Fakeons, Microcausality And The Classical Limit Of Quantum Gravity

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Abstract

We elaborate on the idea of fake particle and its physical consequences. When a theory contains fakeons, the true classical limit is determined by the quantization and a subsequent process of "classicization". One of the major predictions due to the fake particles is the violation of microcausality, which survives the classical limit. This fact gives hope to find ways to detect the violation experimentally. Quantum gravity predicts that at least one fakeon exists in nature, with spin 2 and a mass that could be much smaller than the Planck mass. By classicizing the theory, we work out the corrections to the field equations of general relativity. We show that the finalized equations have, in simple terms, the form $\langle F\rangle = ma$, where $\langle F\rangle$ is an average that includes a little bit of "future".

1 Introduction

A theory of quantum gravity has been recently formulated [1], by means of a new quantization prescription that turns certain poles of the free propagators into fake particles, or "fakeons", which can be consistently projected away from the physical spectrum. The action of the theory is interpreted in a radically new way and some of its physical predictions are quite unexpected, even at the classical level. In particular, the physical space of configurations is a proper subspace of the whole space of configurations. The restriction follows from the projection on the space of states that is required to make sense of the theory at the quantum level. In this paper, we study how these features of quantum gravity affect the classical limit. In particular, we work out the corrections to the field equations of general relativity and investigate their major prediction, which is the violation of microcausality.

The *interim* classical action is

$$S_{\mathrm{QG}}(g,\Phi)=-\frac{1}{2\kappa^2}\int\sqrt{-g}\left[2\Lambda_C+\zeta R+\alpha\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2\right)-\frac{\xi}{6}R^2\right]+S_{\mathfrak{m}}(g,\Phi),\qquad(1.1)$$

where α, ξ, ζ and κ are real positive constants, while Λ_C can be positive or negative, Φ are the matter fields and $S_{\mathfrak{m}}$ is the covariantized action of the standard model, or one of its extensions, equipped with the nonminimal couplings that are compatible with renormalizability. The reduced Planck mass is $\bar{M}_{\mathrm{Pl}}=M_{\mathrm{Pl}}/\sqrt{8\pi}=\sqrt{\zeta}/\kappa$.

In addition to the matter fields, the theory describes the graviton, a scalar ϕ of squared mass $m_\phi^2=\zeta/\xi$ and a spin-2 fakeon $\chi_{\mu\nu}$ of squared mass $m_\chi^2=\zeta/\alpha$ (neglecting a small correction due to the cosmological constant). These fields can be introduced explicitly as auxiliary fields. After simple field redefinitions, we obtain an equivalent form of the interim classical action, which reads [2]

$$\mathcal{S}_{\mathrm{QG}}(g,\phi,\chi,\Phi)=S_{\mathrm{H}}(g)+S_\chi(g,\chi)+S_\phi(\tilde{g},\phi)+S_{\mathfrak{m}}(\tilde{g}\mathrm{e}^{\kappa\phi},\Phi),\qquad(1.2)$$

where $\tilde{g}_{\mu\nu}=g_{\mu\nu}+2\chi_{\mu\nu}$ and

$$S_{\mathrm{H}}(g)=-\frac{\zeta}{2\kappa^2}\int\sqrt{-g}R,\qquad S_\phi(g,\phi)=\frac{3\zeta}{4}\int\sqrt{-g}\left[\nabla_\mu\phi\nabla^\mu\phi-\frac{m_\phi^2}{\kappa^2}\left(1-\mathrm{e}^{\kappa\phi}\right)^2\right],$$
$$S_\chi(g,\chi)=S_{\mathrm{H}}(\tilde{g})-S_{\mathrm{H}}(g)-2\int\chi_{\mu\nu}\frac{\delta S_{\mathrm{H}}(\tilde{g})}{\delta g_{\mu\nu}}+\frac{\zeta^2}{2\alpha\kappa^2}\int\sqrt{-g}(\chi_{\mu\nu}\chi^{\mu\nu}-\chi^2)\Big|_{g\rightarrow\tilde{g}}.\qquad(1.3)$$

We have written these expressions for $\Lambda_C=0$, which is sufficient for most purposes of this paper. The formulas for $\Lambda_C\neq 0$ can be found in ref. [2].

The first thing to point out is that (1.1) and (1.2) do not encode the true classical theory, which is why we have called them "interim" classical actions. Indeed, they are still unprojected. However, the projection comes from the quantization, so we must first quantize the theory and then *classicize* it back, which means investigate the classical backlash of the quantization. Only at the end of this procedure, we obtain the action that describes the classical limit, which we call *finalized classical action*. The interim classical action and the finalized classical action coincide in the absence of gravity (for example, in the case of the standard model in flat space, where the projection is trivial), but do not coincide when quantum gravity is present.

The second major observation is that the fakeons induce violations of microcausality, which survive the classical limit. This fact opens the possibility to investigate such violations, and other predictions of the theory, beyond the scattering processes and possibly beyond the perturbative expansion, by studying the solutions of the finalized classical field equations.

We recall that if the action (1.1) is quantized following the standard procedures (which means using the Feynman prescription for all the fields), the Stelle theory is obtained [3], where $\chi_{\mu\nu}$ is a ghost. This option is not acceptable, because there exists no consistent projection to get rid of a ghost.

The paper is organized as follows. In section 2 we recall what a fakeon is and elaborate on the concept. In section 3 we study the main physical consequences of a fakeon, such as the violation of microcausality, at the level of scattering processes. In section 4 we study how the violations of microcausality survive the classical limit in a toy higher-derivative model. In section 5 we classicize quantum gravity. In particular, we work out the finalized classical action and study its field equations. Section 6 contains the conclusions.

2 Quantization

In this section, we elaborate on the idea of fake particle, or fakeon, introduced in ref. [1]. Unitarity is crucial for the discussion, so we begin by recalling the optical theorem

$$2\mathrm{Im}T = T^{\dagger}T, \tag{2.1}$$

which follows from the unitarity equation $S^{\dagger}S = 1$, once the S matrix is written as $1 + iT$. Let V denote the space of physical states. The transition amplitude $\mathcal{M}(a|b)$ between an initial state $|a\rangle \in V$ of total momentum P_a and a final state $|b\rangle \in V$ of total momentum

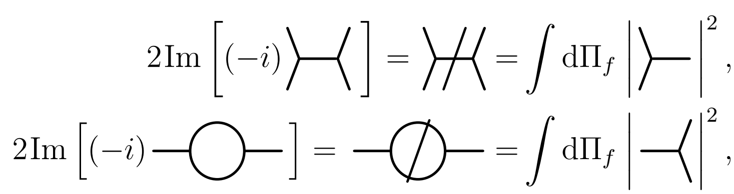
P_b is related to the T matrix element by the identity

$$\langle b|T|a\rangle = (2\pi)^4 \delta^{(4)}(P_a - P_b)\mathcal{M}(a|b).$$

Moreover, $i\mathcal{M}(a|b)$ is the sum of the connected, amputated diagrams that have the external legs determined by $|a\rangle$ and $|b\rangle$. Taking $|b\rangle = |a\rangle$ and inserting a complete set of orthonormal states $|n\rangle \in V$, equation (2.1) implies

$$2\mathrm{Im}\langle a|T|a\rangle = \sum_{|n\rangle \in V} |\langle n|T|a\rangle|^2, \tag{2.2}$$

i.e. the total cross section for production of all final states is proportional to the imaginary part of the forward scattering amplitude. A version of the optical theorem that holds diagram by diagram is provided by the so-called cutting equations [4], which express the imaginary part of a diagram as a sum of "cut diagrams", where the contributions of $\langle n|T|a\rangle$ and $\langle a|T^\dagger|n\rangle$ stand to the left and right sides of the cuts, respectively. The simplest cutting equations are


(2.3)
(2.4)

where the integrals are over the phase spaces Π_f of the final states [5].

Now, consider the propagator

$$G(p,m) = \frac{1}{p^2 - m^2}. \tag{2.5}$$

If we endow it with the Feynman prescription ($p^2 \to p^2 + i\epsilon$), we obtain

$$G_+(p,m,\epsilon) = \frac{1}{p^2 - m^2 + i\epsilon}, \tag{2.6}$$

which describes a particle of mass m. The identity (2.3) implies $\mathrm{Im}[-P] \geqslant 0$, if the vertices are assumed to be equal to $-i$ and P is the propagator of the intermediate line on the left-hand side. Specifically, $P = G_+$ gives

$$\mathrm{Im}\left[-\frac{1}{p^2 - m^2 + i\epsilon}\right] = \pi\delta(p^2 - m^2). \tag{2.7}$$

What happens if we multiply (2.6) by a minus sign? If we do not do anything else, we obtain a ghost, since $P = -G_+$ satisfies $\mathrm{Im}[-P] \leqslant 0$, which violates the optical theorem. However, if we also replace $+i\epsilon$ with $-i\epsilon$, the right-hand side of (2.7) does not change,

$$\mathrm{Im}\left[\frac{1}{p^2 - m^2 - i\epsilon}\right] = \pi\delta(p^2 - m^2), \tag{2.8}$$

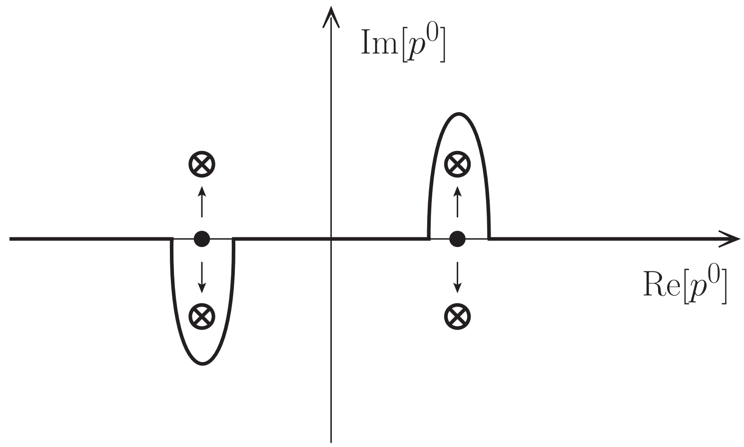


Figure 1: Splitting the poles located on the real axis. The dots denote the poles of (2.5) and the circled crosses denote the poles of (2.9)

and the optical theorem remains valid. The moral of the story is that we can in principle have both propagators

$$G_{\pm}(p,m,\epsilon)=\pm\frac{1}{p^2-m^2\pm i\epsilon},$$

since both fulfill the identity (2.3).

Nevertheless, this is not what normally occurs in quantum field theory, at least within the same loop integral. The reason is that the presence of both G_+ and G_- originates bad nonlocal divergences at $\epsilon \neq 0$ [6] and even worse problems for $\epsilon \rightarrow 0$.

A place where both propagators do appear in the same diagram are the cutting equations already mentioned: one side of a cut diagram is built with the propagators G_+ and the other side is built with the propagators G_-. For the reasons just recalled, a cut diagram gives a loop integral that can be badly divergent, because it contains both G_+ and G_-. However, the sum of the cut diagrams is well defined, because, by the optical theorem, it is equal to the imaginary part of an uncut diagram, which in turn is the sum of a diagram built only with G_+ plus a diagram built only with G_-.

Is there any hope to have $G_{\pm}$ coexist consistently in the same loop integral, i.e. an ordinary, uncut diagram? The first thing to note is that we should not integrate directly on Minkowski spacetime, to avoid the nonlocal divergences of ref. [6]. The only alternative is to come from Euclidean space by means of the Wick rotation [7]. It turns out that the Wick rotation is not analytic. However, this difficulty is not serious enough to prevent us from moving further.

Multiply (2.5) by $\pm$ and, following [1], write the outcome as

$$\pm\frac{p^2-m^2}{(p^2-m^2)^2}.$$

Then, eliminate the singularity by introducing an infinitesimal width $\mathcal{E}$ as follows, to define the new propagators

$$\mathbb{G}_\pm(p,m,\mathcal{E}^2)=\pm\frac{p^2-m^2}{(p^2-m^2)^2+\mathcal{E}^4}=\pm\frac{1}{2}\left[G_+(p,m,\mathcal{E}^2)-G_-(p,m,\mathcal{E}^2)\right]. \tag{2.9}$$

A first reason why $\mathbb{G}_\pm(p,m,\mathcal{E}^2)$ does not truly propagate a particle is that it vanishes on shell at $\mathcal{E}>0$. Alternatively, the relative sign between G_+ and G_- in the last expression suggests that $\mathbb{G}_\pm$ propagates a sort of "siamese particle/ghost pair" and that the particle and the ghost somewhat compensate each other. This is what gives the fake particle.

Doing (2.9), we basically split the poles of (2.5) into pairs of complex conjugates poles, as shown in fig. 1. The construction and the Wick rotation of the Euclidean theory suggest that, when the propagator $\mathbb{G}_\pm$ is used inside the Feynman diagrams, the loop energy p^0 must be integrated along the path shown in the same fig. 1, which passes under the left pair of complex conjugate poles and over the right pair. This is called *Lee-Wick (LW) integration prescription*, because it first appeared in the Lee-Wick models [8]. We comment on the relation between fakeons and Lee-Wick models below.

There exist three types of fakeons. The fakeon with propagator $\mathbb{G}_+(p,m,\mathcal{E}^2)$ is called *fakeon plus*. The fakeon with propagator $\mathbb{G}_-(p,m,\mathcal{E}^2)$ is called *fakeon minus*. For the moment, we ignore the third type of fakeon, to be described later on.

To make the arguments more explicit, we consider the bubble diagram as an example. Integrating the loop energy k^0 along the Lee-Wick integration path and the loop space momentum $\mathbf{k}$ on $\mathbb{R}^3$, we get

$$i\mathcal{M}(p)=c\int_{\mathbb{LW}}\frac{\mathrm{d}k^0}{2\pi}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathbb{G}_\pm(p-k,m_1,\mathcal{E}^2)\mathbb{G}_\pm(k,m_2,\mathcal{E}^2)=\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}w(p,\mathbf{k}) \tag{2.10}$$

for some integrand $w(p,\mathbf{k})$, where c collects the coupling constants and the combinatorial factor. For the arguments that follow, it does not matter whether the propagators are both $\mathbb{G}_+$, or both $\mathbb{G}_-$, or one and one.

The integral is always regular. When p is such that $w(p,\mathbf{k})$ is regular for all $\mathbf{k}$, the function $\mathcal{M}(p)$ is analytic in p. Moreover, when p is real and $\mathcal{M}(p)$ is analytic in p, $\mathcal{M}(p)$ is real. Analyticity has potential problems when $w(p,\mathbf{k})$ is singular. The $w(p,\mathbf{k})$ singularities originate the discontinuity of $\mathcal{M}$, which is also the imaginary part Im$\mathcal{M}$. By the optical

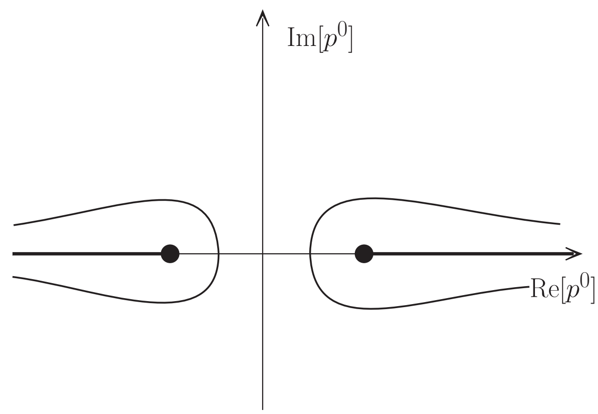


Figure 2: Analyticity fails in extended regions instead of branch cuts

theorem, specifically the identity (2.4), the imaginary part gives the total cross section of a scattering process, where the external particle of momentum p decays into the two particles circulating in the loop, of momenta $p-k$ and k, and the integral on the phase space Π_f of the final states is performed. The conditions for having such a process read

$$|p^0| = \sqrt{(\mathbf{p}-\mathbf{k})^2 + m_1^2 \pm i\mathcal{E}^2} + \sqrt{\mathbf{k}^2 + m_2^2 \pm i\mathcal{E}^2} \tag{2.11}$$

(all four possibilities occurring) [9]. Note that both frequencies on the right-hand side come with positive signs. The other possibilities are excluded by the LW integration prescription.

In the limit $\mathcal{E} \to 0$, analyticity fails in branch cuts, whose branch points are the thresholds of the process. Instead, when $\mathcal{E} \neq 0$ the branch cuts are replaced by extended regions $\tilde{\mathcal{A}}$, obtained by plotting (2.11) for $\mathbf{k} \in \mathbb{R}^3$ with $\mathbf{p}$ fixed. Examples of such regions are shown in fig. 2. Inside those regions the result of the integral (2.10) is not analytic and not Lorentz invariant.

Since all that matters is the limit $\mathcal{E} \to 0$, one might wonder why we pay attention to the properties of the integral at $\mathcal{E} > 0$. The reason is that if we take the limit too quickly we miss the new quantization prescription. Indeed, if we work at $\mathcal{E} = 0$, when the regions $\tilde{\mathcal{A}}$ shrink to branch cuts, we can circumvent the, say, right branch point by coming from above or from below, i.e. from the upper or lower side of the complex plane (check the left picture of fig. 3). The two options correspond to the Feynman prescription and its conjugate, respectively, which give nothing new. It is like replacing the propagators of (2.10) with two G_+ or two G_-, respectively: we have no coexistence of G_+ and G_- in the same loop integral.

Before taking the limit $\mathcal{E} \to 0$ we have a new possibility, which turns out to be the

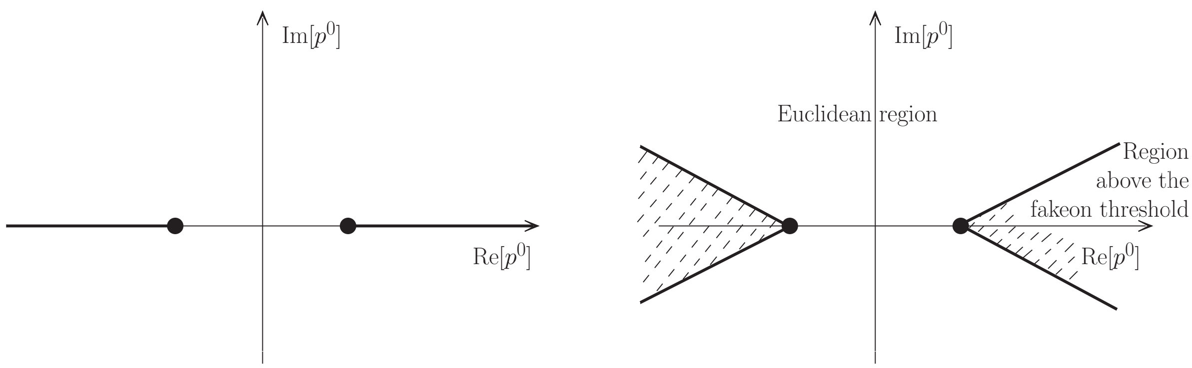


Figure 3: Analyticity versus regionwise analyticity

only way to mix $G_\pm$ in the same loop integral consistently with unitarity: we go inside a region $\tilde{\mathcal{A}}$ (i.e. we choose a p^0 that belongs to the portion of the real axis that is contained in the region), evaluate the integral there and *then* take the limit $\mathcal{E} \to 0$. In such a limit, analyticity and Lorentz invariance are recovered.

Moreover, the discontinuity Im$\mathcal{M}$ of $\mathcal{M}$ disappears, because the operation is symmetric under reflections with respect to the real axis, so it cannot generate an imaginary part. Thanks to this, the fakeon can be projected away from the physical spectrum V. The candidate physical process becomes a fake, with zero probability to occur, and is thrown away as well. Precisely, the identities (2.4) are satisfied as follows: the left-hand side and the middle expression vanish thanks to the fakeon prescription, while the right-hand side vanishes thanks to the fakeon projection. This is how the fakeon prescription/projection turns out to be compatible with unitarity. Instead, no projection can be consistent with the Feynman prescription, since the right-hand side of (2.3) would vanish by definition, but the left-hand side would continue to be nontrivial.

A mathematical theorem [7, 10] ensures that the result of the $\mathcal{M}$ evaluation inside the region is equal to the arithmetic average of the two analytic continuations that circumvent the branch point. This operation is called *average continuation*.

These properties generalize to all diagrams [10]. In conclusion, we can have the propagators G_+ and G_- coexist in the form $\mathbb{G}_\pm$, provided we treat them as just explained. Their poles do not correspond to physical particles, nor ghosts, but fake particles, or fakeons. The scattering processes involving at least one fakeon are fake processes and their thresholds must be circumvented by means of the average continuation. Then, their probability to occur vanishes, which makes it possible to project them away and have unitarity.

The average continuation is an unambiguous, nonanalytic operation to circumvent branch points. It associates an analytic function $f_{\text{AV}}(z)$ to an analytic function $f(z)$. However, in general, $f_{\text{AV}}(z)$ is not analytically related to $f(z)$. As a consequence, the complex hyperplane $\mathcal{P}$ of the complexified external momenta turns out to be divided into disjoint regions of analyticity and the amplitudes $\mathcal{M}$ are separately analytic in each region. We call this property *regionwise analyticity*.

The main region is the Euclidean one, which contains the purely imaginary energies. There, the Wick rotation is analytic, because no average continuation is necessary. The other regions can be reached unambiguously from the main one by means of the average continuation. It is worth to emphasize that the relative simplicity of the average continuation makes calculations doable with not much more effort than usual.

If the theory contains only physical particles, then we have analyticity, which means that it is sufficient to compute a loop diagram in *any* open subset of $\mathcal{P}$ to derive it everywhere in $\mathcal{P}$ by means of the analytic continuation. If the theory contains fakeons, then we have regionwise analyticity. In particular, it is sufficient to compute a loop diagram in any open subset of the Euclidean region to derive it everywhere in $\mathcal{P}$ by means of the average continuation. It is not sufficient to know the amplitude in regions different from the Euclidean one. Indeed, there are many functions $f(z)$ whose average continuation $f_{\text{AV}}(z)$ vanishes identically (for example, $f(z) = \sqrt{z}$), so in general it is impossible to reconstruct an $f(z)$ from its average continuation $f_{\text{AV}}(z)$.

In fig. 3 analyticity and regionwise analyticity are compared. The basic difference is that some branch cuts are replaced by extended regions.

We have anticipated that there exists a third type of fakeon, which is the *thick fakeon*. The fakeon plus and the fakeon minus have infinitesimal widths $\mathcal{E}$ and the limit $\mathcal{E} \to 0$ must be taken as explained above. Instead, the thick fakeon has a finite nonzero width $\mathcal{E}$ and propagator

$$\mathbb{G}_{\text{c}}(p, m, \mathcal{E}^2) = aG_+(p, m, \mathcal{E}^2) - a^* G_-(p, m, \mathcal{E}^2), \tag{2.12}$$

where a is a complex coefficient. The squared masses $m^2 \pm i\mathcal{E}^2$ are also complex, with a nonnegative real part m^2. There is no option to quantize the poles of $\mathbb{G}_{\text{c}}$ as physical particles.

The thick fakeons are also called Lee-Wick fakeons, since they are the ones that appear in the Lee-Wick models [8]. Before moving on, we recall how the quantization of the thick fakeon works, because it requires supplementary operations. Let us start over, from the Lee-Wick prescription of fig. 1 for the integrals on the loop energies. That prescription is not sufficient to define the Lee-Wick models properly, because it leads to violations of Lorentz invariance and ambiguities [11, 12]. Extra prescriptions were proposed right after the papers of Lee and Wick [12], but they did not remove the ambiguities completely.

Since the Lee-Wick integration prescription involves complex values of the loop energies, it is not possible to have Lorentz invariance above the fakeon thresholds at finite $\mathcal{E}$ by integrating on real loop space momenta. It is necessary to deform the integration domain on the loop space momenta to include complex values for them as well [7, 10]. It can be shown that it is possible to arrange the deformation so as to squeeze the extended regions $\tilde{\mathcal{A}}$ onto branch cuts even if $\mathcal{E} > 0$. At the end, interestingly enough, the amplitude above the fakeon threshold is still given by the average continuation. Again, the scattering processes that involve the fakeons have zero probability to occur and can be projected away, as required by unitarity.

The Lee-Wick prescription and the deformations of the integration domains on the loop space momenta are the operations that define the nonanalytic Wick rotation of the Euclidean theory, which is equivalent to the average continuation. What makes the whole construction work, by removing the difficulties that prevented to find a sound definition of the Lee-Wick models decades ago, is the concept of fake particle.

A weak feature of the Lee-Wick models is that they are super-renormalizable. Since there are infinitely many of them, we have no way to decide which is the right one. Having infinitely many theories with finitely many parameters is not so different from having one nonrenormalizable theory with infinitely many parameters, like Einstein gravity, after it is equipped with all the counterterms generated by renormalization. Moreover, nature does not seem to favor super-renormalizable theories for high-energy physics.

Instead, the theory of quantum gravity (1.1)-(1.2) considered here, defined using the fakeon prescription (2.9) for the spin-2 massive field $\chi_{\mu\nu}$, is essentially unique, because it is strictly renormalizable. More precisely, it contains a finite number of independent parameters and can be quantized in a finite number of consistent ways. Under many respects, it is the theory that is most similar to the standard model, to which the matter sector can be attached with no effort.

2.1 Prescription and projection

Summarizing, the quantization prescription is defined by introducing two infinitesimal widths ϵ and $\mathcal{E}$ in the propagators as follows:

(a) replace p^2 with $p^2 + i\epsilon$ everywhere in the denominators, where p denotes the momentum;

(b) treat the poles you want to convert into fakeons by means of replacements of the form

$$\frac{1}{p^2 - m^2 + i\epsilon} \rightarrow \frac{p^2 - m^2}{(p^2 - m^2 + i\epsilon)^2 + \mathcal{E}^4}; \tag{2.13}$$

(c) calculate the diagrams in the Euclidean framework, nonanalytically Wick rotate them as explained above, then make ϵ tend to zero first and $\mathcal{E}$ tend to zero last.

Clearly, these rules are meant to be applied in momentum space. It is harder to work out the quantization rules directly in coordinate space and study the nonanalytic Wick rotation there. Thus, we might also want to specify that

(d) the amplitudes and the loop integrals must be evaluated in momentum space and then Fourier transformed to coordinate space.

An equivalent set of quantization rules is obtained by combining (a) with (d) and the requirement that, in evaluating the loop integrals,

(e) every threshold involving a fakeon must be overcome by means of the average continuation.

Now, let us quantize the actions (1.1) and (1.2). Formula (1.3) shows that the $\chi_{\mu\nu}$ quadratic action has the wrong overall sign, so the field $\chi_{\mu\nu}$ must be quantized as a fakeon, according to step (b). Instead, ϕ can be quantized either as a fakeon or a physical particle. Depending on which option we choose, we have a graviton/fakeon/fakeon (GFF) theory or a graviton/scalar/fakeon (GSF) theory.

Since the processes that involve the fakeons have zero probabilities to occur, the fakeons can be consistently projected away from the physical spectrum. Thus, the physical states are obtained by acting on the vacuum $|0\rangle$ by means of the creation operators of the physical particles only, ignoring the creation operators of the fakeons (which are $a_\chi^\dagger$ and $a_\phi^\dagger$ in the GFF theory, just $a_\chi^\dagger$ in the GSF theory). This defines, after Cauchy completion, the Fock space V of the physical states, which is a proper subspace of the total Fock space W. The projection $W \rightarrow V$ is called *fakeon projection*. The free Hamiltonian is bounded from below in V (but obviously not in W).

It is helpful to make a comparison between the fakeon projection and the *gauge projection*, by which we mean the projection involved in the gauge theories, which concerns the Faddeev-Popov ghosts and the longitudinal and temporal components of the gauge fields. Let us focus our attention on the gauge-fixed action S_{gf}. The main reason why the gauge-fixing condition is not solved explicitly, usually, is that if we inserted its solution into S_{gf} we would obtain a nonlocal action, which is much more difficult to deal with. It is preferable to keep the Faddeev-Popov ghosts and the longitudinal and temporal modes of the gauge fields till the very end, work in a local framework and perform the gauge projection only when strictly needed.

The fakeon projection also introduces nonlocalities (see sections 4 and 5 for details). The virtue of the interim, unprojected actions (1.1) and (1.2) is that they allow us to work within local frameworks, pretty much like the gauge-fixed actions. However, there is a crucial difference between the fakeon projection and the gauge projection. By changing the gauge fixing it is possible to reach a gauge (the Coulomb gauge), where the gauge projection acts not only on the initial and final states, but even inside the loop diagrams. Thanks to this, the gauge modes disappear from everywhere. It is not possible to achieve an analogous result by means of the fakeon projection, which does act on the initial and final states, but cannot reach inside the loop diagrams. The net result is that the fake particles leave an important remnant, which is the violation of causality at energies larger than their masses. This is also the reason, why *the fakeons must be massive*, otherwise causality would be violated at all energies. We stress that the fakeon projection is the only projection known at present that is consistent with unitarity even it does not follow from a gauge or symmetry principle.

3 Microcausality

Thanks to the average continuation, calculating loop diagrams with the fakeon prescription does not require much more effort than calculating diagrams with the ordinary prescriptions [13, 2]. Among the first things to compute, we mention the one-loop self-energy diagrams, which give the physical masses $\bar{m}$ and the physical widths Γ.

In the case of the fakeons, if we resum the bubble diagrams B, we get the dressed propagators

$$\bar{\mathbb{G}}_{\pm} = \mathbb{G}_{\pm} + \mathbb{G}_{\pm} B \mathbb{G}_{\pm} + \mathbb{G}_{\pm} B \mathbb{G}_{\pm} B \mathbb{G}_{\pm} + \cdots = \frac{1}{\mathbb{G}_{\pm}^{-1} - B}. \tag{3.1}$$

After the resummation, we can take $\mathcal{E}$ to zero, which gives, around the physical peak $p^2 = \bar{m}^2$,

$$\bar{\mathbb{G}}_{\pm} \sim \pm \frac{Z}{p^2 - \bar{m}^2 + i\bar{m}\Gamma_{\pm}} = \pm Z G_{+}(p, \bar{m}, \bar{m}\Gamma_{\pm}),$$

where Z is the normalization factor. The optical theorem implies

$$\mathrm{Im}[\mp Z G_{+}(p, \bar{m}, \bar{m}\Gamma_{\pm})] = \frac{\bar{m} Z(\pm\Gamma_{\pm})}{(p^2 - \bar{m}^2)^2 + \bar{m}^2 \Gamma_{\pm}^2} \geqslant 0,$$

i.e. $\Gamma_{+} > 0$, $\Gamma_{-} < 0$. We thus learn that a fakeon plus has a positive width, while a fakeon minus has a negative width. Moreover, the limits $\Gamma_{\pm} \rightarrow 0^{\pm}$ give

$$\lim_{\Gamma_{\pm} \rightarrow 0^{\pm}} \mathrm{Im}[\mp Z G_{+}(p, \bar{m}, \bar{m}\Gamma_{\pm})] \sim \pi Z \delta(p^2 - \bar{m}^2) \tag{3.2}$$

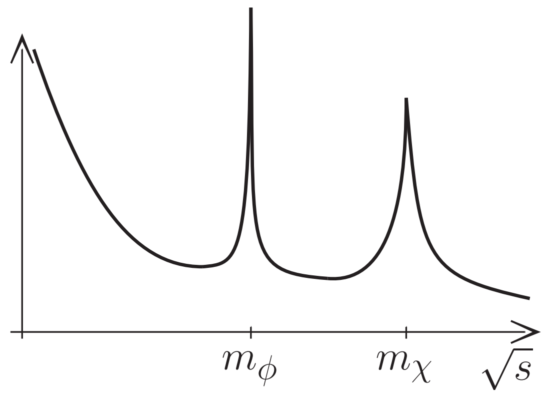


Figure 4: Cross section

in both cases. This result shows that if we just watch the decay products of a fakeon, we have the illusion that a true particle exists, no matter how small its width is and no matter whether the width is positive or negative.

At the same time, the resummation (3.1) is legitimate only if $p^2 - m^2$ is large enough, which means that it misses the contact terms $\delta(p^2 - m^2)$, $\delta'(p^2 - m^2)$, etc. In general, the resummation of the contact terms gives

$$\sigma\pi Z\delta(p^2 - \bar{m}^2) \tag{3.3}$$

for $\Gamma_\pm \to 0^\pm$, where $\sigma = 1, 0, -1$ in the case of a physical particle, a fakeon and a ghost, respectively. For example, at the tree level a physical particle gives (2.7), a ghost gives the opposite of (2.7), while a fakeon gives

$$\mathrm{Im}[-\mathbb{G}_\pm(p, m, \mathcal{E}^2)] = \mathrm{Im}[-\mathbb{G}_c(p, m, \mathcal{E}^2)] = 0. \tag{3.4}$$

This formula tells us what we see if we do not let the fakeons decay, but try to detect them "on the fly". The answer is that we see precisely nothing. Only in the case of a physical particle what we infer from the indirect observation, which is encoded in formula (3.2), coincides with what we get from the direct observation, which is given by formula (3.3).

In other words, if we restrict our attention to the processes where the fakeon does decay, the total cross section gives a plot like the one shown in fig. 4, which coincides with the plot we would see in the case of physical particles. However, a physical particle can also be detected before it decays (at least in principle), when it is still "alive", since it belongs to the physical spectrum. On the other hand, a fakeon cannot be detected directly, because the probability to produce it is zero. The only ways to "see" a fakeon are indirect, by means of its decay products and the interactions it mediates. These facts justify the name, "fake particle", or fakeon, for the new entity.

Since $\chi_{\mu\nu}$ is a fakeon minus, its width Γ_χ is negative. Precisely, the calculation gives [2]

$$\Gamma_\chi = -C\frac{m_\chi^3}{M_{\rm Pl}^2}, \qquad C = \frac{1}{120}(N_s + 6N_f + 12N_v), \tag{3.5}$$

in the case of the GFF theory, where N_s, N_f and N_v are the numbers of (physical) scalars, Dirac fermions (plus one half the number of Weyl fermions) and gauge vectors, respectively. We are assuming that the masses of the matter fields are much smaller than m_χ, otherwise there are corrections [2]. In the case of the GSF theory, there is also a correction due to ϕ, which depends on m_ϕ. The graviton and the fakeons do not contribute to Γ_χ.

The negative sign of Γ_χ signals the violation of microcausality at the quantum level. Consider the Breit-Wigner distribution and its Fourier transform:

$$\frac{i}{E - \bar{m} + i\frac{\Gamma}{2}}, \qquad G_{\rm BW}(t) = \text{sgn}(t)\theta(\Gamma t)\exp\left(-i\bar{m}t - \frac{\Gamma t}{2}\right), \tag{3.6}$$

where sgn(t) is the sign of t. Observe that $\exp(-\Gamma t/2)$ is always a dumping factor. If $J(t)$ denotes an external source, the response reads

$$\int_{-\infty}^{+\infty} \mathrm{d}t' G_{\rm BW}(t-t')J(t') = -\int_t^{\infty} \mathrm{d}t' \,\mathrm{e}^{-\left(i\bar{m}+\frac{\Gamma}{2}\right)(t-t')}J(t') \tag{3.7}$$

for $\Gamma < 0$. This formula shows that when the width is negative it is necessary to know the values of the source J in the future. However, due to the (dumping or oscillating) exponential factors that appear on the right-hand side of (3.7), it is sufficient, at the practical level, to anticipate the source $J(t')$ just for a little bit of future, such that $t' - t \lesssim \tau \equiv \min(1/\bar{m}, 2/|\Gamma|)$.

In other words, time, as well as past, present and future, and the concepts of cause and effect, lose meaning for intervals smaller than τ. However, as long as τ is short enough, the possibility of having violations of microcausality in nature is compatible with experiments.

Formula (3.5) shows that the violation of causality cannot be eliminated, since adding physical matter fields can at most increase $|\Gamma_\chi|$, while adding fakeons leaves $|\Gamma_\chi|$ invariant.

As said, the massive scalar ϕ can be a physical particle or a fakeon plus. In either case, its width is positive.

We have no experimental or logical reason to claim that causality should hold up to infinite energies or zero relative distances. On the contrary, we view the violation of microcausality as a major prediction of quantum gravity and concentrate on finding ways to detect its effects. The fakeon mass m_χ is a free parameter at the moment. Its actual value might be smaller, or even much smaller, than the Planck mass M_{Pl}. Thanks to this, it might be possible to detect the first signs of quantum gravity various orders of magnitude below the Planck scale. Moreover, as we show in the next sections, we can also study the problem noperturbatively, by working out the corrections to the field equations of general relativity.

To make progress in this direction, it is helpful to study the remnants of the fakeons in the classical limit. At the tree level, the steps (c) and (e) can be skipped, so the fakeon has the free propagator

$$\frac{p^2-m^2}{(p^2-m^2)^2+\mathcal{E}^4}=\frac{1}{2}\left[\frac{1}{(p^0+i\epsilon)^2-\mathbf{p}^2-m^2}+\frac{1}{(p^0-i\epsilon)^2-\mathbf{p}^2-m^2}\right]=\mathcal{P}\frac{1}{p^2-m^2}. \quad (3.8)$$

The equalities are meant in the sense of distributions, for $\epsilon \to 0$ and $\mathcal{E} \to 0$. We obtain the Cauchy principal value, which is also the half sum of the retarded and advanced Yukawa potentials. Here, the violation of microcausality is due to the advanced Green function.

Since different quantum prescriptions may have the same classical limit, it is not possible to infer the fakeon prescription from the principal value (3.8). Actually, as it stands, (3.8) leads to wrong results, if applied to the loop integrals, because it misses steps (c) and (e), hence it generates the problems of ref. [6] and violates the optical theorem. We stress again that the loop integrals must be calculated from their Euclidean versions by performing the nonanalytic Wick rotation of ref.s [7, 10], or crossing the fakeon thresholds inside the amplitudes by means of the average continuation.

4 Classicization: a toy model

The next goal is to analyze the effects of the fakeons on causality in the classical limit. In this section, we study the nonrelativistic particle as a toy model to illustrate the main properties. Consider the higher-derivative Lagrangian

$$\mathcal{L}_{\mathrm{HD}}=\frac{m}{2}(v^2-\tau^2a^2)-V(x,t), \quad (4.1)$$

where x is the coordinate ($v=\dot{x}$, $a=\ddot{x}$) and τ is a real constant. To begin with, let us take $V(x,t)=-xF_{\mathrm{ext}}(t)$, where $F_{\mathrm{ext}}(t)$ is an external force. The equation of motion $m(a+\tau^2\ddot{a})=F_{\mathrm{ext}}$ has runaway solutions, unless we restrict the configuration space. The restriction can be achieved by writing

$$ma=\frac{1}{1+\tau^2\frac{d^2}{dt^2}}F_{\mathrm{ext}}\equiv\tilde{G}F_{\mathrm{ext}} \quad (4.2)$$

and giving a prescription for the Green function $\tilde{G}$ that acts on F_{ext}. The distribution $\tilde{G}$ that follows from the classical fakeon prescription (3.8) is

$$G_F(u,\tau) = \frac{\sin(|u|/\tau)}{2\tau}, \tag{4.3}$$

where the subscript F stands for "fakeon". The projected equation of motion is then

$$ma(t) = \int_{-\infty}^{\infty} \mathrm{d}u G_F(u,\tau) F_{\text{ext}}(t-u) \equiv \langle F_{\text{ext}}(t) \rangle \tag{4.4}$$

and its degrees of freedom are just the initial position and the initial velocity of the particle. Since

$$\lim_{\tau \to 0} G_F(u,\tau) = \delta(u), \tag{4.5}$$

the equation becomes $ma = F_{\text{ext}}$ in the limit $\tau \to 0$, as expected[1].

The fakeon Green function (4.3) makes the average $\langle F_{\text{ext}}(t) \rangle$ sensitive to both the past and the future, so the runaway solutions have disappeared at the expenses of microcausality. However, the violation of microcausality is "small", since its effects are averaged away by the oscillating behavior of G_F for $|u| \gg \tau$. As in the previous section, it is sufficient to know just "a little bit of future", say $|u| \lesssim \tau$, to predict the future.

If we introduce an auxiliary coordinate Q and make the redefinition $x = q + Q$, we obtain the equivalent Lagrangian

$$\mathcal{L}(q,Q,t) = \frac{m}{2}\dot{q}^2 - \frac{m}{2}\dot{Q}^2 + \frac{m}{2\tau^2}Q^2 - V(q+Q,t). \tag{4.6}$$

The Q-quadratic part has the wrong sign, as expected. The equations of motion

$$m\ddot{q} = -\frac{\partial V}{\partial q}(q+Q,t), \qquad m\ddot{Q} + \frac{m}{\tau^2}Q = \frac{\partial V}{\partial q}(q+Q,t), \tag{4.7}$$

can be projected by interpreting Q as a fake coordinate. This is achieved by solving its equation of motion by means of the classical fakeon prescription [which gives $Q(q,t) = -\tau^2\langle\ddot{q}\rangle$] and substituting the result back into (4.7). So doing, we obtain the projected equation of motion for q:

$$m\ddot{q} = -\left.\frac{\partial V(x,t)}{\partial x}\right|_{x=\langle q \rangle}, \tag{4.8}$$

where the average $\langle q(t) \rangle$ is defined by the last equality of eq. (4.4) with $F_{\text{ext}} \to q$. Again, the runaway solutions have disappeared at the expenses of microcausality.

[1] The quickest way to prove (4.5), pointed out to us by L. Bracci, is to take the derivative of the distribution $\text{sgn}(u)\cos(u/\tau)$, which tends to zero by the Riemann-Lebesgue theorem.

The projected Lagrangian $\mathcal{L}_r(q,t)$ can be obtained by inserting the solution $Q(q,t) = -\tau^2\langle\ddot{q}\rangle$ back into (4.6), which gives

$$\mathcal{L}_r(q,t) = \mathcal{L}(q,Q(q,t),t) = \frac{m}{2}\left(\langle\dot{q}\rangle^2 + 2\tau^2\langle\dot{q}\rangle\langle\dddot{q}\rangle + \tau^2\langle\ddot{q}\rangle^2\right) - V(\langle q\rangle,t). \tag{4.9}$$

Its Lagrange equations are indeed the projected equations of motion (4.8).

Given an arbitrary Lagrangian $L(q,\dot{q},\ddot{q},\dddot{q},\cdots,t)$, the energy is

$$E = -L + \dot{q}\sum_{n=1}^{\infty}\frac{\overleftarrow{\partial_t}^n - (-1)^n\overrightarrow{\partial_t}^n}{\overleftarrow{\partial_t} + \overrightarrow{\partial_t}}\frac{\partial L}{\partial q^{(n)}}, \tag{4.10}$$

where $\partial_t = d/dt$, the arrows specify whether the derivatives act to the left or the right, and $q^{(n)} = \partial_t^n q$. The ratio of derivative operators appearing in (4.10) must be simplified by means of the polynomial identity $(x^n - (-1)^n y^n)/(x+y) = x^{n-1} + \cdots - (-1)^n y^{n-1}$. It is easy to check that the Lagrange equations imply $dE/dt = -\partial L/\partial t$, so E is conserved if t is a cyclic coordinate. Moreover, under arbitrary changes of variables $u = u(q,\dot{q},\ddot{q},\dddot{q},\cdots)$ that do not depend explicitly on t, E can at most go into a function of E, so we can use formula (4.10) with the variables we want. To apply it to $L = \mathcal{L}_r$ it is convenient to use the variable $u = \langle q\rangle = x$, which easily leads to the expression

$$E' = \frac{m}{2}\left(\langle\dot{q}\rangle^2 + 2\tau^2\langle\dot{q}\rangle\langle\dddot{q}\rangle - \tau^2\langle\ddot{q}\rangle^2\right) + V(\langle q\rangle,t). \tag{4.11}$$

An as example, consider the harmonic oscillator, $V = m\omega^2 x^2/2$. The equations of motion give

$$(\nu^2 - \omega^2 - \tau^2\nu^4)\tilde{x}(\nu) = 0, \tag{4.12}$$

where $\tilde{x}(\nu)$ is the Fourier transform of $x(t)$. If $\omega > 1/(2\tau)$ the polynomial $\nu^2 - \omega^2 - \tau^2\nu^4$ has two pairs of complex conjugate zeros, which correspond to thick fakeons. Thus, the projected subspace of configurations is empty. Instead, for $\omega \leqslant 1/(2\tau)$ the polynomial has four real zeros, but only two of them give solutions that have regular $\tau \to 0$ limits. They read

$$x(t) = A\cos(\Omega t + \varphi_0), \qquad \text{where } \Omega = \frac{1}{\tau\sqrt{2}}\sqrt{1 - \sqrt{1 - 4\tau^2\omega^2}} \tag{4.13}$$

and A, φ_0 are constants. The other two solutions are fakeons minus, as can be seen from the residues of their propagators, obtained by replacing the right-hand side of (4.12) with 1. Projecting the fakeons away, the final result is a harmonic oscillator with the modified frequency Ω. The energy (4.11) is

$$E' = \frac{m}{2}\left(\dot{x}^2 + 2\tau^2\dot{x}\dddot{x} - \tau^2\ddot{x}^2\right) + V(x) = \frac{m}{2}A^2\Omega^2\sqrt{1 - 4\tau^2\omega^2} \tag{4.14}$$

and is positive definite on the solutions (4.13).

To quantize the theory (4.6) in the case $\omega \leqslant 1/(2\tau)$, we define the annihilation operators

$$a_\xi = \sqrt{\frac{m\Omega}{2}}\left(\xi + \frac{i}{m\Omega}P_\xi\right), \qquad a_\eta = \sqrt{\frac{m\tilde{\Omega}}{2}}\left(\eta + \frac{i}{m\tilde{\Omega}}P_\eta\right),$$

where $\tilde{\Omega} = \sqrt{1-\tau^2\Omega^2}/\tau$, $\xi = q\cosh\theta - Q\sinh\theta$, $\eta = Q\cosh\theta - q\sinh\theta$, $P_\xi = p\cosh\theta + P\sinh\theta$, $P_\eta = P\cosh\theta + p\sinh\theta$ and θ is such that $\tanh\theta = \Omega^2/\tilde{\Omega}^2$. The commutation rules $[p,q] = [P,Q] = -i$ lead to $[a_\xi, a_\xi^\dagger] = [a_\eta, a_\eta^\dagger] = 1$, $[a_\eta, a_\xi^\dagger] = [a_\eta, a_\xi^\dagger] = [a_\eta, a_\xi] = [a_\eta^\dagger, a_\xi^\dagger] = 0$. The Hamiltonian is

$$H = \Omega\left(a_\xi^\dagger a_\xi + \frac{1}{2}\right) - \tilde{\Omega}\left(a_\eta^\dagger a_\eta + \frac{1}{2}\right).$$

We see that $a_\eta^\dagger$ are the creation operators of the fakeons minus. The physical subspace V is obtained by projecting them away. Specifically, the vacuum $|0\rangle$ satisfies $a_\xi|0\rangle = a_\eta|0\rangle = 0$. The space V is made of the states $|v\rangle$ such that $a_\eta|v\rangle = 0$. It is generated by $(a_\xi^\dagger)^n|0\rangle$ and contains the wave functions of the form

$$\psi(q,Q) = \psi(\xi)\exp\left(-\frac{m\tilde{\Omega}\eta^2}{2}\right).$$

The reduced Hamiltonian reads $H_V = \Omega a_\xi^\dagger a_\xi + (1/2)(\Omega - \tilde{\Omega})$ and is bounded from below in V. Note that H_V must be shifted by a constant to have a regular limit $\tau \to 0$.

5 The classicization of quantum gravity

In this section we generalize the fakeon projection to the classical limit of quantum gravity. For simplicity, we work at $\Lambda_C = 0$, the generalization to $\Lambda_C \neq 0$ being straightforward. The field equations derived from the interim classical action (1.2) read

$$\begin{aligned} &R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{\kappa^2}{\zeta}\left[\mathrm{e}^{3\kappa\phi}fT_{\mathfrak{m}}^{\mu\nu}(\tilde{g}\mathrm{e}^{\kappa\phi},\Phi) + fT_\phi^{\mu\nu}(\tilde{g},\phi) + T_{\mu\nu}^\chi(g,\chi)\right], \\ &\nabla_\mu\nabla^\mu\phi + \frac{m_\phi^2}{\kappa}\left(\mathrm{e}^{\kappa\phi}-1\right)\mathrm{e}^{\kappa\phi} = \frac{\kappa f\mathrm{e}^{3\kappa\phi}}{3\zeta}T_{\mathfrak{m}}^{\mu\nu}(\tilde{g}\mathrm{e}^{\kappa\phi},\Phi)\tilde{g}_{\mu\nu}, \\ &\frac{1}{\sqrt{-g}}\frac{\delta S_\chi(g,\chi)}{\delta\chi_{\mu\nu}} = \mathrm{e}^{3\kappa\phi}fT_{\mathfrak{m}}^{\mu\nu}(\tilde{g}\mathrm{e}^{\kappa\phi},\Phi) + fT_\phi^{\mu\nu}(\tilde{g},\phi), \end{aligned} \tag{5.1}$$

where $T_A^{\mu\nu}(g) = -(2/\sqrt{-g})(\delta S_A(g)/\delta g_{\mu\nu})$ are the energy-momentum tensors ($A = \mathfrak{m}$, ϕ, χ) and $f = \sqrt{\det\tilde{g}_{\rho\sigma}/\det g_{\alpha\beta}}$.

The projection onto the right subspace of configurations works as follows. Solve the third equations of (5.1) for $\chi_{\mu\nu}$ by means of the half sum of the retarded and advanced Green functions. Then insert the solution $\langle\chi_{\mu\nu}\rangle$ into the other two equations. So doing, χ becomes a classical fakeon and the first two lines of (5.1) with $\chi_{\mu\nu} \rightarrow \langle\chi_{\mu\nu}\rangle$ become the projected equations of the GSF theory. They are also the field equations of the finalized classical action

$$\mathcal{S}_{\text{QG}}^{\text{GSF}}(g,\phi,\Phi) = S_{\text{H}}(g) + S_{\chi}(g,\langle\chi\rangle) + S_{\phi}(\bar{g},\phi) + S_{\mathfrak{m}}(\bar{g}\text{e}^{\kappa\phi},\Phi), \tag{5.2}$$

where $\bar{g}_{\mu\nu} = g_{\mu\nu} + 2\langle\chi_{\mu\nu}\rangle$.

If we want to treat ϕ as a fakeon as well (GFF theory), we solve the second and third equations of (5.1) for ϕ and $\chi_{\mu\nu}$ by means of the half sums of the retarded and advanced Green functions and insert the solutions $\langle\phi\rangle$, $\langle\chi_{\mu\nu}\rangle$ into the first equation. The finalized classical action is then

$$\mathcal{S}_{\text{QG}}^{\text{GFF}}(g,\Phi) = S_{\text{H}}(g) + S_{\chi}(g,\langle\chi\rangle) + S_{\phi}(\bar{g},\langle\phi\rangle) + S_{\mathfrak{m}}(\bar{g}\text{e}^{\kappa\langle\phi\rangle},\Phi). \tag{5.3}$$

We can make these operations more explicit by expanding around flat space. The field equations of the action (1.1) can be written in the form

$$\left(\zeta + \alpha\nabla^2\right) G_{\mu\nu} + \frac{\alpha-\xi}{3}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2\right) G = \kappa^2 T_{\mu\nu}, \tag{5.4}$$

where $G_{\mu\nu}$ is the Einstein tensor and $G = g^{\mu\nu}G_{\mu\nu}$ denotes its trace, while

$$\kappa^2 T_{\mu\nu} \equiv \kappa^2 T_{\mathfrak{m}\mu\nu} + \frac{\alpha}{2} g_{\mu\nu} R^{\rho\sigma} R_{\rho\sigma} - 2\alpha R_{\mu\rho\nu\sigma} R^{\rho\sigma} + \frac{2\alpha+\xi}{3} R\left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\right). \tag{5.5}$$

Now, write $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat-space metric, and decompose $G_{\mu\nu}$ as $G^0_{\mu\nu} + (\kappa^2 J_{\mu\nu}/\zeta)$, where $G^0_{\mu\nu}$ denotes its linear part in $h_{\mu\nu}$. We understand that the indices of $\eta_{\mu\nu}$, ∂_μ, $h_{\mu\nu}$ and $G^0_{\mu\nu}$ are raised and lowered by means of $\eta_{\mu\nu}$. Split the left-hand side of the field equation (5.4) into its linear part plus the rest. Precisely, recalling that $\partial^\mu G^0_{\mu\nu} = 0$, write

$$\left(\zeta + \alpha\nabla^2\right) G_{\mu\nu} + \frac{\alpha-\xi}{3}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2\right) G \equiv \mathbb{Q}_{\mu\nu}{}^{\rho\sigma} G^0_{\rho\sigma} + \kappa^2 U_{\mu\nu}, \tag{5.6}$$

where $U_{\mu\nu}$ collects the corrections that are at least quadratic in $h_{\mu\nu}$ and

$$\mathbb{Q}_{\mu\nu}{}^{\rho\sigma} \equiv (\zeta + \alpha\partial^2)\mathbb{I}_{\mu\nu}{}^{\rho\sigma} - \frac{\alpha-\xi}{3}\partial^2\pi_{\mu\nu}\pi^{\rho\sigma}. \tag{5.7}$$

Here, $\mathbb{I}_{\mu\nu}{}^{\rho\sigma}$ is the identity operator for transverse symmetric tensors with two indices and $\pi_{\mu\nu}$ is the spin-1 projector:

$$\mathbb{I}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}(\pi_\mu^\rho \pi_\nu^\sigma + \pi_\mu^\sigma \pi_\nu^\rho), \qquad \pi_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}.$$

The operators $\mathbb{I}_{\mu\nu}{}^{\rho\sigma}$ and $\mathbb{Q}_{\mu\nu}{}^{\rho\sigma}$ satisfy obvious symmetry properties and are transverse: $\partial^\mu \mathbb{I}_{\mu\nu}{}^{\rho\sigma} = \partial^\mu \mathbb{Q}_{\mu\nu}{}^{\rho\sigma} = 0$. Clearly, $\mathbb{I}_{\mu\nu}{}^{\alpha\beta}\mathbb{I}_{\alpha\beta}{}^{\rho\sigma} = \mathbb{I}_{\mu\nu}{}^{\rho\sigma}$. The inverse of $\mathbb{Q}_{\mu\nu}{}^{\rho\sigma}$ is

$$\mathbb{P}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{\zeta + \alpha\partial^2}\left(\mathbb{I}_{\mu\nu}{}^{\rho\sigma} + \frac{\alpha - \xi}{3}\frac{\partial^2}{\zeta + \xi\partial^2}\pi_{\mu\nu}\pi^{\rho\sigma}\right),$$

i.e. $\mathbb{P}_{\mu\nu}{}^{\alpha\beta}\mathbb{Q}_{\alpha\beta}{}^{\rho\sigma} = \mathbb{Q}_{\mu\nu}{}^{\alpha\beta}\mathbb{P}_{\alpha\beta}{}^{\rho\sigma} = \mathbb{I}_{\mu\nu}{}^{\rho\sigma}$.

The projected equations of the GFF theory can be obtained by using (5.6) in (5.4) and inverting $\mathbb{Q}_{\mu\nu}{}^{\rho\sigma}$. This gives

$$G^0_{\mu\nu} = \kappa^2 \mathbb{P}_{\mu\nu}{}^{\rho\sigma}(T_{\rho\sigma} - U_{\rho\sigma}).$$

Using the transversality of $\kappa^2(T_{\mu\nu} - U_{\mu\nu}) = \mathbb{Q}_{\mu\nu}{}^{\rho\sigma}G^0_{\rho\sigma}$ and inserting the classical fakeon prescription (3.8), we obtain

$$G^0_{\mu\nu} = \frac{1}{\bar{M}^2_{\rm Pl}}\left\langle T_{\mu\nu} - U_{\mu\nu} + \frac{r_{\phi\chi}}{3}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle T - U\rangle_\phi\right\rangle_\chi, \tag{5.8}$$

where $T = \eta^{\mu\nu}T_{\mu\nu}$, $U = \eta^{\mu\nu}U_{\mu\nu}$, $r_{\phi\chi} = (m_\phi^2 - m_\chi^2)/(m_\phi^2 m_\chi^2)$ and the average $\langle\cdots\rangle_F$ associated with the fakeon F of mass m_F is defined as

$$\langle \mathcal{O}\rangle_F \equiv \frac{m_F^2}{2}\left[\left.\frac{1}{m_F^2 + \partial^2}\right|_{\rm ret} + \left.\frac{1}{m_F^2 + \partial^2}\right|_{\rm adv}\right]\mathcal{O}. \tag{5.9}$$

In covariant form, the field equations of the GFF theory read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{\bar{M}^2_{\rm Pl}}T^{\rm GFF}_{\mu\nu}, \tag{5.10}$$

where

$$T^{\rm GFF}_{\mu\nu} = \left\langle T_{\mu\nu} - U_{\mu\nu} + \frac{r_{\phi\chi}}{3}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle T - U\rangle_\phi\right\rangle_\chi + J_{\mu\nu}.$$

Note that although $T^{\rm GFF}_{\mu\nu}$ is not manifestly covariant, it is covariant, because the left-hand side of (5.10) is.

Now we consider the GSF theory. Tracing (5.8) with the flat-space metric and using the definition (5.9), we obtain

$$\frac{r_{\phi\chi}}{\bar{M}^2_{\rm Pl}}\langle T - U\rangle_{\phi\chi} = \frac{1}{\bar{M}^2_{\rm Pl}m_\chi^2}\langle T - U\rangle_\chi - \frac{1}{m_\phi^2}G^0,$$

where $G^0 = G^0_{\mu\nu}\eta^{\mu\nu}$. Inserting this formula back into (5.8), we also get

$$G^0_{\mu\nu} + \frac{1}{3m_\phi^2}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)G^0 = \frac{1}{\bar{M}_{\text{Pl}}^2}\left(\langle T_{\mu\nu} - U_{\mu\nu}\rangle_\chi + \frac{1}{3m_\chi^2}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle T - U\rangle_\chi\right).$$

Then, calling

$$K_{\mu\nu} = \frac{\bar{M}_{\text{Pl}}^2}{3m_\phi^2}\left[\left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)G - \left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)G^0\right],$$

we find the covariant form of the projected field equations of the GSF theory, which is

$$G_{\mu\nu} + \frac{1}{3m_\phi^2}\left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)G = \frac{1}{\bar{M}_{\text{Pl}}^2}T^{\text{GSF}}_{\mu\nu}, \tag{5.11}$$

where

$$T^{\text{GSF}}_{\mu\nu} = \langle T_{\mu\nu} - U_{\mu\nu}\rangle_\chi + \frac{1}{3m_\chi^2}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle T - U\rangle_\chi + J_{\mu\nu} + K_{\mu\nu}.$$

The left-hand side of equation (5.11) coincides with the left-hand side of (5.4), divided by ζ, in the limit $\alpha \to 0$, i.e. $m_\chi \to \infty$. Moreover, the right-hand side of (5.11) contains only the average $\langle \cdots \rangle_\chi$. These facts show that the field $\chi_{\mu\nu}$ is integrated out. It is easy to prove that the surviving degrees of freedom are the graviton and the massive scalar ϕ, and that their poles have positive residues.

Instead, the left-hand side of equation (5.10) coincides with the left-hand side of (5.4), divided by ζ, in the limit where both m_χ and m_ϕ are sent to infinity. The right-hand side of (5.10) contains both the averages $\langle \cdots \rangle_\chi$ and $\langle \cdots \rangle_\phi$, which means that both $\chi_{\mu\nu}$ and ϕ are integrated out.

The equations (5.10) and (5.11) are the classical field equations of quantum gravity. They are written so that the initial conditions, or the boundary conditions, are those dictated by their left-hand sides. The right-hand sides must be treated perturbatively in $1/\bar{M}_{\text{Pl}}^2$. The cosmological constant can be reinstated by replacing $T_{\mu\nu}$ with $T_{\mu\nu} + g_{\mu\nu}(\Lambda_C/\kappa^2)$.

The averages $\langle \cdots \rangle_\chi$ and $\langle \cdots \rangle_\phi$ that appear on the right-hand sides of the equations are the main remnants of the classicization. They show that the violations of microcausality and their intrinsic nonlocalities survive the classical limit of quantum gravity. By searching for exact solutions of physical interest and comparing them with the solutions of the Einstein equations, we may identify complex systems and nonperturbative configurations where the effects of the violations of microcausality get amplified enough to become detectable. The masses m_χ and m_ϕ might be sufficiently small to let us uncover the first signs of quantum gravity without having to reach the Planck scale.

6 Conclusions

The finalized classical action of quantum gravity is (5.2) in the case of the GSF theory and (5.3) in the case of the GFF theory. The classical field equations derived from such actions are (5.11) and (5.10), respectively. Among the other things, they show that the violations of microcausality survive the classical limit.

The corrections to general relativity become important at energies comparable with the masses m_χ and m_ϕ. The values of such masses in nature could be much smaller than the Planck mass and still be compatible with every experimental observations made so far. Yet, those values might still be too large to detect new effects in scattering processes and other elementary or perturbative phenomena. For this reason, it may be interesting to study the classicized actions derived here, because they give us the chance to investigate collective, nonperturbative effects. A possibility is to study how the solutions of the field equations of general relativity get modified and search for situations where the violations of microcausality get amplified enough to become detectable. This kind of investigation might also help us put experimental bounds on the values of m_χ and m_ϕ.

It is by no means easy to detect the violations of causality directly, since when we solve a self-consistent system (no external sources being involved), we know in advance what the interactions will be in the future (as functions of the fields), which makes it hard to discriminate what is expected from what is unexpected. However, the corrections to general relativity predicted by the equations (5.10) and (5.11) might help us detect the violations indirectly or test other nontrivial predictions of quantum gravity, maybe from the observations of black holes or by studying the consequences on cosmology.

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