15A4 Renorm

Background Field Method And The Cohomology Of Renormalization

Damiano Anselmi

Dipartimento di Fisica "Enrico Fermi", Università di Pisa,
and INFN, Sezione di Pisa,
Largo B. Pontecorvo 3, 56127 Pisa, Italy

damiano.anselmi@unipi.it

Abstract

Using the background field method and the Batalin-Vilkovisky formalism, we prove a key theorem on the cohomology of perturbatively local functionals of arbitrary ghost numbers, in renormalizable and nonrenormalizable quantum field theories whose gauge symmetries are general covariance, local Lorentz symmetry, non-Abelian Yang-Mills symmetries and Abelian gauge symmetries. Interpolating between the background field approach and the usual, nonbackground approach by means of a canonical transformation, we take advantage of the properties of both approaches and prove that a closed functional is the sum of an exact functional plus a functional that depends only on the physical fields and possibly the ghosts. The assumptions of the theorem are the mathematical versions of general properties that characterize the counterterms and the local contributions to the potential anomalies. This makes the outcome a theorem on the cohomology of renormalization, rather than the whole local cohomology. The result supersedes numerous involved arguments that are available in the literature.

1 Introduction

Locality and gauge invariance allow us to prove that the divergences of perturbative quantum field theory can be subtracted in a renormalization-group invariant way, preserving the cancellation of gauge anomalies to all orders when they vanish at one loop.

Common tricks to handle certain difficulties, e.g. to fix the gauge in a local way, consist of extending the set of the physical fields ϕ to a larger set Φ^α, which includes the Faddeev-Popov ghosts C [1], the antighosts $\bar{C}$ and suitable Lagrange multipliers B for the gauge fixing. Moreover, to keep track of the effects of renormalization on the gauge symmetries, external sources K_α are coupled to the transformations of the fields. The extra fields and the sources simplify several arguments and derivations, but enlarge the set of counterterms and potential anomalies. It is then necessary to show that the enlargement has no impact on the physical quantities.

To ease this task, a canonical formalism is introduced, known as Batalin-Vilkovisky formalism [2], which collects the Ward-Takahashi-Slavnov-Taylor (WTST) identities [3] in a compact form and generalizes the BRST symmetry [4]. The basic properties of the gauge symmetries are incorporated into an extended action $S(\Phi, K) = S_c(\phi) - \int R^\alpha(\Phi)K_\alpha$, where $S_c(\phi)$ is the classical action and the functions $R^\alpha(\Phi)$ are the infinitesimal transformations of the fields Φ^α. The great advantage of the BV formalism is that it relates in a simple way the identities satisfied by the action $S(\Phi, K)$ to the WTST identities satisfied by the generating functional Γ of the one-particle irreducible correlation functions.

A notion of antiparentheses (X, Y) for functionals X, Y is defined, where the fields Φ and the sources K are viewed as conjugate variables. Gauge invariance is lifted to a certain identity obeyed by S, the *master equation* $(S, S) = 0$, and a certain cohomology. The counterterms and the local contributions to the potential anomalies are characterized by being cohomologically closed, i.e. they are local functionals X that satisfy $(S, X) = 0$. A local functional is said to be cohomologically exact, or trivial, if it has the form (S, Y), Y being another local functional. Two local functionals X and Y are said to be cohomologically equivalent if $X - Y = (S, Z)$, Z being another local functional. Throughout this paper, when we speak of local functionals we include the perturbatively local ones. A perturbatively local functional is a functional that can be written as a perturbative expansion where each term of the sum is equal to the spacetime integral of a polynomial function of the fields, the sources and their derivatives, evaluated in the same point.

To show that the enlargement mentioned above does not extend the set of cohomological classes, we must prove that the new local functionals X_{new} that can be built with the extra fields and the sources are all trivial. This problem has been widely studied in the literature.

Kluberg-Stern and Zuber conjectured in ref. [5] that the solution of $(S, X) = 0$ for local functionals X of vanishing ghost number has the expected form in non-Abelian Yang-Mills theory, i.e. it is the sum of a local functional $\mathcal{G}(\phi)$ of the physical fields ϕ plus a trivial term (S, Y).

Motivated by the Kluberg-Stern–Zuber (KSZ) conjecture, several people, starting from Dixon and Taylor [6] and Joglekar and Lee [7], embarked in the brave task of classifying all the local functionals and operators that are cohomologically closed. A strong motivation was to work out the most general solutions of the Wess-Zumino consistency conditions [8] for the classification of anomalies [9, 10, 11]. In refs. [12] several results were generalized and formulated in the context of the BV formalism. In ref. [13] the classification was worked out in detail in the physically interesting case of Einstein–Yang-Mills theories. For a review of this approach, see ref. [14].

Unfortunately, the assumptions under which the Kluberg-Stern–Zuber conjecture holds are too restrictive. In important cases, including the standard model, coupled to quantum gravity or not, there exist nontrivial cohomological classes $X_{\text{new}}(\Phi, K)$ that depend on the sources K. The reasons are the presence of global symmetries and the $U(1)$ factor in the gauge group. It is well known that the hypercharges of the matter fields are not uniquely fixed (up to the overall normalization) by the tree-level standard-model Lagrangian. The extra terms X_{new} are precisely those associated with the free hypercharges. If some counterterms proportional to X_{new} were generated by renormalization, they would jeopardize the cancellation of gauge anomalies. Indeed, the cancellation of gauge anomalies at one loop imposes further constraints on the hypercharges and often fixes them uniquely [15]. Thus, it is crucial to show that renormalization cannot generate the terms X_{new}, even if such terms are cohomologically allowed. In several situations, this result can be achieved with supplementary *ad hoc* arguments [16, 17]. However, a deeper understanding is most welcome.

It is clear that the cohomology we must consider is not the whole cohomology of the local functionals of the fields and the sources, but the *cohomology of renormalization*, that is to say the cohomology of the local functionals that can be generated by renormalization. This is a sort of cohomology with constraints. The crucial point is that it is not enough to characterize the counterterms and the local contributions to the potential anomalies as being cohomologically closed. Indeed, they satisfy more restrictive conditions. In this paper, we convert those conditions into mathematical assumptions and prove a general theorem that bypasses most of the involved arguments offered so far in the literature and provides a better understanding of the matter.

In power-counting renormalizable theories, it is relatively easy to list all the local terms and solve the cohomological problem with a few algebraic manipulations. On the other hand, in nonrenormalizable theories, such as the standard model coupled to quantum gravity, the potential counterterms and the local contributions to anomalies can have arbitrarily large dimensions, which makes their cohomological classification rather involved. The problem is equally hard in renormalizable theories, when we include composite fields of higher dimensions. Most of the arguments that can be found in the literature are extremely involved and unfit to become part of a quantum field theory textbook.

The theorem we prove here is simpler and more to the point. We show that the nontrivial sector of the cohomology of renormalization just depends on the physical fields ϕ and (in the case of functionals of nonvanishing ghost numbers) the ghosts C. Precisely, in general gauge theories whose gauge symmetries possibly include general covariance, local Lorentz symmetry, Abelian gauge symmetries and non-Abelian Yang-Mills symmetries, the solution of the problem $(S, X) = 0$, where $X(\Phi, K)$ is a local functional generated by renormalization, has the form

$$\mathcal{G}(\phi, C) + (S, Y), \tag{1.1}$$

where $\mathcal{G}$ and $Y(\Phi, K)$ are other local functionals.

The key ingredient of the proof is the use of the background field method [18] and the comparison with the usual, nonbackground approach. The background field method was formulated in the context of the BV formalism by Binosi and Quadri in refs. [19][1] and by the present author in ref. [22]. The two approaches differ in some respects and highlight different properties. Here we take advantage of the approach of [22], which offers, in particular, an exhaustive characterization of the counterterms and the local contributions to the potential anomalies. Specifically, it allows us to translate the assumption that X is "generated by renormalization" into simple mathematical requirements. In the end, we manage to prove the theorem with a relatively small effort.

The result (1.1) implies that the antighosts $\bar{C}$, the Lagrange multipliers B and the sources K cannot alter the cohomology of renormalization and offers a better understanding of why renormalization cannot generate the extra terms $X_{\text{new}}(\Phi, K)$.

We stress the main differences between the results of this paper and those of the previous literature, in particular refs. [12, 13, 14]. Those references contain theorems about the *algebraic cohomology* of local functionals. Precisely, they classify the local solutions X of the cohomological problem $(S, X) = 0$ from the purely algebraic point of view. In various cases, K-dependent solutions $X_{\text{new}}(\Phi, K)$ are present, and it is necessary to find *ad hoc* arguments to prove that they are actually not generated by renormalization. Instead, the theorem proved here goes straight to the point and deals directly with the local functionals that can be generated by renormalization, which satisfy $(S, X) = 0$ plus a few other assumptions. The K-dependent functionals are automatically excluded from the cohomology, and the results are much easier to prove.

We mention an important application of our theorem, to be presented in detail in a separate publication. The basic tool to prove the cancellation of gauge anomalies to all orders, once they vanish at one loop, is the Adler-Bardeen theorem [23, 24], which was recently generalized to nonrenormalizable general gauge theories in ref. [17]. A step of the proof given in [17] requires the knowledge of the cohomological properties satisfied by the counterterms. There, a variant of the KSZ conjecture for functionals of vanishing ghost number was proved using the results of refs. [12, 13, 14] and some *ad hoc* arguments. Now, it is possible to upgrade the proof of [17] by using the background field method and the theorem proved here.

[1] See also ref. [20] for a similar approach in the language of WTST identities and the Zinn-Justin equation [21].

The paper is organized as follows. In section 2 we build the background field action and work out its relation with the ordinary, nonbackground action. In section 3 we state and prove the theorem, after listing the mathematical assumptions and motivating them physically. The proof is split into four main steps to make the reader better appreciate the tricks we use and the guiding philosophy. Section 4 contains the conclusions. The appendix collects some useful formulas from previous references.

2 Background field method in the Batalin-Vilkovisky formalism

We assume that the gauge symmetries of the theory are general covariance, local Lorentz symmetry, Abelian gauge symmetries and non-Abelian Yang-Mills symmetries, or a subset of these. What such symmetries have in common is that (*i*) the infinitesimal gauge transformations of the physical fields ϕ are linear functions of ϕ, and (*ii*) the closure relations are ϕ independent. These features are crucial, because they endow the background field method with the renormalization properties we need. An example of symmetry that does not obey these assumptions is local supersymmetry.

The Batalin-Vilkovisky formalism is convenient for studying general gauge theories. It is a type of canonical formalism, where the conjugate variables are the fields Φ^α and certain external sources K_α coupled to the Φ transformations. The fields Φ^α and the sources K_α have statistics ε_α and $\varepsilon_\alpha + 1$, respectively, where ε_α is to 0 mod 2 for bosons and 1 mod 2 for fermions. A notion of *antiparentheses*

$$(X, Y) \equiv \int \left(\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K_\alpha} - \frac{\delta_r X}{\delta K_\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} \right) \tag{2.1}$$

is introduced, where X and Y are functionals of Φ and K, the subscripts l and r in δ_l and δ_r denote the left and right functional derivatives, respectively, and the integral is over spacetime points associated with repeated indices.

The set of fields $\Phi^\alpha = \{\phi^i, C^I, \bar{C}^I, B^I\}$ contains the classical fields ϕ^i, the Faddeev-Popov ghosts C^I, the antighosts $\bar{C}^I$ and the Lagrange multipliers B^I for the gauge fixing. The action $S(\Phi, K)$ is a local functional that solves the master equation $(S, S) = 0$ and coincides with the classical action $S_c(\phi)$ at $\bar{C} = B = K = 0$. The terms that are linear in $K_\alpha = \{K_\phi^i, K_C^I, K_{\bar{C}}^I, K_B^I\}$ collect the infinitesimal transformations $R^\alpha(\Phi)$ of the fields. With the gauge symmetries we are considering here, the master equation admits the simple solution

$$\mathcal{S}(\Phi, K) = S_c(\phi) - \int R^\alpha(\Phi) K_\alpha = S_c(\phi) - \int R_\phi^i(\phi, C) K_\phi^i - \int R_C^I(C) K_C^I - \int B^I K_{\bar{C}}^I, \tag{2.2}$$

which is linear in K. Explicit expressions of the functions $R^\alpha(\Phi)$ can be found in the appendix. Note that each $R^\alpha(\Phi)$ is at most quadratic in Φ.

Several operations, such as the gauge fixing, can be performed by means of canonical transformations, which are the transformations $\Phi, K \rightarrow \Phi', K'$ that preserve the antiparentheses (2.1). They can be derived from a generating functional $F(\Phi, K')$ of fermionic statistics, by means of the formulas

$$\Phi^{\alpha\prime} = \frac{\delta F}{\delta K'_\alpha}, \qquad K_\alpha = \frac{\delta F}{\delta \Phi^\alpha}.$$

Given a functional $\chi(\Phi, K)$ that behaves as a scalar, we write its transformation law $\chi'(\Phi', K') = \chi(\Phi, K)$ in a compact form as $\chi' = F\chi$.

Background field action

In the framework of the BV formalism, the background field method can be implemented as follows. Let $\underline{\Phi}$ and $\underline{K}$ denote the background fields and the background sources. We associate background fields with just the physical fields ϕ and the ghosts C, but not the antighosts $\bar{C}$ and the Lagrange multipliers B. Thus, we have $\underline{\Phi}^\alpha = \{\underline{\phi}^i, \underline{C}^I, 0, 0\}$, $\underline{K}_\alpha = \{\underline{K}^i_\phi, \underline{K}^I_C, 0, 0\}$ and $R^\alpha(\underline{\Phi}) = \{R^i_\phi(\underline{\phi}, \underline{C}), R^I_C(\underline{C}), 0, 0\}$.

One starts from the action

$$S(\Phi, K, \underline{\Phi}, \underline{K}) = S_c(\phi) - \int R^\alpha(\Phi) K_\alpha - \int R^\alpha(\underline{\Phi}) \underline{K}_\alpha, \tag{2.3}$$

which is obtained from (2.2) by adding a background copy with vanishing classical action. Obviously, the action (2.3) satisfies the master equation

$$[\![S, S]\!] = 0, \tag{2.4}$$

where the squared antiparentheses are defined as

$$[\![X, Y]\!] \equiv \int \left(\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K_\alpha} - \frac{\delta_r X}{\delta K_\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} + \frac{\delta_r X}{\delta \underline{\Phi}^\alpha} \frac{\delta_l Y}{\delta \underline{K}_\alpha} - \frac{\delta_r X}{\delta \underline{K}_\alpha} \frac{\delta_l Y}{\delta \underline{\Phi}^\alpha} \right)$$

and act on functionals X and Y of Φ, K, $\underline{\Phi}$ and $\underline{K}$.

The background shift is the canonical transformation generated by[2]

$$F_{\mathrm{b}}(\Phi, \underline{\Phi}, K', \underline{K}') = \int (\Phi^\alpha - \underline{\Phi}^\alpha) K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha, \tag{2.5}$$

which gives the action $F_{\mathrm{b}} S$. The new fields Φ^α are called quantum fields and the sources K_α are called quantum sources.

The symmetry transformations $R^i(\phi, C)$ of ϕ^i are turned into the transformations $R^i(\phi + \underline{\phi}, C + \underline{C})$ of $\phi^i + \underline{\phi}^i$, which decompose into the sum of

[2]In this paper, the fields and sources with primes are the transformed ones. This choice is opposite to the one of [22], which is why there are some sign differences with respect to the formulas of that paper.

(i) the background transformations $R^i(\underline{\phi},\underline{C})$ of $\underline{\phi}^i$, plus

(ii) the transformations $R^i(\phi+\underline{\phi},C+\underline{C})-R^i(\underline{\phi},\underline{C})$ of ϕ^i.

Recalling that the functions $R^i(\phi,C)$ are proportional to C, the transformations (ii) split into the sum of

(a) the quantum transformations $R^i(\phi+\underline{\phi},C)$ of ϕ^i, which are given by the $\underline{C}$-independent contributions, plus

(b) the background transformations $R^i(\phi+\underline{\phi},\underline{C})-R^i(\underline{\phi},\underline{C})$ of ϕ^i, which are given by the $\underline{C}$-dependent contributions.

Something similar happens to the symmetry transformations $R^I(C)$ of the ghosts C. Using the fact that the functions $R^I(C)$ depend quadratically on C, the quantum transformations of C^I are just $R^I(C)$, and the background transformations of C^I are $\int \underline{C}^J(\delta_l R^I(C)/\delta C^J)$, while the background transformations of $\underline{C}^I$ are $R^I(\underline{C})$. The background fields have trivial quantum transformations, because they are external fields from the quantum field theoretical point of view.

The background transformations of $\bar{C}$ and B remain trivial after the shift due to F_{b}, which is not what we want. We can adjust them by making the further transformation generated by

$$F_{\text{nm}}(\Phi,\underline{\Phi},K',\underline{K}')=\int\Phi^\alpha K'_\alpha+\int\underline{\Phi}^\alpha\underline{K}'_\alpha-\int\mathcal{R}^I_{\bar{C}}(\bar{C},\underline{C})K^{I\prime}_B, \tag{2.6}$$

where $\mathcal{R}^I_{\bar{C}}(\bar{C},\underline{C})$ is the antighost background transformation. More explicitly, the last term of (2.6) (together with the minus sign in front of it) is

$$\int(gf^{abc}\underline{C}^b\bar{C}^c+\underline{C}^\rho\partial_\rho\bar{C}^a)K^{a\prime}_B+\int(2\underline{C}^{\hat{a}\hat{c}}\eta_{\hat{c}\hat{d}}\bar{C}^{\hat{d}\hat{b}}+\underline{C}^\rho\partial_\rho\bar{C}^{\hat{a}\hat{b}})K'_{\hat{a}\hat{b}B}+\int\left(\underline{C}^\rho\partial_\rho\bar{C}_\mu-\bar{C}_\rho\partial_\mu\underline{C}^\rho\right)K^{\mu\prime}_B, \tag{2.7}$$

for Yang-Mills gauge symmetries (including the Abelian ones), local Lorentz symmetry and diffeomorphisms, where the indices $\hat{a},\hat{b},\ldots$ are local Lorentz indices.

At this point, all the quantum fields transform as matter fields under the background transformations. For example, a Yang-Mills gauge potential A^a_μ behaves as a vector in the adjoint representation, instead of a connection. Complete formulas of the background transformations are given below.

The theory must be gauge fixed in a background invariant way. This can be achieved by means of a canonical transformation generated by

$$F_{\text{gf}}(\Phi,\underline{\Phi},K',\underline{K}')=\int\ \Phi^\alpha K'_\alpha+\int\ \underline{\Phi}^\alpha\underline{K}'_\alpha-\Psi(\Phi,\underline{\phi}),$$

where the gauge fermion $\Psi(\Phi,\underline{\phi})$ is invariant under background transformations. Typically, we choose

$$\Psi(\Phi,\underline{\phi})=\int\bar{C}^I\left(G^{Ii}(\underline{\phi},\partial)\phi^i+\zeta_{IJ}(\underline{\phi},\partial)B^J\right), \tag{2.8}$$

where $G^{Ii}(\underline{\phi},\partial)\phi^i$ are the gauge-fixing functions. The operator matrix $\zeta_{IJ}(\underline{\phi},\partial)$ is nonsingular at $\underline{\phi}=0$ and symmetric. Invariance under background transformations can be easily ensured, since $\bar{C}^I$ and B^I transform as matter fields, while $\underline{\phi}$ and the plain derivative ∂ can be combined into the background covariant derivative.

For example, we can take

$$\Psi = \int \sqrt{|\underline{g}|}\left[\bar{C}^a\left(\underline{g}^{\mu\nu}\underline{D}_\mu A^a_\nu + \zeta_{ab}B^b\right) + \bar{C}_{\hat{a}\hat{b}}\left(\underline{e}^{\rho\hat{a}}\underline{g}^{\mu\nu}\underline{D}_\mu\underline{D}_\nu f^{\hat{b}}_\rho + \frac{\zeta_2}{2}B^{\hat{a}\hat{b}} + \frac{\zeta_3}{2}\underline{g}^{\mu\nu}\underline{D}_\mu\underline{D}_\nu B^{\hat{a}\hat{b}}\right)\right]$$
$$+\int \sqrt{|\underline{g}|}\bar{C}_\mu\left[\underline{g}^{\mu\nu}\underline{g}^{\rho\sigma}\left(\underline{D}_\rho h_{\sigma\nu} + \zeta_4\underline{D}_\nu h_{\rho\sigma}\right) + \frac{\zeta_5}{2}\underline{g}^{\mu\nu}B_\nu\right].$$

Here $\underline{A}^a_\mu$, $\underline{e}^a_\mu$ and $\underline{g}_{\mu\nu}$ denote the background gauge fields, vielbein and metric, respectively, while A^a_μ, f^a_μ and $h_{\mu\nu}$ are the respective quantum fluctuations. The tensor ζ_{ab} is constant and proportional to the identity in every simple subgroup of the Yang-Mills gauge group, while ζ_i are other constants. Finally, $\underline{D}$ denotes the covariant derivative on the background fields.

The three canonical transformations listed so far can be easily composed using the theorems of ref. [25], recalled in the appendix. In our case, it suffices to use formula (A.3), because the nontrivial sectors of the canonical transformations do not contain any pairs of conjugate variables besides B and K'_B, on which they depend linearly. The corrections to (A.3) contain second or higher derivatives, as shown in (A.2), so they vanish. The composition gives the generating functional

$$F_{\rm gf}F_{\rm nm}F_{\rm b} = \int(\Phi^\alpha - \underline{\Phi}^\alpha)K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha - \int \mathcal{R}^I_{\bar{C}}(\bar{C},\underline{C})K^{I\prime}_B - \Psi(\Phi-\underline{\Phi},\underline{\phi}) + \int \bar{C}^I\zeta_{IJ}(\underline{\phi},\partial)\mathcal{R}^J_{\bar{C}}(\bar{C},\underline{C}).$$

Applying this canonical transformation to the action (2.3), we obtain the background field gauge-fixed action

$$S_{\rm b} = F_{\rm gf}F_{\rm nm}F_{\rm b}S, \tag{2.9}$$

which can be decomposed as the sum of a quantum action $\hat{S}_{\rm b}$ plus a background action $\bar{S}_{\rm b}$. Precisely, we have

$$S_{\rm b}(\Phi,\underline{\Phi},K,\underline{K}) = \hat{S}_{\rm b}(\Phi,\underline{\phi},K) + \bar{S}_{\rm b}(\Phi,\underline{\Phi},K,\underline{K}), \tag{2.10}$$

where

$$\hat{S}_{\rm b}(\Phi,\underline{\phi},K) = S_c(\phi+\underline{\phi}) - \int R^\alpha(\phi+\underline{\phi},C,\bar{C},B)\tilde{K}_\alpha, \tag{2.11}$$

$$\bar{S}_{\rm b}(\Phi,\underline{\Phi},K,\underline{K}) = -\int \mathcal{R}^\alpha(\Phi,\underline{C})K_\alpha - \int R^\alpha(\underline{\Phi})\underline{K}_\alpha. \tag{2.12}$$

Here the tilde sources $\tilde{K}_\alpha$ coincide with K_α apart from $\tilde{K}^i_\phi$ and $\tilde{K}^I_{\bar{C}}$, which are given by

$$\tilde{K}^i_\phi = K^i_\phi - \bar{C}^I G^{Ii}(\underline{\phi},-\overleftarrow{\partial}), \qquad \tilde{K}^I_{\bar{C}} = K^I_{\bar{C}} - G^{Ii}(\underline{\phi},\partial)\phi^i - \zeta_{IJ}(\underline{\phi},\partial)B^J.$$

The source-dependent sectors of the actions $\hat{S}_{\text{b}}$ and $\bar{S}_{\text{b}}$ encode the quantum transformations and the background transformations, respectively. The curly R denotes the background transformations of the quantum fields. Those of the antighosts are $\mathcal{R}_{\bar{C}}^{I}$, and those of ϕ and C are given by the formula

$$\mathcal{R}^{\alpha}(\Phi,\underline{C})=R^{\alpha}(\Phi+\underline{\Phi})-R^{\alpha}(\underline{\Phi})-R^{\alpha}(\phi+\underline{\phi},C,\bar{C},B), \tag{2.13}$$

while those of the Lagrange multipliers B are given by

$$\mathcal{R}_{B}^{I}(B,C)=-\int B^{J}\frac{\delta_{l}}{\delta\bar{C}^{J}}\mathcal{R}_{\bar{C}}^{I}(\bar{C},C). \tag{2.14}$$

We recall that the transformations R^{α} and $\mathcal{R}^{\alpha}$ obey the identity [22]

$$\int\left(R_{C}^{J}\frac{\delta_{l}}{\delta C^{J}}+\mathcal{R}_{\bar{C}}^{J}(\bar{C},C)\frac{\delta_{l}}{\delta\bar{C}^{J}}\right)\mathcal{R}_{\bar{C}}^{I}(\bar{C},C)=0, \tag{2.15}$$

which can be easily checked in the three cases (2.7) and is useful to work out the formulas (2.11) and (2.12).

For example, in the case of Yang-Mills symmetry the gauge fields A_{μ}^{a} have $R_{\mu}^{a}(\Phi)=\partial_{\mu}C^{a}+gf^{abc}A_{\mu}^{b}C^{c}$, so formula (2.13) gives $\mathcal{R}_{\mu}^{a}(\Phi,\underline{C})=-gf^{abc}\underline{C}^{b}A_{\mu}^{c}$. The ghosts C^{a} have $R_{C}^{a}(\Phi)=-(g/2)f^{abc}C^{b}C^{c}$, so $\mathcal{R}_{C}^{a}(\Phi,\underline{C})=-gf^{abc}\underline{C}^{b}C^{c}$.

Clearly, (2.4) and (2.9) imply that the background field action satisfies the master equation

$$[\![S_{\text{b}},S_{\text{b}}]\!]=0, \tag{2.16}$$

which splits into

$$[\![\hat{S}_{\text{b}},\hat{S}_{\text{b}}]\!]=[\![\hat{S}_{\text{b}},\bar{S}_{\text{b}}]\!]=[\![\bar{S}_{\text{b}},\bar{S}_{\text{b}}]\!]=0. \tag{2.17}$$

To prove the splitting, rescale $\underline{C}$ and $\underline{K}_{C}$ by τ and $1/\tau$, respectively. Then, $\hat{S}_{\text{b}}\rightarrow\hat{S}_{\text{b}}$ and $\bar{S}_{\text{b}}\rightarrow\tau\bar{S}_{\text{b}}$, so $[\![S_{\text{b}},S_{\text{b}}]\!]$ is a quadratic polynomial in τ. Setting the three coefficients of the polynomial to zero gives (2.17).

A way to characterize a functional X that is invariant under the background transformations is to say that it satisfies $[\![\bar{S}_{\text{b}},X]\!]=0$. In particular, the gauge fermion (2.8) must satisfy $[\![\bar{S}_{\text{b}},\Psi]\!]=0$.

Nonbackground action

The nonbackground gauge-fixed action is $F_{\text{gf}}^{\prime}S$, where

$$F_{\text{gf}}^{\prime}(\Phi,\underline{\Phi},K^{\prime},\underline{K}^{\prime})=\int\ \Phi^{\alpha}K_{\alpha}^{\prime}+\int\ \underline{\Phi}^{\alpha}\underline{K}_{\alpha}^{\prime}-\Psi^{\prime}(\Phi), \tag{2.18}$$

is the canonical transformation that performs the gauge fixing. The background fields and sources are inert here. We have included them just for comparison with the background field action. To simplify the renormalization, we take a quadratic gauge fermion $\Psi^{\prime}$, such as

$$\Psi^{\prime}(\Phi)=\int\bar{C}^{I}\left(G^{Ii}(0,\partial)\phi^{i}+\zeta_{IJ}(0,\partial)B^{J}\right).$$

For convenience, we define the nonbackground action S_{nb} by making a further background shift through the transformation F_{b}. So doing, we have

$$S_{\text{nb}} = F_{\text{b}} F'_{\text{gf}} S. \tag{2.19}$$

We easily find the decomposition

$$S_{\text{nb}}(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}_{\text{nb}}(\Phi + \underline{\Phi}, K) + \bar{S}_{\text{nb}}(\Phi, \underline{\Phi}, K, \underline{K}), \tag{2.20}$$

where

$$\hat{S}_{\text{nb}}(\Phi + \underline{\Phi}, K) = S_c(\phi + \underline{\phi}) - \int R^\alpha(\Phi + \underline{\Phi}) \bar{K}_\alpha, \tag{2.21}$$

$$\bar{S}_{\text{nb}}(\underline{\Phi}, K, \underline{K}) = -\int R^\alpha(\underline{\Phi})(\underline{K}_\alpha - K_\alpha), \tag{2.22}$$

and the sources $\bar{K}_\alpha$ with bars coincide with K_α apart from $\bar{K}^i_\phi$ and $\bar{K}^I_{\bar{C}}$, which are

$$\bar{K}^i_\phi = K^i_\phi - \bar{C}^I G^{Ii}(0, -\overleftarrow{\partial}), \qquad \bar{K}^I_{\bar{C}} = K^I_{\bar{C}} - G^{Ii}(0, \partial)(\phi^i + \underline{\phi}^i) - \zeta_{IJ}(0, \partial) B^J. \tag{2.23}$$

The nonbackground action satisfies the master equation

$$[\![S_{\text{nb}}, S_{\text{nb}}]\!] = 0, \tag{2.24}$$

which splits into

$$[\![\hat{S}_{\text{nb}}, \hat{S}_{\text{nb}}]\!] = [\![\hat{S}_{\text{nb}}, \bar{S}_{\text{nb}}]\!] = [\![\bar{S}_{\text{nb}}, \bar{S}_{\text{nb}}]\!] = 0. \tag{2.25}$$

The splitting is a direct consequence of (2.21) and (2.22).

Interpolation between the background and nonbackground actions

To switch back and forth between the background and nonbackground approaches we must make the canonical transformation $F_{\text{b}} F'_{\text{gf}} F_{\text{b}}^{-1} F_{\text{nm}}^{-1} F_{\text{gf}}^{-1}$, since formulas (2.9) and (2.19) give

$$S_{\text{nb}} = F_{\text{b}} F'_{\text{gf}} F_{\text{b}}^{-1} F_{\text{nm}}^{-1} F_{\text{gf}}^{-1} S_{\text{b}}.$$

Using again formula (A.3), we easily find the generating function

$$F_{\text{b}} F'_{\text{gf}} F_{\text{b}}^{-1} F_{\text{nm}}^{-1} F_{\text{gf}}^{-1} = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + \Delta\Psi(\Phi, \underline{\Phi}) + \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C}) K^{I\prime}_B - \int \bar{C}^I \zeta_{IJ}(0, \partial) \mathcal{R}^J_{\bar{C}}(\bar{C}, \underline{C}), \tag{2.26}$$

where

$$\Delta\Psi(\Phi, \underline{\Phi}) = \int \bar{C}^I \left[G^{Ii}(\underline{\phi}, \partial)\phi^i - G^{Ii}(0, \partial)(\phi^i + \underline{\phi}^i) + (\zeta_{IJ}(\underline{\phi}, \partial) - \zeta_{IJ}(0, \partial)) B^J \right].$$

It is convenient to express the canonical transformation (2.26) by means of the componential map $\mathcal{C}$ of ref. [25], also recalled in the appendix. We can write

$$F_{\mathrm{b}}F_{\mathrm{gf}}^{\prime}F_{\mathrm{b}}^{-1}F_{\mathrm{nm}}^{-1}F_{\mathrm{gf}}^{-1}=\mathcal{C}(Q),$$

where

$$Q(\Phi,\underline{\Phi},K^{\prime})=\Delta\Psi(\Phi,\underline{\Phi})+\int\mathcal{R}_{\bar{C}}^{I}(\bar{C},\underline{C})K_{B}^{I\prime}-\frac{1}{2}\int\bar{C}^{I}\left[\zeta_{IJ}(\underline{\phi},\partial)+\zeta_{IJ}(0,\partial)\right]\mathcal{R}_{\bar{C}}^{J}(\bar{C},\underline{C}). \tag{2.27}$$

Again, this functional does not contain any pairs of conjugate variables besides B and $K_{B}^{\prime}$, and depends linearly on them. Thus, the expansion (A.4) effectively reduces to (A.6), which gives (2.26).

We can continuously interpolate between the two approaches by introducing a parameter ξ that varies from 0 to 1 and make the canonical transformation generated by $\mathcal{C}(\xi Q)$, whose inverse is $\mathcal{C}(-\xi Q)$. We find

$$\begin{aligned}\mathcal{C}(\xi Q)=&\int\Phi^{\alpha}K_{\alpha}^{\prime}+\int\underline{\Phi}^{\alpha}\underline{K}_{\alpha}^{\prime}+\xi\Delta\Psi+\xi\int\mathcal{R}_{\bar{C}}^{I}(\bar{C},\underline{C})K_{B}^{I\prime}\\&-\frac{\xi}{2}\int\bar{C}^{I}\left[(1-\xi)\zeta_{IJ}(\underline{\phi},\partial)+(1+\xi)\zeta_{IJ}(0,\partial)\right]\mathcal{R}_{\bar{C}}^{J}(\bar{C},\underline{C}).\end{aligned} \tag{2.28}$$

The form of this transformation is important to simplify some arguments that follow. We define the interpolating action as

$$S_{\xi}=\mathcal{C}(-\xi Q)S_{\mathrm{nb}}=\mathcal{C}(-\xi Q)F_{\mathrm{b}}F_{\mathrm{gf}}^{\prime}S. \tag{2.29}$$

Explicitly, we have

$$S_{\xi}(\Phi,\underline{\Phi},K,\underline{K})=S_{c}(\phi+\underline{\phi})-\int R^{\alpha}(\Phi+\underline{\Phi})\tilde{K}_{\alpha}(\xi)-\xi\int\mathcal{R}_{\bar{C}}^{I}(\bar{C},\underline{C})\tilde{K}_{\bar{C}}^{I}(\xi)-\int R^{\alpha}(\underline{\Phi})(\underline{\tilde{K}}_{\alpha}(\xi)-\tilde{K}_{\alpha}(\xi)), \tag{2.30}$$

where $\tilde{K}_{C}^{I}(\xi)=K_{C}^{I}$,

$$\tilde{K}_{\phi}^{i}(\xi)=K_{\phi}^{i}-\xi\bar{C}^{I}G^{Ii}(\underline{\phi},-\overleftarrow{\partial})-(1-\xi)\bar{C}^{I}G^{Ii}(0,-\overleftarrow{\partial}), \tag{2.31}$$

while the other ξ-dependent tilde sources have expressions that we do not need to report here. It suffices to say that they are linear functions of the quantum fields Φ, apart from $\underline{\tilde{K}}_{\phi}^{i}(\xi)$ and $\underline{\tilde{K}}_{C}^{I}(\xi)$, which are quadratic. Moreover, the differences $\tilde{K}_{\alpha}(\xi)-K_{\alpha}$ and $\underline{\tilde{K}}_{\alpha}(\xi)-\underline{K}_{\alpha}$ are independent of the sources K and $\underline{K}$ other than K_{B}^{I}. In particular, $\delta_{r}S_{\xi}/\delta\underline{K}_{\alpha}=-R^{\alpha}(\underline{\Phi})$. Note that the B-dependent terms of the interpolating action (2.30) are quadratic functions of the quantum fields. Moreover, the $K_{\bar{C}}^{I}$-dependent terms are just

$$-\int\left(B^{I}+\xi\mathcal{R}_{\bar{C}}^{I}(\bar{C},\underline{C})\right)K_{\bar{C}}^{I} \tag{2.32}$$

and the K_B-dependent terms are linear in the quantum fields.

Obviously, S_ξ satisfies the master equation

$$[\![S_\xi, S_\xi]\!] = 0, \tag{2.33}$$

which is a direct consequence of (2.4) and (2.29).

3 The theorem

In this section we state and prove the main theorem of this paper. Since the parameter ξ is introduced by means of the canonical transformation (2.28), the action S_ξ satisfies the differential equation

$$\frac{\partial S_\xi}{\partial \xi} - [\![S_\xi, \tilde{Q}]\!] = 0, \tag{3.1}$$

where $\tilde{Q}(\Phi, \underline{\Phi}, K)$ coincides with $Q(\Phi, \underline{\Phi}, K)$. Note that the sources have no primes in the last expression. We use different symbols for the two functionals $\tilde{Q}$ and Q, since the natural variables of $\tilde{Q}$ are the fields and the sources without primes, while the natural variables of Q are the fields without primes and the sources with primes.

The result (3.1) can be proved as follows. The nonbackground action $S_{\text{nb}} = \mathcal{C}(\xi Q) S_\xi$ obviously satisfies $\partial S_{\text{nb}}/\partial \xi = 0$. By formula (A.1), recalled in the appendix, the transformed action S_ξ satisfies (3.1) as long as $\tilde{Q}$ coincides with the derivative $\partial \mathcal{C}(\xi Q)/\partial \xi$, after the sources with primes are expressed in terms of the fields and the sources without primes. This fact, which can be checked directly in our case, is a general property of the componential map [25]. Indeed, $\tilde{Q}$ plays the role of the Hamiltonian associated with a fictitious time evolution parametrized by ξ and $\mathcal{C}(\xi Q)$ solves the Hamilton-Jacobi equation.

Assumptions

We consider local functionals X_ξ and Y_ξ of Φ, K and $\underline{\Phi}$ that satisfy the equations

$$[\![S_\xi, X_\xi]\!] = 0, \tag{3.2}$$

$$\frac{\partial X_\xi}{\partial \xi} - [\![X_\xi, \tilde{Q}]\!] = [\![S_\xi, Y_\xi]\!]. \tag{3.3}$$

We use the notation X_0, X_1 to denote the functional X_ξ at $\xi = 0$ and $\xi = 1$, respectively. We assume that

(i) X_0 just depends on $\Phi + \underline{\Phi}$ and K.

(ii) X_1 just depends on $\underline{\Phi}$ at $\phi = C = 0$.

(iii) X_ξ and Y_ξ are independent of B, $K_{\bar{C}}$ and K_B.

Let g denote the ghost number of X_ξ. Then, Y_ξ has ghost number $g-1$. For future use, we show that assumption (ii) can be replaced by

(ii') X_1 is a sum of terms that contain g background ghosts $\underline{C}$ at $\phi = C = 0$.

When we set $\phi = C = 0$, no objects of positive ghost number remain, besides $\underline{C}$. If (ii') holds, we can drop all objects that have negative ghost numbers inside X_1, which means $\bar{C}$ and all the sources K except $K_{\bar{C}}$. Since (iii) tells us that X_1 does not depend on B and $K_{\bar{C}}$, (ii) follows. On the other hand, if (ii) holds, X_1 just depends on $\underline{\phi}$ and $\underline{C}$ at $\phi = C = 0$. Then it must be a sum of terms that contain g background ghosts $\underline{C}$, because it has ghost number g. This implies (ii').

Now we explain the meaning of the assumptions listed above. In most applications, the functionals X_ξ and Y_ξ are originated by the counterterms and the local contributions to the potential anomalies. In particular, in the case $g = 0$ the functional X_ξ typically comes from the renormalization of the action, while Y_ξ comes from the renormalization of the average $\langle \tilde{Q} \rangle$.

Without entering into details, we recall that the formulas (3.2) and (3.3) are typical consequences of the regularized and partially renormalized versions of the equations of gauge invariance and gauge independence [16], which are

$$[\![\Gamma_R, \Gamma_R]\!] = 0, \qquad \frac{\partial \Gamma_R}{\partial \xi} - [\![\Gamma_R, \langle \tilde{Q}_R \rangle]\!] = 0, \tag{3.4}$$

where Γ_R is the renormalized Γ functional and $\tilde{Q}_R$ is the renormalized $\tilde{Q}$. When Γ_R and $\tilde{Q}_R$ are just partially renormalized, say up to and including n loops, the equations of gauge invariance and gauge independence give conditions on the $(n+1)$-loop counterterms, which have the forms (3.2) and (3.3).

Condition (i) is dictated by the properties of the nonbackground action (2.20), encoded in the formulas (2.22) and (2.21). If the regularized action $S_{\text{nb}}^{\text{reg}}$ has a similar structure, that is to say

$$S_{\text{nb}}^{\text{reg}}(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}_{\text{nb}}^{\text{reg}}(\Phi + \underline{\Phi}, K) + \bar{S}_{\text{nb}}(\underline{\Phi}, K, \underline{K}),$$

where $\bar{S}_{\text{nb}}$ is unmodified, then the renormalized action also has this structure. Note that $\bar{S}_{\text{nb}}$ is just made of external fields, so it cannot contribute to the nontrivial Feynman diagrams. Moreover, the counterterms do not depend on Φ and $\underline{\Phi}$ separately, but only on their sum, as specified in (i). Finally, $\bar{S}_{\text{nb}}$ is not renormalized, for this very reason.

Similarly, conditions (ii) and (ii') are dictated by the properties of the background field side. Indeed, we know that the action S_{b} of formula (2.10) splits into (2.11) plus (2.12). If the regularized action $S_{\text{b}}^{\text{reg}}$ has a similar structure, that is to say

$$S_{\text{b}}^{\text{reg}}(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}_{\text{b}}^{\text{reg}}(\Phi, \underline{\phi}, K) + \bar{S}_{\text{b}}(\Phi, \underline{\Phi}, K, \underline{K}),$$

the counterterms just depend on Φ, $\underline{\phi}$ and K, so they satisfy property (ii') for $g = 0$. Note that $\bar{S}_{\text{b}}$, which is linear in the quantum fields, does not contribute to any nontrivial one-particle irreducible diagrams. It is also not renormalized, because it vanishes at $\underline{C}=0$, while the counterterms are independent of $\underline{C}$.

The case $g=1$ is also important, because it concerns the potential gauge anomalies. When the dimensional regularization – or any regularization that embeds the dimensional one – is used, the gauge anomalies are encoded in the functional $\langle\llbracket S_{\text{b}}^{\text{reg}},S_{\text{b}}^{\text{reg}}\rrbracket\rangle=\langle\llbracket\hat{S}_{\text{b}}^{\text{reg}},\hat{S}_{\text{b}}^{\text{reg}}\rrbracket\rangle+2\langle\llbracket\hat{S}_{\text{b}}^{\text{reg}},\bar{S}_{\text{b}}\rrbracket\rangle$ (for the proof see, for example, the appendix of ref. [24]), so they are linear functions of $\underline{C}$. Indeed, $\langle\llbracket\hat{S}_{\text{b}}^{\text{reg}},\hat{S}_{\text{b}}^{\text{reg}}\rrbracket\rangle$ is $\underline{C}$ independent, while $\langle\llbracket\hat{S}_{\text{b}}^{\text{reg}},\bar{S}_{\text{b}}\rrbracket\rangle$ contains one power of $\underline{C}$. Thus, $\langle\llbracket S_{\text{b}}^{\text{reg}},S_{\text{b}}^{\text{reg}}\rrbracket\rangle$ must be proportional to $\underline{C}$ at $\phi=C=0$, because it has ghost number 1 and no other fields or sources of positive ghost numbers are present in that case. This ensures that the local contributions to the potential anomalies satisfy property (ii') for $g=1$.

Condition (iii) is also suggested by the properties of renormalization. If the regularization preserves the basic properties of the structure of S_{ξ}, then it is easy to show that the counterterms have the same properties. In particular, we can arrange the regularization so that no nontrivial one-particle irreducible diagrams can be constructed with external legs of types B, $K_{\bar{C}}$ and K_B. It is even easier to ensure that X_{ξ} and Y_{ξ} are independent of $\underline{K}$, since the $\underline{K}$-dependent terms of the action are just made of external fields and do not even need to be regularized.

To make some steps of the derivation clearer, we write $Y_{\xi}=Y_{\xi}(\phi,C,\bar{C},\underline{\phi},\underline{C},\tilde{K}_{\phi}(\xi),K_C,\xi)$, where $\tilde{K}_{\phi}(\xi)$ is given in (2.31). It is not necessary to organize the variables of X_{ξ} in a similar way, so we just write $X_{\xi}=X_{\xi}(\Phi,\underline{\Phi},K,\xi)$.

The assumption (3.2) is actually necessary for just one value of ξ, since then equation (3.3) implies (3.2) for every ξ. Indeed, taking the derivative of $\llbracket S_{\xi},X_{\xi}\rrbracket$ with respect to ξ and using (3.1) and (3.3), we get

$$\frac{\partial}{\partial\xi}\llbracket S_{\xi},X_{\xi}\rrbracket-\llbracket\llbracket S_{\xi},X_{\xi}\rrbracket,\tilde{Q}\rrbracket=0. \tag{3.5}$$

This equation can be easily integrated [16, 22, 25]. The result is that the ξ dependence of $\llbracket S_{\xi},X_{\xi}\rrbracket$ is just due to the canonical transformation generated by $\mathcal{C}(-\xi Q)$. Thus, if $\llbracket S_{\xi},X_{\xi}\rrbracket$ vanishes for some ξ, it vanishes for all ξ.

Statement

We want to prove that there exist local functionals $\mathcal{G}(\phi,C)$ and $\chi_{\xi}(\Phi,\underline{\Phi},K,\xi)$ such that

$$X_{\xi}(\Phi,\underline{\Phi},K,\xi)=\mathcal{G}(\phi+\underline{\phi},C+\underline{C})+\llbracket S_{\xi},\chi_{\xi}\rrbracket. \tag{3.6}$$

Moreover, we show that χ_{ξ} is independent of B, $K_{\bar{C}}$ and K_B, while $\chi_{\xi}(\{0,0,\bar{C},B\},\underline{\Phi},K,1)=0$ and $\chi_{\xi}(\Phi,\underline{\Phi},K,0)$ just depends on $\Phi+\underline{\Phi}$ and K. We also find the explicit expression of χ_{ξ}.

Note that $\mathcal{G}$ is independent of ξ. Taking the squared antiparentheses $\llbracket\cdots\rrbracket$ of both sides of (3.6) with S_{ξ} and using (2.33) and (3.2), we get $\llbracket S_{\xi},\mathcal{G}\rrbracket=0$. In the case $g=0$, the functional $\mathcal{G}$ is C independent and gauge invariant.

For convenience, we divide the proof into four steps.

Step 1 of proof: Interpolation between the background and nonbackground sides

Consider the functionals

$$S_{R\xi}=S_{\xi}+wX_{\xi},\qquad \tilde{Q}_{R\xi}=\tilde{Q}+wY_{\xi}, \tag{3.7}$$

where w is a constant parameter such that $w^2=0$. If the ghost number g of X_{ξ} is odd, we can take $w=\varpi$, where ϖ is constant and anticommuting. If g is even, we can take $w=\varpi\varpi'$, where ϖ' is constant, anticommuting and different from ϖ. We introduce the parameter w to make the first order of the w Taylor expansion exact, which simplifies several arguments. We have used the subscript R in (3.7), because in several practical applications X_{ξ} and Y_{ξ} are counterterms, while $S_{R\xi}$ and $\tilde{Q}_{R\xi}$ are renormalized functionals.

Combining $[\![S_{\xi},S_{\xi}]\!]=0$ with (3.2) and (3.1) with (3.3), we get

$$[\![S_{R\xi},S_{R\xi}]\!]=0,\qquad \frac{\partial S_{R\xi}}{\partial\xi}=[\![S_{R\xi},\tilde{Q}_{R\xi}]\!], \tag{3.8}$$

which are the analogues of the equations in (3.4). The second formula implies that the canonical transformation generated by

$$F_{\xi}(\Phi,\underline{\Phi},K',\underline{K}',\xi)=\mathcal{C}(\xi Q)+w\int_0^{\xi}\mathrm{d}\xi' Y_{\xi}(\phi,C,\bar{C},\underline{\phi},\underline{C},\tilde{K}'_{\phi},K'_C,\xi'), \tag{3.9}$$

where

$$\tilde{K}^{i\prime}_{\phi}=K^{i\prime}_{\phi}-\bar{C}^I G^{Ii}(0,-\overleftarrow{\partial}), \tag{3.10}$$

interpolates between the nonbackground and background values of the action $S_{R\xi}$, which are $S_{\mathrm{nb}}+wX_0\equiv S_{\mathrm{nb}R}$ and $S_{\mathrm{b}}+wX_1\equiv S_{\mathrm{b}R}$, respectively. In compact notation, we have

$$S_{\mathrm{nb}R}=S_{\mathrm{nb}}+wX_0=F_{\xi}S_{R\xi}. \tag{3.11}$$

To check that (3.9) implies (3.8), it is sufficient to apply formula (A.1) to (3.11) and express $\partial F_{\xi}/\partial\xi$ in terms of the fields and the sources without primes.

The composition formulas recalled in the appendix allow us to write the relation

$$F_{\xi}=F_{Y\xi}\mathcal{C}(\xi Q), \tag{3.12}$$

where $F_{Y\xi}$ denotes the canonical transformation generated by

$$F_{Y\xi}(\Phi,\underline{\Phi},K',\underline{K}',\xi)=\int\Phi^{\alpha}K'_{\alpha}+\int\underline{\Phi}^{\alpha}\underline{K}'_{\alpha}+w\int_0^{\xi}\mathrm{d}\xi' Y_{\xi}(\phi,C,\bar{C},\underline{\phi},\underline{C},\tilde{K}^{i\prime}_{\phi},K'_C,\xi'). \tag{3.13}$$

Indeed, formula (A.2) reduces to $C=A+B$ in this case. All the corrections vanish, since the nontrivial part of $\mathcal{C}(\xi Q)$ does not depend on any sources K', $\underline{K}'$ other than K'_B, while Y_{ξ} is B independent.

Now we compare the background and nonbackground "renormalized" actions $S_{\mathrm{b}R}$ and $S_{\mathrm{nb}R}$. Evaluating (3.11) at $\xi = 1$, we get the equality

$$\hat{S}_{\mathrm{nb}R}(\Phi' + \underline{\Phi}, K') - \int R^{\alpha}(\underline{\Phi})(\underline{K}'_{\alpha} - K'_{\alpha}) = \hat{S}_{\mathrm{b}R}(\Phi, \underline{\Phi}, K) - \int \mathcal{R}^{\alpha}(\Phi, \underline{C})K_{\alpha} - \int R^{\alpha}(\underline{\Phi})\underline{K}_{\alpha}, \quad (3.14)$$

where the fields and the sources with primes are related to those without primes by the canonical transformation F_{ξ} with $\xi = 1$, while

$$\begin{aligned}\hat{S}_{\mathrm{nb}R}(\Phi + \underline{\Phi}, K) &= \hat{S}_{\mathrm{nb}}(\Phi + \underline{\Phi}, K) + wX_0(\Phi + \underline{\Phi}, K),\\ \hat{S}_{\mathrm{b}R}(\Phi, \underline{\Phi}, K) &= \hat{S}_{\mathrm{b}}(\Phi, \underline{\phi}, K) + wX_1(\Phi, \underline{\Phi}, K).\end{aligned} \quad (3.15)$$

Using (2.28) and (3.9), we find, among other things, $\underline{\Phi}^{\alpha\prime} = \underline{\Phi}^{\alpha}$ and

$$\phi^{i\prime} = \phi^{i} + \int_0^1 \mathrm{d}\xi \frac{\delta(wY_{\xi})}{\delta K_{\phi}^{i\prime}}, \quad C^{I\prime} = C^{I} + \int_0^1 \mathrm{d}\xi \frac{\delta(wY_{\xi})}{\delta K_{C}^{I\prime}}, \quad \bar{C}^{I\prime} = \bar{C}^{I}, \quad B^{I\prime} = B^{I} + \mathcal{R}^{I}_{\bar{C}}(\bar{C}, \underline{C}). \quad (3.16)$$

Step 2 of proof: The background field trick

The basic trick of the background field method is to switch the quantum fields off and resume them later with the help of the background fields. Now we explain how this trick is implemented when the background field method is used together with the BV formalism, and we show how this helps us achieve our goal.

First, we need to express both sides of equation (3.14) in terms of the fields Φ, $\underline{\Phi}$ and the sources K', $\underline{K}'$. Once this is done, we set $\phi = C = \underline{K}' = 0$ and keep $\underline{\phi}, \underline{C}, \bar{C}, B$ and K' as independent fields and sources. We denote the fields $\Phi^{\alpha\prime}$ and the sources K_{α}, $\underline{K}_{\alpha}$ obtained by applying these operations by $\Phi_0^{\alpha\prime}$ and $K_{\alpha 0}$, $\underline{K}_{\alpha 0}$, respectively. We find

$$\begin{aligned}\hat{S}_{\mathrm{nb}R}(\Phi'_0 + \underline{\Phi}, K') = {}& \hat{S}_{\mathrm{b}R}(\{0, 0, \bar{C}, B\}, \underline{\Phi}, K_0)\\ &- \int \mathcal{R}^{I}_{\bar{C}}(\bar{C}, \underline{C})K^{I}_{\bar{C}0} - \int \mathcal{R}^{I}_{B}(B, \underline{C})K^{I}_{B0} - \int R^{\alpha}(\underline{\Phi})(\underline{K}_{\alpha 0} + K'_{\alpha}).\end{aligned} \quad (3.17)$$

Moreover, (3.15), (2.11) and assumption (ii) give

$$\hat{S}_{\mathrm{b}R}(\{0, 0, \bar{C}, B\}, \underline{\Phi}, K_0) = S_c(\underline{\phi}) + wX_1(0, \underline{\Phi}, 0) - \int B^{I}K^{I}_{\bar{C}0} + \int B^{I}\zeta_{IJ}(\underline{\phi}, \partial)B^{J}. \quad (3.18)$$

Consider the canonical transformation $\{\underline{\phi}, \underline{C}, \bar{C}, B\}, \breve{K} \rightarrow \Phi'', K'$ defined by the generating functional

$$\begin{aligned}F(\{\underline{\phi}, \underline{C}, \bar{C}, B\}, K') = {}& \int \underline{\phi}^{i}K_{\phi}^{i\prime} + \int \underline{C}^{I}K_{C}^{I\prime} + F_{\xi}(\{0, 0, \bar{C}, B\}, \underline{\Phi}, K', 0, 1) = \int \underline{\phi}^{i}K_{\phi}^{i\prime} + \int \underline{C}^{I}K_{C}^{I\prime}\\ &+ \int \bar{C}^{I}K_{\bar{C}}^{I\prime} + \int B^{I}K_{B}^{I\prime} - \int \bar{C}^{I}G^{Ii}(0, \partial)\underline{\phi}^{i} + \int \bar{C}^{I}(\zeta_{IJ}(\underline{\phi}, \partial) - \zeta_{IJ}(0, \partial))B^{J} + \int \mathcal{R}^{I}_{\bar{C}}(\bar{C}, \underline{C})K_{B}^{I\prime}\\ &- \int \bar{C}^{I}\zeta_{IJ}(0, \partial)\mathcal{R}^{J}_{\bar{C}}(\bar{C}, \underline{C}) + w\int_0^1 \mathrm{d}\xi\, Y_{\xi}(0, 0, \bar{C}, \underline{\phi}, \underline{C}, \tilde{K}_{\phi}^{i\prime}, K'_{C}, \xi).\end{aligned} \quad (3.19)$$

Using (3.16) and the other transformation rules not reported in that formula, we find

$$\Phi^{\alpha\prime\prime} = \underline{\Phi}^{\alpha} + \Phi_0^{\alpha\prime}, \qquad \breve{K}_{\phi}^{i} = K_{\phi}^{i\prime} + \underline{K}_{\phi 0}^{i}, \qquad \breve{K}_{C}^{I} = K_{C}^{I\prime} + \underline{K}_{C0}^{I}, \qquad \breve{K}_{\bar{C}}^{I} = K_{\bar{C}0}^{I}, \qquad \breve{K}_{B}^{I} = K_{B0}^{I},$$

which, with the help of (3.18), turn the identity (3.17) into

$$\begin{aligned}\hat{S}_{\text{nb}R}(\Phi^{\prime\prime}, K^{\prime}) = & S_c(\underline{\phi}) + \int B^I \zeta_{IJ}(\underline{\phi}, \partial) B^J + w X_1(0, \underline{\Phi}, 0) - \int R_{\phi}^{i}(\underline{\Phi}) \breve{K}_{\phi}^{i} - \int R_{C}^{I}(\underline{\Phi}) \breve{K}_{C}^{I} \\ & - \int B^I \breve{K}_{\bar{C}}^{I} - \int \mathcal{R}_{\bar{C}}^{I}(\bar{C}, \underline{C}) \breve{K}_{\bar{C}}^{I} - \int \mathcal{R}_{B}^{I}(B, \underline{C}) \breve{K}_{B}^{I}. \qquad (3.20)\end{aligned}$$

This is the key result we need. Now we elaborate it further and express it in ways that make its contents more transparent.

Step 3 of proof: Simplification of the result

First, we check that the master equation $(\hat{S}_{\text{nb}R}, \hat{S}_{\text{nb}R}) = 0$ is satisfied. Assumption (3.2) at $\xi = 1$ gives

$$0 = [\![S_{\text{b}}, X_1]\!] = \int \left(\frac{\delta_r S_{\text{b}}}{\delta \Phi^{\alpha}} \frac{\delta_l X_1}{\delta K_{\alpha}} - \frac{\delta_r S_{\text{b}}}{\delta K_{\alpha}} \frac{\delta_l X_1}{\delta \Phi^{\alpha}} - \frac{\delta_r \bar{S}_{\text{b}}}{\delta \underline{K}_{\alpha}} \frac{\delta_l X_1}{\delta \underline{\Phi}^{\alpha}} \right),$$

having used $\delta_l X_1 / \delta \underline{K}_{\alpha} = 0$ and $\delta_r S_{\text{b}} / \delta \underline{K}_{\alpha} = \delta_r \bar{S}_{\text{b}} / \delta \underline{K}_{\alpha}$. Now, we set $\phi = C = 0$ in this equation. Recalling that X_1 is independent of K, $\bar{C}$ and B at $\phi = C = 0$, while $\delta_r S_{\text{b}} / \delta K_{\phi}^{i}$ and $\delta_r S_{\text{b}} / \delta K_{C}^{I}$ vanish there, we get

$$0 = -\int \frac{\delta_r \bar{S}_{\text{b}}}{\delta \underline{K}_{\alpha}} \frac{\delta_l X_1(0, \underline{\Phi}, 0)}{\delta \underline{\Phi}^{\alpha}} = [\![\bar{S}_{\text{b}}, X_1(0, \underline{\Phi}, 0)]\!], \qquad (3.21)$$

which states that $X_1(0, \underline{\Phi}, 0)$ is invariant under background transformations. The conclusion obviously also applies to $S_c(\underline{\phi})$. The background invariance of the gauge fermion (2.8) implies that the second term on the right-hand side of (3.20) is also invariant under background transformations. Thanks to these facts, we can easily check, from the right-hand side of formula (3.20), that $\hat{S}_{\text{nb}R}$ does satisfy the master equation $(\hat{S}_{\text{nb}R}, \hat{S}_{\text{nb}R}) = 0$.

It is convenient to relabel the fields $\{\underline{\phi}, \underline{C}, \bar{C}, B\}$ as Φ^{α}, the sources $\breve{K}_{\alpha}$ as K_{α} and the fields $\Phi^{\alpha\prime\prime}$ as $\Phi^{\alpha\prime}$. Then formulas (3.19) and (3.20) tell us that the canonical transformation

$$F(\Phi, K^{\prime}) = \int \phi^i K_{\phi}^{i\prime} + \int C^I K_{C}^{I\prime} + F_{\xi}(\{0, 0, \bar{C}, B\}, \{\phi, C\}, K^{\prime}, 0, 1)$$

is such that

$$\tilde{F}_{\text{b}}^{-1} \hat{S}_{\text{nb}R} = F \hat{S}_{\text{b}R}^{\prime}, \qquad (3.22)$$

where

$$\begin{aligned}\hat{S}_{\text{b}R}^{\prime}(\Phi, K) = & S_c(\phi) + \int B^I \zeta_{IJ}(\phi, \partial) B^J + w X_1(0, \{\phi, C\}, 0) - \int R^{\alpha}(\Phi) K_{\alpha} \\ & - \int \mathcal{R}_{\bar{C}}^{I}(\bar{C}, C) K_{\bar{C}}^{I} - \int \mathcal{R}_{B}^{I}(B, C) K_{B}^{I}, \qquad (3.23)\end{aligned}$$

and $\tilde{F}_{\text{b}}^{-1}$ is the canonical transformation that undoes the background shift. Precisely, $\tilde{F}_{\text{b}}^{-1}$ is the inverse of the transformation generated by

$$\tilde{F}_{\text{b}}(\Phi, \underline{\Phi}, K') = \int (\Phi^{\alpha} - \underline{\Phi}^{\alpha}) K'_{\alpha},$$

which is obtained from (2.5) by reducing the set of fields and sources and downgrading the background fields $\underline{\Phi}$ to the role of mere spectators.

Making the further canonical transformation F_{ζ}, where

$$F_{\zeta}(\Phi, K') = \int \Phi^{\alpha} K'_{\alpha} + \int \bar{C}^{I} \zeta_{IJ}(\phi, \partial) B^{J},$$

we get $F_{\zeta} \hat{S}'_{\text{b}R} = \hat{S}''_{\text{b}R}$, where

$$\hat{S}''_{\text{b}R}(\Phi, K) = S_{c}(\phi) + w X_{1}(0, \{\phi, C\}, 0) - \int R^{\alpha}(\Phi) K_{\alpha} - \int \mathcal{R}_{\bar{C}}^{I}(\bar{C}, C) K_{\bar{C}}^{I} - \int \mathcal{R}_{B}^{I}(B, C) K_{B}^{I}.$$

In this derivation, it is helpful to recall that the last term of F_{ζ} with $\phi \rightarrow \underline{\phi}$ is invariant under the background transformations. Thus, F_{ζ} simply cancels the second term on the right-hand side of (3.23).

Finally, the canonical transformation generated by [check (2.6)]

$$\tilde{F}_{\text{nm}}(\Phi, K') = \int \Phi^{\alpha} K'_{\alpha} - \int \mathcal{R}_{\bar{C}}^{I}(\bar{C}, C) K_{B}^{\prime I},$$

gives $\tilde{F}_{\text{nm}}^{-1} \hat{S}''_{\text{b}R} = \hat{S}'''_{\text{b}R}$, where

$$\hat{S}'''_{\text{b}R}(\Phi, K) = S_{c}(\phi) + w X_{1}(0, \{\phi, C\}, 0) - \int R^{\alpha}(\Phi) K_{\alpha} = \mathcal{S}(\Phi, K) + w X_{1}(0, \{\phi, C\}, 0).$$

In the last step, we have used (2.2). Collecting the results found so far, we get

$$\tilde{F}_{\text{b}}^{-1} \hat{S}_{\text{nb}R} = F F_{\zeta}^{-1} \tilde{F}_{\text{nm}} \hat{S}'''_{\text{b}R}.$$

More explicitly, the canonical transformation $\Phi, K \rightarrow \Phi', K'$ generated by $F F_{\zeta}^{-1} \tilde{F}_{\text{nm}}$ is such that

$$\hat{S}_{\text{nb}R}(\Phi', K') = \mathcal{S}(\Phi, K) + w X_{1}(0, \{\phi, C\}, 0). \tag{3.24}$$

Note that formula (3.21) can be recast in the form

$$(\mathcal{S}, X_{1}(0, \{\phi, C\}, 0)) = 0. \tag{3.25}$$

Together with $(\mathcal{S}, \mathcal{S}) = 0$, this formula and (3.24) give back $(\hat{S}_{\text{nb}R}, \hat{S}_{\text{nb}R}) = 0$.

Let $\mathcal{G}_i(\phi, C)$ denote a basis of local functionals of ghost number g that satisfy $(\mathcal{S}, \mathcal{G}_i) = 0$ and are constructed with the physical fields ϕ and the ghosts C. For $g = 0$ they are just the usual gauge invariant local functionals. We have the expansion

$$X_1(0, \{\phi, C\}, 0) = \sum_i \tau_i \mathcal{G}_i(\phi, C), \tag{3.26}$$

where τ_i are constants, and formula (3.24) gives

$$\hat{S}_{\mathrm{nb}}(\Phi', K') + wX_0(\Phi', K') = \mathcal{S}(\Phi, K) + w\sum_i \tau_i \mathcal{G}_i(\phi, C). \tag{3.27}$$

This formula relates the background field functionals with the nonbackground functionals. With the help of a few other manipulations, we can work out the interpolation between the two sides and conclude the proof of the theorem.

Step 4 of proof: Interpolation of the result

Using (2.28) and (3.9), it is easy to check the composition rule $F = \bar{F}_Y \bar{F}$, where F is the transformation encoded in (3.19), while $\bar{F}$ and $\bar{F}_Y$ are the transformations generated by

$$\begin{aligned} \bar{F}(\Phi, K') = &\int \Phi^\alpha K'_\alpha - \int \bar{C}^I G^{Ii}(0, \partial)\phi^i + \int \bar{C}^I(\zeta_{IJ}(\phi, \partial) - \zeta_{IJ}(0, \partial))B^J \\ &+ \int \mathcal{R}^I_{\bar{C}}(\bar{C}, C)K^{I\prime}_B - \int \bar{C}^I \zeta_{IJ}(0, \partial)\mathcal{R}^J_{\bar{C}}(\bar{C}, C), \end{aligned} \tag{3.28}$$

$$\bar{F}_Y(\Phi, K') = \int \Phi^\alpha K'_\alpha + w\int_0^1 \mathrm{d}\xi\, Y_\xi(0, 0, \bar{C}, \phi, C, \tilde{K}^{i\prime}_\phi, K'_C, \xi), \tag{3.29}$$

respectively. Using this property and collecting the canonical transformations made so far, we can write the identity (3.27) in the compact form

$$\tilde{F}^{-1}_{\mathrm{nm}} F_\zeta \bar{F}^{-1} \bar{F}^{-1}_Y \tilde{F}^{-1}_{\mathrm{b}}(\hat{S}_{\mathrm{nb}} + wX_0) = \mathcal{S} + w\sum_i \tau_i \mathcal{G}_i. \tag{3.30}$$

Applying the composition rules recalled in the appendix, in particular formula (A.3), it is easy to check that

$$\bar{F} F^{-1}_\zeta \tilde{F}_{\mathrm{nm}} = \tilde{F}'_{\mathrm{gf}},$$

where

$$\tilde{F}'_{\mathrm{gf}}(\Phi, K') = \int \Phi^\alpha K'_\alpha - \Psi'(\Phi)$$

is the reduced version of (2.18). Thus, formula (3.30) simplifies to

$$\tilde{F}'^{-1}_{\mathrm{gf}} \bar{F}^{-1}_Y \tilde{F}^{-1}_{\mathrm{b}}(\hat{S}_{\mathrm{nb}} + wX_0) = \mathcal{S} + w\sum_i \tau_i \mathcal{G}_i. \tag{3.31}$$

The terms independent of w give

$$\tilde{F}_{\text{gf}}'^{-1}\tilde{F}_{\text{b}}^{-1}\hat{S}_{\text{nb}}=\mathcal{S}, \tag{3.32}$$

which can be easily verified. Indeed, by formula (2.21), if we undo the background shift and the gauge fixing on the action $\hat{S}_{\text{nb}}$, we obtain the starting action $\mathcal{S}$.

The terms of (3.31) that are proportional to w can be better read after acting on both sides with $\bar{F}_{Y}\tilde{F}_{\text{gf}}'$, which gives

$$w\tilde{F}_{\text{b}}^{-1}X_{0}=(\bar{F}_{Y}-1)\tilde{F}_{\text{b}}^{-1}\hat{S}_{\text{nb}}+w\sum_{i}\tau_{i}\mathcal{G}_{i}. \tag{3.33}$$

Define the functional

$$\Upsilon(\Phi,K)=(-1)^{g+1}\int_{0}^{1}\mathrm{d}\xi' Y_{\xi}(0,0,\bar{C},\phi,C,\bar{K}_{\phi},K_{C},\xi'), \tag{3.34}$$

where $\bar{K}_{\phi}$ is defined in formula (2.23). Then from (3.33) we can write

$$\tilde{F}_{\text{b}}^{-1}X_{0}=\sum_{i}\tau_{i}\mathcal{G}_{i}+(\tilde{F}_{\text{b}}^{-1}\hat{S}_{\text{nb}},\Upsilon). \tag{3.35}$$

At this point, we start looking at the functionals as functionals of $\Phi,\underline{\Phi},K,\underline{K}$ again. We obviously have $\tilde{F}_{\text{b}}^{-1}\hat{S}_{\text{nb}}=F_{\text{b}}^{-1}\hat{S}_{\text{nb}}$ and $\tilde{F}_{\text{b}}^{-1}X_{0}=F_{\text{b}}^{-1}X_{0}$, since $\hat{S}_{\text{nb}}$ and X_{0} are independent of $\underline{K}$. Moreover, $(\tilde{F}_{\text{b}}^{-1}\hat{S}_{\text{nb}},\Upsilon)=[\![F_{\text{b}}^{-1}\hat{S}_{\text{nb}},\Upsilon]\!]=[\![F_{\text{b}}^{-1}S_{\text{nb}},\Upsilon]\!]=F_{\text{b}}^{-1}[\![S_{\text{nb}},F_{\text{b}}\Upsilon]\!]$, where we have used $[\![F_{\text{b}}^{-1}\bar{S}_{\text{nb}},\Upsilon]\!]=0$, which is due to the fact that $F_{\text{b}}^{-1}\bar{S}_{\text{nb}}$ just depends on $\underline{\Phi}$, $\underline{K}$, while Υ just depends on Φ,K. Applying F_{b} to both sides of (3.35), we obtain

$$X_{0}(\Phi+\underline{\Phi},K)=\sum_{i}\tau_{i}\mathcal{G}_{i}(\phi+\underline{\phi},C+\underline{C})+[\![S_{\text{nb}},F_{\text{b}}\Upsilon]\!]. \tag{3.36}$$

Using this result together with formula (3.11), we find

$$F_{\xi}S_{R\xi}=S_{\text{nb}}+w\sum_{i}\tau_{i}\mathcal{G}_{i}(\phi+\underline{\phi},C+\underline{C})+w[\![F_{\xi}S_{R\xi},F_{\text{b}}\Upsilon]\!].$$

Now we apply F_{ξ}^{-1} to both sides and use $S_{R\xi}=S_{\xi}+wX_{\xi}$ from the first formula of (3.7) and $F_{\xi}^{-1}=\mathcal{C}(-\xi Q)F_{Y\xi}^{-1}$ from (3.12). Noting that $\mathcal{C}(-\xi Q)$ leaves $\mathcal{G}_{i}(\phi+\underline{\phi},C+\underline{C})$ invariant, because it does not affect the fields besides B^{I}, we find

$$S_{\xi}+wX_{\xi}=\mathcal{C}(-\xi Q)F_{Y\xi}^{-1}S_{\text{nb}}+w\sum_{i}\tau_{i}\mathcal{G}_{i}(\phi+\underline{\phi},C+\underline{C})+w[\![S_{\xi},\mathcal{C}(-\xi Q)F_{\text{b}}\Upsilon]\!]. \tag{3.37}$$

Using (3.13) and (2.29), we observe that

$$F_{Y\xi}^{-1}S_{\text{nb}}=S_{\text{nb}}-w[\![S_{\text{nb}},\Upsilon_{\xi}]\!],\qquad \mathcal{C}(-\xi Q)F_{Y\xi}^{-1}S_{\text{nb}}=S_{\xi}-w[\![S_{\xi},\mathcal{C}(-\xi Q)\Upsilon_{\xi}]\!],$$

where

$$\Upsilon_\xi(\Phi,\underline{\Phi},K,\xi)=(-1)^{g+1}\int_0^\xi \mathrm{d}\xi' Y_\xi(\phi,C,\bar{C},\underline{\phi},\underline{C},\bar{K}_\phi,K_C,\xi'). \tag{3.38}$$

Finally, formula (3.37) gives

$$X_\xi(\Phi,\underline{\Phi},K,\xi)=\sum_i \tau_i \mathcal{G}_i(\phi+\underline{\phi},C+\underline{C})+\llbracket S_\xi,\chi_\xi\rrbracket, \tag{3.39}$$

where

$$\chi_\xi(\Phi,\underline{\Phi},K,\xi)=\mathcal{C}(-\xi Q)\left(F_{\mathrm{b}}\Upsilon-\Upsilon_\xi\right). \tag{3.40}$$

Formula (3.39) coincides with (3.6) once we set $\mathcal{G}(\phi,C)=\sum_i \tau_i\mathcal{G}_i(\phi,C)$. Using (2.28), (3.34), (3.38) and (2.31), we find

$$\begin{aligned}\chi_\xi(\Phi,\underline{\Phi},K,\xi)=(-1)^{g+1}&\int_0^1 \mathrm{d}\xi' Y_\xi(0,0,\bar{C},\phi+\underline{\phi},C+\underline{C},\tilde{K}_\phi(\xi),K_C,\xi')\\ &-(-1)^{g+1}\int_0^\xi \mathrm{d}\xi' Y_\xi(\phi,C,\bar{C},\underline{\phi},\underline{C},\tilde{K}_\phi(\xi),K_C,\xi').\end{aligned}$$

This functional is local and independent of B, $K_{\bar{C}}$, K_B and $\underline{K}$. Moreover, $\chi_\xi(\Phi,\underline{\Phi},K,0)$ just depends on $\Phi+\underline{\Phi}$ and K. Finally, $\chi_\xi(\{0,0,\bar{C},B\},\underline{\Phi},K,1)=0$. This concludes the proof of the theorem.

Observe that χ_ξ satisfies the equation

$$\frac{\partial\chi_\xi}{\partial\xi}-\llbracket\chi_\xi,\tilde{Q}\rrbracket=(-1)^g Y_\xi,$$

which can be proved by using (A.1) on (3.40) and noting that $\mathcal{C}(-\xi Q)\tilde{Q}=\tilde{Q}$.

Formulas (3.39) and (3.26) tell us that, in the end, the nontrivial cohomological content of the functional X_ξ is just $X_1(0,\underline{\Phi},0)$, which can be worked out with the background field method by evaluating Feynman diagrams that just have background fields $\underline{\phi}$ and background ghosts $\underline{C}$ on their external legs.

4 Conclusions

We have studied the cohomology of the local functionals of arbitrary ghost numbers generated by renormalization in quantum field theories whose gauge symmetries are general covariance, local Lorentz symmetry, non-Abelian Yang-Mills symmetries and Abelian gauge symmetries. The case of ghost number 0 is important to characterize the divergent parts of Feynman diagrams. The case of ghost number 1 is important for the local contributions to the potential anomalies.

Using the Batalin-Vilkovisky formalism and the background field method, we have proved that a closed local functional can always be written as a local functional that just depends on the physical fields ϕ and the Faddeev-Popov ghosts C, plus an exact functional, which is equal to the BV antiparentheses between the action and another local functional. Every term that depends nontrivially on the antighosts $\bar{C}$, the Lagrange multipliers B for the gauge fixing and/or the sources K coupled to the symmetry transformations of the fields is cohomologically trivial.

The basic idea of the proof is to interpolate between the background field approach and the usual, nonbackground approach by means of canonical transformations. This allows us to take advantage of the virtues of both approaches, and highlight, among other things, that the counterterms and the local contributions to the potential anomalies are not just cohomologically closed, but satisfy more restrictive conditions. We have managed to translate those conditions into simple mathematical assumptions and obtain a general theorem. The result supersedes numerous involved arguments that exist in the literature and offers a better understanding of the matter. It can be used to upgrade the recent proof [17] of the Adler-Bardeen theorem in nonrenormalizable theories.

Appendix. Useful formulas

In this appendix we collect a few reference formulas that are used in the paper.

The functional $-\int R^{\alpha}(\Phi)K_{\alpha}$ that appears in (2.2) reads

$$\int (C^{\rho}\partial_{\rho}A_{\mu}^{a}+A_{\rho}^{a}\partial_{\mu}C^{\rho}-\partial_{\mu}C^{a}-gf^{abc}A_{\mu}^{b}C^{c})K_{A}^{\mu a}+\int \left(C^{\rho}\partial_{\rho}C^{a}+\frac{g}{2}f^{abc}C^{b}C^{c}\right)K_{C}^{a}$$
$$+\int (C^{\rho}\partial_{\rho}e_{\mu}^{\hat{a}}+e_{\rho}^{\hat{a}}\partial_{\mu}C^{\rho}+C^{\hat{a}\hat{b}}e_{\mu\hat{b}})K_{\hat{a}}^{\mu}+\int C^{\rho}(\partial_{\rho}C^{\mu})K_{\mu}^{C}+\int (C^{\hat{a}\hat{c}}\eta_{\hat{c}\hat{d}}C^{\hat{d}\hat{b}}+C^{\rho}\partial_{\rho}C^{\hat{a}\hat{b}})K_{\hat{a}\hat{b}}^{C}$$
$$+\int \left(C^{\rho}\partial_{\rho}\bar{\psi}_{L}-\frac{i}{4}\bar{\psi}_{L}\sigma^{\hat{a}\hat{b}}C_{\hat{a}\hat{b}}+g\bar{\psi}_{L}T^{a}C^{a}\right)K_{\psi}+\int K_{\bar{\psi}}\left(C^{\rho}\partial_{\rho}\psi_{L}-\frac{i}{4}\sigma^{\hat{a}\hat{b}}C_{\hat{a}\hat{b}}\psi_{L}+gT^{a}C^{a}\psi_{L}\right)$$
$$+\int \left(C^{\rho}(\partial_{\rho}\varphi)+g\mathcal{T}^{a}C^{a}\varphi\right)K_{\varphi}-\int B^{a}K_{\bar{C}}^{a}-\int B_{\mu}K_{\bar{C}}^{\mu}-\int B_{\hat{a}\hat{b}}K_{\bar{C}}^{\hat{a}\hat{b}}.$$

Here $\hat{a},\hat{b},\ldots$ are local Lorentz indices, while C^{μ}-$\bar{C}_{\mu}$-B_{μ}, $C^{\hat{a}\hat{b}}$-$\bar{C}_{\hat{a}\hat{b}}$-$B^{\hat{a}\hat{b}}$ and C^{a}-$\bar{C}^{a}$-B^{a} denote the ghosts, the antighosts and the Lagrange multipliers of diffeomorphisms, local Lorentz symmetry and Yang-Mills symmetry, respectively. Moreover, φ are scalar fields and ψ_L are left-handed fermions, while $\mathcal{T}^a$ and T^a are the anti-Hermitian matrices associated with their representations.

A helpful identity tells us that if $Y(\Phi,K)$ is a functional that behaves as a scalar [i.e. such that $Y'(\Phi',K')=Y(\Phi,K)$] under the canonical transformation $\Phi,K\rightarrow\Phi',K'$, generated by the functional $F(\Phi,K')$, then we have [26, 22]

$$\frac{\partial Y'}{\partial\zeta}=\frac{\partial Y}{\partial\zeta}-(Y,\tilde{F}_{\zeta}), \tag{A.1}$$

where $\tilde{F}_{\zeta}(\Phi,K)=F_{\zeta}(\Phi,K'(\Phi,K))$ and $F_{\zeta}(\Phi,K')=\partial F/\partial\zeta$. In this formula, the ζ derivative of each functional is evaluated while the natural arguments of the functional are kept constant.

Precisely, Φ' and K' are constant in the ζ derivative of Y', while Φ and K are constant in the ζ derivative of Y, and Φ and K' are constant in the ζ derivative of F.

In ref. [25], various formulas for the manipulation of canonical transformations at the level of their generating functions have been given. In particular, if $F_A(q,P) = q^i P^i + A(q,P)$ and $F_B(q,P) = q^i P^i + B(q,P)$ are the generating functions of two canonical transformations $q,p \rightarrow Q,P$, the generating function of the composed transformation $F_C = F_B F_A$ is written as $F_C(q,P) = q^i P^i + C(q,P)$, where the function $C(q,P)$ is expressed as a sum of monomials built with A, B and their derivatives

$$A_{i_1 \cdots i_n} = \frac{\partial^n A(q,P)}{\partial P_{i_1} \cdots \partial P_{i_n}}, \qquad B^{i_1 \cdots i_n} = \frac{\partial^n B(q,P)}{\partial q_{i_1} \cdots \partial q_{i_n}}.$$

The first contributions to $C(q,P)$ are

$$C = A + B + A_i B^i + \frac{1}{2} A_i B^{ij} A_j + \frac{1}{2} B^i A_{ij} B^j + \cdots \tag{A.2}$$

These results extend straightforwardly to the BV formalism. When we compose the canonical transformations $\Phi, K \rightarrow \Phi', K'$ generated by $F_A(\Phi, K') = \int \Phi^\alpha K'_\alpha + A(\Phi, K')$ and $F_B(\Phi, K') = \int \Phi^\alpha K'_\alpha + B(\Phi, K')$, we write the result as $F_C = F_B F_A$, where $F_C(\Phi, K') = \int \Phi^\alpha K'_\alpha + C(\Phi, K')$. In the applications of this paper, due to the simple structures of the functionals A and B, formula (A.2) effectively reduces to

$$C = A + B + \int \frac{\delta A}{\delta K'_\alpha} \frac{\delta B}{\delta \Phi^\alpha}. \tag{A.3}$$

Several operations on canonical transformations can be handled more practically by means of the componential map $\mathcal{C}$ [25], which expresses the generating function $F(q,P)$ in terms of a another function $X(q,P)$ as $\mathcal{C}(X(q,P))$, with the expansion

$$\mathcal{C}(X) = I + X + \frac{1}{2} X_i X^i + \frac{1}{3!} \left(X_{ij} X^i X^j + X^j X^i_j X_i + X^{ij} X_i X_j \right) + \cdots , \tag{A.4}$$

where

$$X^{i_1 \cdots i_n}_{j_i \cdots j_m} \equiv \frac{\partial^{n+m} X(q,P)}{\partial q^{i_1} \cdots \partial q^{i_n} \partial P^{j_1} \cdots \partial P^{j_m}}.$$

The advantage of the componential map $\mathcal{C}(X)$ is that it satisfies the Baker-Campbell-Hausdorff (BCH) formula, like the exponential map $\mathrm{e}^{\mathrm{ad}(X)}$, where $\mathrm{ad}(X)Y = \{X, Y\}$ is the adjoint map and $\{X, Y\}$ are the Poisson brackets of X and Y. Precisely, if we define $X \triangle Y$ from the BCH formula

$$\mathrm{e}^{\mathrm{ad}(X)} \mathrm{e}^{\mathrm{ad}(Y)} = \mathrm{e}^{\mathrm{ad}(X+Y+X\triangle Y)}, \tag{A.5}$$

then the componential map satisfies

$$\mathcal{C}(X)\mathcal{C}(Y) = \mathcal{C}(X + Y + X \triangle Y),$$

where the product on the left-hand side is the composition of the canonical transformations. In particular, the inverse of $\mathcal{C}(X)$ is just $\mathcal{C}(-X)$.

Again, the generalization to the BV formalism is straightforward and, due to the simple structures of the functionals $X(\Phi, K')$, in most applications of this paper the expansion (A.4) of the componential map effectively reduces to

$$\mathcal{C}(X(\Phi, K')) = \int \Phi^{\alpha} K'_{\alpha} + X(\Phi, K') + \frac{1}{2} \int \frac{\delta X}{\delta K'_{\alpha}} \frac{\delta X}{\delta \Phi^{\alpha}}. \tag{A.6}$$

References

[1] L.D. Faddeev and V. Popov, Feynman diagrams for the Yang-Mills field, Phys. Lett. B 25 (1967) 29.

[2] I.A. Batalin and G.A. Vilkovisky, Gauge algebra and quantization, Phys. Lett. B 102 (1981) 27-31;

Quantization of gauge theories with linearly dependent generators, Phys. Rev. D 28 (1983) 2567, Erratum-ibid. D 30 (1984) 508;

see also S. Weinberg, *The quantum theory of fields*, vol. II, Cambridge University Press, Cambridge 1995.

[3] J.C. Ward, An identity in quantum electrodynamics, Phys. Rev. 78, (1950) 182;

Y. Takahashi, On the generalized Ward identity, Nuovo Cimento, 6 (1957) 371;

A.A. Slavnov, Ward identities in gauge theories, Theor. Math. Phys. 10 (1972) 99;

J.C. Taylor, Ward identities and charge renormalization of Yang-Mills field, Nucl. Phys. B33 (1971) 436.

[4] C. Becchi, A. Rouet and R. Stora, The Abelian Higgs Kibble model, unitarity of the S-operator, Phys. Lett. B52 (1974) 344;

Renormalization of the Abelian Higgs-Kibble model, Commun. Math. Phys. 42 (1975) 127.

Renormalization of gauge theories, Ann. Phys. 98, 2 (1976) 287.

I.V. Tyutin, Gauge invariance in field theory and statistical physics in operator formalism, Lebedev Physics Institute preprint 39 (1975) and arXiv:0812.0580.

[5] H. Kluberg-Stern and J.B. Zuber, Renormalization of non-Abelian gauge theories in a background field gauge. 1. Green functions, Phys. Rev. D12 (1975) 482;

Renormalization of nonabelian gauge theories in a background field gauge. 2. Gauge invariant operators, Phys. Rev. D 12 (1975) 3159.

[6] J.A. Dixon and J.C. Taylor, Renormalization of Wilson operators in gauge theories, Nucl.Phys. B78 (1974) 552.

[7] S.D. Joglekar and B.W. Lee, General theory of renormalization of gauge invariant operators, Ann. Phys. (NY) 97 (1976) 160.

[8] J. Wess and B. Zumino, Consequences of anomalous Ward identities, Phys. Lett. B 37 (1971) 95.

[9] L. Bonora and P. Cotta-Ramusino, Some remarks on BRS transformations, anomalies and the cohomology of the Lie algebra of the group of gauge transformations, Commun. Math. Phys. 87 (1983) 589.

[10] F. Brandt, N. Dragon and M. Kreuzer, Lie algebra cohomology, Nucl. Phys. B332 (1990) 250;

The gravitational anomalies, Nucl. Phys. B 340 (1990) 187.

[11] O. Piguet and S.P. Sorella, *Algebraic renormalization*, Lecture Notes in Physics (Springer, New York, 1995), Vol. 28.

[12] G. Barnich, F. Brandt and M. Henneaux, Local BRST cohomology in the antifield formalism. I. General theorems, Commun. Math. Phys. 174 (1995) 57 and arXiv:hep-th/9405109;

Local BRST cohomology in the antifield formalism. II. Application to Yang-Mills theory, Commun. Math. Phys. 174 (1995) 93 and arXiv:hep-th/9405194.

[13] G. Barnich, F. Brandt and M. Henneaux, Local BRST cohomology in Einstein–Yang-Mills theory, Nucl. Phys. B 455 (1995) 357 and arXiv:hep-th/9505173.

[14] G. Barnich, F. Brandt and M. Henneaux, Local BRST cohomology in gauge theories, Phys. Reports 338 (2000) 439 and arXiv:hep-th/0002245.

[15] R. Foot, H. Lew and R.R. Volkas, Electric charge quantization, J. Phys. G19 (1993) 361; Erratum-ibid. G19 (1993) 1067, and arXiv:hep-ph/9209259.

[16] D. Anselmi, Ward identities and gauge independence in general chiral gauge theories, Phys. Rev. D 92 (2015) 025027, 15A1 Renormalization.com and arXiv:1501.06692 [hep-th].

[17] D. Anselmi, Adler-Bardeen theorem and cancellation of gauge anomalies to all orders in nonrenormalizable theories, Phys. Rev. D 91 (2015) 105016, 15A2 Renormalization.com and arXiv:1501.07014 [hep-th].

[18] B.S. De Witt, Quantum theory of gravity. II. The manifestly covariant theory, Phys. Rev. 162 (1967) 1195;

Dynamic theory of groups and fields (Gordon and breach, New York, 1965);

L.F. Abbott, The background field method beyond one loop, Nucl. Phys. B 185 (1981) 189.

[19] D. Binosi and A. Quadri, Slavnov-Taylor constraints for nontrivial backgrounds, Phys. Rev. D84 (2011) 065017 and arXiv:1106.3240 [hep-th];

Canonical transformations and renormalization group invariance in the presence of nontrivial backgrounds, Phys. Rev. D85 (2012) 085020 and arXiv:1201.1807 [hep-th];

The background field method as a canonical transformation, Phys.Rev. D85 (2012) 121702 and arXiv:1203.6637 [hep-th].

[20] P.A. Grassi, Stability and renormalization of Yang-Mills theory with background field method: a regularization independent proof, Nucl. Phys. B 462 (1996) 524 and arXiv:hep-th/9505101;

Renormalization of nonsemisimple gauge models with the background field method, Nucl. Phys. B 560 (1999) 499 and arXiv:hep-th/9908188.

[21] J. Zinn-Justin, Renormalization of gauge theories, in *Trends in Elementary Particle Physics*, Lecture Notes in Physics, edited by H. Rollnik and K. Dietz (Springer-Verlag, Berlin), Vol. 37.

[22] D. Anselmi, Background field method, Batalin-Vilkovisky formalism and parametric completeness of renormalization, Phys. Rev. D 89 (2014) 045004, 13A3 Renormalization.com and arXiv:1311.2704 [hep-th].

[23] S.L. Adler and W.A. Bardeen, Absence of higher order corrections in the anomalous axial vector divergence, Phys. Rev. 182 (1969) 1517;

A. Zee, Axial-vector anomalies and the scaling property of field theory, Phys. Rev. Lett. 29 (1972) 1198;

J. Collins, *Renormalization*, Cambridge University Press (Cambridge, England, 1984), Chap. 13;

T. Marinucci and M. Tonin, Dimensional regularization and anomalies, Il Nuovo Cimento A 31 (1976) 381;

G. Costa, J. Julve, T. Marinucci and M. Tonin, Non-Abelian gauge theories and triangle anomalies, Nuovo Cimento A 38 (1977) 373;

C. Lucchesi, O. Piguet and K. Sibold, The Adler-Bardeen theorem for the axial U(1) anomaly in a general non-Abelian gauge theory, Int. J. Mod. Phys. A 02 (1987) 385;

O. Piguet and S. Sorella, Adler-Bardeen theorem and vanishing of the gauge beta function, Nucl.Phys. B 395 (1993) 661 and arXiv:hep-th/9302123;

E. Witten, Global aspects of current algebra, Nucl. Phys. B 223 (1983) 422;

E. Kraus, Anomalies in quantum field theory: properties and characterization, Talk presented at the Hesselberg workshop "Renormalization and regularization", 2002, arXiv:hep-th/0211084;

for a review till 2004, see S.L. Adler, Anomalies to all orders, in "*Fifty Years of Yang-Mills Theory*", G. 't Hooft ed., World Scientific, Singapore, 2005, p. 187-228, and arXiv:hep-th/0405040.

[24] D. Anselmi, Adler-Bardeen theorem and manifest anomaly cancellation to all orders in gauge theories, Eur. Phys. J. C74 (2014) 3083, 14A1 Renormalization.com and arXiv:1402.6453 [hep-th].

[25] D. Anselmi, Some reference formulas for the generating functions of canonical transformations, Eur. Phys. J. C 76 (2016) 49, 15A3 Renormalization.com and arXiv:1511.00828 [hep-th].

[26] D. Anselmi, Removal of divergences with the Batalin-Vilkovisky formalism, Classical Quantum Gravity 11 (1994) 2181-2204, 93A2 Renormalization.com and arXiv:hep-th/9309085;

More on the subtraction algorithm, Classical, Quantum Gravity 12 (1995) 319, 94A1 Renormalization.com and arXiv:hep-th/9407023.