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Fakeons, Unitarity, Massive Gravitons And The Cosmological Constant

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Abstract

We give a simple proof of perturbative unitarity in gauge theories and quantum gravity using a special gauge that allows us to separate the physical poles of the free propagators, which are quantized by means of the Feynman prescription, from the poles that belong to the gauge-trivial sector, which are quantized by means of the fakeon prescription. The proof applies to renormalizable theories, including the ultraviolet complete theory of quantum gravity with fakeons formulated recently, as well as low-energy (nonrenormalizable) theories. We clarify a number of subtleties related to the study of scattering processes in the presence of a cosmological constant Λ. The scattering amplitudes, defined by expanding the metric around flat space, obey the optical theorem up to corrections due to Λ, which are negligible for all practical purposes. Problems of interpretation would arise if such corrections became important. In passing, we obtain local, unitary (and "almost" renormalizable) theories of massive gravitons and gauge fields, which violate gauge invariance and general covariance explicitly.

1 Introduction

Unitarity, i.e. the statement that the S matrix satisfies $S^{\dagger}S = 1$, is a key principle of perturbative quantum field theory, together with locality and renormalizability. It can be proved diagrammatically by means of the so-called cutting equations [1, 2], which are sums of diagrams made of two parts, one associated with S and the other associated with $S^{\dagger}$. The proof is relatively straightforward in theories of scalar fields and fermions. Gauge theories are more demanding, since they require to show that the Faddeev-Popov ghosts and the longitudinal and temporal components of the gauge fields mutually compensate and can be projected away. A direct analysis of this compensation dates back to the early '70s and is due to 't Hooft [3].

In this paper we study these issues by means of more modern techniques. The first goal is to simplify and generalize the proof of perturbative unitarity by using the concept of fake particle, or fakeon [4, 5]. The fakeon is a degree of freedom that can only be virtual and must be consistently projected away from the physical spectrum to have unitarity. The consistency of the fakeon projection does not follow from a gauge principle, but from a new quantization prescription. Under certain assumptions, fakeons can make sense of higher-derivative theories. They provide a better understanding of the Lee-Wick models [6] and actually lead to the completion of their formulation [7], which had ambiguities [8] and issues related to Lorentz invariance [9]. Fakeons can also be applied to non-higher-derivative theories and allow us to formulate a consistent theory of quantum gravity [4, 10, 11].

The fakeons are introduced by quantizing some poles of the free propagators in momentum space by means of the fakeon prescription, which works as follows:

(*i*) at the tree level, the free fakeon propagator coincides with the Cauchy principal value of the unprescribed propagator;

(*ii*) inside the Feynman diagrams, the thresholds (and the cuts associated with them) coincide with those determined by the Feynman prescription (or by Wick rotating the Euclidean diagram), but they are bypassed in different ways:

(*ii-a*) the thresholds associated with the processes that involve at least one fakeon (which we call fake thresholds) are circumvented by means of the average continuation [7, 5], which is the arithmetic average of the two analytic continuations;

(*ii-b*) instead, the physical thresholds (those that do not involve fakeons) are circumvented analytically, as usual.

The fakeon prescription is consistent with unitarity for every nonzero value (positive, negative or complex) of the residue at the pole, as long as the real part of the squared mass is nonnegative.

A special gauge [12] allows us to separate the poles corresponding to the physical helicities of the graviton and the gauge fields from the poles that belong to the gauge-trivial sector. We quantize the former by means of the Feynman prescription and the latter as fakeons, i.e. by means of the fakeon prescription. A gauge-fixing parameter λ is conveniently kept free. We use the λ dependence inside the loop diagrams to distinguish the physical thresholds, which are overcome analytically, from the fake thresholds, which are overcome by means of the average continuation. So doing, the proof of unitarity in gauge theories and quantum gravity is relatively straightforward, comparable to the one of scalar-fermion models. Our results apply to ordinary renormalizable theories, low-energy (nonrenormalizable) effective theories, as well as the ultraviolet complete theory of quantum gravity formulated in 2017 in ref. [4].

A nontrivial issue is due to the cosmological constant Λ, which cannot be completely turned off in realistic models of quantum gravity. A consistent formulation of the theory of scattering at $\Lambda \neq 0$ is currently unavailable and might even not exist [13]. Yet, we know that, when we make scattering experiments in our laboratories, we do not care whether the universe has a cosmological constant or not. Since the value of Λ is very small in nature, we expect that its effects are negligible for all practical purposes. While it is obvious that Λ can be treated perturbatively at the classical level, it is not equally obvious that we can do so in quantum field theory, where divergences can simplify small quantities and return finite results.

We overcome these obstacles by showing that if we formulate the theory of scattering in the presence of the cosmological constant by expanding around flat space, perturbative unitarity holds up to corrections due to the cosmological constant itself, which are indeed negligible for all practical purposes. In the (unrealistic) situations where such corrections were not negligible, our approach gives well-defined cutting equations, but does not provide a physical interpretation for them in the realm of a theory of scattering.

For various purposes, it is necessary to equip the gauge fields and the graviton with small artificial masses, which we call gauge masses. Gauge invariance, Lorentz invariance and general covariance are violated when the gauge masses are nonzero and recovered when they are sent to zero. An unexpected feature of our approach is that unitarity holds (up to the corrections due to Λ) even when the gauge masses are nonvanishing. Contrary to the common lore that gauge invariance and general covariance cannot be explicitly broken without violating unitarity, the fakeons allow us to achieve precisely that goal. Specifically, we can formulate theories of massive gauge fields and gravitons that are local and unitary (in the sense explained above). In the case of gauge fields, they are also renormalizable. In the case of gravitons, they are "almost" renormalizable. We briefly compare such massive theories with the known approaches to massive gravitons [14, 15, 16].

The paper is organized as follows. In section 2 we prove unitarity in Yang-Mills theories. In section 3 we extend the proof to the low-energy (nonrenormalizable) theory of quantum gravity, at vanishing cosmological constant. In section 4 we formulate the theory of scattering in the presence of a cosmological constant. In section 5 we extend the proof of unitarity to the ultraviolet complete theory of quantum gravity of [4] and its higher-dimensional versions [17]. In section 6 we discuss the properties of the theories of massive gravitons and gauge fields that emerge from our approach. Section 7 contains the conclusions. Whenever it is necessary to specify a regularization technique, we use the dimensional one.

2 Yang-Mills theories

In this section we prove unitarity in Abelian and non-Abelian gauge theories in dimensions $d > 2$, by quantizing the gauge-trivial sector with the fakeon prescription. The generalization of the arguments to the coupling to matter is straightforward (if the theory is manifestly anomaly free, which we assume here), so we focus on pure gauge theories.

We start from four dimensions. Consider the gauge-fixed Lagrangian

$$\mathcal{L}_{\text{gf}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\lambda}\mathcal{G}^a(A)\mathcal{G}^a(A) - \bar{C}^a\mathcal{G}^a(DC) + \frac{m_0^2}{2}A_0^{a2} - \frac{m_\gamma^2}{2}\mathbf{A}^{a2} - \bar{m}^2\bar{C}^aC^a, \quad (2.1)$$

where $A^{a\mu} = (A^{a0}, \mathbf{A}^a)$, λ is a positive gauge-fixing parameter, C^a and $\bar{C}^a$ are the Faddeev-Popov ghosts and antighosts, respectively, D is the covariant derivative and $\mathcal{G}^a(A)$ denotes the gauge-fixing functions, which we assume to be linear in A. Gauge masses m_0, m_γ, $\bar{m}$ are included to regulate the on-shell infrared divergences of the cutting equations. Lorentz invariance and gauge invariance are explicitly broken at nonvanishing gauge masses. They are smoothly recovered in the limit of vanishing gauge masses.

We work in the "special gauge" of ref. [12], which amounts to take[1]

$$\mathcal{G}^a(A) = \lambda\partial_0 A_0 + \nabla \cdot \mathbf{A}. \quad (2.2)$$

[1]Note that the mass terms of (2.1) are slightly different from those of [12]. Indeed, the approach of the present paper is more versatile than the one of [12] and allows us to make important simplifications.

The propagators derived from (2.1) are then

$$\begin{aligned} \left\langle A^0(k)A^0(-k)\right\rangle_0 &= -\left.\frac{i}{\lambda E^2-\mathbf{k}^2-m_0^2}\right|_{\mathrm{f}}, \qquad \left\langle A^i(k)A^0(-k)\right\rangle_0 = 0, \\ \left\langle A^i(k)A^j(-k)\right\rangle_0 &= \frac{i\Pi^{ij}}{E^2-\mathbf{k}^2-m_\gamma^2+i\epsilon}+\left.\frac{i\lambda}{\lambda E^2-\mathbf{k}^2-\lambda m_\gamma^2}\right|_{\mathrm{f}}\frac{k^ik^j}{\mathbf{k}^2}, \\ \left\langle C(k)\bar{C}(-k)\right\rangle_0 &= \left.\frac{i}{\lambda E^2-\mathbf{k}^2-\bar{m}^2}\right|_{\mathrm{f}}, \end{aligned} \tag{2.3}$$

where $k^\mu = (E, \mathbf{k})$ and

$$\Pi^{ij} = \delta^{ij} - \frac{k^ik^j}{\mathbf{k}^2} \tag{2.4}$$

is the transverse projector.

We have already inserted the quantization prescriptions we need. Specifically, we quantize the physical poles (which are given by the transverse components of A^i) by means of the Feynman $i\epsilon$ prescription and all the unphysical poles as fakeons, i.e. by means of the fakeon prescription. The latter is denoted by the subscript "f" and, as recalled in the introduction, amounts to circumvent the fake thresholds (those that involve at least one fakeon) inside the loop diagrams by means of the average continuation. The thresholds (and the cuts associated with them) coincide with those determined by the Feynman prescription or by Wick rotating the Euclidean version of the diagram. At the tree level, the free fakeon propagator coincides with the principal value of the unprescribed propagator.

A virtue of the special gauge is that the propagators have only simple poles for arbitrary λ. Instead, the usual Lorenz gauge-fixing function $\mathcal{G}(A) = \partial^\mu A_\mu$ leads to double poles whenever $\lambda \neq 1$.

A caveat concerns the situations where two or more thresholds coincide, which must be treated as limits of distinct thresholds [5]. For example, the square of the principal value distribution is ill defined. But if we split the singularities and make them coincide at the end, we obtain (minus) the derivative of the principal value, which is well defined. See details in ref. [18]. This is the right method to evaluate the loop integrals.

We proceed as follows. First, we deform the masses inside the loop diagrams (independently for every propagator), to eliminate the coinciding thresholds. In what follows, the parameter that measures this deformation will be called η. Second, we complexify the external momenta p and compute the integrals in the Euclidean region, where the prescriptions are immaterial (since no thresholds appear) and analyticity holds. Then we move towards the subspace of real external momenta. When we do so, we find physical and fake thresholds. The propagators (2.3) ensure that the physical thresholds are λ independent, while the fake ones do depend on λ. This allows us to keep them distinct and overcome them in different ways. Specifically, the physical thresholds are circumvented by means of the Feynman prescription, that is to say analytically. Instead, the fake thresholds are circumvented by means of the average continuation, that is to say by taking the arithmetic average of the two analytic continuations. The average continuation is safe at $\eta \neq 0$, since there are no coinciding thresholds by construction. At the end, we remove the η deformation by taking the limit $\eta \to 0$.

Renormalizability

We recall that it is sufficient to prove the renormalizability of the theory in the Euclidean framework [4, 5], because the average continuation of convergent functions is obviously convergent. In other words, the prescriptions do not affect the divergent parts of Feynman diagrams.

The diagrams G that contain coinciding thresholds are deformed as explained above into diagrams $G_{\text{split}}(\eta)$. The counterterms are modified consistently. Once we subtract the subdivergences and the overall divergence, we obtain a function

$$G_{\text{split}}(\eta) - \sum_i G_{\text{split}}^{(i)\text{sub}}(\eta) - G_{\text{split}}^{\text{ovrll}}(\eta) \tag{2.5}$$

that is convergent in the Euclidean region. Then we move from the Euclidean region to any other region, by taking the average continuation where necessary. After that, we take the limit $\eta \to 0$. Clearly, the result of these operations is convergent. It is easy to prove by using standard tricks in the Euclidean region that every deformed counterterm is polynomial in η (which can be treated as a mass here) and tends to the right counterterm for $\eta \to 0$.

A possible source of worry comes from the denominators $\mathbf{k}^2$. In principle, they could lead to violations of the locality of counterterms (see [12] for details). In fact, they do not, because they cancel out when the prescriptions are neglected. Indeed,

$$\left\langle A^i(k)A^j(-k)\right\rangle_0 \to \frac{i\delta^{ij}}{E^2 - \mathbf{k}^2 - m_\gamma^2} + \frac{i(1-\lambda)k^ik^j}{(E^2 - \mathbf{k}^2 - m_\gamma^2)(\lambda E^2 - \mathbf{k}^2 - \lambda m_\gamma^2)}.$$

To get rid of the denominators $\mathbf{k}^2$ and the coinciding thresholds at the same time, we can make the same η deformation in both terms of each propagator $\langle A^i(k)A^j(-k)\rangle_0$. In the end, the quantization (2.3) ensures that the ultraviolet divergences are local and the counterterms obey the usual rules of power counting, so the proof of renormalizability reveals no surprises. Moreover, the counterterms are polynomial in the masses (and η).

At $m_0 = m_\gamma = \bar{m} = 0$, we have a renormalization constant Z_g for the gauge coupling g and wave-function renormalization constants Z_0, Z_γ and $\bar{Z}$ for A_0, $\mathbf{A}$ and $\bar{C}$-C, respectively. At nonvanishing gauge masses, the renormalized Lagrangian coincides with the one at $m_0 = m_\gamma = \bar{m} = 0$ plus the counterterms

$$\Delta\mathcal{L}_{m_\gamma} = \frac{\Delta m_0^2}{2}A_0^2 - \frac{\Delta m_\gamma^2}{2}\mathbf{A}^2 - \Delta\bar{m}^2\bar{C}C,$$

where Δm_0^2, Δm_γ^2 and $\Delta\bar{m}^2$ are divergent constants.

In the limit $\lambda \rightarrow 1$ we can choose Lorentz invariant mass terms ($m_0 = m_\gamma$). In that case, the action (2.1) is Lorentz invariant, as well as its renormalization, so $Z_0 = Z_\gamma$ and $\Delta m_0^2 = \Delta m_\gamma^2$. However, the finite parts of the amplitudes are not exactly Lorentz invariant, because different quantization prescriptions are used for the physical and unphysical poles of the propagators (which are distinguished from one another in a non-Lorentz invariant way). The Lorentz violations appear starting from the imaginary parts of the one-loop diagrams, above the fake thresholds. Lorentz symmetry is recovered in the limit of vanishing gauge masses (see below).

Unitarity

The theory is perturbatively unitary, even at nonvanishing gauge masses, because both the Feynman prescription and the fakeon prescription are manifestly consistent with unitarity [5]. The loop integrals are evaluated at $\eta \neq 0$ as explained above. It is crucial to observe that the cutting equations, which are identities that can be written down for every diagram separately, hold for arbitrary $\eta \neq 0$. Then, they still hold in the limit $\eta \rightarrow 0$, which proves the optical theorem.

Gauge invariance and gauge independence

The next task is to prove that gauge invariance is recovered in the limit of vanishing gauge masses. Gauge invariance is expressed by means of the Slavnov-Taylor-Ward-Takahashi (STWT) identities [19], which establish relations among (off-shell) amplitudes and loop diagrams. Such identities can be collected into the Zinn-Justin equation [20], also-called master equation, which can be written as $(\Gamma, \Gamma) = 0$ (assuming that we use the dimensional regularization), where Γ is the generating functional of the one-particle irreducible diagrams and $(.,.)$ denotes the Batalin-Vilkovisky antiparentheses [21].

In the absence of fakeons, the limit of vanishing gauge masses is smooth off-shell, so we can set them directly to zero in the integrands of the loop diagrams. When fakeons are present we have to be more careful, because we need to work at $\eta \neq 0$ to avoid the coinciding thresholds, which in turn requires nonvanishing gauge masses.

Recall that the STWT identities stem from simple, polynomial relations among the Feynman rules. The famous QED Ward identity, for example, follows from

$$\gamma^\mu k_\mu - [\gamma^\mu (p+k)_\mu - m] + \gamma^\mu p_\mu - m = 0. \tag{2.6}$$

In other words, even the more complicated STWT identity can be phrased as the loop integral of a rational function $r(q)$ that factorizes a polynomial that vanished identically, such as the left-hand side of (2.6), where q denotes all the momenta involved. At $\eta \neq 0$, $m_g \neq 0$ (where m_g denotes the gauge masses), $r(q) = 0$ turns into a corrected algebraic relation of the form

$$r(q, \eta, m_g) = \eta r'(q, \eta, m_g) + m_g^2 r''(q, \eta, m_g), \tag{2.7}$$

where both sides are rational functions, $r(q, 0, 0) = r(q)$ and $r'(q, \eta, m_g)$ and $r''(q, \eta, m_g)$ are regular for $\eta \to 0$, $m_g \to 0$. For instance, in the case of (2.6), if we deform the masses we obtain

$$\gamma^\mu k_\mu - [\gamma^\mu (p+k)_\mu - m_1] + [\gamma^\mu p_\mu - m_2] = m_1 - m_2 \equiv \eta, \tag{2.8}$$

where the left-hand side stands for $r(q, \eta)$ and $r' = 1$, $r'' = 0$.

When we integrate on the loop momenta, both sides of (2.7) have no coinciding thresholds. We start again from the Euclidean region, then move to the other regions by taking the average continuation where necessary and finally take the limit $\eta \to 0$ of coinciding thresholds. The first term on the right hand side of (2.7) disappears in the limit. The second term describes the violation of gauge invariance at nonvanishing gauge masses m_g and disappears in the limit $m_g \to 0$. This proves the STWT identities.

Normally, when we manipulate identities like (2.7), $r(q, \eta, m_g)$ is a sum of terms that end up being part of different diagrams, which are calculated separately. Thus, it is important to overcome the thresholds consistently in all of them. The prescription formulated so far ensures this, by treating all the λ-dependent thresholds by means of the average continuation and all the λ-independent thresholds by means of the analytic continuation.

To show that gauge independence is also recovered in the limit of vanishing gauge masses, we can argue similarly. Indeed, gauge independence also stems from simple polynomial identities obeyed by the Feynman rules.

Lorentz invariance is broken by the quantization prescription (2.3). However, it is recovered in the limit of vanishing gauge masses. Precisely, once gauge invariance and gauge independence are restored, the Lorentz violation is confined to the gauge-trivial sector of the theory, which does not affect the physical quantities.

On the other hand, when the gauge masses are nonvanishing, the physical quantities are not Lorentz invariant. As stressed above, the Lorentz violation can be "minimized" by taking the limit $\lambda \to 1$ (which can be done only at the end of the calculations, since the λ dependence is crucial to distinguish the fake thresholds from the physical ones) and choosing $m_0 = m_\gamma$.

In conclusion, the quantization formulated here is manifestly unitary for arbitrary gauge masses. It is gauge and Lorentz invariant in the limit of vanishing gauge masses.

Observe that the set of physical degrees of freedom is always the same, at vanishing and nonvanishing gauge masses, since the fakeons are always projected away from the physical spectrum. It is evident that the proof of unitarity we have just provided is much more economic than any other proof given so far [3, 12].

Normally, an explicit breaking of gauge invariance, such as the one due to the gauge masses, is expected to break unitarity, by making unphysical degrees of freedom propagate. This does not happen, if we quantize the would-be unphysical degrees of freedom as fakeons. A byproduct of our construction is that we can build manifestly unitary, local, renormalizable theories of massive gauge fields.

Higher dimensions

In higher dimensions the theory (2.1) is nonrenormalizable. The gauge-fixing procedure, the propagators and the quantization prescriptions (2.3) are the same. The only part that changes is the set of counterterms, which are infinitely many.

At nonvanishing gauge masses, Lorentz violating counterterms appear in both the physical and gauge sectors, multiplied by the gauge masses. When we include them, we basically have a theory of scalar fields and space vector fields. The quadratic terms can be resummed into "dressed" propagators

$$\langle A_\mu A_\nu \rangle_{\text{dressed}} = \langle A_\mu A_\nu \rangle_0 + \langle A_\mu A_\rho \rangle_0 V^{\rho\sigma} \langle A_\sigma A_\nu \rangle_0 + \cdots , \qquad (2.9)$$

where $V^{\rho\sigma}$ is local and collects the quadratic terms of higher dimensions turned on by renormalization. What is important is that the dressed propagators still have the properties we need to prove unitarity along the guidelines explained above. In particular, using the arguments of ref. [22] we can remove all the higher time derivatives from the quadratic action (and so $V^{\rho\sigma}$) by means of field redefinitions: this ensures that the resummation (2.9) generates no new poles. The physical poles remain λ independent and the unphysical poles remain λ dependent, so we can distinguish the physical thresholds from the fake ones inside the loop diagrams and treat them accordingly. In the end, the proof of unitarity works as above. In the limit of vanishing gauge masses, the counterterms are gauge and Lorentz invariant in the physical sector and rotationally invariant in the gauge-trivial sector.

By means of the fakeon quantization prescription, it is possible to build local, unitary, strictly renormalizable Yang-Mills theories in arbitrary higher spacetime dimensions $d \geqslant 6$ [17]. Their interim classical actions read

$$S_{\mathrm{YM}}^{d} = -\frac{1}{4}\int \mathrm{d}^d x F_{\mu\nu}^{a} P(D^2) F^{a\mu\nu} + \mathcal{O}(F^3), \tag{2.10}$$

where D is the covariant derivative, $P(x)$ is a real polynomial of degree $(d-4)/2$ in x such that $P(0) > 0$, while $\mathcal{O}(F^3)$ are the Lagrangian terms that have dimensions smaller than or equal to d and are built with at least three field strengths and their covariant derivatives. The quadratic terms can always be reduced to the form (2.10) by means of Bianchi identities and partial integrations. The coefficients of the polynomial P must be such that the poles of $1/P$ are massive and the squared masses have nonnegative real parts.

The special gauge can be built by choosing the gauge-fixed Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{gf}} = & -\frac{1}{4}F_{\mu\nu}P(D^2)F^{\mu\nu} + \mathcal{O}(F^3) - \frac{1}{2\lambda}\mathcal{G}(A)P(\partial^2)\mathcal{G}(A) - \bar{C}P(\partial^2)\mathcal{G}(DC) \\ & + \frac{1}{2}A^{\mu}\left(m_0^2\delta_{\mu 0}\delta_{0\nu} - m_\gamma^2\delta_{\mu i}\delta_{i\nu}\right)P(\partial^2)A^{\nu} - \bar{m}^2\bar{C}P(\partial^2)C.\end{aligned} \tag{2.11}$$

We have chosen convenient "mass terms", to simplify the propagators, which then coincide with the ones of (2.3), multiplied by $1/P(-k^2)$. The quantization prescription follows from the replacements

$$\begin{aligned}\frac{1}{(\lambda E^2 - \mathbf{k}^2 - m^2)P(-k^2)} &\rightarrow \left.\frac{1}{(\lambda E^2 - \mathbf{k}^2 - m^2)P(-k^2)}\right|_{\mathrm{f}}, \\ \frac{1}{(k^2 - m^2)P(-k^2)} &\rightarrow \frac{1}{(k^2 - m^2 + i\epsilon)P(-m^2)} - \left.\frac{P(-k^2) - P(-m^2)}{P(-k^2)(k^2 - m^2)P(-m^2)}\right|_{*},\end{aligned}$$

where m is m_0, m_γ or $\bar{m}$, depending on the case. The star in the second line means that the poles with negative or complex residues, as well as those with positive residues but complex masses, must be quantized as fakeons. Instead, the poles with positive residues and nonvanishing real masses can be quantized either as fakeons or physical particles.

Renormalization generates mass terms of lower dimensionalities, but we do not need to include them at the tree level, since they are going to disappear when we take the gauge masses to zero. The proof of unitarity proceeds as above, as well as the recovery of gauge invariance and Lorentz invariance at vanishing gauge masses.

3 Quantum gravity: low-energy theory

In this section and the next ones we generalize the proof to quantum gravity in arbitrary dimensions $d > 3$. We start from the low-energy nonrenormalizable theory at vanishing cosmological constant. In the next section we formulate the theory of scattering at $\Lambda \neq 0$ and in section 5 we generalize the results to ultraviolet complete theories.

The gauge-fixed Hilbert-Einstein Lagrangian is

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\kappa^{d-2}}\sqrt{|g|}R + \frac{1}{4\lambda_1\kappa^{d-2}}\mathcal{G}_0^2(g) - \frac{1}{4\lambda_2\kappa^{d-2}}\mathcal{G}_i^2(g) + \bar{C}_0\mathcal{G}_0(\overline{DC}) - \bar{C}_i\mathcal{G}_i(\overline{DC}), \quad (3.1)$$

where $\mathcal{G}_0(g)$ and $\mathcal{G}_i(g)$ are the gauge-fixing functions, assumed to be linear in the metric $g_{\mu\nu}$, while C_μ and $\bar{C}_\mu$ are the Faddeev-Popov ghosts and antighosts, respectively, and $\overline{DC}$ stands for $D_\mu C_\nu + D_\nu C_\mu$, D_μ denoting the covariant derivative. The constant κ is chosen to have dimension -1 in units of mass for every d.

The special gauge is obtained by choosing [12]

$$\mathcal{G}_0(g) = \frac{\lambda}{2}\partial_0 g_{00} + \frac{1}{2}\partial_0 g_{ii} - \partial_i g_{0i}, \qquad \mathcal{G}_i(g) = -\lambda_1\partial_j g_{ij} + \frac{1}{2}\left(2\lambda_1 - 1\right)\partial_i g_{jj} + \lambda\partial_0 g_{0i} - \frac{\lambda}{2}\partial_i g_{00}, \quad (3.2)$$

with

$$\lambda_1 = \frac{\lambda(d-3)+d-1}{2(d-2)}, \qquad \lambda_2 = \lambda\lambda_1.$$

We expand around flat space by writing $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa^{(d/2)-1}h_{\mu\nu}$. With the prescriptions

$$\bar{P}_{\text{phys}} = \frac{1}{E^2 - \mathbf{k}^2 + i\epsilon}, \qquad \bar{P}_{\text{f}} = \left.\frac{1}{\lambda E^2 - \mathbf{k}^2}\right|_{\text{f}}, \qquad \bar{P}_{\text{f}}' = \left.\frac{1}{\lambda E^2 - \lambda_1\mathbf{k}^2}\right|_{\text{f}},$$

we find the ghost propagators

$$\langle C^0\bar{C}^0\rangle_0 = -i\bar{P}_{\text{f}}, \qquad \langle C^0\bar{C}^i\rangle_0 = \langle C^i\bar{C}^0\rangle_0 = 0, \qquad \langle C^i\bar{C}^j\rangle_0 = i\bar{P}_{\text{f}}'\Pi^{ij} + i\bar{P}_{\text{f}}\frac{k^ik^j}{\mathbf{k}^2}, \quad (3.3)$$

and the $h_{\mu\nu}$ propagators

$$\begin{aligned}
\langle h_{00}h_{00}\rangle_0 &= \frac{d-3}{d-2}i\bar{P}_{\text{f}}, \qquad \langle h_{00}h_{ij}\rangle_0 = \frac{i\delta_{ij}\bar{P}_{\text{f}}}{d-2}, \\
\langle h_{0i}h_{0j}\rangle_0 &= -\frac{i\lambda_1}{2}\left(\bar{P}_{\text{f}}'\Pi_{ij} + \bar{P}_{\text{f}}\frac{k_ik_j}{\mathbf{k}^2}\right), \qquad \langle h_{00}h_{0i}\rangle_0 = \langle h_{0i}h_{jk}\rangle_0 = 0, \qquad (3.4)\\
\langle h_{ij}h_{mn}\rangle_0 &= \frac{i\bar{P}_{\text{phys}}}{2}\left(\Pi_{im}\Pi_{jn} + \Pi_{in}\Pi_{jm} - \frac{2}{d-2}\Pi_{ij}\Pi_{mn}\right) - \frac{\lambda}{\mathbf{k}^2}\frac{i\bar{P}_{\text{f}}}{d-2}\left(\Pi_{ij}k_mk_n + k_ik_j\Pi_{mn}\right) \\
&\quad + \frac{\lambda i\bar{P}_{\text{f}}'}{2\mathbf{k}^2}\left(\Pi_{im}k_jk_n + \Pi_{in}k_jk_m + \Pi_{jm}k_ik_n + \Pi_{jn}k_ik_m\right) + \lambda i\bar{P}_{\text{f}}\frac{d-3}{d-2}\frac{k_ik_jk_mk_n}{(\mathbf{k}^2)^2}.
\end{aligned}$$

As in the case of gauge theories, the denominators proportional to $\mathbf{k}^2$ and $(\mathbf{k}^2)^2$ cancel out, if the quantization prescriptions are ignored. This ensures that the locality of counterterms works as usual, since the ultraviolet divergences do not depend on the prescriptions.

To have control on the on-shell infrared divergences, we add the most general mass terms that are invariant under rotations,

$$\Delta\mathcal{L}_m = -\frac{m_1^2}{4}h_{00}^2 - \frac{m_2^2}{2}h_{ij}^2 + m_3^2 h_{0i}^2 + \frac{m_4^2}{4}h_{ii}h_{jj} - \frac{m_5^2}{2}h_{00}h_{ii} + \frac{\bar{m}_1^2}{2}\bar{C}^0 C^0 - \frac{\bar{m}_2^2}{2}\bar{C}^i C^i. \quad (3.5)$$

The coefficients are labeled so that when all the gauge masses m_a, $a = 1, 2, 3, 4$, are equal to m and the ghost masses $\bar{m}_b$, $b = 1, 2$, are equal to $\bar{m}$, we obtain the Lorentz invariant combination

$$\Delta\mathcal{L}_m = -\frac{m^2}{2}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right) + \frac{\bar{m}^2}{2}\bar{C}^\mu C_\mu. \quad (3.6)$$

As far as the graviton propagator is concerned, the cosmological constant can be seen as a correction to m^2 (see next section).

The propagators for the most general mass terms (3.5) are rather involved. We just report that in both cases (3.5) and (3.6) they have no simple poles. Moreover, the poles have squared masses with positive real parts if

$$m_1^2 > 0, \qquad m_2^2 > 0, \qquad m_3^2 > 0, \qquad (d-1)m_4^2 > 2m_2^2, \qquad m_5^2 > 0$$

(in addition to $\lambda > 0$, $d > 3$). From now on, we assume that such inequalities hold. As in (3.4), the unphysical poles are λ dependent and the physical poles are λ independent.

Without making involved calculations, the λ dependence can be studied as follows. The massive propagators

$$\langle h_{\mu\nu}h_{\rho\sigma}\rangle_m = \langle h_{\mu\nu}h_{\rho\sigma}\rangle_0 + \langle h_{\mu\nu}h_{\alpha\beta}\rangle_0 V_m^{\alpha\beta\gamma\delta}\langle h_{\gamma\delta}h_{\rho\sigma}\rangle_0 + \cdots, \quad (3.7)$$

can be obtained by resumming the corrections due to the two-leg vertices $V_m^{\alpha\beta\gamma\delta}$ provided by $\Delta\mathcal{L}_m$. The projector

$$\frac{1}{2}\left(\Pi_{im}\Pi_{jn} + \Pi_{in}\Pi_{jm} - \frac{2}{d-2}\Pi_{ij}\Pi_{mn}\right),$$

which multiplies the physical pole in (3.4), is orthogonal to every term we may build for $\langle h_{\mu\nu}h_{\rho\sigma}\rangle_m$, apart from the identity $(\delta^{im}\delta^{jn} + \delta^{in}\delta^{jm})/2$. Moreover, it cannot be generated by multiplying terms that do not contain the identity. For this reason, when we perform the resummation (3.7), the λ-dependent poles do not affect the physical pole, and vice versa. Note that new poles may appear in the resummation, because some invariants on the right-hand sides of (3.4) are missing. By the arguments just given, such new poles are necessarily λ dependent and must be quantized as fakeons.

The theory is nonrenormalizable. At vanishing gauge masses, we must include all the local, generally covariant terms that can be built with at least three Weyl tensors and their covariant derivatives [23], multiplied by independent parameters. Then, the divergent parts of the Feynman diagrams are subtracted by means of redefinitions of the parameters and the fields. Note that, by power counting, the cosmological term is not generated, since the theory contains no parameters of positive dimensions in units of mass. When the gauge masses are nonvanishing, extra counterterms proportional to the squared gauge masses must be added. They do not need to be general covariant, but just invariant under space rotations.

In the evaluation of the loop diagrams, the λ dependent thresholds of the fake processes can be distinguished from the thresholds of the physical processes, which are λ independent. As in case of gauge theories, this allows us to circumvent the former by means of the average continuation and the latter by means of the Feynman prescription, thereby proving perturbative unitarity.

General covariance is recovered in the limit of vanishing gauge masses. When $\lambda = 1$ we can choose the $\Delta\mathcal{L}_m$ of formula (3.6) to have a Lorentz invariant renormalization. The finite parts of the amplitudes, however, are not Lorentz and general covariant. They become so only when the gauge masses are sent to zero.

So far, we have set the cosmological constant Λ to zero, which is consistent only in special, unrealistic models. The problem of defining the theory of scattering in the presence of a cosmological constant must be discussed apart.

4 Theory of scattering in the presence of a cosmological constant

In this section we formulate the theory of scattering in the presence of a small, but nonvanishing cosmological constant Λ. By expanding around flat space, we obtain scattering amplitudes that satisfy perturbative unitarity up to corrections due to Λ. Such corrections are negligible for all practical purposes. In the academic case they were non negligible, our analysis provides well-defined cutting equations, which however do not have a clear physical interpretation in the context of a theory of scattering. For definiteness, we assume to work in four dimensions, but the arguments work in arbitrary dimensions $d > 3$.

Let us first address the main aspects of the problem we have to deal with. In some models, the cosmological constant Λ can be turned off consistently, since the Λ beta function vanishes when Λ vanishes. The simplest example is pure gravity, whose Lagrangian is the sum of the Hilbert term, plus the counterterms built with at least three Weyl tensors and their covariant derivatives [23]. There, power counting ensures that the cosmological constant is not turned on by renormalization, because the theory contains no parameters of positive dimensions in units of mass. In the realm of ultraviolet complete theories, Λ can be consistently switched off in super-renormalizable models with more higher derivatives [4]. Both types of models, however, are not realistic, since Λ is turned on by renormalization as soon as massive or self-interacting matter fields are included.

Thus, it is compulsory to study the case $\Lambda \neq 0$ in detail. However, a consistent theory of scattering is available only in flat space and might not even exist at $\Lambda \neq 0$ [13], where we cannot talk about asymptotic states and scattering amplitudes in a strict sense. At the same time, flat space is not a solution of the classical field equations (in the absence of matter) at $\Lambda \neq 0$, and the perturbative expansion around nonflat backgrounds is extremely inconvenient.

When we study scattering experiments for our laboratories, we do not care whether the universe has a cosmological constant or not. We just expand around flat space and move on. Since the value of Λ is very small in nature, we expect that its effects are negligible for all practical purposes. Thus, in the presence of a cosmological constant it should be possible to formulate a theory of scattering that makes physical sense up to the corrections due to Λ. Put it differently, we demand that the theory be "as unitary as it can be" at $\Lambda \neq 0$.

At the classical level, it is obvious that Λ can be treated perturbatively and neglected for most purposes. It is not obvious that we can do the same in quantum field theory. Indeed, often quantities that are classically negligible become important due to quantum effects. For example, the axial anomalies and the renormalization group flow are originated by conflicts between classically negligible quantities and ultraviolet divergences. In the case of the cosmological constant, a possible source of conflict is provided by the infrared divergences.

For these reasons, we need to investigate the matter carefully. We insist on expanding around flat space and our results show that, in the end, this is the right choice.

When we expand around flat space, the cosmological term

$$-\frac{\Lambda}{\kappa^2}\int \mathrm{d}^4x\sqrt{-g}$$

generates

i) tadpole (one-leg) vertices, which allow us to build infinitely many connected diagrams of the same order;

ii) quadratic terms (two-leg vertices), which can be resummed to give the graviton a sort of "mass";

iii) super-renormalizable vertices, which cause the appearance of (off-shell) infrared divergences in loop diagrams (for Λ small).

Specifically, point (*ii*) leads to a graviton propagator that reads, in the De Donder gauge,

$$\langle h_{\mu\nu}(k)h(-k)\rangle_0 = \frac{i}{2}\frac{\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}}{k^2+2\Lambda+i\epsilon}. \tag{4.1}$$

This may look okay when $\Lambda<0$, but is tachyonic when $\Lambda>0$.

Let us see how to overcome the problems just listed one by one. The problems (*i*) appear because the propagator (4.1) is proportional to $1/\Lambda$ in the infrared limit $k\rightarrow 0$ and the tadpole vertices are of order Λ. To better illustrate the issue, consider the Lagrangian

$$\mathcal{L}_\Lambda = -\frac{\Lambda}{\kappa^2}\sqrt{-g}. \tag{4.2}$$

Expanding the metric around flat space, by writing $g_{\mu\nu}=\eta_{\mu\nu}+2\kappa h_{\mu\nu}$, the Legendre transform $\mathcal{F}$ of $\mathcal{L}_\Lambda$ with respect to $h_{\mu\nu}$ gives

$$\mathcal{F}(J)=\mathcal{L}_\Lambda - J^{\mu\nu}h_{\mu\nu}=\frac{1}{\Lambda}\sqrt{-\det J^{\mu\nu}}+\frac{1}{2\kappa}J^{\mu\nu}\eta_{\mu\nu},$$

where

$$J^{\mu\nu}=\frac{\partial\mathcal{L}_\Lambda}{\partial h_{\mu\nu}}=-\frac{\Lambda}{\kappa}\sqrt{-g}g^{\mu\nu}. \tag{4.3}$$

Now, if $\mathcal{L}_\Lambda$ is viewed as a classical Lagrangian, it provides well-defined Feynman rules. Moreover, since the $\mathcal{L}_\Lambda$ quadratic part is non dynamic, all loop diagram vanish (in dimensional regularization), so $\mathcal{L}_\Lambda$ coincides with the generating functional of the one-particle irreducible diagrams. Then, $\mathcal{F}$ should be the generating functional of the corresponding connected diagrams, which would be the derivatives of $\mathcal{F}$ with respect to $J^{\mu\nu}$, evaluated at $J^{\mu\nu}=0$. However, such derivatives are generically singular. The reason is that, by means of one-leg vertices, it is possible to build infinitely many connected tree diagrams of the same order, with the same set of external legs.

The point is that setting $J^{\mu\nu}=0$ is not the right thing to do, since it does not correspond to flat space. Actually, (4.3) shows that the condition $J^{\mu\nu}=0$ implies $\sqrt{-g}g^{\mu\nu}=0$, which leads to singularities. Since we insist on expanding around flat space, we should impose conditions that correspond to flat space both before and after the Legendre transform. The right source is then

$$j^{\mu\nu}=J^{\mu\nu}+\frac{\Lambda}{\kappa}\eta^{\mu\nu} \tag{4.4}$$

and the connected diagrams are the derivatives of $\mathcal{F}$ with respect to $j^{\mu\nu}$, calculated at $j^{\mu\nu}=0$. As a check, it is straightforward to verify that the propagator, obtained by inverting the second derivative of $\mathcal{L}_{\Lambda}$ with respect to $h_{\mu\nu}$, calculated at $h_{\mu\nu}=0$, is equal to the second derivative of $\mathcal{F}$ with respect to $j^{\mu\nu}$, calculated at $j^{\mu\nu}=0$.

In conclusion, the definition (4.4) removes problem (i). In practice, it removes the connected diagrams that contain legs attached to tadpole vertices.

The problem (ii) of the tachyonic propagator (4.1) for $\Lambda>0$ can be overcome by introducing a mass term for the graviton. Let us consider the Lagrangian

$$\mathcal{L}_{\Lambda}^{\prime}=-\frac{\Lambda}{\kappa^{2}}\sqrt{-g}-\frac{1}{2}h_{\mu\nu}(\Box+m_{g}^{2})h^{\mu\nu}+\frac{1}{4}h(\Box+m_{g}^{2})h.$$

and treat Λ perturbatively with respect to m_g^2. The graviton propagator in the De Donder gauge,

$$\langle h_{\mu\nu}(k)h(-k)\rangle_{0}=\frac{i}{2}\frac{\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}}{k^{2}-m_{g}^{2}+2\Lambda+i\epsilon}, \tag{4.5}$$

is not tachyonic as long as $m_g^2>2\Lambda$. If this inequality is satisfied, we face no further obstruction to derive the cutting equations.

One might object that the graviton is massless in nature, so at the end m_g should tend to zero. Hence, it does not seem to make sense to take $m_g^2>2\Lambda$. However, we stress again that our purpose is not to formulate a theory of scattering at nonzero Λ in a strict sense, which is likely impossible. We just want to formulate a theory of scattering at $\Lambda\neq 0$ that is meaningful up to corrections due to Λ itself. If m_g^2 is sufficiently small, larger than 2Λ and such that $|m_g^2-2\Lambda|\sim|\Lambda|$ (for definiteness, it may be useful to assume $m_g^2\sim 3|\Lambda|$), we can keep it nonvanishing as well, since its corrections are not so different from the ones due to Λ, and whenever the latter are negligible, so are the former. In this sense, the solution (4.5) removes problem (ii).

As said, we have well-defined cutting equations whenever $m_g^2>2\Lambda$. However, the tools that allow us to prove unitarity from the cutting equations, which are asymptotic states, scattering amplitudes and reduction formulas, are well-defined only at $\Lambda=0$. Thus, we can at most prove unitarity up to the corrections due to Λ.

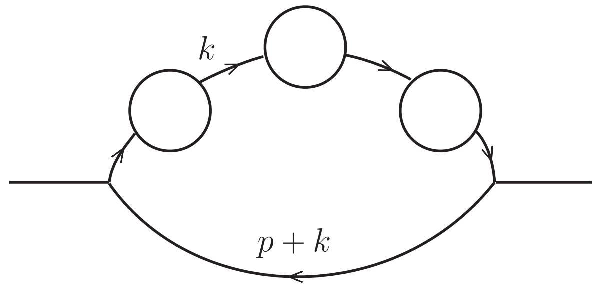

Figure 1: Multibubble diagrams

To ensure that the effects of Λ are indeed small enough to be negligible, we study the behavior of the physical quantities when both m_g^2 and Λ tend to zero at the same time and $|m_g^2 - 2\Lambda| \sim |\Lambda|$. We avoid problems with tachyonic poles by keeping $m_g^2 > 2\Lambda$, or analogous variants of such an inequality [which apply when $\lambda \neq 1$ and (3.5) is used]. We know that in realistic models the limit $|m_g^2 - 2\Lambda| \sim |\Lambda| \to 0$ cannot be pushed to the very end, since it is not consistent to turn Λ identically off. Thus, we stop short of doing that, which is anyway enough to estimate the corrections due to $\Lambda \neq 0$.

The main source of worry is problem (*iii*), since certain off-shell infrared divergences are generated inside the loop diagrams in the limit. Specifically, the divergences appear when propagators with the same loop momentum p are raised to high powers, which happens in multibubble diagrams like the one of fig. 1.

Let

$$B(p^2) \sim \sum_i C_i \int_0^1 \mathrm{d}x \ln\left[m_i^2 - i\epsilon - p^2 x(1-x)\right]$$

denote the bubble diagram with external graviton legs and external momentum p, where C_i are factors due to the vertices. For simplicity, we are assuming that the fields circulating in the loop have the same masses m_i and the vertices contributing to $B(p^2)$ are non derivative, since our estimates will not depend on such assumptions.

If P denotes the graviton propagator, the loop integral of the multibubble diagrams have the following infrared behaviors:

$$\int_{\mathrm{IR}} PB\cdots BP \sim \int_{\mathrm{IR}} \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{B(p^2)^n}{(p^2 - m_g^2 + 2\Lambda + i\epsilon)^{n+1}} \sim \frac{B(0)^n}{(m_g^2 - 2\Lambda - i\epsilon)^{n-1}} \qquad \text{for } n > 1,$$
$$\int_{\mathrm{IR}} PBP \sim B(0)\ln(m_g^2 - 2\Lambda - i\epsilon). \qquad (4.6)$$

Again, we assume that the vertices are non derivative, since extra powers of p carried by them can only improve the infrared behaviors. The $i\epsilon$ prescription is kept to emphasize that we have a well-defined way to cross $m_g^2 = 2\Lambda$, so it does not really matter whether Λ is positive or negative, as long as $|m_g^2 - 2\Lambda| \sim |\Lambda|$.

To estimate the corrections due to m_g and Λ, when they are small, it is convenient to resum the bubble diagram into the corrected propagator

$$\frac{1}{p^2 - M^2(p^2) + i\epsilon}, \tag{4.7}$$

where

$$M^2(p^2) \equiv m_g^2 - 2\Lambda + B(p^2). \tag{4.8}$$

It is easy to show that the M corrections to a loop diagram calculated with the propagators (4.7) are of order $M^2 \ln M^2$ for M small [the logarithm being due to integrals $\sim \mathrm{d}^4 p/(p^2)^2$ originated by the small-M expansion]. Now, in the limit we want to study, the absolute value of M^2 is of order $|\Lambda_R|$, where Λ_R is the running cosmological constant, so in the end the corrections due to Λ are

$$|\Lambda_R| \ln |\Lambda_R|, \tag{4.9}$$

to be divided by an energy squared.

The radiative corrections $\Delta\Lambda$ of $\Lambda_R = \Lambda + \Delta\Lambda$, due to the one-loop diagrams, are

$$\Delta\Lambda \sim \frac{m^4}{M_{\mathrm{Pl}}^2} \ln \frac{E^2}{\mu^2}, \tag{4.10}$$

where m is the mass of the particle circulating in the loop, E is the typical energy of the process of interest and μ is a reference energy. Taking one hundredth of an electronvolt as the mass m_ν of the lightest neutrino and expressing all quantities as energies, we have

$$\begin{aligned} &\sqrt{|\Lambda|} \sim 10^{-42}\mathrm{GeV}, \qquad \frac{m_\nu^2}{M_{\mathrm{Pl}}} \sim 10^{-41}\mathrm{GeV}, \qquad \frac{m_e^2}{M_{\mathrm{Pl}}} \sim 10^{-26}\mathrm{GeV}, \\ &\frac{m_\mu^2}{M_{\mathrm{Pl}}} \sim 10^{-21}\mathrm{GeV}, \qquad \frac{m_Z^2}{M_{\mathrm{Pl}}} \sim \frac{m_t^2}{M_{\mathrm{Pl}}} \sim 10^{-15}\mathrm{GeV}. \end{aligned} \tag{4.11}$$

Taking $\Lambda_R \sim \Delta\Lambda$ [since, by (4.11), Λ is comparable to or smaller than its radiative corrections], the corrections due to the cosmological constant are

$$\frac{|\Delta\Lambda|}{\Delta m^2} \ln |\Delta\Lambda|, \tag{4.12}$$

where Δm is the energy resolution of the instruments employed in the processes we want to study. Typical values for the experimental errors of the particle masses are

$$\begin{aligned} &\Delta m_e \sim 10^{-13}\mathrm{GeV}, \qquad \Delta m_\mu \sim 10^{-9}\mathrm{GeV}, \qquad \Delta m_Z \sim 10^{-3}\mathrm{GeV}, \\ &\Delta m_W \sim 10^{-2}\mathrm{GeV}, \qquad \Delta m_t \sim 1\mathrm{GeV}. \end{aligned} \tag{4.13}$$

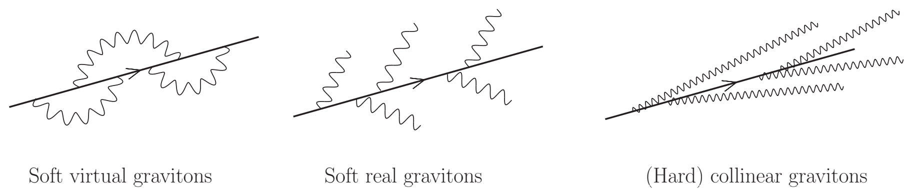

Figure 2: Soft and collinear gravitons. Both can be virtual or real. The collinear ones can be both soft and hard.

The radiative corrections due to a particle of mass m are negligible at energies $E \ll m$, where the particle is effectively integrated out. Formula (4.13) shows that the best energy resolution is achieved in the measurement of the electron mass, which however does not involve high energies. This means that, say, m_t^2/M_{Pl} is not relevant to Δm_e.

Even if we take the ratio between the size of the universe and the Planck distance, the logarithms of (4.9), (4.10) and (4.12) provide a couple of orders of magnitude at most. In the end, formula (4.12) gives

$$\frac{|\Delta\Lambda|}{\Delta m^2}\ln|\Delta\Lambda| \sim 10^{-20} \tag{4.14}$$

or less, in all the relevant cases. This makes the corrections due to the cosmological constant negligible for all practical purposes, as promised.

To complete our analysis, we also consider the on-shell infrared divergences (see fig. 2), which are logarithmic [i.e. $\sim \ln(m_g^2 - 2\Lambda)$] and of two types: soft or collinear. The soft infrared divergences are due to gravitons of small momenta, which can be real or virtual. The collinear divergences are due to the emissions of gravitons at small angles with respect to the incoming or outgoing particles.

Detectors have a finite resolution ΔE, which means that they cannot distinguish a particle from a "jet" made of the particle plus gravitons of momenta smaller than ΔE. From the detector's viewpoint, all such states are equivalent, which makes it necessary to sum over them. It can be proved [24, 25] that the result of the resummation over the real soft gravitons with momenta smaller than ΔE cancels the infrared divergences due to the resummation of the virtual soft gravitons. The net result is infrared regular and depends on the resolution ΔE. This means that the logarithms $\ln(m_g^2 - 2\Lambda)$ cancel out, so there is no problem to take the limit of vanishing gauge masses after the cancellation.

The collinear infrared divergences have similar, but also different properties. If the angular resolution $\Delta\theta$ is finite, the divergences due to the virtual and the real collinear gravitons mutually cancel out and the final result depends on $\Delta\theta$. If the collinear divergences (such as those associated with incoming particles) do not cancel out, they can be removed by studying Altarelli-Parisi evolution equations. So doing, they are practically buried under the initial conditions of such equations and eliminated by means of reference measurements. All the other measurements are then predictive and free of collinear logarithms $\ln(m_g^2 - 2\Lambda)$. There, the limit of vanishing gauge masses can be safely taken.

As said, we can keep the gauge masses nonvanishing throughout the calculation, as long as $|m_g^2 - 2\Lambda| \sim |\Lambda|$. Alternatively, we can take the limit $m_g \to 0$ at the end. Note that, although the logarithm $\ln(m_g^2 - 2\Lambda - i\epsilon)$ generates imaginary parts for $m_g^2 < 2\Lambda$, the optical theorem, expressed by the cutting equations, involves also the opposite prescription, coming from the shadowed portions of the cut diagrams, and the final result is real.

It is worth to stress that the diagrammatic cutting equations are satisfied at $\Lambda < 0$ as well as $\Lambda > 0$, although they imply unitarity only up to the corrections due to Λ. We might want to know whether they imply unitarity even if we included such corrections. This issue is a bit academic, since at the practical level it may require detectors that can resolve wavelengths comparable to the size of the universe.

The situation is as follows. The procedure outlined so far leads to well-defined cutting equations and tentative "mathematical cross sections". If we assume that hypothetical experiments are sensitive to the Λ corrections, they also give well-defined results: the numbers of events counted by their detectors. The point is that we can no longer state that the two numbers – the mathematical prediction and the result of the experiment – must coincide, because the theory that relates them (made of asymptotic states, reduction formulas and scattering amplitudes) is available only at $\Lambda = 0$. A problem of interpretation survives in this case.

In conclusion, we have built a theory of scattering that is unitary up to the corrections due to Λ and its running, which are negligible for all practical purposes. Luckily, it is correct to expand the metric tensor around flat spacetime, which makes the calculations doable.

5 Quantum gravity: ultraviolet complete theory

In this section we generalize the proof of unitarity up to corrections due to the cosmological constant to the ultraviolet complete theory of quantum gravity formulated in 2017 in ref. [4], as well as its higher-dimensional variants (listed in ref. [17]). We show that the problem can be reduced to the one of the previous two sections, by separating the Hilbert-Einstein sector from the higher-derivative sector.

A unique unitary and strictly renormalizable theory exists in four dimensions. Its classical action, coupled to matter, can be written in two basic ways. If we use higher derivatives, it reads

$$S_{\rm QG}(g,\Phi)=-\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[2\Lambda+\zeta R+\alpha\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2\right)-\frac{\xi}{6}R^2\right]+S_{\mathfrak{m}}(g,\Phi), \quad (5.1)$$

where α, ξ, ζ and κ are real positive constants, the reduced Planck mass is $\bar{M}_{\rm Pl}=M_{\rm Pl}/\sqrt{8\pi}=\sqrt{\zeta}/\kappa$, $S_{\mathfrak{m}}$ is the covariantized action of the standard model (or an extension of it), equipped with the nonminimal couplings required by renormalization, and Φ are the matter fields.

For various purposes, it is convenient to eliminate the higher derivatives by adding extra fields. Then the action reads [11]

$$S'_{\rm QG}(g,\phi,\chi,\Phi)=\tilde{S}_{\rm HE}(g)+S_\chi(g,\chi)+S_\phi(\tilde{g},\phi)+S_{\mathfrak{m}}(\tilde{g}\mathrm{e}^{\kappa\phi},\Phi), \quad (5.2)$$

where

$$\begin{aligned}\tilde{S}_{\rm HE}(g)&=-\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left(2\tilde{\Lambda}+\tilde{\zeta}R\right),\\ S_\phi(g,\phi)&=\frac{3\hat{\zeta}}{4}\int \mathrm{d}^4x\sqrt{-g}\left[\nabla_\mu\phi\nabla^\mu\phi-\frac{m_\phi^2}{\kappa^2}\left(1-\mathrm{e}^{\kappa\phi}\right)^2\right], \quad (5.3)\\ S_\chi(g,\chi)&=\tilde{S}_{\rm HE}(\tilde{g})-\tilde{S}_{\rm HE}(g)+\int \mathrm{d}^4x\left[-2\chi_{\mu\nu}\frac{\delta\tilde{S}_{\rm HE}(g)}{\delta g_{\mu\nu}}+\frac{\tilde{\zeta}^2}{2\alpha\kappa^2}\sqrt{-g}(\chi_{\mu\nu}\chi^{\mu\nu}-\chi^2)\right]_{g\rightarrow\tilde{g}},\end{aligned}$$

and

$$\tilde{g}_{\mu\nu}=g_{\mu\nu}+2\chi_{\mu\nu}+\chi_{\mu\nu}\chi-2\chi_{\mu\rho}\chi_\nu^\rho, \qquad \hat{\zeta}=\zeta\left(1+\frac{4}{3}\frac{\xi\Lambda}{\zeta^2}\right),$$
$$\tilde{\Lambda}=\Lambda\left(1+\frac{2}{3}\frac{(\alpha+2\xi)\Lambda}{\zeta^2}\right), \qquad \tilde{\zeta}=\zeta\frac{\tilde{\Lambda}}{\Lambda}.$$

In addition to the matter fields Φ, the theory describes the graviton, through the metric tensor $g_{\mu\nu}$, a scalar field ϕ of squared mass $m_\phi^2=\zeta/\xi$ and a spin-2 field $\chi_{\mu\nu}$ of squared mass $m_\chi^2=\tilde{\zeta}/\alpha$. Making formula (5.3) more explicit, it is easy to show that the $\chi_{\mu\nu}$ quadratic action is a covariantized Pauli-Fierz action [26] with the wrong overall sign, plus nonminimal terms [11]. This means that, to have unitarity, the field $\chi_{\mu\nu}$ must be quantized as a fakeon. Instead, the ϕ action has the correct sign, so ϕ can be quantized either as a fakeon or a physical particle, leading to two physically inequivalent theories. We recall that if the Feynman quantization prescription is used for all the fields, the Stelle theory is obtained [27], where $\chi_{\mu\nu}$ is a ghost. In that case, unitarity is violated at energies larger than m_χ.

Renormalizability can be straightforwardly proved from the action (5.1), because it does not depend on the quantization prescription [5]. Therefore, the beta functions coincide with those of the Stelle theory [28].

Working with the action (5.2), equipped with the gauge-mass terms (3.5), the theory of scattering in the presence of a cosmological constant can be formulated as explained in the previous two sections. The massive fields ϕ and $\chi_{\mu\nu}$ can be viewed as additional matter fields. As before, we can focus on the Hilbert-Einstein sector, described by the action $\tilde{S}_{\mathrm{HE}}(g)$, and use the special gauge (3.2) to quantize the gauge-trivial poles as fakeons and the physical poles by means of the Feynman prescription. Note that we have two types of fakeons, here: those due to the higher-derivatives of (5.1), such as $\chi_{\mu\nu}$, and those that belong to the gauge-trivial sector, due to (3.2). The former will be called hard fakeons, while the latter will be called gauge fakeons. The λ-independent thresholds may be physical or involve hard fakeons. If they are physical, they are circumvented by means of the Feynman prescription. If they involve hard fakeons they are circumvented by means of the average continuation. The λ dependent thresholds always involve gauge fakeons and require the average continuation.

The gauge mass terms (3.5) make the massive theory nonrenormalizable. This does not pose obstacles to the proof of unitarity, as we know. As far as potential infrared divergences are concerned, nonrenormalizable terms are also not a problem, since they can only improve the behaviors (4.6). The nonrenormalizable sector disappears altogether when the gauge masses are sent to zero, so it is not necessary to include new vertices at the tree level, multiplied by independent parameters, to subtract the divergent parts that belong to that sector. Nevertheless, in the next section we show how to modify the tree-level Lagrangian in a simple way, to include all the counterterms we need.

Strictly renormalizable theories with analogous features exist in every even dimensions d greater than or equal to six [17]. Their classical actions read

$$S_{\mathrm{QG}}^d = -\frac{1}{2\kappa^2}\int \mathrm{d}^d x\sqrt{-g}\left[2\Lambda + \zeta R + \hat{G}_{\mu\nu}P(D^2)\hat{G}^{\mu\nu} - \hat{G}P'(D^2)\hat{G} + \mathcal{O}(R^3)\right] + S_{\mathfrak{m}}^d(g,\Phi), \tag{5.4}$$

where

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R}, \qquad \hat{G} = g^{\mu\nu}\hat{G}_{\mu\nu}, \qquad \hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{\Lambda}{\zeta}g_{\mu\nu},$$

$S_{\mathfrak{m}}^d$ is the action of the matter fields Φ, P and P' denote real polynomials of degree $(d-4)/2$

and $\mathcal{O}(R^3)$ collects the local Lagrangian terms that have dimensions smaller than or equal to d and are built with at least three curvature tensors and their covariant derivatives.

As before, we separate the Hilbert-Einstein sector by introducing extra fields. We first illustrate the procedure in a simple toy model. Consider the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - (\Box \varphi) Q \Box \varphi - V(\varphi), \tag{5.5}$$

where φ is a scalar field, $V(\varphi)$ is an interaction potential (containing vertices that are at least cubic in φ) and Q is a possibly field-dependent polynomial of the partial derivatives ∂_μ. First, we add extra fields χ and $\bar{\chi}$ of bosonic statistics and fields ξ and $\bar{\xi}$ of fermionic statistics, to rewrite the Lagrangian in the equivalent form

$$L' = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + \bar{\chi} Q \chi - (\bar{\chi} + \chi) Q \Box \varphi - \bar{\xi} Q \xi - V(\varphi). \tag{5.6}$$

The propagators of the extra fields χ, $\bar{\chi}$, ξ and $\bar{\xi}$ contain old and new poles. The old poles match poles of the φ propagator of the initial Lagrangian (5.5) and must be quantized like those. The new poles can be quantized with the prescription we want, as long as it is the same for all of them, since they have to compensate one another. For definiteness, we assume that they are quantized with the fakeon prescription. The equivalence between L and L' is easily proved by integrating over χ, $\bar{\chi}$, ξ and $\bar{\xi}$, noting that the Jacobian determinants cancel out.

The next step is to diagonalize the Lagrangian L' by means of the field redefinition

$$\varphi \rightarrow \varphi - Q(\bar{\chi} + \chi) \equiv \tilde{\varphi}, \tag{5.7}$$

which gives

$$L' = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + \bar{\chi} Q \chi + \frac{1}{2}(\bar{\chi} + \chi) Q \Box Q (\bar{\chi} + \chi) - \bar{\xi} Q \xi - V(\tilde{\varphi}).$$

We could introduce further fields of fermionic statistics to account for the Jacobian determinant of the redefinition (5.7). We do not do so, because we work with the dimensional-regularization technique, where such a determinant is identically one. In the end, L' has a standard φ quadratic action and all the higher-derivative terms act on the extra fields.

Let us now come to the action (5.4). We introduce extra fields $\chi_{\mu\nu}$, $\bar{\chi}_{\mu\nu}$, ϕ and $\bar{\phi}$ of bosonic statistics, as well as extra fields $\xi_{\mu\nu}$, $\bar{\xi}_{\mu\nu}$, ξ and $\bar{\xi}$ of fermionic statistics, to obtain

$$S_{\mathrm{QG}}^{d\prime} = S_g^d + S_\chi^d + S_\phi^d + S_{\mathfrak{m}}^d,$$

where

$$
\begin{aligned}
S_g^d &= -\frac{1}{2\kappa^2}\int \mathrm{d}^d x\sqrt{-g}\left[2\Lambda+\zeta R+\mathcal{O}(R^3)\right], \\
S_\chi^d &= -\frac{1}{2\kappa^2}\int \mathrm{d}^d x\sqrt{-g}\left[-\bar{\chi}_{\mu\nu}P(D^2)\chi^{\mu\nu}+(\chi_{\mu\nu}+\bar{\chi}_{\mu\nu})P(D^2)\hat{G}^{\mu\nu}-\bar{\xi}_{\mu\nu}P(D^2)\xi^{\mu\nu}\right], \\
S_\phi^d &= -\frac{1}{2\kappa^2}\int \mathrm{d}^d x\sqrt{-g}\left[\bar{\phi}P'(D^2)\phi-(\phi+\bar{\phi})P'(D^2)\hat{G}-\bar{\xi}P'(D^2)\xi\right].
\end{aligned}
$$

Then we diagonalize the quadratic part by means of the redefinition

$$
g_{\mu\nu}\rightarrow g_{\mu\nu}+\frac{1}{\zeta}P(D^2)(\chi_{\mu\nu}+\bar{\chi}_{\mu\nu})-\frac{1}{\zeta}g_{\mu\nu}P'(D^2)(\phi+\bar{\phi})\equiv\tilde{g}_{\mu\nu}.
$$

Again, we do not need to take care of the Jacobian determinant of this field redefinition if we use the dimensional regularization. The resulting action is ready to be expanded around flat space. The $h_{\mu\nu}$ quadratic terms are diagonal, apart from the tadpole vertex originated by the cosmological term, which must be treated as explained in section 4.

In the end, we manage to isolate the Hilbert-Einstein action from the rest, so the proof of unitarity up to corrections due to the cosmological constant works as before. The polynomials P and P' must satisfy suitable restrictions, so that the poles of the free propagators have squared masses with nonnegative real parts. The poles with negative or complex residues, as well as those with positive residues but complex masses, must be quantized as fakeons. Instead, the poles with positive residues and real, positive masses can be quantized either as fakeons or physical particles.

Super-renormalizable ultraviolet complete theories also exist (see [4]) and can be treated similarly. They are less interesting, from the physical point of view, because they are not unique.

6 Massive gravitons?

The masses we have introduced for the gauge fields and the gravitons are artifacts to carry out the proofs of unitarity to the end. In particular, in the case of gravity they allow us to treat the cosmological constant as explained. At the very end, the gauge masses must tend to zero, so that gauge invariance, Lorentz invariance and general covariance are recovered. However, in some cases it is interesting to keep the graviton masses different from zero and study the compatibility of such an assumption with the experimental data. In this section, we point out that our approach does allow us to formulate a unitary theory of massive gravitons. We also compare it with other approaches to massive gravitons available in the literature.

Normally, it is believed that gauge invariance cannot be broken explicitly without violating unitarity. Thanks to the fakeon prescription, used for the quantization of the poles that belong to the gauge-trivial sector, our construction achieves precisely that goal. In gauge theories, we have been able to keep the gauge masses nonvanishing without renouncing unitarity, locality and renormalizability. In gravity, so far, the graviton mass terms we have added preserve unitary and locality, but not renormalizability. Now we elaborate more on this issue.

We start by adding an arbitrary potential $V(\kappa h)$ (which, in some sense, corrects the cosmological term) to the higher-derivative Lagrangian (5.1). In general, V is just invariant under rotations, but in particular cases it may be Lorentz invariant. Its quadratic part is made of the mass terms (3.5) or (3.6). The resulting action

$$S_{m\mathrm{QG}}(g,\Phi)=S_{\mathrm{QG}}(g,\Phi)-\frac{1}{\kappa^2}\int \mathrm{d}^4x V(\kappa h) \tag{6.1}$$

is renormalizable by power counting (with infinitely many independent couplings). Indeed, the renormalization of the theory is governed by a power counting that makes κ and $h_{\mu\nu}$ dimensionless at high-energies, so the vertices of V are multiplied by parameters of dimension four. Since the divergent parts of the loop diagrams depend on those parameters polynomially, and the theory contains no parameters of negative dimensions, the counterterms generated by V have the same form as the monomials contained in V. Thus, if the coefficients of the V monomials are independent, the action (6.1) is renormalizable. The action may equally well be considered nonrenormalizable, due to the presence of infinitely many couplings in V. To avoid confusion and stress that the nonrenormalizability is of a peculiar type, we say that the action (6.1) is hard-renormalizable, or soft-nonrenormalizable, or almost renormalizable.

The reason why we have not used the action (6.1) in the previous section is that an analogue of the special gauge is not available at present for the higher-derivative action S_{QG}, which means that (6.1) leads to extremely involved propagators.

Starting from (6.1), it is still convenient to switch to the non-higher-derivative form of the action by inserting the extra fields ϕ and $\chi_{\mu\nu}$ explicitly. Mimicking the steps of ref. [11], the metric redefinition is

$$g_{\mu\nu}\rightarrow(g_{\mu\nu}+2\chi_{\mu\nu}+\chi_{\mu\nu}\chi-2\chi_{\mu\rho}\chi_{\nu}^{\rho})\mathrm{e}^{\kappa\phi}.$$

After the redefinition, the action we get is $S_{\mathrm{QG}}^{\prime}$ plus the transformed potential, which contains, among the other things, extra mass terms for ϕ and $\chi_{\mu\nu}$ and off-diagonal quadratic terms.

The quadratic corrections can be included into modified propagators, by means of a resummation like (3.7). First, it is convenient to resum the h-h mass terms, following the guidelines explained below formula (3.7). Second, it is convenient to resum the corrections to the χ-χ mass terms. Since such corrections are generically not of the Pauli-Fierz type, they turn on a new pole in the χ propagator, which describes a scalar field π. The residue of the π pole is positive (because the χ Pauli-Fierz action is multiplied by the wrong sign), so π can be quantized as a physical particle or a hard fakeon. Third, we resum the ϕ-ϕ terms, which just correct the ϕ mass. Finally, we resum the off-diagonal h-ϕ-χ mass terms following the lines explained below formula (3.7). The physical poles, as well as those associated with the hard fakeons, remain λ independent up to corrections due to the gauge masses. The poles associated with the gauge fakeons remain λ dependent. As usual, the physical thresholds are treated by means of the Feynman prescription, while the fake thresholds are treated by means of the average continuation. We recall that coinciding thresholds must be treated as limits of distinct thresholds.

Ultimately, we obtain a local, unitary and almost renormalizable theory of massive gravitons. The theory violates general covariance and, in particular, Lorentz symmetry. We can reduce the effects of the Lorentz violation by choosing a Lorentz invariant potential V and sending λ to one after the computations of the loop diagrams. Then, the surviving Lorentz violations start from one loop.

Due to the violation of general covariance, the gauge fakeons have physical effects, like the hard fakeons. It is known that causality is violated at energies larger than the fakeon masses [10, 11, 29, 18]. To make the violations small and compatible with the data, the masses of the hard fakeons must be large, while the potential V must be small. The latter condition follows from the fact that the gauge fakeons do not contribute to the physical quantities in the limit of vanishing V (since they get buried into the gauge-trivial sector). A simple way to see this is to note that the physical quantities are gauge independent in that limit, while the sector of the gauge fakeons is λ dependent. It is easy to check [see formulas (6.3) and (6.5) below] that the π mass is large when the gauge masses are small, so if π is quantized as a fakeon, the violation of causality due to it is small, as desired.

One of the reasons why the theories of massive gravitons attract interest is that in many cases they remove the van Dam-Veltman-Zacharov (vDVZ) discontinuity [30]. The massive theories we have just built also achieve this goal, since the limit of vanishing gauge masses is smooth. Not only: the fakeon quantization prescription provides a further, Lorentz invariant option. Consider the Pauli-Fierz Lagrangian

$$L_{\rm PF} = \frac{1}{2}\left[\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \partial_\rho h \partial^\rho h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\rho\nu}\partial_\rho h^\mu_\nu - m^2(h_{\mu\nu}h^{\mu\nu} - h^2)\right] \tag{6.2}$$

for a symmetric tensor $h_{\mu\nu}$ and add the unconventional correction

$$L_m' = -\frac{3m^4}{4(2m^2 + \bar{m}^2)}h^2 \tag{6.3}$$

to its mass term. The propagator $-i\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle$ of the resulting Lagrangian

$$L_{\rm PF} + L_m' \tag{6.4}$$

is the sum of the Pauli-Fierz one, which describes a spin-2 particle of mass m, plus

$$-\frac{1}{6(k^2 - \bar{m}^2)}\left(\eta_{\mu\nu} + 2\frac{k_\mu k_\nu}{m^2}\right)\left(\eta_{\rho\sigma} + 2\frac{k_\rho k_\sigma}{m^2}\right), \tag{6.5}$$

which has a scalar pole of mass $\bar{m}$. Since the residue of this pole in negative, the Feynman quantization prescription turns it into the Boulware-Deser (BD) ghost [31] and violates unitarity. However, if we quantize the pole (6.5) as a fakeon, unitarity holds and the vDVZ discontinuity is removed in a Lorentz invariant way. At the observative level $\bar{m}$ should be small and since $1/\bar{m}$ is the range of the violation of causality, the theory with (6.5) as a fakeon has the problem of explaining why that violation is not observed.

Rubakov has shown that (if we do not make use of fakeons) the vDVZ discontinuity can be removed in a unitary way if Lorentz invariance is broken and certain restrictions on the masses are imposed [14]. If Rubakov's would-be ghosts are quantized as fakeons, it might be possible to relax some of the Rubakov conditions on the masses.

Like Rubakov's theory, the theories of massive gravitons of the previous two sections are not Lorentz invariant. The main difference between our propagators and the Rubakov's ones are that we have included the gauge-fixing terms, to ensure a smooth limit of vanishing V, where Lorentz invariance and general covariance are recovered.

The Dvali-Gabadadze-Porrati (DGP) model [15] overcomes the vDVZ discontinuity by obtaining four-dimensional gravity from a five-dimensional theory. In the de Rham-Gabadadze-Tolley (dRGT) model [16] the Boulware-Deser ghost is removed by adding nonderivative interaction terms for the metric $g_{\mu\nu}$, which requires to introduce an extra, reference metric $f_{\mu\nu}$. The extra metric is also present in our model: it is the flat-space metric used to build the potential V.

Finally, we stress that the theory of massive gravitons (6.1) is unitary (up to corrections due to the cosmological constant) and almost renormalizable. The problem of renormalizability remains open in the theories of refs. [14, 15, 16], as well as in any Lorentz invariant massive theory with quadratic Lagrangian (6.4).

7 Conclusions

In this paper we have worked out simple proofs of perturbative unitarity in gauge theories and quantum gravity. The special gauge allows us to separate the physical poles, which are quantized by means of the standard Feynman prescription, from the poles that belong to the gauge sector, which are quantized by means of the fakeon prescription. Inside the loop diagrams, the dependence on a gauge-fixing parameter λ allows us to distinguish the physical thresholds, which are overcome analytically, from the fake thresholds, which are overcome non analytically by means of the average continuation. The proof works for nonrenormalizable and ultraviolet complete theories.

We also clarified a number of nontrivial issues about the formulation of the theory of scattering in the presence of a cosmological constant. The scattering amplitudes are defined by expanding the metric around flat space. They obey unitarity up to corrections due to the cosmological constant, which can be neglected in all practical situations. In the (unrealistic) case that such corrections became important, the cutting equations hold, but their physical meaning remains unclear.

We have introduced nonvanishing gauge masses for various practical purposes. Gauge invariance, Lorentz invariance and general covariance are recovered in the limit where the gauge masses vanish. If we keep the gauge masses different from zero, our construction provides a way of building local, unitary and almost renormalizable theories of massive gauge fields and gravitons (which violate Lorentz invariance, gauge invariance and general covariance). Usually, it is believed that the explicit breaking of a gauge symmetry leads to the violation of unitarity. We have shown that once fakeons are employed, unitarity and gauge invariance are ultimately disentangled from each other and it is possible to break the latter without breaking the former.

The theories with fakeons violate causality at energies larger than the fakeon masses. The masses of the hard fakeons must be large enough, to have compatibility with the data, while the gauge masses must be small enough, because in the limit where they vanish the gauge fakeons do not contribute to the physical quantities.

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References

[1] R.E. Cutkosky, Singularities and discontinuities of Feynman amplitudes, J. Math. Phys. (NY) 1 (1960) 429;

M. Veltman, Unitarity and causality in a renormalizable field theory with unstable particles, Physica 29 (1963) 186.

[2] G. 't Hooft and M. Veltman, *Diagrammar*, report No. CERN-73-09, available at this link;

M. Veltman, *Diagrammatica. The path to Feynman rules* (Cambridge University Press, New York, 1994).

[3] G. 't Hooft, Renormalization of massless Yang-Mills fields, Nucl. Phys. B 33 (1971) 173;

G. 't Hooft, Renormalizable Lagrangians for massive Yang-Mills fields, Nucl. Phys. B 35 (1971) 167.

[4] D. Anselmi, On the quantum field theory of the gravitational interactions, J. High Energy Phys. 06 (2017) 086, 17A3 Renormalization.com and arXiv: 1704.07728 [hep-th].

[5] D. Anselmi, Fakeons and Lee-Wick models, J. High Energy Phys. 02 (2018) 141, 18A1 Renormalization.com and arXiv:1801.00915 [hep-th].

[6] T.D. Lee and G.C. Wick, Negative metric and the unitarity of the S-matrix, Nucl. Phys. B 9 (1969) 209;

T.D. Lee and G.C. Wick, Finite theory of quantum electrodynamics, Phys. Rev. D 2 (1970) 1033.

[7] D. Anselmi and M. Piva, A new formulation of Lee-Wick quantum field theory, J. High Energy Phys. 06 (2017) 066, 17A1 Renormalization.com and arXiv:1703.04584 [hep-th];

D. Anselmi and M. Piva, Perturbative unitarity in Lee-Wick quantum field theory, Phys. Rev. D 96 (2017) 045009 and 17A2 Renormalization.com and arXiv:1703.05563 [hep-th].

[8] R.E. Cutkosky, P.V Landshoff, D.I. Olive, J.C. Polkinghorne, A non-analytic S matrix, Nucl. Phys. B12 (1969) 281.

[9] N. Nakanishi, Lorentz noninvariance of the complex-ghost relativistic field theory, Phys. Rev. D 3, 811 (1971).

[10] D. Anselmi and M. Piva, The ultraviolet behavior of quantum gravity, J. High Energ. Phys. 05 (2018) 27, 18A2 Renormalization.com and arXiv:1803.07777 [hep-th].

[11] D. Anselmi and M. Piva, Quantum gravity, fakeons and microcausality, J. High Energy Phys. 11 (2018) 21, 18A3 Renormalization.com and arXiv:1806.03605 [hep-th].

[12] D. Anselmi, Aspects of perturbative unitarity, Phys. Rev. D 94 (2016) 025028, 16A1 Renormalization.com and arXiv:1606.06348 [hep-th].

[13] See for example the discussions in

S.B. Giddings, The boundary S-matrix and the AdS to CFT dictionary, Phys. Rev. Lett. 83 (1999) 2707 and arXiv:hep-th/9903048;

V. Balasubramanian, S.B. Giddings, A. Lawrence, What do CFTs tell us about anti-de Sitter spacetimes?, J. High Energy Phys. 9903 (1999) 001 and arXiv:hep-th/9902052;

J. Bros, H. Epstein, M. Gaudin, U. Moschella and V. Pasquier, Triangular invariants, three-point functions and particle stability on the de Sitter universe, Commun. Math. Phys. 295 (2010) 261 and arXiv:0901.4223 [hep-th];

E.T. Akhmedov and P. V. Buividovich, Interacting field theories in de Sitter space are non-unitary, Phys. Rev. D 78 (2008) 104005 and arXiv:0808.4106 [hep-th];

E.T. Akhmedov, Lecture notes on interacting quantum fields in de Sitter space, Int. J. Mod. Phys. D 23 (2014) 1430001 and arXiv:1309.2557 [hep-th]

D. Marolf, I. A. Morrison, and M. Srednicki, Perturbative S-matrix for massive scalar fields in global de Sitter space, Class. Quant. Grav. 30 (2013) 155023 and arXiv:1209.6039 [hep-th].

[14] V. Rubakov, Lorentz-violating graviton masses: getting around ghosts, low strong coulping scale and VDVZ discontinuity, arXiv:hep-th/0407104.

[15] G. Dvali, G. Gabadadze and M. Porrati, 4D Gravity on a brane in 5D Minkowski space, Phys. Lett. B 485 (2000) 2082 and arXiv:hep-th/0005016.

[16] C. de Rham, G. Gabadadze and A. Tolley, Resummation of massive gravity, Phys. Rev. Lett. 106 (23) (2011) 231101 and arXiv:1011.1232 [hep-th].

[17] D. Anselmi, The correspondence principle in quantum field theory and quantum gravity, 18A5 Renormalization.com, PhilSci 15287, OSF preprints d2nj7, Preprints 2018110213 and hal-01900207;

D. Anselmi, Fakeons, quantum gravity and the correspondence principle, in "*Progress and Visions in Quantum Theory in View of Gravity: Bridging foundations of physics and mathematics*", edited by F. Finster, D. Giulini, J. Kleiner and J. Tolksdorf, Birkhäuser Verlag (2019), 19R2 Renormalization.com and arXiv:1911.10343 [hep-th].

[18] D. Anselmi, Fakeons and the classicization of quantum gravity: the FLRW metric, J. High Energy Phys. 04 (2019) 61, 19A1 Renormalization.com and arXiv:1901.09273 [gr-qc].

[19] J.C. Ward, An identity in quantum electrodynamics, Phys. Rev. 78, (1950) 182;

Y. Takahashi, On the generalized Ward identity, Nuovo Cimento, 6 (1957) 371;

A.A. Slavnov, Ward identities in gauge theories, Theor. Math. Phys. 10 (1972) 99;

J.C. Taylor, Ward identities and charge renormalization of Yang-Mills field, Nucl. Phys. B33 (1971) 436.

[20] J. Zinn-Justin, Renormalization of gauge theories, in *Trends in Elementary Particle Physics*, Lecture Notes in Physics, edited by H. Rollnik and K. Dietz (Springer-Verlag, Berlin), Vol. 37.

[21] I.A. Batalin and G.A. Vilkovisky, Gauge algebra and quantization, Phys. Lett. B 102 (1981) 27-31;

I.A. Batalin and G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generators, Phys. Rev. D 28 (1983) 2567; 30 (1984) 508(E);

See also S. Weinberg, *The Quantum Theory of Fields*, vol. II, Cambridge University Press, Cambridge (1995).

[22] D. Anselmi, Absence of higher derivatives in the renormalization of propagators in quantum field theories with infinitely many couplings, Class. and Quantum Grav. 20 (2003) 2355, 02A4 Renormalization.com and arXiv:hep-th/0212013.

[23] D. Anselmi, Properties of the classical action of quantum gravity, J. High Energy Phys. 05 (2013) 028, 13A2 Renormalization.com and arXiv:1302.7100 [gr-qc].

[24] F. Bloch and A. Nordsieck, Note on the radiation field of the electron, Phys. Rev. 52 (1937) 54;

T. Kinoshita, Mass singularities of Feynman amplitudes, J. Math. Phys. 3 (1962) 650;

T. D. Lee and M. Nauenberg, Degenerate systems and mass singularities, Phys. Rev. 133 (1964) B1549;

R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and collider physics*, Cambridge monographs on particle physics, nuclear physics, and cosmology, 8, Cambridge University Press, Cambridge (2003);

Y.L. Dokshitzer, V.A. Khoze, A.H. Mueller and S.I. Troian, *Basics of perturbative QCD*, World Scientific Publishing (1991);

T.D. Lee, *Particle physics and introduction to field theory*, Contemporary Concepts in Physics, Vol. 1, Harwood Academic (1981).

[25] S. Weinberg, Infrared photons and gravitons, Phys. Rev. 140 (1965) B516.

[26] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173, 211 (1939).

[27] K.S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D 16 (1977) 953.

[28] J. Julve, M. Tonin, Quantum gravity with higher derivative terms, Nuovo Cim. B 46 (1978) 137;

E.S. Fradkin, A.A. Tseytlin, Renormalizable asymptotically free quantum theory of gravity, Nucl. Phys. B 201 (1982) 469;

I. G. Avramidi and A. O. Barvinsky, Asymptotic freedom in higher derivative quantum gravity, Phys. Lett. B 159 (1985) 269;

N. Ohta, R. Percacci and A.D. Pereira, Gauges and functional measures in quantum gravity II: Higher-derivative gravity, Eur. Phys. J. C 77 (2017) 611 and arXiv:1610.07991 [hep-th];

A. Salvio and A. Strumia, Agravity up to infinite energy, Eur. Phys. C 78 (2018) 124 and arXiv:1705.03896 [hep-th].

[29] D. Anselmi, Fakeons, microcausality and the classical limit of quantum gravity, Class. and Quantum Grav. 36 (2019) 065010, 18A4 Renormalization.com and arXiv:1809.05037 [hep-th].

[30] H. van Dam and M.J.G. Veltman, Massive and massless Yang-Mills and gravitational fields, Nucl. Phys. B 22 (1970) 397;

V.I. Zakharov, Linearized gravitation theory and the graviton mass, JETP Letters (Sov. Phys.) 12 (1970) 312.

[31] D.G. Boulware and S. Deser, Can gravitation have a finite range?, Phys. Rev. D 6, 3368.