Quantum gravity from fakeons

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• How the idea of fake particle ("fakeon") leads to what I claim is the right theory of quantum gravity
• Physical implications and predictions
• Classical limit
• FLRW solution
• Cosmology
The problem of quantum gravity is to make it renormalizable and unitary at the same time.

\[
\frac{1}{(p^2-m_1^2)(p^2-m_2^2)} = \frac{1}{m_1^2-m_2^2} \left[ \frac{1}{p^2-m_1^2} - \frac{1}{p^2-m_2^2} \right]
\]

Higher derivatives lead to renormalizability, but violate unitarity, unless ...
Unitarity: \[ S^\dagger S = 1 \]

\[ S = 1 + i T \]

\[ 2 \text{Im} T = T^\dagger T \quad \text{Optical theorem} \]

\[
\frac{1}{2i} \left[ \langle b | T | a \rangle - \langle b | T^\dagger | a \rangle \right] = \sum_{|n\rangle \in W} \langle b | T^\dagger | n \rangle \langle n | T | a \rangle, \quad |a\rangle, |b\rangle \in W
\]

In general, \( W \) may contain unphysical states in higher-derivative theories (those with \( \sigma_n = 1 \)).

Feynman prescription \( \rightarrow \) pseudo unitarity equation
Diagrammatic version of the optical theorem:

\[
2 \text{Im} \left[ (-i) \right] = \frac{d \Pi_f}{d \Pi_f} \left| \frac{\partial}{\partial \Pi_f} \right|^2
\]

\[
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\]

Cutting equations

Propagator: \( G(p, m) = \frac{1}{p^2 - m^2} \)

A prescription is needed

\( G_+(p, m, \epsilon) = \frac{1}{p^2 - m^2 + i\epsilon} \)

The Feynman prescription gives
The optical theorem is ok:

\[ 2\text{Im} \left[ (-i) \langle \rangle \right] = \langle \rangle = \int \text{d}\Pi_f \left| \langle \rangle \right|^2 \geq 0 \]

Indeed:

\[ \text{Im} \left[ -\frac{1}{p^2 - m^2 + i\epsilon} \right] = \pi\delta(p^2 - m^2) \geq 0 \]

Ghost: opposite residue \(- \frac{1}{p^2 - m^2 + i\epsilon}\)

The optical theorem is violated

Note that the prescription is crucial

\(- \frac{1}{p^2 - m^2 - i\epsilon}\) would be ok
Correct answer: the fermion

\[ G(p, m) = \frac{1}{p^2 - m^2} \]

Write \( \pm \frac{p^2 - m^2}{(p^2 - m^2)^2} \) and split the poles into pairs.
This is achieved by inserting an infinitesimal width as follows:

\[ G_\pm(p, m, \mathcal{E}^2) = \pm \frac{p^2 - m^2}{(p^2 - m^2)^2 + \mathcal{E}^4} \]

Note that the residue is zero on shell:

\[ \Rightarrow \text{NO PARTICLE} \]

However, the story is not that simple...
Example: bubble diagram

\[ i \mathcal{M} \propto \int \frac{dk^0}{2\pi} \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} G_\pm (p - k, m_1, \mathcal{E}^2) G_\pm (k, m_2, \mathcal{E}^2) \]

\[ = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} w(p, k) \]

\( w(p, \mathcal{E}) \) is singular for

\[ |p^0| = \sqrt{(p - k)^2 + m_1^2 \pm i\mathcal{E}^2} + \sqrt{k^2 + m_2^2 \pm i\mathcal{E}^2} \]

Here Lorentz invariance & analyticity are violated.
The integration domain on the loop space momentum must be deformed

\[ \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} w(p, k) \quad \rightarrow \quad \int_{\mathcal{D}_3} \frac{d^3k}{(2\pi)^3} w(p, k) \]

Deformation: \( \mathbb{R} \rightarrow \mathcal{D}_1 \) (\( D = 2 \))
Lorentz invariance and analyticity are recovered in the limit.

Result: average continuation

\[ \frac{1}{2} (\mathcal{M}^+ + \mathcal{M}^-) \]
Result:

\[ \text{Re} \left[ (i-0-) \right] \text{threshold} \]

\[ \text{Im} \left[ (i-0-) \right] = 0 \]

Feynman:

\[ \int_0^1 \! dx \ln \left[ -p^2 x(1-x) + m^2 - i\epsilon \right] \]

Feynman:

\[ \frac{1}{2} \int_0^1 \! dx \ln \left[ \left( -p^2 x(1-x) + m^2 \right)^2 \right] \]

Since the prescription is symmetric w.r.t. the real axis, the imaginary part of the amplitude vanishes:

\[ 0 = 2 \text{Im} \left[ (-i) \begin{array}{c} F \\ F \end{array} \right] = \begin{array}{c} F \\ F \end{array} = \sum_{\Gamma} \left( \int d\Pi_f \right) \begin{array}{c} \Gamma \\ \Gamma \end{array}^2 = 0 \]

\[ \Rightarrow \text{the fakeon } F \text{ MUST be projected away.} \]
From
\[
\frac{1}{2i} \left[ \langle b|T|a \rangle - \langle b|T^\dagger|a \rangle \right] = \sum_{|n\rangle \in W} \langle b|T^\dagger|n \rangle \langle n|T|a \rangle, \quad |a\rangle, |b\rangle \in W
\]

to
\[
\frac{1}{2i} \left[ \langle b|T|a \rangle - \langle b|T^\dagger|a \rangle \right] = \sum_{|n\rangle \in V} \langle b|T^\dagger|n \rangle \langle n|T|a \rangle, \quad |a\rangle, |b\rangle \in V
\]

with \( V \subset W \), \(|f\rangle \in W\), \(|f\rangle \notin V\)

To all orders:
D. Anselmi (2018), Fakeons & Lee-Wick models, JHEP
Quantum gravity

Consider the (renormalizable) higher-derivative action \( S_{QG} \) \( \forall \lambda, \xi > 0 \)

\[
S_{QG} = -\frac{1}{2\kappa^2} \int \sqrt{-g} \left[ 2\Lambda_G + \xi R + \alpha \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) - \frac{\xi}{6} R^2 \right] + S_m(g, \Phi)
\]

Eliminate the higher derivatives by means of extra fields:

\[
S_{QG}(g, \phi, \chi, \Phi) = S_H(g) + S_{\chi}(g, \chi) + S_\phi(\tilde{g}, \phi) + S_m(\tilde{g} e^{\kappa \phi}, \Phi)
\]

Where \( \tilde{g}_{\mu\nu} = g_{\mu\nu} + 2\chi_{\mu\nu} \)

\[
S_H(g) = -\frac{\xi}{2\kappa^2} \int \sqrt{-g} R, \quad S_\phi(g, \phi) = \frac{3\xi}{4} \int \sqrt{-g} \left[ \nabla_\mu \phi \nabla^\mu \phi - \frac{m_\phi^2}{\kappa^2} (1 - e^{\kappa \phi})^2 \right]
\]

\[
S_{\chi}(g, \chi) = S_H(\tilde{g}) - S_H(g) - 2 \int \chi_{\mu\nu} \frac{\delta S_H(\tilde{g})}{\delta g_{\mu\nu}} + \frac{\xi^2}{2\alpha\kappa^2} \int \sqrt{-g} (\chi_{\mu\nu} \chi^{\mu\nu} - \chi^2) \bigg|_{g \to \tilde{g}}
\]


Now, the $X_{\mu\nu}$ action has the wrong overall sign:

\[ S_X(g, \chi) = -\frac{\zeta}{\kappa^2} S_{PF}(g, \chi, m_{\chi}^2) - \frac{\zeta}{2\kappa^2} \int \sqrt{-g} R^{\mu\nu}(\chi\chi_{\mu\nu} - 2\chi_{\mu\rho}\chi_{\nu}^\rho) + S_{X}^{(>2)}(g, \chi) \]

where $S_{PF}$ is the covariantized Pauli-Fierz action.

This means that $X_{\mu\nu}$ must be quantized as a fakeon. This way, we have both renormalizability and unitarity.

Instead, $\phi$ can be quantized either as a fakeon or as a physical particle.
Graviton multiplet: $\{ h_{\mu \nu}, \phi, X_{\mu \nu} \}$

$g_{\mu \nu} = \eta_{\mu \nu} + 2k h_{\mu \nu}$

Spin-2 fakeon of mass $m_x$

Massive scalar

Fakeon width:

$\Gamma_x = -\alpha_x C m_x$

$\Gamma_x < 0$: causality is violated by $X_{\mu \nu}$

$C = \frac{N_S + 6 N_f + 12 N_V}{120}$

$\alpha_x = \left( \frac{m_x}{M_{\text{pl}}} \right)^2$

---

The fields of the standard model and quantum gravity could explain everything we know.
At the level of generating functionals: \( \Gamma(\varphi, \chi) \)

\( \varphi = \text{physical fields} \)

\( \chi = \text{fakeons} \)

Solve \( \frac{\delta \Gamma(\varphi, \chi)}{\delta \chi} = 0 \) by means of the fakeon prescription.

Let \( \langle \chi \rangle \) denote the solution.

Projected functionals:

\( \Gamma_{\text{pr}}(\varphi) = \Gamma(\varphi, \langle \chi \rangle) \)

\( Z_{\text{pr}}(J) = \int [d\varphi d\chi] \exp \left( iS(\varphi, \chi) + i \int J \varphi \right) = \exp (iW_{\text{pr}}(J)) \)

NO SOURCE \( J_\chi \) FOR \( \chi \)

Projection = integrating out the fakeons with the fakeon prescription.
Classical limit: Set of tree diagrams with NO fakeon external legs

The starting action

\[ S_{\text{QG}}(g, \phi, \chi, \Phi) = S_H(g) + S_X(g, \chi) + S_\phi(\tilde{g}, \phi) + S_m(\tilde{g}e^{\kappa \phi}, \Phi) \]

is NOT the classical limit, because it is unprojected

INTERIM classical action: LOCAL

E.g.: \[ L_{gf} = -\frac{1}{4} F_{\mu \nu}^2 + \frac{1}{2\lambda} (\partial A)^2 + \mathcal{E} \mathcal{E} \]

\[ \text{gauge-fixed Lagrangian: unprojected, but LOCAL} \]
Unprojected field equations:

\[ g_{\mu \nu} : \quad R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R = \frac{\kappa^2}{\zeta} \left[ e^{3\kappa \phi} f T^\mu_\mu (\tilde{g} e^{\kappa \phi}, \Phi) + f T^\mu_\phi (\tilde{g}, \phi) + T^\mu_\chi (g, \chi) \right]. \]

\[ \phi : \quad -\frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left( \sqrt{-\tilde{g}} \tilde{g}^{\mu \nu} \partial_\nu \phi \right) - \frac{m^2}{\kappa} (e^{\kappa \phi} - 1) e^{\kappa \phi} = \frac{\kappa e^{3\kappa \phi}}{3\zeta} T^\mu_\mu (\tilde{g} e^{\kappa \phi}, \Phi) \tilde{g}_{\mu \nu}, \]

\[ \chi_{\mu \nu} : \quad \frac{1}{\sqrt{-g}} \frac{\delta S_\chi (g, \chi)}{\delta \chi_{\mu \nu}} = e^{3\kappa \phi} f T^\mu_\mu (\tilde{g} e^{\kappa \phi}, \Phi) + f T^\mu_\phi (\tilde{g}, \phi), \]

Projection: solve the \( \chi \) field equation (with the fakon prescription) and insert the solution into the other equations.
At the tree level, the subtleties about integration paths and average continuations are not important, so we can take

$$\frac{p^2 - m^2}{(p^2 - m^2)^2 + \mathcal{E}^4} = \mathcal{P} \frac{1}{p^2 - m^2}$$

Use it to solve the faddeev equations.

The projected classical action is then

$$S_{\text{QG}}(g, \phi, \Phi) = S_H(g) + S_X(g, \langle \chi \rangle) + S_\phi(\bar{g}, \phi) + S_m(\bar{g}e^{\kappa\phi}, \Phi)$$

where $\langle \chi \rangle$ is the solution

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + 2\langle \chi_{\mu\nu} \rangle$$
Example:

\[ \mathcal{L}_{HD} = \frac{m}{2}(\dot{x}^2 - \tau^2 \ddot{x}^2) + x F_{\text{ext}}(t) \]

\[ m \frac{d^2 x}{dt^2} \left( 1 + \tau^2 \frac{d^2}{dt^2} \right) x = F_{\text{ext}} \]

The projected equation is

\[ m \ddot{x} = \int_{-\infty}^{\infty} du \frac{\sin(|u|/\tau)}{2\tau} F_{\text{ext}}(t - u) \]

\[ \tau \leq \frac{1}{m \chi} \]

\[ \Rightarrow \text{violation of microcausality} \]

\[ F \neq ma \]

\[ \langle F \rangle = ma \]

\[ |u| \sim \tau \]
The FLRW metric

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) d\sigma^2 \]

\[ d\sigma^2 = \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

Unprojected equations \(( \frac{L}{\sqrt{-g}} \sim R + R^2 )\)

\[ \Sigma \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = \frac{4\pi G}{3} (\rho - 3p), \quad \Upsilon \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \right) = -4\pi G (\rho + p), \]

Where

\[ \Sigma = 1 + \frac{1}{m^2_\phi} \left( 3 \frac{\dot{a}}{a} + \frac{d}{dt} \right) \frac{d}{dt}, \quad \Upsilon = \Sigma + \frac{2}{m^2_\phi} \left[ \frac{k}{a^2} + 3 \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) \right]. \]
Projection. Define

\[ \langle A \rangle_X = \frac{1}{2} \left[ \frac{1}{X} \right]_{\text{rit}} + \frac{1}{X} \right]_{\text{adv}} A \]

and use it to define \( \frac{1}{\Sigma} \) and \( \frac{1}{\gamma} \)

(Partially) projected equations:

\[ \begin{align*}
\ddot{a} + \frac{\ddot{a}^2}{a^2} + \frac{k}{a^2} &= \frac{4\pi G}{3} \langle \rho - 3p \rangle_\Sigma \\
\ddot{a} - \frac{\ddot{a}^2}{a^2} - \frac{k}{a^2} &= -4\pi G \langle \rho + p \rangle_\gamma
\end{align*} \]

(it is NOT over yet....!!)
\[ \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3} \tilde{\rho}, \]

\[ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G \tilde{p}. \]

where

\[ \tilde{\rho} = \frac{1}{4} \langle \rho - 3p \rangle_\Sigma + \frac{3}{4} \langle \rho + p \rangle_\gamma \]

\[ \tilde{p} = \frac{1}{4} \langle \rho + p \rangle_\gamma - \frac{1}{4} \langle \rho - 3p \rangle_\Sigma \]
The projection can be handled exactly for radiation combined with the vacuum energy:

\[ p = \frac{\rho}{3} + p_0 \quad p_0 = \text{constant} \]

**Exact solution:**

\[ \rho_0 = \frac{3\sigma^2}{8\pi G} \quad \tilde{p} = \frac{(\tilde{\rho} - 4\rho_0)}{3} \]

\( \sigma, \sigma' = \text{constants} \)

\[ \rho(t) = \frac{3}{8\pi G} \left( \sigma^2 + \frac{\sigma''^2}{4a^4} \right), \quad \sigma'' = \sigma'^2 \left( 1 + \frac{4\sigma^2}{m^2_\phi} \right) \quad \tilde{\rho}(t) = \frac{3}{8\pi G} \left( \sigma^2 + \frac{\sigma'^2}{4a^4} \right) \]

\[ a(t) = \sqrt{\frac{\sinh(\sigma t)}{\sigma}} \left( \sigma' \cosh(\sigma t) - \frac{k}{\sigma} \sinh(\sigma t) \right) \]
But in general the projection is defined perturbatively (since it comes from quantum gravity, which is defined perturbatively)

$\rightarrow$ The classical equations are defined perturbatively: one may have to face asymptotic series and non-p perturbative effects (just to write the equations).

Example: cosmic dust ($p = 0$) or $p = m\rho$

$\omega = \frac{4}{3}, -1$
Example:

\[ L_{HD} = \frac{m}{2}(\dot{x}^2 - \tau^2 \ddot{x}^2) - V \quad V = \frac{m}{2} \omega^2 x^2 + \frac{\lambda}{4!} x^4 \]

Unprojected equations of motion:

\[ m \ddot{\tilde{K}} \left( \frac{d^2}{dt^2} + \Omega^2 \right) x = -\frac{\lambda x^3}{3!} \]

where

\[ \Omega = \frac{1}{\tau \sqrt{2}} \sqrt{1 - \sqrt{1 - 4\tau^2 \omega^2}}, \quad \tilde{\Omega} = \frac{1}{\tau \sqrt{2}} \sqrt{1 + \sqrt{1 - 4\tau^2 \omega^2}}, \quad \tilde{K} = \tau^2 \tilde{\Omega}^2 + \tau^2 \frac{d^2}{dt^2}. \]

Projected equations of motion:

\[ m \left( \frac{d^2}{dt^2} + \Omega^2 \right) x = -\frac{\lambda}{3!} \langle x^3 \rangle \tilde{K} \]

\[ \langle A \rangle x \equiv \frac{1}{2} \left[ \left| \frac{1}{X} \right|_{\text{rit}} + \left| \frac{1}{X} \right|_{\text{adv}} \right] A \]
By iterating the projection, we get
\[ \omega = 0 \quad \tilde{\lambda} = \frac{\lambda}{m} \]
\[
\ddot{x} = -\frac{\tilde{\lambda} x}{6} (x^2 - 6\tau^2 \dot{x}^2) - \frac{\tilde{\lambda}^2 \tau^2 x}{12} (x^4 - 48\tau^2 x^2 \dot{x}^2 + 372\tau^4 \dot{x}^4)
- \frac{\tilde{\lambda}^3 \tau^4 x}{6} (x^6 - 156\tau^2 x^4 \dot{x}^2 + 4572\tau^4 x^2 \dot{x}^4 - 31152\tau^6 \dot{x}^6) + \mathcal{O}(\tilde{\lambda}^4)
\]
\[
\mathcal{L} = \frac{\dot{x}^2}{2} - \frac{\tilde{\lambda} x^2}{4!} (x^2 + 12\tau^2 \dot{x}^2) + \frac{\tau^2 \tilde{\lambda}^2 x^2}{72} (x^4 - 54\tau^2 x^2 \dot{x}^2 + 372\tau^4 \dot{x}^4) + \mathcal{O}(\tilde{\lambda}^3)
\]

Growth of the coefficients:

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<th>5</th>
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<th>15</th>
<th>20</th>
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<td>10^{28}</td>
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Solution with $x(0) = 1$, $\dot{x}(0) = 0$, $m = \tau = 1$, $\lambda = \frac{1}{10}$

$n = 1$ : fairly good

$n = 2$ : good

$n = 3$ to $n = 9$ : excellent

$n > 9$ : meaningless
The fakeon prescription can be applied universally.

We should talk about "quantum field theory of particles and fakeons".

Is H a fakeon? Maybe...

\[ m_H + m_z = 216 \text{ GeV} \]

Every diagram, when multiplied by the appropriate source functions and integrated over all $x$ contributes to the S-matrix. The contribution to the T-matrix, defined by

$$S = 1 + iT$$  \hspace{1cm} (6.7)

is obtained by multiplying by a factor $-i$. Unitarity of the S-matrix implies an equation for the imaginary part of the so defined T matrix

$$T - T^\dagger = iT^\dagger T.$$  \hspace{1cm} (6.8)

The T-matrix, or rather the diagrams, are also constrained by the requirement of causality. As yet nobody has found a definition of causality that corresponds directly to the intuitive notions; instead formulations have been proposed involving the off-mass-shell Green's functions. We will employ the causality requirement in the form proposed by Bogoliubov that has at least some intuitive appeal and is most suitable in connection with a diagrammatic analysis. Roughly speaking Bogoliubov's condition can be put as follows: if a space-time point $x_1$ is in the future with respect to some other space-time point $x_2$, then the diagrams involving $x_1$ and $x_2$ can be rewritten in terms of functions that involve positive energy flow from $x_2$ to $x_1$ only.

The trouble with this definition is that space-time points cannot be accurately pinpointed with relativistic wave packets corresponding to particles on mass-shell. Therefore this definition cannot be formulated as an S-matrix constraint. It can only be used for the Green's functions.

Other definitions refer to the properties of the fields. In particular there is the proposal of Lehmann, Symanzik and Zimmermann that the fields commute outside the light cone. Defining fields in terms of diagrams, this definition can be shown to reduce to Bogoliubov's definition. The formulation of Bogoliubov causality in terms of cutting rules for diagrams will be given in Section 6.4.
Comments on alternative approaches to the problem of quantum gravity

--- **string theory** is criticized for being nonpredictive. Moreover, its calculations often require mathematics that is not completely understood.

--- **loop quantum gravity** is even more challenging, because it is at an earlier stage of development.

--- **holography** (AdS/CFT correspondence) do not admit a weakly coupled expansion.

--- **asymptotic safety** in nonperturbative as well.

None is as close to the standard model as the solution based on the fakeon idea, which is a quantum field theory, admits a perturbative expansion in terms of Feynman diagrams and allows us to make calculations with a comparable effort. It can be coupled to the standard model straightforwardly.

Our solution bests its competitors in calculatibility, predictivity and falsifiability. It is also rather rigid, because it contains only two new parameters. It could turn out to be the most predictive theory ever.
QG is the "last mile" of the path to the infinitesimally small, right after the SM.