticing that

$$J^{\frac{1}{2}} = \det M^{-1}, \tag{134}$$

and that

$$d \ln J^{\frac{1}{2}} = (-1)^{\varepsilon_B+1} dN_B{}^D (N^{-1})_D{}^B, \tag{135}$$

we conclude

$$\tilde{\Delta} S = \left.\frac{\partial \ln J^{\frac{1}{2}}}{\partial g}\right|_{\Phi,K} + (-1)^{\varepsilon_B+1} (M^{-1})^B{}_C F^{CA} (N^{-1})_A{}^D \left.\frac{\partial F_{DB}}{\partial g}\right|_{\Phi,K} . \tag{136}$$

The last term vanishes due to symmetry properties and the statistics of the various factors. This concludes the proof.

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