

Renormalization-Group Techniques for Single-Field Inflation in Primordial Cosmology and Quantum Gravity

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Abstract

We study inflation as a “cosmic” renormalization-group flow. The flow, which encodes the dependence on the background metric, is described by a running coupling α , which parametrizes the slow roll, a de Sitter free, analytic beta function and perturbation spectra that are RG invariant in the superhorizon limit. Using RG invariance as a guiding principle, we classify the main types of flows according to the properties of their spectra, without referring to their origins from specific actions or models. Novel features include spectra with essential singularities in α and violations of the relation $r + 8n_t = 0$ to the leading order. Various classes of potentials studied in the literature can be described by means of the RG approach, even when the action includes a Weyl-squared term, while others are left out. In known cases, the classification helps identify the models that are ruled out by data. The RG approach is also able to generate spectra that cannot be derived from standard Lagrangian formulations.

1 Introduction

Inflation explains the approximate isotropy and homogeneity of the cosmic background radiation by means of a primordial accelerated expansion [1, 2, 3, 4, 5, 6, 7, 8]. The quantum fluctuations are sources of the perturbations that originate the present large-scale structure of the universe [9, 10, 11, 12, 13, 14, 15].

The expansion of the universe can be driven by scalar fields rolling down a potential [16]. Various scenarios of this type lead to a scalar perturbation spectrum that is compatible with observations [17, 18]. In a “geometric” approach, instead, inflation is driven by gravity itself, as in the Starobinsky $R + R^2$ model [2] and the $f(R)$ theories [19, 20]. A third approach is to study inflation as a “cosmic” renormalization-group (RG) flow [21, 22], which is generated by the dependence on the background metric. The spectra of the cosmological perturbations satisfy equations of the Callan-Symanzik type in the superhorizon limit.

In this paper, we study the RG approach in a general setting and relate it, when possible, to the “inflaton” approach, based on a scalar field rolling down a potential. We classify the main types of flows according to their properties, such as the structures of their perturbation spectra.

The flow should be such that

- a) the beta function is perturbative in the coupling α , starts quadratically in α and is de Sitter free in the infinite past (i.e., $\alpha = 0$ gives the de Sitter background for $t \rightarrow -\infty$);
- b) the power spectra are perturbative, apart from overall factors, and RG invariant in the superhorizon limit.

Requirement *a)* implies that the beta function must behave like the one of an asymptotically free quantum field theory, such as QCD, in the perturbative region. The overall factors mentioned in point *b)* can be negative or fractional powers of α , or even essential singularities. Negative powers are already familiar from the scalar spectra.

Defining the coupling α as $\sqrt{4\pi G/3}\dot{\phi}/H$, where ϕ is the inflaton and H is the Hubble parameter, we can describe a large number of potentials studied in the literature by means of RG flows. However, many others are left out. To cover larger classes of models, it might be necessary to satisfy requirements *a)* and *b)* with a different definition of coupling, related to α in a non perturbative way. This possibility is not explored here. On the other hand, the approach based on the cosmic RG flow allows us to study cases that cannot be described by means of more common approaches.

The strategy is to start from a general Mukhanov-Sasaki action

$$S_{\text{MS}} = \frac{1}{2} \int d\eta \left[w'^2 - (1 + \Delta h) w^2 + \frac{2 + \sigma}{\eta^2} w^2 \right], \quad (1.1)$$

for some perturbation $w(\eta)$, where η is a rescaled conformal time. Here Δh and σ are analytic functions of α that vanish for $\alpha = 0$. We refer to Δh as a “mass renormalization”, which affects how we impose the Bunch-Davies vacuum condition. We do not need to be specific about the theory or model that gives (1.1). We can build the spectra associated with (1.1) by solely requiring that they be RG invariant in the superhorizon limit. This property is sufficient to determine them up to an α -independent normalization constant, which can be fixed in the de Sitter limit $\alpha = 0$.

In several cases, the spectra exhibit essential singularities in α . Moreover, they often violate the “consistency condition” $r + 8n_t = 0$ [23], which is known to hold to the leading order in single-field slow-roll models, independently of the scalar potential $V(\phi)$. This suggests that the cosmic RG approach is essentially different from the other approaches, although it intersects them in a number of cases.

The Starobinsky $R + R^2$ model works well phenomenologically. However, once we add R^2 it is worth to consider the inclusion of the square $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \equiv C^2$ of the Weyl tensor $C_{\mu\nu\rho\sigma}$ as well, because C^2 and R^2 have the same dimensions in units of mass. So doing, we obtain the Lagrangian $R + R^2 + C^2$, which is renormalizable (once the cosmological constant is switched on), but leads to a theory plagued by ghosts, if quantized by means of the standard methods [24]. The problem of ghosts can be overcome by formulating a physically different theory [25, 26, 27] in terms of purely virtual particles, or fakeons [28]. The properties of the new theory can be appreciated particularly well in primordial cosmology [29], where, due to a bound relating the coefficients of C^2 and R^2 , which must be satisfied to make the formulation in terms of fake particles consistent, the physics changes even on superhorizon scales [29]. The main outcome is a stringent prediction for the tensor-to-scalar ratio r , not available in other formulations. The models with C^2 provide an interesting arena for theoretical investigations, since they are the only ones known to date which lead to a nontrivial mass renormalization Δh .

We stress that in order to be able to talk about renormalization-group flow, it is not enough to identify a flow that is governed by an autonomous first order differential equation

$$\frac{dx}{dt} = f(x). \quad (1.2)$$

It is not even sufficient to show that there exists an equilibrium point x_0 ($f(x_0) = 0$) that is Lyapunov stable (i.e., such that the solutions that start close enough to x_0 remain close

enough forever), or one that is asymptotically stable (i.e., such that the solutions that start close enough to x_0 remain close enough to x_0 and converge to x_0). As realized in ref.s [21, 22], the key ingredient is having correlation functions (the perturbation spectra, in our case) that satisfy equations of the Callan-Symanzik type, $f(x)$ being the beta function for the coupling x .

Note that the RG techniques studied here and in ref.s [21, 22] are essentially different from the “beta function formalism” studied in [30]. The autonomous differential equation associated with the first class of models was discussed in [31] in the context of holography, but not related to the RG properties of the correlation functions and spectra. Among earlier studies on the running behaviors of the spectral indices we mention those of refs. [32]. Calculations of subleading corrections to the perturbation spectra (in models without fakeons and without C^2) are done in refs. [33].

The paper is organized as follows. In section 2 we recall the main properties of the cosmic RG flow and show how it originates from specific actions. In section 3 we classify the flows by ignoring their origins from specific actions or models. We also relate them to classes of known and less known potentials, when possible. In section 4 we build the perturbation spectra in this general setting, by means of RG invariance. In section 5 we calculate the spectra for the classes of flows identified in section 3, in the absence of a mass renormalization Δh . In section 6 we apply the results to the RG flows associated with standard actions. In section 7 we investigate the reversed approach, which means search for the flow associated with a given potential. We also list the cases that we cannot treat at the moment. In section 8 we extend the results to the flows with a nontrivial Δh and the presence of purely virtual particles. Section 9 contains the conclusions and appendix A collects reference formulas and higher-order corrections.

2 Inflationary beta function and cosmic RG flow

In this section we recall the formulation of inflation as an RG flow. We call it “cosmic” RG flow to emphasize that it is originated by the dependence on the background metric and not by the radiative corrections.

The starting point is to consider actions such as

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \frac{1}{2} \int d^4x \sqrt{-g} (D_\mu \phi D^\mu \phi - 2V(\phi)), \quad (2.1)$$

or

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R + \frac{1}{2m_\chi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) + \frac{1}{2} \int d^4x \sqrt{-g} (D_\mu \phi D^\mu \phi - 2V(\phi)), \quad (2.2)$$

where $V(\phi)$ is an arbitrary potential. For convenience, the cosmological term is switched off. We do not add a term proportional to R^2 to (2.2), because it would lead to models of double-scalar inflation, which are beyond the scope of this paper. Yet, the theory $R + R^2 + C^2$ is equivalent to a particular case of (2.2) when $V(\phi)$ is the Starobinsky potential.

The FLRW metric

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2),$$

where $a(t)$ is the scale factor, leads to the equations

$$\dot{H} = -4\pi G \dot{\phi}^2, \quad H^2 = \frac{4\pi G}{3} (\dot{\phi}^2 + 2V(\phi)), \quad \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (2.3)$$

where $H = \dot{a}/a$ is the Hubble parameter, in both cases (2.1) and (2.2).

We introduce the coupling¹

$$\alpha = \frac{\hat{\kappa} \dot{\phi}}{2H} = \sqrt{-\frac{\dot{H}}{3H^2}}, \quad (2.4)$$

where $\hat{\kappa} = \sqrt{16\pi G/3}$. Using $\dot{\phi} = 2\alpha H/\hat{\kappa}$ inside the second equation (2.3), we obtain the potential $V(\phi(\alpha))$ as a function of α :

$$V = \frac{2H^2}{\hat{\kappa}^2} (1 - \alpha^2). \quad (2.5)$$

Eliminating $\ddot{\phi}$ from the last equation (2.3), it is easy to show that α satisfies

$$\frac{\dot{\alpha}}{H} = -3\alpha(1 - \alpha^2) - \frac{\hat{\kappa}}{2} \frac{V'}{H^2}. \quad (2.6)$$

Introducing the conformal time

$$\tau = - \int_t^{+\infty} \frac{dt'}{a(t')}, \quad (2.7)$$

equation (2.6) can be converted into the beta function

$$\beta_\alpha \equiv \frac{d\alpha}{d \ln|\tau|} = -\frac{1}{v} \frac{\dot{\alpha}}{H}, \quad (2.8)$$

¹For the purposes of this paper, we can assume $\dot{\phi} > 0$, $\alpha > 0$.

of the cosmic RG flow, where $v \equiv -(aH\tau)^{-1}$.

By differentiating its definition with respect to τ , it is easy to show that the function v satisfies the linear differential equation

$$\beta_\alpha \frac{dv}{d\alpha} = 1 - 3\alpha^2 - v, \quad (2.9)$$

which can be integrated by quadratures and has solution

$$v(\alpha) = 1 - 3\alpha^2 + 6 \int_{\alpha_0}^{\alpha} d\alpha' \alpha' \exp\left(-\int_{\alpha'}^{\alpha} \frac{d\alpha''}{\beta_\alpha(\alpha'')}\right), \quad (2.10)$$

where α_0 must be chosen to eliminate the essential singularity.

From the definition of α , we have the equation $\dot{H} = -3\alpha^2 H^2$. If H is viewed as a function of α , the equation can be written as

$$\frac{dH}{d\alpha} = \frac{3\alpha^2 H}{v\beta_\alpha},$$

which is solved by

$$H(\alpha) = H_0 \exp\left(\int_{\alpha_0}^{\alpha} \frac{3\alpha'^2 d\alpha'}{v(\alpha')\beta_\alpha(\alpha')}\right), \quad (2.11)$$

where H_0 is an arbitrary constant.

Dividing $\dot{\phi} = 2\alpha H/\hat{\kappa}$ by $\dot{\alpha} = -Hv\beta_\alpha$, we obtain the equation satisfied by ϕ , also viewed as a function of α , which reads

$$\frac{d\phi}{d\alpha} = \frac{\dot{\phi}}{\dot{\alpha}} = -\frac{2\alpha}{\hat{\kappa}v\beta_\alpha} \quad (2.12)$$

and is solved by

$$\phi(\alpha) = -\frac{2}{\hat{\kappa}} \int^{\alpha} \frac{\alpha' d\alpha'}{v(\alpha')\beta_\alpha(\alpha')}, \quad (2.13)$$

where the lower limit of integration remains arbitrary.

By inserting the inverse $\alpha(\phi)$ of this function inside (2.5), we can fully reconstruct the potential $V(\phi)$ from the beta function β_α .

In this paper, we assume a generic de-Sitter free beta function

$$\beta_\alpha(\alpha) = \alpha^2(b_0 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + \dots) \quad (2.14)$$

that behaves like the one of a quantum field theory around a free-field fixed point. This means, in practice, that the linear term is missing in (2.14). In most cases, the first

nonvanishing coefficient is negative, so the de Sitter fixed point $\alpha = 0$ corresponds to the infinite past.

In the next section we classify the types of potentials obtained from (2.14). The other way around (i.e., search for the cosmic RG flow associated with a given potential) is considered in section 7. Although it is always possible to associate a potential to a flow with beta function (2.14), it is not always possible to generate a beta function of the form (2.14) from a potential, with α defined as in (2.4).

The running coupling $\alpha(-\tau)$ is defined from

$$\ln \frac{\tau}{\tau'} = \int_{\alpha(-\tau')}^{\alpha(-\tau)} \frac{d\alpha'}{\beta_\alpha(\alpha')}.$$

We often denote $\alpha(-\tau)$ by α and $\alpha(1/k)$ by α_k , where k is just a constant for the moment (later on it will denote the absolute value of the space momentum \mathbf{k} of the quantum fluctuations of the metric). Writing $\tau' = -1/k$, we have

$$\ln(-k\tau) = \int_{\alpha_k}^{\alpha} \frac{d\alpha'}{\beta_\alpha(\alpha')}.$$

It is possible to show that the spectra \mathcal{P} of the tensor and scalar fluctuations satisfy RG evolution equations in the superhorizon limit, with vanishing anomalous dimensions [21]. Viewing \mathcal{P} as functions of τ and α , the equations read

$$\frac{d\mathcal{P}}{d \ln |\tau|} = \left(\frac{\partial}{\partial \ln |\tau|} + \beta_\alpha(\alpha) \frac{\partial}{\partial \alpha} \right) \mathcal{P} = 0. \quad (2.15)$$

Viewing \mathcal{P} as functions of α and α_k , the dependence on α drops out and we have

$$\mathcal{P} = \tilde{\mathcal{P}}(\alpha_k), \quad \frac{d\tilde{\mathcal{P}}(\alpha_k)}{d \ln k} = -\beta_\alpha(\alpha_k) \frac{d\tilde{\mathcal{P}}(\alpha_k)}{d\alpha_k}, \quad (2.16)$$

which means that the spectra depend on the momentum k only through the running coupling α_k . A third option is to view the spectra as functions of k/k_* and $\alpha_* = \alpha(1/k_*)$, where k_* is the pivot scale. Then the equations read

$$\left(\frac{\partial}{\partial \ln k} + \beta_\alpha(\alpha_*) \frac{\partial}{\partial \alpha_*} \right) \mathcal{P}(k/k_*, \alpha_*) = 0. \quad (2.17)$$

Most RG techniques known from particle physics in flat spacetime apply to the cosmic RG flow. They allow us to work out RG improved perturbation spectra \mathcal{P} , resum the leading and subleading logs, simplify the computations of tilts and running coefficients

and easily push the calculations to high orders [22]. In the log expansion the spectra are expanded in powers of α_k , with the caveat that a certain product $\alpha_k^n \ln(-k\tau)$ is considered of order unity and treated exactly, where n is a positive integer that depends on the properties of the beta function.

3 From the flow to the potential

In this section we classify the main types of cosmic RG flows and relate them to classes of known and less known potentials.

The expansion

$$v = 1 - 3\alpha^2 + 6b_0\alpha^3 + 6(b_1 - 3b_0^2)\alpha^4 + 6(b_2 - 7b_0b_1 + 12b_0^3)\alpha^5 + \mathcal{O}(\alpha^6) \quad (3.1)$$

of the function $v = -(aH\tau)^{-1}$ can be obtained straightforwardly from formula (2.10) to arbitrary orders. The other basic quantities H , V and ϕ are derived as follows. Once v is known, the Hubble parameter H is derived from (2.11), while the field $\phi(\alpha)$ is obtained from (2.13). Then formula (2.5) gives the potential V as a function of α . By inverting $\phi(\alpha)$ we find $\alpha(\phi)$, which then allows us to work out V as a function of ϕ .

We distinguish various cases, corresponding to different classes of potentials: 1) $b_0 \neq 0$; 2) $b_0 = 0$, $b_1 \neq 0$; 3) $b_0 = b_1 = 0$, $b_2 \neq 0$; etc.

3.1 Flows of class I: $b_0 \neq 0$

If $b_0 \neq 0$ we find the expansions

$$\begin{aligned} H &= H_0 \left[1 + \frac{3\alpha}{b_0} + \frac{3(3-b_1)\alpha^2}{2b_0^2} + \frac{\alpha^3}{2b_0^3}(9 + 6b_0^2 - 9b_1 + 2b_1^2 - 2b_0b_2) + \mathcal{O}(\alpha^4) \right], \\ \hat{\kappa}\phi &= -\frac{2}{b_0} \ln \frac{\alpha}{\Phi_0} + \frac{2\alpha b_1}{b_0^2} - \frac{\alpha^2}{b_0^3}(3b_0^2 + b_1^2 - b_0b_2) + \mathcal{O}(\alpha^3), \\ V &= \frac{2H_0^2}{\hat{\kappa}^2} \left[1 + \frac{6\alpha}{b_0} + \frac{\alpha^2}{b_0^2}(18 - 3b_1 - b_0^2) + \frac{2\alpha^3}{b_0^3}(18 - 9b_1 + b_1^2 - b_0b_2) + \mathcal{O}(\alpha^4) \right], \end{aligned}$$

where H_0 and Φ_0 are arbitrary constants. We see that $\Phi \equiv \Phi_0 e^{-b_0 \hat{\kappa} \phi / 2}$ is a power series in α that starts from $\mathcal{O}(\alpha)$. Inverting for α , we obtain

$$\alpha(\phi) = \Phi + \frac{b_1}{b_0} \Phi^2 - \frac{3b_0^2 - 2b_1^2 - b_0b_2}{2b_0^2} \Phi^3 + \mathcal{O}(\Phi^4).$$

Inserting $\alpha(\phi)$ into V , we get the potential

$$V = \frac{2H_0^2}{\hat{\kappa}^2} \left[1 + 6\frac{\Phi}{b_0} + (18 - b_0^2 + 3b_1)\frac{\Phi^2}{b_0^2} + (b_0b_2 - b_0^2(9 + 2b_1) + 2(18 + 9b_1 + b_1^2))\frac{\Phi^3}{b_0^3} + \mathcal{O}(\Phi^4) \right].$$

The conclusion is that when $b_0 \neq 0$, the potential is an expansion in powers of $\Phi_0 e^{-b_0 \hat{\kappa} \phi / 2}$. The de Sitter limit is $\phi \rightarrow \infty \times \text{sign}(b_0)$, where V tends to a nonvanishing positive constant $V_0 = 2H_0^2 / \hat{\kappa}^2$, H_0 being the limiting value of H .

The autonomous first order differential equation (1.2) associated with this type of potentials was also discussed in [31] in the context of holography, but not related to the RG properties of the spectra.

Examples of potentials that fall into this class are the α -attractor E-Models [34], which include the Starobinsky potential

$$V(\phi) = \frac{m_\phi^2}{2\hat{\kappa}^2} (1 - e^{\hat{\kappa}\phi})^2, \quad (3.2)$$

the α -attractor T-Models [34], which include $V(\phi) = V_0 \tanh^2(c\phi)$ and $V(\phi) = V_0 \tanh^4(c\phi)$ (which can also be obtained in the context of Palatini inflation [35]), the potential $V(\phi) = V_0(1 - e^{c\phi})$ of exponential SUSY inflation [36] and the potential $V(\phi) = V_0(1 - \text{sech}(c\phi))$ of mutated hilltop inflation [37], where V_0 and c are constants.

The running coupling can be worked out from

$$d \log \eta = \frac{d\alpha}{\beta_\alpha(\alpha)}, \quad (3.3)$$

where $\eta = -k\tau$. Integrating both sides from $\eta = 1$ to generic η , we find

$$\alpha(-\tau) = \frac{\alpha_k}{\lambda}, \quad \lambda \equiv 1 - b_0 \alpha_k \log \eta, \quad (3.4)$$

to the leading log order and

$$\alpha(-\tau) = \frac{\alpha_k}{\lambda} \prod_{n=1}^{\infty} (1 + \alpha_k^n \gamma_n(\lambda)), \quad (3.5)$$

in general, where

$$\gamma_1(\lambda) = -\frac{b_1 \ln \lambda}{b_0 \lambda}, \quad \gamma_2(\lambda) = \frac{(\lambda - 1)(b_1^2 - b_0 b_2) + b_1^2 (\ln \lambda - 1) \ln \lambda}{\lambda^2 b_0^2}, \quad (3.6)$$

etc.

3.2 Flows of class II: $b_0 = 0$, $b_1 \neq 0$

If $b_0 = 0$, $b_1 \neq 0$, we find

$$\begin{aligned}
 H &= H_0 \alpha^{3/b_1} \left[1 - \frac{3b_2 \alpha}{b_1^2} + \frac{3(3b_1^3 + 3b_2^2 + b_1 b_2^2 - b_1^2 b_3) \alpha^2}{2b_1^4} + \mathcal{O}(\alpha^3) \right], \\
 \hat{\kappa} \phi &= \hat{\kappa} \phi_0 + \frac{2}{b_1 \alpha} + \frac{2b_2}{b_1^2} \ln \alpha - \frac{2\alpha}{b_1^3} (3b_1^2 + b_2^2 - b_1 b_3) + \mathcal{O}(\alpha^2).
 \end{aligned} \tag{3.7}$$

The de Sitter fixed point is $\phi \times \text{sign}(b_1) \rightarrow \infty$. With no loss of generality, we set the arbitrary constant ϕ_0 to zero. We see that $\phi(\alpha)$ involves both $\ln \alpha$ and $1/\alpha$. We can invert $\alpha(\phi)$ as an expansion in powers of $1/\phi$ and logarithms of ϕ . A special case where we can proceed straightforwardly is $b_2 = 0$, where we find (for $b_1 \phi > 0$)

$$\begin{aligned}
 \alpha &= \frac{2}{b_1 \hat{\kappa} \phi} \left[1 - \frac{4(3b_1 - b_3)}{b_1^3 \hat{\kappa}^2 \phi^2} + \frac{4b_4}{b_1^4 \hat{\kappa}^3 \phi^3} + \mathcal{O}((\hat{\kappa} \phi)^{-4}) \right], \\
 V(\phi) &= \frac{2H_0^2}{\hat{\kappa}^2} \left(\frac{b_1 \hat{\kappa} \phi}{2} \right)^{-6/b_1} \left[1 - \frac{4(9b_1 + b_1^2 - 3b_3)}{b_1^4 \hat{\kappa}^2 \phi^2} + \frac{8b_4}{b_1^5 \hat{\kappa}^3 \phi^3} + \mathcal{O}((\hat{\kappa} \phi)^{-4}) \right].
 \end{aligned} \tag{3.8}$$

We see that this class of potentials is equal to a nontrivial (possibly fractional) overall power of ϕ times a powers series in $1/\phi$. When $b_2 \neq 0$ we obtain corrections of the form $(\hat{\kappa} \phi)^{-n} \ln^m(\hat{\kappa} \phi)$ with $n \geq m \geq 1$ inside the square brackets of formulas (3.8).

From (3.3), we find the running coupling

$$\alpha(-\tau) = \frac{\alpha_k}{\lambda}, \quad \lambda \equiv \sqrt{1 - 2b_1 \alpha_k^2 \ln \eta}, \tag{3.9}$$

to the leading log order, which suggests that the log expansion must be defined as the expansion in powers of α_k , with the caveat that $\alpha_k^2 \ln \eta$ is considered of order unity and resummed exactly. In general, α is still of the form (3.5). For instance, when $b_2 = 0$ we have

$$\begin{aligned}
 \gamma_1(\lambda) &= 0, & \gamma_2(\lambda) &= -\frac{b_3 \ln \lambda}{\lambda^2 b_1}, & \gamma_3(\lambda) &= \frac{(1 - \lambda) b_4}{\lambda^3 b_1}, \\
 \gamma_4(\lambda) &= \frac{(\lambda^2 - 1)(b_3^2 - b_1 b_5) + b_3^2 (3 \ln \lambda - 2) \ln \lambda}{2\lambda^4 b_1^2},
 \end{aligned} \tag{3.10}$$

etc.

Examples of potentials of class II are the powerlike potentials $V(\phi) = V_0 \phi^n$ of large field inflation [38], where V_0 is a constant (see section 7). More general polynomial potentials are included as well, such as $V(\phi) = V_0 + V_1 \phi^2$, $V(\phi) = V_1 \phi^2 + V_2 \phi^4$ (mixed large field inflation), $V(\phi) = V_0 + V_1 \phi^2 + V_2 \phi^4$ (Hilltop quartic model and double well inflation [39]), MSSM inflation [40], etc., where V_i are constants.

3.3 Flows of class III: $b_0 = b_1 = 0$, $b_2 \neq 0$

Under the assumptions $b_0 = b_1 = 0$, $b_2 \neq 0$ we find

$$\hat{\kappa}\phi = \frac{1}{b_2\alpha^2} - \frac{2b_3}{b_2^2\alpha} - \frac{2(3b_2^2 + b_3^2 - b_2b_4)}{b_2^3} \ln \alpha + \mathcal{O}(\alpha).$$

Again, the inverse function $\phi(\alpha)$ is quite involved, unless we make some further assumptions, such as $b_4 = 3b_2 + (b_3^2/b_2)$. In that case, we obtain

$$\begin{aligned} H &= H_0\alpha^{-3b_3/b_2^2} \exp\left(-\frac{3}{b_2\alpha}\right) \left[1 + \frac{3(3b_2^2b_3 + b_3^3 - b_2^2b_5)\alpha^2}{2b_2^4} + \mathcal{O}(\alpha^3)\right], \\ \alpha &= \frac{1}{\sqrt{b_2\hat{\kappa}\phi}} \left[1 - \frac{b_3}{b_2\sqrt{b_2\hat{\kappa}\phi}} + \frac{b_3^2}{2b_2^3\hat{\kappa}\phi} - \frac{3b_2^2b_3 + b_3^3 - b_2^2b_5}{b_2^3(b_2\hat{\kappa}\phi)^{3/2}} + \mathcal{O}((\hat{\kappa}\phi)^{-2})\right], \\ V(\phi) &= \frac{2H_0^2 (b_2\hat{\kappa}\phi)^{3b_3/b_2^2}}{\hat{\kappa}^2} \exp\left(-\frac{6}{b_2}\sqrt{b_2\hat{\kappa}\phi} - 6\frac{b_3}{b_2^2}\right) \left[1 + \frac{3b_3^2}{b_2^3\sqrt{b_2\hat{\kappa}\phi}} + \mathcal{O}((\hat{\kappa}\phi)^{-1})\right]. \end{aligned} \quad (3.11)$$

Setting $b_3 = 0$ and choosing the coefficients b_{2n} , $n > 1$, appropriately, we may obtain an exponential times a power series:

$$V(\phi) = \frac{2H_0^2}{\hat{\kappa}^2} \exp\left(-\frac{6}{b_2}\sqrt{b_2\hat{\kappa}\phi}\right) \left[1 - \frac{b_2^2 - 3b_5}{b_2^3\hat{\kappa}\phi} + \mathcal{O}((\hat{\kappa}\phi)^{-2}) \times (\text{power series in } (\hat{\kappa}\phi)^{-1})\right].$$

The simplest example is $V = V_0 e^{\sqrt{c}\phi}$. To our knowledge, the potentials of class III have not been considered in the literature so far [17].

The running coupling to the leading log order is

$$\alpha(-\tau) = \frac{\alpha_k}{(1 - 3b_2\alpha_k^3 \ln \eta)^{1/3}}$$

and the subleading orders can be worked out as before.

4 Spectral RG invariance

In this section we outline the strategy of the calculations of the perturbation spectra. We start from a generic Mukhanov-Sasaki action of the form

$$S_{\text{MS}} = \frac{1}{2} \int d\eta \left[w'^2 - hw^2 + \frac{2 + \sigma}{\eta^2} w^2 \right], \quad (4.1)$$

where the function $w(\eta)$ stands for a generic fluctuation and $h(\alpha) = 1 + \mathcal{O}(\alpha)$, $\sigma(\alpha) = \mathcal{O}(\alpha)$ are generic power series in α . The equation of motion is

$$w'' + hw - \frac{2 + \sigma}{\eta^2} w = 0. \quad (4.2)$$

The superhorizon limit of its solution and the RG properties of the spectra can be studied by decomposing $\eta w(\eta)$ as the sum of a power series $Q(\ln \eta)$ in $\ln \eta$ plus a power series $W(\eta)$ in η and $\ln \eta$, such that $W(\eta) \rightarrow 0$ term-by-term for $\eta \rightarrow 0$:

$$\eta w = Q(\ln \eta) + W(\eta). \quad (4.3)$$

Inserting (4.3) into (4.2), we find

$$\frac{d^2 Q}{(d \ln \eta)^2} - 3 \frac{dQ}{d \ln \eta} - \sigma Q = -\eta^2 \frac{d^2 W}{d \eta^2} + 2\eta \frac{dW}{d \eta} - h\eta^2(W + Q) + \sigma W.$$

The right-hand side is negligible in the superhorizon limit, so both sides of this equations must be separately zero. We end up with the Q equation

$$\left(\frac{d}{d \ln \eta} - 3 \right) \frac{dQ}{d \ln \eta} = \sigma Q. \quad (4.4)$$

The operator in parenthesis must be inverted perturbatively, to eliminate contributions proportional to η^3 , which do not belong to Q . So doing, we find

$$\frac{dQ}{d \ln \eta} = -\frac{1}{3} \frac{1}{1 - \frac{1}{3} \frac{d}{d \ln \eta}} \sigma Q$$

with an arbitrary initial condition $Q(0)$, or, as in ref. [21],

$$\frac{dQ}{d \ln \eta} = -\frac{\sigma}{3} Q - \frac{1}{3} \sum_{n=1}^{\infty} 3^{-n} \frac{d^n(\sigma Q)}{d \ln^n \eta}, \quad (4.5)$$

where the higher-derivative terms on the right-hand side have to be handled perturbatively.

Choosing a reference scale k , we can also view $Q(\ln \eta)$ as a function $\tilde{Q}(\alpha, \alpha_k)$ of α and α_k , satisfying

$$\beta_\alpha \frac{\partial \tilde{Q}}{\partial \alpha} = -\frac{\sigma \tilde{Q}}{3} - \frac{1}{3} \sum_{n=1}^{\infty} 3^{-n} \left(\beta_\alpha \frac{\partial}{\partial \alpha} \right)^n (\sigma \tilde{Q}). \quad (4.6)$$

The general solution of this equation can be written in the form

$$\tilde{Q}(\alpha, \alpha_k) = \tilde{Q}(\alpha_k) \frac{J(\alpha)}{J(\alpha_k)}, \quad (4.7)$$

where $Q(0) = \tilde{Q}(\alpha_k)$ is called ‘‘spectral normalization’’.

We divide the calculation of the spectrum in three steps. The first step is to determine to spectral normalization $\tilde{Q}(\alpha_k)$ by solving equation (4.2) perturbatively in α_k with the

Bunch-Davies vacuum condition². To this purpose, we expand the running coupling α in powers of α_k and write

$$h(\alpha) = 1 + \alpha_k \sum_{j=0}^{\infty} \tilde{h}_j \alpha_k^j, \quad \sigma(\alpha) = \alpha_k \sum_{j=0}^{\infty} \sigma_j \alpha_k^j, \quad (4.8)$$

where σ_j, \tilde{h}_j are functions of η . Expanding the function w as well,

$$w(\eta) = w_0(\eta) + \sum_{n=1}^{\infty} \alpha_k^n w_n(\eta), \quad (4.9)$$

equation (4.2) gives $w_0'' + w_0 - 2(w_0/\eta^2) = 0$ and

$$w_n'' + w_n - 2\frac{w_n}{\eta^2} = \frac{1}{\eta^2} \sum_{j=0}^{n-1} \sigma_j w_{n-1-j} - \sum_{j=0}^{n-1} \tilde{h}_j w_{n-1-j}, \quad n \geq 1. \quad (4.10)$$

The solution w must agree with (4.7) in the superhorizon limit by means of (4.3). The comparison between (4.7) and (4.9) allows us to derive $Q(0) = \tilde{Q}(\alpha_k)$. This calculation can be done for all the classes of cosmic RG flows at once, because the results are analytic in the coefficients b_i of the beta function.

The second step is to solve the Q equation (4.6) to determine the function $J(\alpha)$. Note that $J(\alpha)$ is independent of the mass renormalization h of (4.1) and the Bunch-Davies vacuum condition. This part of the calculation must be done class by class.

The third step is to build the spectra, which we can achieve by taking advantage of RG invariance. We quantize (4.1) by introducing the operator

$$\hat{w}_{\mathbf{k}}(\eta) = w_{\mathbf{k}}(\eta) \hat{a}_{\mathbf{k}} + w_{-\mathbf{k}}^*(\eta) \hat{a}_{-\mathbf{k}}^\dagger,$$

where $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ are creation and annihilation operators satisfying $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$. We know that the spectra are RG invariant in the superhorizon limit. However, the w two-point function does not have this property. The right perturbation is

$$w_{\text{RG}}(\eta) \equiv \frac{C}{k^{3/2}} \frac{\eta w(\eta)}{J(\alpha)}, \quad (4.11)$$

where C is a constant that depends on the model and the type of perturbation, while the factor $k^{-3/2}$ is introduced to match the known cases (see sections 6 and 8). It is easy to check that w_{RG} is indeed RG invariant in the superhorizon limit, where

$$w^{\text{RG}} \simeq \frac{C}{k^{3/2}} \frac{\tilde{Q}(\alpha, \alpha_k)}{J(\alpha)} = \frac{C}{k^{3/2}} \frac{\tilde{Q}(\alpha_k)}{J(\alpha_k)}. \quad (4.12)$$

²When $h \neq 1$ it is necessary to first switch to different variables $\tilde{\eta}$ and $\tilde{w}(\tilde{\eta})$ (see section 8).

The power spectrum \mathcal{P} of the fluctuations w^{RG} is defined by the two-point function

$$\langle \hat{w}_{\mathbf{k}}^{\text{RG}}(\eta) \hat{w}_{\mathbf{k}'}^{\text{RG}}(\eta) \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{\varsigma k^3} \mathcal{P}. \quad (4.13)$$

Here ς is equal to 16 and 1 for the tensor and scalar perturbations, respectively, and takes into account the normalization of the polarizations and the sum over them. Using (4.12), we find

$$\mathcal{P} = \frac{\varsigma k^3}{2\pi^2} |w_{\mathbf{k}}^{\text{RG}}|^2 \simeq \varsigma \frac{|C|^2}{2\pi^2} \left| \frac{\tilde{Q}(\alpha_k)}{J(\alpha_k)} \right|^2, \quad (4.14)$$

which is RG invariant in the superhorizon limit, by construction. The tilts $n_{\mathcal{P}}$ and the running coefficients can be calculated straightforwardly by differentiating the spectra:

$$n_{\mathcal{P}} - \theta = -\beta_{\alpha}(\alpha_k) \frac{\partial \ln \mathcal{P}}{\partial \alpha_k}, \quad \frac{d^n n_{\mathcal{P}}}{d \ln k^n} = \left(-\beta_{\alpha}(\alpha_k) \frac{\partial}{\partial \alpha_k} \right)^n n_{\mathcal{P}}, \quad (4.15)$$

where $\theta = 0$ and $\theta = 1$ for the tensor and scalar perturbations, respectively.

Some remarks are in order before proceeding. Typically, the mass renormalization h is identically one when the Weyl-squared term C^2 is absent and the action is (2.1). Instead, h is nontrivial when the action is (2.2) and C^2 is treated by means of the fakeon prescription and projection. In the presence of a mass renormalization, a supplementary step is required to impose the Bunch-Davies vacuum condition. For this reason, we postpone this part of the investigation to section 8 and first derive the spectra in the case $h \equiv 1$.

The leading behavior of σ for α small determines the leading behavior of the spectrum. Inspired by most common scenarios of class I, which are investigated in section 6, we distinguish two cases. In the first case, σ starts from order α^2 ,

$$\sigma_{\text{t}}(\alpha) = \alpha^2 \sum_{n=0}^{\infty} s_n^{\text{t}} \alpha^n, \quad (4.16)$$

while in the second case σ starts linearly in α ,

$$\sigma_{\text{s}}(\alpha) = \bar{s}\alpha + \alpha^2 \sum_{n=0}^{\infty} s_n^{\text{s}} \alpha^n. \quad (4.17)$$

We adopt the subscripts “t” and “s” to distinguish the two cases, because a σ like (4.16) is typical of the tensor perturbations of class I, while a σ like (4.17) is typical of the scalar perturbations, also of class I. Apart from this, the general approach we adopt here makes no real distinction between tensor fluctuations and scalar fluctuations. Note that in the

common cases of class II σ is $\mathcal{O}(\alpha^2)$ for both tensor and scalar perturbations (see section 6).

The first step of the calculation, which is the derivation of $\tilde{Q}(\alpha_k)$, can be performed right away and, as said, is the same for all classes of RG flows. If we assume (4.16), it is easy to check that to, the next-to-next-to-leading (NNL) order, the solution of the equation (4.2) that satisfies the Bunch-Davies vacuum condition

$$w(\eta) \simeq \frac{e^{i\eta}}{\sqrt{2}} \quad \text{for } \eta \rightarrow \infty, \quad (4.18)$$

is (4.9) with

$$w_0 = W_0, \quad w_1 = 0, \quad w_2 = \frac{s_0}{9}W_2, \quad w_3 = -b_0s_0\frac{W_3}{18} + (8b_0s_0 + 3s_1)\frac{W_2}{27}, \quad (4.19)$$

where the recurring functions W_i are defined in formula (A.1) of the appendix. The comparison between w and Q in the superhorizon limit, done by means of (4.3), gives

$$\begin{aligned} \tilde{Q}_t(\alpha_k) = & \frac{i}{\sqrt{2}} \left[1 + \frac{s_0}{3}\alpha_k^2(2 - \tilde{\gamma}_M) \right. \\ & \left. + \frac{\alpha_k^3}{18}(b_0s_0(32 - 28\tilde{\gamma}_M + 6\tilde{\gamma}_M^2 + \pi^2) + 6s_1(2 - \tilde{\gamma}_M)) \right] + \mathcal{O}(\alpha_k^4), \end{aligned} \quad (4.20)$$

where $\tilde{\gamma}_M$ is also defined in the appendix.

If, instead, we assume (4.17), the solution of (4.2) to the NNL order is (4.9) with

$$w_0 = W_0, \quad w_1 = \frac{\bar{s}}{9}W_2, \quad w_2 = \frac{\bar{s}^2}{36}W_4 - \frac{\bar{s}(\bar{s} + 3b_0)}{108}W_3 + \frac{6s_0 + 8b_0\bar{s} - \bar{s}^2}{54}W_2, \quad (4.21)$$

which gives

$$\begin{aligned} \tilde{Q}_s(\alpha_k) = & \frac{i}{\sqrt{2}} \left[1 + \frac{\bar{s}}{3}(2 - \tilde{\gamma}_M)\alpha_k + \frac{s_0}{3}(2 - \tilde{\gamma}_M)\alpha_k^2 + \frac{b_0\bar{s}}{36}(32 - 28\tilde{\gamma}_M + 6\tilde{\gamma}_M^2 + \pi^2)\alpha_k^2 \right. \\ & \left. - \frac{\bar{s}^2}{108}(8 + 20\tilde{\gamma}_M - 6\tilde{\gamma}_M^2 - 3\pi^2)\alpha_k^2 \right] + \mathcal{O}(\alpha_k^3). \end{aligned} \quad (4.22)$$

The second and third steps of the calculation, which are the derivations of the functions $J(\alpha)$ and the spectra, respectively, are performed in the next section class by class.

5 From the flow to the spectra

In this section we derive the functions $J(\alpha)$ and the spectra for the RG flows of classes I, II and III with $h \equiv 1$, without referring to the origin of the flows from specific actions

or models. We give enough details to derive the results to the next-to-next-to-leading log (NNLL) order. However, once the procedure is clear enough we just report them to the NLL order.

5.1 Class I

If we assume the expansion (4.16), the Q function

$$\tilde{Q}_t(\alpha, \alpha_k) = \tilde{Q}_t(\alpha_k) \frac{J_t(\alpha)}{J_t(\alpha_k)} \quad (5.1)$$

is determined by solving (4.6). We obtain

$$J_t(\alpha) = 1 - \frac{s_0 \alpha}{3b_0} + (-2b_0^2 s_0 + 3b_1 s_0 + s_0^2 - 3b_0 s_1) \frac{\alpha^2}{18b_0^2} + \alpha^3 \Delta J_t^I + \mathcal{O}(\alpha^4),$$

where the NNLL corrections ΔJ_t^I are given in formula (A.2). The tensor spectrum is, from (4.14) and (4.20),

$$\mathcal{P}_t^I = \frac{4|C_t|^2}{\pi^2} \left\{ 1 + \frac{2s_0 \alpha_k}{3b_0} + \frac{\alpha_k^2}{9b_0^2} [2b_0^2 (7 - 3\gamma_M) s_0 + s_0(2s_0 - 3b_1) + 3b_0 s_1] + \alpha_k^3 \Delta P_t^I + \mathcal{O}(\alpha_k^4) \right\}, \quad (5.2)$$

where ΔP_t^I is also given in formula (A.2).

If we assume the expansion (4.17), we find

$$\tilde{Q}_s(\alpha, \alpha_k) = \tilde{Q}_s(\alpha_k) \frac{J_s(\alpha)}{J_s(\alpha_k)} = \tilde{Q}_s(\alpha_k) \frac{\alpha^{-\bar{s}/(3b_0)} \tilde{J}_s(\alpha)}{\alpha_k^{-\bar{s}/(3b_0)} \tilde{J}_s(\alpha_k)}, \quad (5.3)$$

where

$$\tilde{J}_s(\alpha) = 1 + \frac{\alpha}{27b_0^2} (-3b_0^2 \bar{s} + 9b_1 \bar{s} + b_0 \bar{s}^2 - 9b_0 s_0) + \alpha^2 \Delta \tilde{J}_s^I + \mathcal{O}(\alpha^3), \quad (5.4)$$

and $\Delta \tilde{J}_s^I$ is given in formula (A.3). Using (4.22), the spectrum is

$$\mathcal{P}_s^I = \frac{|C_s|^2}{4\pi^2} \alpha_k^{2\bar{s}/(3b_0)} \left\{ 1 - \frac{2\alpha_k}{27b_0^2} [9b_1 \bar{s} - 3b_0^2 \bar{s}(7 - 3\gamma_M) + b_0(\bar{s}^2 - 9s_0)] + \alpha_k^2 \Delta P_s^I + \mathcal{O}(\alpha_k^3) \right\}. \quad (5.5)$$

The NNLL corrections ΔP_s^I , which we do not report explicitly, can be easily derived from (5.3), (4.22) and (5.4).

Assuming that the subscripts “t” and “s” stand for the tensor and scalar perturbations, respectively, which is what happens in typical cases, we can define the tensor-to-scalar ratio r . The main results to the leading log order are

$$\mathcal{P}_t^I \simeq \frac{4|C_t|^2}{\pi^2}, \quad \mathcal{P}_s^I \simeq \frac{|C_s|^2}{4|\pi^2} \alpha_k^{2\bar{s}/(3b_0)}, \quad r^I = \frac{\mathcal{P}_t^I}{\mathcal{P}_s^I} \simeq \frac{16|C_t|^2}{|C_s|^2} \alpha_k^{-2\bar{s}/(3b_0)},$$

$$n_t = -\beta_\alpha(\alpha_k) \frac{\partial \ln \mathcal{P}_t^I}{\partial \alpha_k} \simeq -\frac{2s_0^t}{3} \alpha_k^2, \quad n_s - 1 = -\beta_\alpha(\alpha_k) \frac{\partial \ln \mathcal{P}_s^I}{\partial \alpha_k} \simeq -\frac{2\bar{s}}{3} \alpha_k,$$

where we have distinguished the coefficients of the two σ expansions by means of superscripts “t” and “s”, when necessary.

The novel feature of the general spectra just found is that they can violate the “consistency condition” $r^I + 8n_t = 0$ to the lowest order. Moreover, the overall factor of \mathcal{P}_s^I can be any power of α_k , possibly fractional. At present we do not know how to obtain these results from more standard approaches. In the typical situations (see section 6) we have $\bar{s} = -3b_0$, so the overall power of \mathcal{P}_s^I is just $1/\alpha_k^2$. Moreover, the other coefficients conspire to make $r^I + 8n_t = 0$ hold to the lowest order.

Usually, the physical predictions are expressed through the number of e-foldings N . We prefer not to do so, because N is not a perturbative quantity. Besides, in the RG approach it is more convenient to use quantities that have clear RG evolution properties. The coupling α_k is not physical (like the coupling constants of quantum field theory), since we can always make arbitrary perturbative redefinitions

$$\alpha_k \rightarrow \alpha_k + c_1 \alpha_k^2 + c_2 \alpha_k^3 + \dots,$$

without spoiling the basic properties of the flow, where c_i are constants. A way out is to eliminate α_k from one physical quantity (say $|n_s - 1|$, which is the best measured one) and then express every other predictions in terms of it. For example, to the leading order we find the relation

$$r^I (1 - n_s)^{2\bar{s}/(3b_0)} \simeq \frac{16|C_t|^2}{|C_s|^2} \left(\frac{2\bar{s}}{3}\right)^{2\bar{s}/(3b_0)} = \text{constant}. \quad (5.6)$$

5.2 Class II

We assume the beta function (2.14) with $b_0 = b_2 = 0$, $b_1 \neq 0$. The expansion (4.16) leads to the Q function

$$\tilde{Q}_t(\alpha, \alpha_k) = \tilde{Q}_t(\alpha_k) \frac{J_t(\alpha)}{J_t(\alpha_k)} = \tilde{Q}_t(\alpha_k) \frac{\alpha^{-s_0/(3b_1)} \tilde{J}_t(\alpha)}{\alpha_k^{-s_0/(3b_1)} \tilde{J}_t(\alpha_k)},$$

where

$$\tilde{J}_t(\alpha) = 1 - \frac{s_1\alpha}{3b_1} + \frac{\alpha^2(9b_3s_0 - 9b_1s_2 - 6b_1^2s_0 + b_1s_0^2 + 3s_1^2)}{54b_1^2} + \alpha^3\Delta\tilde{J}_t^\Pi + \mathcal{O}(\alpha^4),$$

and $\Delta\tilde{J}_t^\Pi$ is given in formula (A.4). Then formula (4.22) leads to the spectrum

$$\mathcal{P}_t^\Pi = \frac{4|C_t|^2}{\pi^2}\alpha_k^{2s_0/(3b_1)} \left[1 + \frac{2s_1}{3b_1}\alpha_k + \frac{2s_0}{9}(7 - 3\gamma_M)\alpha_k^2 + \frac{6s_1^2 - b_1s_0^2 + 9(b_1s_2 - b_3s_0)}{27b_1^2}\alpha_k^2 + \alpha_k^3\Delta P_t^\Pi + \mathcal{O}(\alpha_k^4) \right],$$

where ΔP_t^Π is easy to derive from $\tilde{Q}_t(\alpha_k)$ and $\Delta\tilde{J}_t^\Pi$, when needed.

From this point onwards we report the results to the NLL order only, since the procedure is now clear. The expansion (4.17) gives

$$\tilde{Q}_s(\alpha, \alpha_k) = \tilde{Q}_s(\alpha_k) \frac{J_s(\alpha)}{J_s(\alpha_k)} = \tilde{Q}_s(\alpha_k) \frac{\alpha^{(\bar{s}^2 - 9s_0)/(27b_1)} \exp\left(\frac{\bar{s}}{3\alpha b_1}\right) \tilde{J}_s(\alpha)}{\alpha_k^{(\bar{s}^2 - 9s_0)/(27b_1)} \exp\left(\frac{\bar{s}}{3\alpha_k b_1}\right) \tilde{J}_s(\alpha_k)},$$

with

$$\tilde{J}_s(\alpha) = 1 - \frac{81s_1 + 27b_1\bar{s} - 18s_0\bar{s} + 2\bar{s}^3}{243b_1}\alpha + \frac{b_3\bar{s}}{3b_1^2}\alpha + \mathcal{O}(\alpha^2).$$

The spectrum is

$$\mathcal{P}_s^\Pi = \frac{|C_s|^2}{4\pi^2}\alpha_k^{2(9s_0 - \bar{s}^2)/(27b_1)} e^{-2\bar{s}/(3\alpha_k b_1)} \left[1 + \frac{2s_1}{3b_1}\alpha_k + \frac{4\bar{s}\alpha_k}{243b_1}(\bar{s}^2 - 9s_0) + \frac{2\bar{s}}{9}(7 - 3\gamma_M)\alpha_k - \frac{2b_3\bar{s}}{3b_1^2}\alpha_k + \mathcal{O}(\alpha_k^2) \right].$$

The main predictions to the leading order are

$$r^\Pi = \frac{\mathcal{P}_t^\Pi}{\mathcal{P}_s^\Pi} \simeq \frac{16|C_t|^2}{|C_s|^2} \exp\left(\frac{2\bar{s}}{3\alpha_k b_1}\right), \quad n_t \simeq -\frac{2s_0^t}{3}\alpha_k^2, \quad n_s - 1 \simeq -\frac{2\bar{s}}{3}\alpha_k. \quad (5.7)$$

Again, we see that the relation $r^I + 8n_t = 0$ is in general violated to the lowest order. A new feature here is the presence of essential singularities in the overall factors of \mathcal{P}_s^Π .

If we assume $\bar{s} = 0$, which is what occurs in most known cases (see sections 6 and 7), then σ_s has an expansion of the form (4.16) and we find

$$r^\Pi \simeq \frac{16|C_t|^2}{|C_s|^2}\alpha_k^{2(s_0^t - s_0^s)/(3b_1)}, \quad n_t \simeq -\frac{2s_0^t}{3}\alpha_k^2, \quad n_s - 1 \simeq -\frac{2s_0^s}{3}\alpha_k^2,$$

$$r^\Pi(1 - n_s)^{(s_0^s - s_0^t)/(3b_1)} \simeq \frac{16|C_t|^2}{|C_s|^2} \left(\frac{2s_0^s}{3}\right)^{(s_0^s - s_0^t)/(3b_1)} = \text{constant}.$$

5.3 Class III

We assume $b_0 = b_1 = 0$, $b_2 \neq 0$, $b_4 = 3b_2 + (b_3^2/b_2)$. We just report the leading order and the overall factors for $b_3 = 0$, since it is now clear how to compute the corrections, if necessary.

Assuming the expansion (4.16), the Q function is

$$\tilde{Q}_t(\alpha, \alpha_k) = \tilde{Q}_t(\alpha_k) \frac{\alpha^{-s_1/(3b_2)} \exp\left(\frac{s_0}{3\alpha b_2}\right) \tilde{J}_t(\alpha)}{\alpha_k^{-s_1/(3b_2)} \exp\left(\frac{s_0}{3\alpha_k b_2}\right) \tilde{J}_t(\alpha_k)},$$

where $\tilde{J}_t(\alpha) = 1 + \mathcal{O}(\alpha)$, so the spectrum is

$$\mathcal{P}_t^{\text{III}} = \frac{4|C_t|^2}{\pi^2} \alpha_k^{2s_1/(3b_2)} \exp\left(-\frac{2s_0}{3\alpha_k b_2}\right) (1 + \mathcal{O}(\alpha_k)).$$

Instead, if we assume (4.17), we find

$$\mathcal{P}_s^{\text{III}} = \frac{|C_s|^2}{4\pi^2} \alpha_k^{2(81s_1+2\bar{s}^3-18s_0\bar{s}-243\bar{s})/(243b_2)} \exp\left(-\frac{\bar{s}}{3\alpha_k^2 b_2} + \frac{2(\bar{s}^2 - 9s_0)}{27\alpha_k b_2}\right) (1 + \mathcal{O}(\alpha_k)).$$

We see that the essential singularity has become more severe.

The main results to the leading order are

$$r^{\text{III}} \simeq \frac{16|C_t|^2}{|C_s|^2} \exp\left(\frac{\bar{s}}{3\alpha_k^2 b_2}\right), \quad n_t \simeq -\frac{2s_0^t}{3} \alpha_k^2, \quad n_s - 1 \simeq -\frac{2\bar{s}}{3} \alpha_k.$$

Mimicking the known cases, where typically $\bar{s} = 0$ and $s_0^t = s_0^s$, we obtain

$$r^{\text{III}} \simeq \frac{16|C_t|^2}{|C_s|^2} \alpha_k^{2(s_1^t - s_1^s)/(3b_2)}, \quad n_t \simeq -\frac{2s_0}{3} \alpha_k^2, \quad n_s - 1 \simeq -\frac{2s_0}{3} \alpha_k^2.$$

In general, the relation $r + 8n_t = 0$ is violated again.

6 From the action to the spectra

In this section we study the RG flows associated with the action (2.1). We parametrize the metric as

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2) + 2\text{diag}(\Phi, a^2\Psi, a^2\Psi, a^2\Psi) - \delta_\mu^0 \delta_\nu^i \partial_i B - \delta_\mu^i \delta_\nu^0 \partial_i B, \\ -2a^2 (u\delta_\mu^1 \delta_\nu^1 - u\delta_\mu^2 \delta_\nu^2 + v\delta_\mu^1 \delta_\nu^2 + v\delta_\mu^2 \delta_\nu^1) \quad (6.1)$$

and expand (2.1) to the quadratic order in the fluctuations. These are the graviton modes $u = u(t, z)$ and $v = v(t, z)$ (chosen to have a space momentum \mathbf{k} oriented along the z axis after the Fourier transform) and the scalar mode Ψ , while Φ and B are auxiliary fields, which can be integrated out straightforwardly. The ϕ fluctuation $\delta\phi$ is set to zero by working in the comoving gauge, where the curvature perturbation \mathcal{R} coincides with Ψ . For reviews on the parametrizations of the metric fluctuations and their properties, see [41, 16].

To study the tensor fluctuations we set $\Phi = \Psi = B = 0$. The quadratic Lagrangian obtained from (2.1) is

$$(8\pi G)\frac{\mathcal{L}_t}{a^3} = \dot{u}_{\mathbf{k}}\dot{u}_{-\mathbf{k}} - \frac{k^2}{a^2}u_{\mathbf{k}}u_{-\mathbf{k}}, \quad (6.2)$$

plus an identical contribution for $v_{\mathbf{k}}$, where $u_{\mathbf{k}}(t)$ is the Fourier transform of $u(t, z)$ with respect to z , \mathbf{k} denotes the space momentum and $k = |\mathbf{k}|$. We drop the subscripts \mathbf{k} and $-\mathbf{k}$ when no confusion can arise.

To study the scalar fluctuations we set $u = v = 0$. Then (2.1) gives the quadratic Lagrangian

$$(8\pi G)\frac{\mathcal{L}_s}{a^3} = -3(\dot{\Psi} + H\Phi)^2 + 4\pi G\dot{\phi}^2\Phi^2 + \frac{k^2}{a^2}\left[2B(\dot{\Psi} + H\Phi) + \Psi(\Psi - 2\Phi)\right],$$

having Fourier transformed the space coordinates to momentum space and omitted the subscripts \mathbf{k} and $-\mathbf{k}$. As said, B and Φ appear as auxiliary fields. Once they are integrated out, we obtain $\Phi = -\dot{\Psi}/H$ and

$$(8\pi G)\frac{\mathcal{L}_s}{a^3} = 3\alpha^2\left(\dot{\Psi}^2 - \frac{k^2}{a^2}\Psi^2\right). \quad (6.3)$$

The Lagrangians (6.2) and (6.3) are then converted to the form (4.1) by defining

$$w_t = au\sqrt{\frac{k}{4\pi G}}, \quad w_s = \alpha a\Psi\sqrt{\frac{3k}{4\pi G}}, \quad (6.4)$$

and switching to the variable $\eta = -k\tau$. We obtain

$$S_{t,s} = \frac{1}{2}\int d\eta\left[w_{t,s}'^2 - w_{t,s}^2 + (2 + \sigma_{t,s})\frac{w_{t,s}^2}{\eta^2}\right], \quad (6.5)$$

where [21]

$$\begin{aligned} \sigma_t &= \frac{2}{v^2}\left(1 - \frac{3}{2}\alpha^2\right) - 2, \\ \sigma_s &= \left(\frac{\beta_\alpha}{\alpha} - \frac{1}{v} - 1 + \beta_\alpha\frac{d}{d\alpha}\right)\left(\frac{\beta_\alpha}{\alpha} - \frac{1}{v}\right) - 2. \end{aligned} \quad (6.6)$$

Using (2.14) and (3.1), we find the expansions

$$\begin{aligned}\sigma_t &= 9\alpha^2 - 24b_0\alpha^3 + 12(3 + 6b_0^2 - 2b_1)\alpha^4 + \mathcal{O}(\alpha^5), \\ \sigma_s &= -3b_0\alpha + (9 + 2b_0^2 - 3b_1)\alpha^2 - (30b_0 - 5b_0b_1 + 3b_2)\alpha^3 + \mathcal{O}(\alpha^4).\end{aligned}\quad (6.7)$$

We know that the fluctuations $w_{\text{RG}} = u$ and $w_{\text{RG}} = \Psi$ are RG invariant in the super-horizon limit. Comparing (4.11) with (6.4), we find

$$J_t(\alpha) = \frac{C_t}{vH\sqrt{4\pi G}}, \quad J_s(\alpha) = \frac{C_s\alpha}{vH}\sqrt{\frac{3}{4\pi G}}. \quad (6.8)$$

The constants C_t and C_s can be read from the leading behaviors of these functions around $\alpha \sim 0$. In all classes we find

$$C_t = H_0\sqrt{4\pi G}, \quad C_s = \frac{C_t}{\sqrt{3}}. \quad (6.9)$$

From (4.14) we derive the spectra

$$\mathcal{P}_t = \frac{32G}{\pi}v^2(\alpha_k)H^2(\alpha_k)\left|\tilde{Q}_t(\alpha_k)\right|^2, \quad \mathcal{P}_s = \frac{2G}{3\pi\alpha_k^2}v^2(\alpha_k)H^2(\alpha_k)\left|\tilde{Q}_s(\alpha_k)\right|^2. \quad (6.10)$$

The tensor-to-scalar ratio is

$$r = 48\alpha_k^2 \frac{\left|\tilde{Q}_t(\alpha_k)\right|^2}{\left|\tilde{Q}_s(\alpha_k)\right|^2}.$$

Since $\tilde{Q}(\alpha_k) = i(1 + \mathcal{O}(\alpha_k))/\sqrt{2}$ in every case, the leading contribution to r is always $\simeq 48\alpha_k^2$. Assuming $\beta_\alpha = b_n\alpha^{n+2} + \mathcal{O}(\alpha^{n+3})$ to cover all classes of RG flows at the same time, the tilts to the leading order are

$$n_t = -\beta_\alpha(\alpha_k)\frac{\partial \ln \mathcal{P}_t}{\partial \alpha_k} \simeq -6\alpha_k^2, \quad n_s - 1 = -\beta_\alpha(\alpha_k)\frac{\partial \ln \mathcal{P}_s}{\partial \alpha_k} \simeq \begin{cases} 2b_0\alpha_k & \text{for } n = 0 \\ 2(b_1 - 3)\alpha_k^2 & \text{for } n = 1 \\ -6\alpha_k^2 & \text{for } n > 1 \end{cases} \quad (6.11)$$

and the relation $r + 8n_t \simeq 0$ is always satisfied to the same order. Since $n_s - 1 \sim -0.035$ at the pivot scale $k = 0.05\text{Mpc}^{-1}$ [18], we have $r \simeq 0.3$ for $n > 1$, which is ruled out by data [18]. Thus, all the models described by the action (2.1) and belonging to classes greater than or equal to III are ruled out. The models of class II must have $b_1 < -5.4$ to guarantee $r < 0.1$, while the models of class I must have $b_0 < -0.38$.

The investigation of the flows for the action (2.2) is postponed to section 8, because the presence of C^2 typically turns on a nontrivial mass renormalization Δh .

7 From the potential to the flow

Formula (2.8) does not give a beta function of the form (2.14) for any potential $V(\phi)$, with the coupling α defined in (2.4). Here we consider two further models of classes II and III and list the most important potentials that are studied in the literature, but are not described by our present approach.

7.1 Powerlike potentials $V(\phi) = \phi^{2n}$

The simplest representatives of class II are the powerlike potentials

$$V = \frac{m^{4-2n}}{(2n)!} \phi^{2n}, \quad (7.1)$$

where m is a constant of dimension one. We find

$$\begin{aligned} H &= \frac{m(\hat{\kappa}m)^{1-n} (2n)^n}{(3\alpha)^n \sqrt{2(2n)!}} \left[1 - \frac{\alpha^2}{2} + \frac{12-5n}{8n} \alpha^4 + \mathcal{O}(\alpha^6) \right], \\ v &= 1 - 3\alpha^2 - \frac{18}{n} \alpha^4 - \frac{18}{n^2} (3n+11) \alpha^6 + \mathcal{O}(\alpha^8), \\ \hat{\kappa}\phi(\alpha) &= c_0 - \frac{2n}{3\alpha} + \frac{2\alpha}{3} + \frac{2}{3n} (n-2) \alpha^3 + \mathcal{O}(\alpha^4), \end{aligned}$$

where c_0 is a constant. The beta function reads

$$\beta_\alpha = -\frac{3}{n} \alpha^3 - \frac{3}{n^2} (3n-1) \alpha^5 - \frac{3}{n^3} (9n^2 + 12n + 7) \alpha^7 + \mathcal{O}(\alpha^9).$$

The results of subsection 5.2 give the predictions (5.7), to the leading order, at least for generic h and σ in (4.1). With suitable values of b_1 and $\bar{s} \neq 0$ it is not difficult to fit the data available at present. However, in the standard cases of section 6, formulas (6.7) give $\bar{s} = 0$, so the main predictions are actually (6.11) to the leading order, i.e.

$$r \simeq 48\alpha_k^2, \quad n_t \simeq -6\alpha_k^2, \quad n_s - 1 \simeq -\frac{6}{n} (n+1) \alpha_k^2, \quad \frac{r}{1-n_s} \simeq \frac{8n}{n+1},$$

which are ruled out because r is too large. We do not know at present how to obtain $\bar{s} \neq 0$ from conventional Lagrangian models.

7.2 Class III potential $V(\phi) = \exp(\sqrt{-\hat{\kappa}\phi})$

The simplest example of potential of class III is

$$V(\phi) = V_0 \exp\left(c\sqrt{-\hat{\kappa}\phi}\right), \quad (7.2)$$

where V_0 and c are positive constants. It can be obtained from the formulas of subsection 3.3 with

$$b_0 = b_1 = b_3 = 0, \quad b_2 = -\frac{36}{c^2}, \quad b_4 = 3b_2, \quad b_5 = \frac{b_2^2}{3}, \quad V_0 = \frac{2H_0^2}{\hat{\kappa}^2},$$

etc. We find

$$H = H_0 \exp\left(\frac{c^2}{12\alpha}\right) (1 + \mathcal{O}(\alpha^2)), \quad \hat{\kappa}\phi = -\frac{c^2}{36\alpha^2} + \mathcal{O}(\alpha).$$

In the standard cases, where formula (6.7) gives

$$\sigma_t = 9\alpha^2 + 36\alpha^4 + \mathcal{O}(\alpha^5), \quad \sigma_s = 9\alpha^2 + \frac{108}{c^2}\alpha^3 + \mathcal{O}(\alpha^4),$$

the main predictions to the leading log order are, using (6.9),

$$\begin{aligned} \mathcal{P}_t &\simeq \frac{16GH_0^2}{\pi} \exp\left(\frac{c^2}{6\alpha_k}\right), & \mathcal{P}_s &\simeq \frac{GH_0^2}{3\pi\alpha_k^2} \exp\left(\frac{c^2}{6\alpha_k}\right), & r &\simeq 48\alpha_k^2, \\ n_t &\simeq -6\alpha_k^2, & n_s - 1 &\simeq -6\alpha_k^2, & \frac{r}{1 - n_s} &\simeq 8. \end{aligned}$$

Note the essential singularity in both spectra. The model is ruled out by present data, since it predicts $r \sim 8(1 - n_s) \sim 0.28$, which is too large [18].

In general, it seems that raising the class raises the prediction for r and makes it more difficult to fit the data, unless we find ways to obtain $\bar{s} \neq 0$.

7.3 Potentials not described by the present approach

It is worth to emphasize that many potentials studied in the literature cannot be described straightforwardly by the approach pursued so far. This may mean that they do not have an interpretation in terms of a perturbative RG flow or simply that they need an upgraded approach, which may, for example, make use of a more sophisticated definition of coupling α . Examples of such cases are the potentials obtained by including the radiative corrections proportional to $\ln(\hat{\kappa}\phi)$. We mention the radiatively corrected massive inflation, the radiatively corrected quartic inflation, the Coleman-Weinberg inflation and the radiatively corrected Higgs inflation [42, 6]. We recall that, instead, powers of $\ln(\hat{\kappa}\phi)/\phi$ can be obtained in class II when $b_2 \neq 0$.

Other important cases that are left out are (for a more complete list, see [17]):

- the potential $V(\phi) = V_0[1 + \alpha \ln(\hat{\kappa}\phi)]$ of loop inflation [43];

- the potential $V = V_0 [1 + \cos(\hat{\kappa}\phi)]$ of natural inflation [44];
- the potential $V(\phi) = V_0 e^{-c\phi}$ of power law inflation [45];
- the potential $V(\phi) = V_0 (\hat{\kappa}\phi)^2 / [1 + c(\hat{\kappa}\phi)^2]$ of radion gauge inflation [46].

It is also unclear how to describe the potentials associated with generic $f(R)$ theories, since they do not easily fall into our classification. For example, $f(R) = R + cR^3$ is equivalent to the potential $V(\phi) = V_0 e^{\hat{\kappa}\phi/2} (1 - e^{\hat{\kappa}\phi})^{3/2}$ [19].

8 Nontrivial mass renormalization

In this section we study the Mukhanov-Sasaki action (4.1) in the presence of a nontrivial mass renormalization

$$h(\alpha) = 1 + \bar{h}\alpha + \alpha^2 \sum_{n=0}^{\infty} h_n \alpha^n, \quad (8.1)$$

where \bar{h} and h_n are constants. The function $J(\alpha)$ of equation (4.7) remains the same, while $\tilde{Q}(\alpha_k)$ changes.

Following [21], we rewrite the action (4.1) as

$$\tilde{S}_t^{\text{prj}} = \frac{1}{2} \int d\tilde{\eta} \left(\tilde{w}'^2 - \tilde{w}^2 + (2 + \tilde{\sigma}) \frac{\tilde{w}^2}{\tilde{\eta}^2} \right), \quad (8.2)$$

where the new variable $\tilde{\eta}(\eta)$ is defined as the solution of the differential equation $\tilde{\eta}'(\eta) = \sqrt{h(\eta)}$ with the initial condition $\tilde{\eta}(0) = 0$. We have

$$\tilde{w}(\tilde{\eta}(\eta)) = h(\eta)^{1/4} w(\eta), \quad \tilde{\sigma} = \frac{\tilde{\eta}^2(\sigma + 2)}{\eta^2 h} + \frac{\tilde{\eta}^2}{16h^3} (4hh'' - 5h'^2) - 2. \quad (8.3)$$

Using (8.1) we find

$$\begin{aligned} \tilde{\eta}(\eta) = \eta \left[1 + \frac{\bar{h}}{2}\alpha + \frac{\alpha^2}{8}(4h_0 - 4b_0\bar{h} - \bar{h}^2) + b_0^2\bar{h}\alpha^3 \right. \\ \left. + \frac{\alpha^3}{2}(h_1 - b_1\bar{h}) + \frac{\alpha^3}{16}(\bar{h} + 4b_0)(\bar{h}^2 - 4h_0) + \mathcal{O}(\alpha^4) \right]. \end{aligned} \quad (8.4)$$

The advantage of (8.2) is that in the new variables the Bunch-Davies vacuum condition is the usual one,

$$\tilde{w}(\tilde{\eta}) \simeq \frac{e^{i\tilde{\eta}}}{\sqrt{2}} \quad \text{for } \tilde{\eta} \rightarrow \infty, \quad (8.5)$$

while it is not so for (4.1). We first work out the solution

$$\tilde{w}(\tilde{\eta}) = \tilde{w}_0(\tilde{\eta}) + \sum_{n=1}^{\infty} \alpha_k^n \tilde{w}_n(\tilde{\eta})$$

of the problem in the new variables, which looks similar to the solutions (4.19) and (4.21). Then we switch back to the original variables by means of (8.4), to derive $w(\eta)$. From $w(\eta)$ we can read $\tilde{Q}(\alpha_k)$ by means of (4.3) and (4.7). Finally, we obtain the spectrum from formula (4.14).

If we assume the tensorial sigma (4.16), the solutions to the NNL order are

$$\begin{aligned} \tilde{w}_0(\tilde{\eta}) &= W_0(\tilde{\eta}), & \tilde{w}_1(\tilde{\eta}) &= 0, & \tilde{w}_2(\tilde{\eta}) &= \frac{s'_0}{9} W_2(\tilde{\eta}), \\ \tilde{w}_3(\tilde{\eta}) &= -b_0 s'_0 \frac{W_3(\tilde{\eta})}{18} + (8b_0 s'_0 + 3s'_1) \frac{W_2(\tilde{\eta})}{27}, \end{aligned}$$

where

$$s'_0 = s_0 - \frac{9}{4} b_0 \bar{h}, \quad s'_1 = s_1 + \frac{9}{4} [2b_0^2 \bar{h} - b_1 \bar{h} + b_0 (\bar{h}^2 - 2h_0)].$$

If we assume the scalar sigma (4.17), the solutions are

$$\begin{aligned} \tilde{w}_0(\tilde{\eta}) &= W_0(\tilde{\eta}), & \tilde{w}_1(\tilde{\eta}) &= \frac{\bar{s}}{9} W_2(\tilde{\eta}), \\ \tilde{w}_2(\tilde{\eta}) &= \frac{\bar{s}^2}{36} W_4(\tilde{\eta}) - \frac{\bar{s}(\bar{s} + 3b_0)}{108} W_3(\tilde{\eta}) + \frac{12s_0 - 27b_0 \bar{h} + 16b_0 \bar{s} - 2\bar{s}^2}{108} W_2(\tilde{\eta}). \end{aligned}$$

As before, these results do not depend on the class of RG flow, as is apparent from the absence of singularities when some coefficients of the beta function vanish. When we want to calculate the spectra, instead, we must work class by class. We report the corrections to the spectra to the same orders as in section 5. However, since the NNLL contributions are rather lengthy and in most practical applications \bar{h} vanishes, we report those just for $\bar{h} \neq 0$. In class I we have

$$\begin{aligned} \mathcal{P}_t^{\text{I,h}} &= \mathcal{P}_t^{\text{I}} + \frac{4|C_t|^2}{\pi^2} \left[-\frac{3}{2} \bar{h} \alpha_k + \frac{\bar{h}}{8b_0} (15\bar{h}b_0 - 4b_0^2(4 - 3\gamma_M) - 8s_0) \alpha_k^2 \right. \\ &\quad \left. - \frac{3}{2} h_1 \alpha_k^3 + \mathcal{O}(\bar{h}) \mathcal{O}(\alpha_k^3) + \mathcal{O}(\alpha_k^4) \right], \\ \mathcal{P}_s^{\text{I,h}} &= \mathcal{P}_s^{\text{I}} + \frac{|C_s|^2}{4\pi^2} \alpha_k^{2\bar{s}/(3b_0)} \left[-\frac{3}{2} \bar{h} \alpha_k + \mathcal{O}(\bar{h}) \mathcal{O}(\alpha_k^2) + \mathcal{O}(\alpha_k^3) \right]. \end{aligned} \quad (8.6)$$

The spectra of class II are

$$\begin{aligned}\mathcal{P}_t^{\text{II,h}} &= \mathcal{P}_t^{\text{II}} + \frac{4|C_t|^2}{\pi^2} \alpha_k^{2s_0/(3b_1)} \left[-\frac{3}{2} \bar{h} \alpha_k + \frac{1}{8b_1} (15b_1 \bar{h}^2 - 12b_1 h_0 - 8\bar{h}s_1) \alpha_k^2 \right. \\ &\quad \left. - \left(\frac{3}{2} h_1 + \frac{h_0 s_1}{b_1} \right) \alpha_k^3 + \mathcal{O}(\bar{h}) \mathcal{O}(\alpha_k^3) + \mathcal{O}(\alpha_k^4) \right], \\ \mathcal{P}_s^{\text{II,h}} &= \mathcal{P}_s^{\text{II}} + \frac{|C_s|^2}{4\pi^2} \alpha_k^{2(9s_0 - \bar{s}^2)/(27b_1)} e^{-2\bar{s}/(3\alpha_k b_1)} \left[-\frac{3}{2} \bar{h} \alpha_k + \mathcal{O}(\alpha_k^2) \right].\end{aligned}$$

The spectra of class III are unmodified to the order they were reported in section 5.

The main applications of these results concern the flows associated with the action (2.2), since the presence of C^2 typically turns on a nontrivial mass renormalization Δh , at least in the case of the tensor perturbations. Repeating the steps of refs. [21, 22, 29] we can generalize the NNLL results found there from the Starobinsky potential (3.2) to a generic flow of class I. Here we describe the main steps of the calculation and report the results to the NLL order.

To study the tensor perturbations, we expand the action (2.2) by means of (6.1), to the quadratic order in the fluctuations u and v , after setting $\Phi = \Psi = B = 0$. We obtain a higher-derivative Lagrangian for the polarization u (plus an identical copy for the other polarization v). Then we eliminate the higher-derivatives by introducing an auxiliary field and finally diagonalize the Lagrangian in the de Sitter limit by means of a change of variables. At the end, we obtain a new Lagrangian $\mathcal{L}_t(U_1, U_2)$ that contains a physical field U_1 and a fakeon U_2 , is diagonal at $\alpha = 0$ and, as said, is free of higher derivatives. As in [21, 22, 29], the fakeon projection sets $U_2 = \mathcal{O}(\alpha^2)$, so the w action is just given by $\mathcal{L}_t(U_1, 0)$ to the NLL order. Working it out, we obtain (4.1) with

$$\begin{aligned}h_t &= 1 - 3\xi\zeta^2\alpha^2 + \frac{3\xi\zeta^3\alpha^3}{2b_0} (6\xi - 12 + 2\xi b_0^2 + b_0^2 \xi^2) + \mathcal{O}(\alpha^4), \\ \sigma_t &= 9\zeta\alpha^2 - \frac{3\zeta^2\alpha^3}{2b_0} (16b_0^2 + 18\xi + 17\xi b_0^2) + \mathcal{O}(\alpha^4),\end{aligned}\tag{8.7}$$

where

$$\xi = \frac{4H_0^2}{m_\chi^2}, \quad \zeta = \left(1 + \frac{\xi}{2}\right)^{-1}, \quad H_0 = \sqrt{\frac{8\pi G V_0}{3}},$$

and V_0 is the value of the potential $V(\phi)$ in the de Sitter limit. The constant $C_t = H_0 \sqrt{4\zeta\pi G}$ of (4.11) can be computed in such a limit, where U_1 and U_2 decouple and U_2 can be dropped. At this point, formulas (8.7) can be used to work out the spectrum $\mathcal{P}_t^{\text{I,h}}$ given in (8.6).

Although the fakeon U_2 and its projection participate in the relation between u and w , RG invariance allows us to get directly to the final result. Indeed, formula (4.11) gives w_{RG} , that is to say u , and allows us to build the RG invariant spectrum directly from w . Since the fakeon projection is RG invariant [21], the result coincides with the one found by performing the projection explicitly.

It is easy to show that the ABP bound $m_\chi > m_\phi/4$ of ref. [29], which is obtained by requiring that the fakeon projection be consistent throughout the RG flow, now becomes $m_\chi > H_0/2$.

In the case of the scalar perturbations, we find the same ABP bound, together with

$$h_s = 1, \quad \sigma_s = -3b_0\alpha + (9 + 2b_0^2 - 3b_1)\alpha^2 + \mathcal{O}(\alpha^3) \quad (8.8)$$

and $C_s = H_0\sqrt{4\pi G/3}$, so $\mathcal{P}_s^{\text{l,h}}$ coincides with \mathcal{P}_s^{I} at this level.

The tensor-to-scalar ratio to the leading order is

$$r \simeq 48\zeta\alpha_k^2.$$

This means that, due to the ABP bound $m_\chi > H_0/2$, all the flows of class I predict r in the interval $0.4 \lesssim 1000r \lesssim 3$ [29]. Nevertheless, the Starobinsky potential (3.2) remains special, since it is the only one that makes the action (2.2) renormalizable.

The NNLL corrections can be worked out with the methods of [22] for the tensor and scalar perturbations of class I. In classes II and III the nontrivial H expansions of formulas (3.7) and (3.11) pose new challenges and the strategy just described needs to be upgraded with the help of further redefinitions.

9 Conclusions

By formulating inflation as an RG flow, it is possible to build a bridge between primordial cosmology and perturbative quantum field theory. The RG techniques allow us to calculate RG improved perturbation spectra to high orders. Moreover, RG invariance can be used as a guiding principle to build the spectra from a general Mukhanov-Sasaki action without referring to specific models. The resulting spectra are expansions in powers of the running coupling times certain prefactors, which may carry essential singularities and evade the relation $r + 8n_t = 0$ to the leading order. The classification emerging from the RG analysis helps identify the classes of models that have more chances to fit the data.

Not all classes of potentials considered in the literature can be described efficiently by means of the RG approach. Those left out require further investigations or do not have a beta function of the form considered here.

The RG approach also applies to models that contain purely virtual particles, such as the theory of quantum gravity $R+R^2+C^2$ [25], as well as more general models with actions (2.2) and potentials $V(\phi)$. In class I, for example, we find that although the ABP bound required by the consistency of the formulation depends on the model, the final prediction for r is universal to the leading order.

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A Appendix. Reference formulas

In this appendix we collect some formulas used in the paper, starting from the recurring functions [22]

$$\begin{aligned}
 W_0 &= \frac{i(1-i\eta)}{\eta\sqrt{2}} e^{i\eta}, & W_2 &= 3 [\text{Ei}(2i\eta) - i\pi] W_0^* + \frac{6W_0}{(1-i\eta)}, \\
 W_3 &= [6(\ln \eta + \tilde{\gamma}_M)^2 + 24i\eta F_{2,2,2}^{1,1,1}(2i\eta) + \pi^2] W_0^* + \frac{24W_0}{(1-i\eta)} - 4(\ln \eta + 1)W_2, & (\text{A.1}) \\
 W_4 &= -\frac{16W_0}{1+\eta^2} + \frac{2(13+i\eta)W_2}{9(1+i\eta)} + \frac{W_3}{3} + 4G_{2,3}^{3,1}(-2i\eta |_{0,0,0}^{0,1}) W_0,
 \end{aligned}$$

where Ei denotes the exponential-integral function, $F_{b_1, \dots, b_q}^{a_1, \dots, a_p}(z)$ is the generalized hypergeometric function ${}_pF_q(\{a_1, \dots, a_p\}, \{b_1, \dots, b_q\}; z)$ and $G_{p,q}^{m,n}$ is the Meijer G function. Moreover, $\tilde{\gamma}_M = \gamma_M - (i\pi/2)$, $\gamma_M = \gamma_E + \ln 2$, γ_E being the Euler-Mascheroni constant.

We also give the NNLL contributions to the J functions and the spectra. For class I we find

$$\begin{aligned}
 162b_0^3 \Delta J_t^I &= -12b_0^4 s_0 - 18b_1^2 s_0 + 18b_0 b_2 s_0 + 8b_0^2 s_0^2 - 9b_1 s_0^2 - s_0^3 - 18b_0^3 s_1 \\
 &\quad + 18b_0 b_1 s_1 + 9b_0 s_0 s_1 - 18b_0^2 s_2, \\
 162b_0^3 \Delta P_t^I &= 3(200 - 168\gamma_M + 36\gamma_M^2 - 3\pi^2) s_0 b_0^4 + 36(7 - 3\gamma_M) s_1 b_0^3 + 36s_2 b_0^2 \\
 &\quad + 4(41 - 18\gamma_M) s_0^2 b_0^2 - 36b_0 b_2 s_0 + 36(s_0 b_1 - b_0 s_1)(b_1 - s_0) + 8s_0^3, & (\text{A.2})
 \end{aligned}$$

for the tensorial sigma (4.16) and

$$\begin{aligned}
1458b_0^4\Delta\tilde{J}_s^I &= -54b_0^5\bar{s} + 243b_0\bar{s}(b_0b_2 - b_1^2) + 243b_0^2(b_1s_0 - b_0s_1) + 81b_1(b_1 - b_0^2)\bar{s}^2 \\
&+ 18b_0\bar{s}^2(b_1\bar{s} - b_0s_0) - 162b_0s_0(b_0^3 + b_1\bar{s}) + 27b_0^2s_0(3s_0 + 4b_0\bar{s}) \\
&+ b_0^2\bar{s}^2(45b_0^2 - 12b_0\bar{s} + \bar{s}^2), \tag{A.3}
\end{aligned}$$

for the scalar sigma (4.17).

For class II we find

$$162b_1^3\Delta\tilde{J}_t^{II} = 9b_1(2b_4s_0 - 2b_1^2s_1 + 2b_3s_1 + s_1s_2) - s_1(9b_3s_0 + b_1s_0^2 + s_1^2) + 2b_1^2(5s_0s_1 - 9s_3), \tag{A.4}$$

for the tensorial sigma.

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