Ward Identities And Gauge Independence In General Chiral Gauge Theories

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Abstract

Using the Batalin-Vilkovisky formalism, we study the Ward identities and the equations of gauge dependence in potentially anomalous general gauge theories, renormalizable or not. A crucial new term, absent in manifestly nonanomalous theories, is responsible for interesting effects. We prove that gauge invariance always implies gauge independence, which in turn ensures perturbative unitarity. Precisely, we consider potentially anomalous theories that are actually free of gauge anomalies thanks to the Adler-Bardeen theorem. We show that when we make a canonical transformation on the tree-level action, it is always possible to re-renormalize the divergences and re-fine-tune the finite local counterterms, so that the renormalized Γ functional of the transformed theory is also free of gauge anomalies, and is related to the renormalized Γ functional of the starting theory by a canonical transformation. An unexpected consequence of our results is that the beta functions of the couplings may depend on the gauge-fixing parameters, although the physical quantities remain gauge independent. We discuss nontrivial checks of high-order calculations based on gauge independence and determine how powerful they are.

1 Introduction

The Ward-Takahashi [1, 2] and Slavnov-Taylor [3, 4] identities are relations among the correlation functions of quantum field theory, and follow from gauge and global symmetries. They are usually studied in theories that are manifestly nonanomalous, that is to say admit a manifestly gauge invariant regularization technique, for example QED and nonchiral Yang-Mills theories. Chiral gauge theories, such as the standard model, are potentially anomalous, because they do not admit a manifestly gauge invariant regularization technique. The Adler-Bardeen (AB) theorem [5, 6, 7] is the main tool that can establish whether a potentially anomalous theory is in the end truly anomalous or nonanomalous. It ensures that, if the gauge anomalies are trivial at one loop, they can be cancelled to all orders.

The potentially anomalous theories that are actually free of gauge anomalies thanks to the Adler-Bardeen theorem will be called *AB nonanomalous*. In this paper, we study the Ward identities of the AB nonanomalous general gauge theories, including the nonrenormalizable ones, and clarify the relation between gauge invariance and gauge independence. Our investigation upgrades the ones available in the literature in several respects.

Gauge invariance and gauge independence are two different concepts, to the extent that a functional can be gauge invariant and gauge dependent at the same time. For example, the renormalized action of non-Abelian Yang-Mills theory contains a term propotional to $Z_A \int F_{\mu\nu}^a F^{a\mu\nu}$, where $F_{\mu\nu}^a$ is the field strength and Z_A is the wave function renormalization constant of the gauge field. This expression is gauge invariant, but not gauge independent, because Z_A may depend on the gauge-fixing parameters.

Yet, the two concepts are related to each other, and crucial to prove perturbative unitarity. Gauge invariance is necessary, because its violation makes unphysical degrees of freedom, such as the longitudinal photons, propagate. On the other hand, gauge independence is important, because it allows us to switch back and forth between gauges that exhibit perturbative unitarity, but do not have good power-counting behaviors (such as the Coulomb gauge), and gauges that have good power-counting behaviors, but do not exhibit unitarity (such as the Lorenz gauge). The Lorenz gauges are very convenient to make calculations and prove theorems to all orders. They make renormalizability manifest, when the theory is power-counting renormalizable. When the theory is nonrenormalizable, they make the locality of counterterms manifest. However, the Lorenz gauges hide unitarity, because they introduce unphysical, propagating degrees of freedom, such as the longitudinal components of the gauge fields and the Fadeev-Popov ghosts. This is where gauge independence plays a key role, because it ensures that every physical quantity can be equivalently defined by using the Coulomb gauge, where the propagators have no unphysical poles and perturbative unitarity is manifest. The equivalence of the two gauges allows us to loosely say that "the unphysical degrees of freedom of the Lorenz gauges compensate one another and drop out of the physical quantities".

Thus, in quantum field theory we need both gauge invariance and gauge independence. If a theory is AB nonanomalous, it is by definition gauge invariant. It is not obvious that the Adler-Bardeen theorem also ensures that the physical quantities are ultimately gauge independent. Is it so, or do we need extra assumptions to ensure that the physics does not depend on the gauge fixing? Among other things, in this paper we answer this question by proving that gauge invariance *always* implies gauge independence.

In our approach, the Ward identities of AB nonanomalous general gauge theories are corrected by a term that is absent in manifestly nonanomalous theories. The correction is evanescent at the bare level, but can generate finite corrections at the renormalized level, by simplifying some divergences. One of the main consequences is that the beta functions of the couplings can depend on the parameters introduced by means of the gauge fixing. However, the physical quantities are protected, and remain gauge independent.

We study how the renormalized Γ functional Γ_R depends on the parameters introduced by the canonical transformations of fields and sources. Canonical transformations encode field redefinitions and changes of the gauge fixing, both of which are expected to have no effect on the physical quantities. When we speak of "gauge dependence" we refer to the dependence on all types of parameters introduced by a canonical transformation, including those associated with field redefinitions.

We work out how a canonical transformation on the (bare) action S affects the renormalized Γ functional Γ_R. After the transformation, the theory must be renormalized anew. We show that in this process of re-renormalization, it is always possible to redefine the subtraction scheme, by fine-tuning the finite local counterterms, so that the transformed theory is also AB nonanomalous. Moreover, the gauge dependence of the transformed Γ_R is encoded into a canonical transformation, up to evanescent corrections.

This result allows us to prove that the physical quantities are gauge independent. However, quantities that are useful for intermediate purposes, such as the beta functions of the couplings, are normally gauge dependent. Their gauge dependence can be absorbed inside finite redefinitions of the couplings.

In manifestly nonanomalous theories we are, to a large extent, free to use a preferred subtraction scheme, such as the minimal one, both before and after the canonical transformation. The physical quantities and the beta functions of the couplings are unaffected by the transformation (see for example [8]). In AB nonanomalous theories, instead, we can use a preferred subtraction scheme neither before, nor after the transformation. Before the transformation, we need to choose a specific class of subtraction schemes to take advantage of the Adler-Bardeen theorem and cancel the gauge anomalies to all orders. After the transformation, we need to choose (another) specific class of subtraction schemes, to enforce the cancellation of gauge anomalies again. In this process, some gauge-fixing parameters move out of the gauge-fixing sector into another unphysical sector, the one encoded by the choice of the subtraction scheme. The result is that the beta functions are gauge dependent, in general. Nevertheless, we can make their gauge dependences disappear, if we specify the new subtraction scheme even further.

Both gauge invariance and gauge independence can be used to make powerful checks of high-order calculations. As said, a consequence of our investigation is that in AB nonanomalous theories, including the standard model, the beta functions of the couplings are not completely gauge independent. We show that, in spite of this, sufficiently powerful checks of high-order calculations are still available. The reason is that the gauge dependence cannot be arbitrary, because it cannot affect the physical quantities.

To keep track of gauge invariance through renormalization, we use the Batalin-Vilkovisky (BV) formalism [9]. The gauge invariant regularization techniques commonly used for manifestly nonanomalous theories are also convenient to treat AB nonanomalous theories, because they minimize the number of terms that are potentially anomalous. In this paper we use the dimensional regularization [10], or any regularization technique that underlies the dimensional one, such as the chiral dimensional (CD) regularization of ref. [11] and the (chiral)dimensional/higher-derivative regularization of refs. [6, 7, 11, 12], obtained by merging the (chiral) dimensional one with the covariant higher-derivative regularization of ref. [13]. We recall that the CD regularization is particularly convenient for studying nonrenormalizable theories, to avoid certain ambiguities that show up when we extract the divergent parts of the BV antiparentheses (X, Y) of two functionals X and Y, as well as other nuisances that the ordinary dimensional regularization is responsible for.

We also take the chance to revisit some known issues under our perspective.

Before presenting our results in more detail, we comment on the existing literature on related subjects, and explain the upgrades we make. Most studies of gauge dependence have been focused on renormalizable theories [14], or nonrenormalizable, but nonchiral, theories [15, 16], where the problem is much simpler (see appendix E). We want to develop an approach that also applies to nonrenormalizable chiral theories, to include the standard model coupled to quantum gravity. In our opinion, it is not necessary to wait for the ultimate theory of quantum gravity to prove general statements about it. The other investigations of gauge dependence we are aware of use the so-called algebraic approach to renormalization [17]. The main feature of the algebraic approach is that it does not make use of an explicit regularization technique. Instead, it relies on tools such as the "quantum action principle"[18].

We think that it is important to develop more standard approaches to the problem of gauge dependence, like the one of the present paper, which uses the dimensional regularization or modified versions of it. For example, anomalies have taught us that working without an explicit regularization may not be completely safe. Another advantage of using an explicit regularization is that we can identify convenient subtraction schemes, where simplifications occur and several properties are easier to deal with, to all orders in the perturbative expansion. Examples are those provided by refs. [6, 7], where it was shown that in suitable subtraction schemes the gauge anomalies automatically vanish from two loops onwards, if they cancel out at one loop. By construction, it is not possible to identify special subtraction schemes in regularization-independent approaches.

Now we state the main results of our investigation. We study canonical transformations that are continuously connected with the identity. Their generating functionals have the form

$$F(\Phi, K', \theta) = \int \Phi^{\alpha} K'_{\alpha} + \mathcal{O}(\theta), \tag{1.1}$$

where θ denotes the "gauge parameters", which are associated with both changes of field variables and changes of the gauge fixing. In the first part of our analysis, we prove the main theorem, which states that if the theory is AB nonanomalous at $\theta = 0$, after making the canonical transformation (1.1) it is always possible to re-renormalize the divergences and re-fine-tune the finite local counterterms, continuously in θ, so that the equations

$$\text{(ABT)} \qquad (\Gamma_{R\theta}, \Gamma_{R\theta}) = \mathcal{O}(\varepsilon), \tag{1.2}$$

$$\text{(GDE)} \qquad \frac{\partial \Gamma_{R\theta}}{\partial \theta} - (\Gamma_{R\theta}, \langle \tilde{Q}_{R\theta} \rangle) = \mathcal{O}(\varepsilon) \tag{1.3}$$

hold for arbitrary θ, where $\Gamma_{R\theta}$ is the renormalized Γ functional of the transformed theory and $\tilde{Q}_{R\theta}$ is a suitable renormalized local functional. The right-hand sides of both equations are (generically nonlocal) functionals that vanish when the continued spacetime dimension $D = d - \varepsilon$ tends to the physical spacetime dimension d. We denote such functionals by $\mathcal{O}(\varepsilon)$ and call them "evanescent".

Equation (1.2) ensures that the theory is AB nonanomalous for arbitrary values of the gauge parameters. Thus, it encodes gauge invariance. Formula (1.3) is the equation of gauge dependence, and follows from the generalized Ward identities. The equations (GDE) can be integrated to show that the entire gauge dependence of $\Gamma_{R\theta}$ can be absorbed inside a (convergent, but generically nonlocal) canonical transformation, up to $\mathcal{O}(\varepsilon)$. The results encoded in formulas (ABT) and (GDE) are so general that they do not require any particular assumption (see section 3).

We also derive the equations of gauge dependence at the level of the renormalized action and show that RG invariance is preserved by the canonical transformation.

A simple, but important application of the theorem is to power-counting renormalizable chiral gauge theories gauge-fixed by means of a nonrenormalizable gauge fixing. We show that the theory remains renormalizable in a nonmanifest form, because the parameters of negative dimensions introduced by the gauge fixing do not propagate into the physical sector. Another application of the theorem is a crucial step in the proof of the Adler-Bardeen theorem for nonrenormalizable theories [7].

In some situations, we can prove formula (ABT) for arbitrary values of a certain gauge parameter θ within a given class of subtraction schemes. Then, it is not necessary to re-renormalize the divergences and the re-fine-tune the finite local counterterms. Under the assumption that the theory satisfies a certain cohomological property, which is a generalized version of the well-known Kluberg-Stern–Zuber conjecture [19], we can derive an more specific version of equations (GDE), which reads

$$\text{(GDE2)} \qquad \frac{\partial \Gamma_{R\theta}}{\partial \theta} - (\Gamma_{R\theta}, \langle H_{R\theta} \rangle) - \sum_i \rho_i \frac{\partial \Gamma_{R\theta}}{\partial \lambda_i} = \mathcal{O}(\varepsilon), \tag{1.4}$$

where λ_i are the independent parameters of the classical action, ρ_i are constants that depend on λ_i and the other parameters of the theory and $H_{R\theta}$ is a renormalized local functional.

We can write (1.4) in the form (1.3) by suitably "evolving the parameters λ in the θ direction". Such redefinitions encode how the beta functions of the couplings depend on θ.

So far, the Adler-Bardeen theorem has been proved in a variety of cases. The original proof given by Adler and Bardeen [5] was designed to work in QED. Most generalizations to renormalizable non-Abelian gauge theories used arguments based on the renormalization group [20, 21, 22, 23], which work well unless the first coefficients of the beta functions satisfy peculiar conditions [23] (for example, they should not vanish). Then there exist algebraic/geometric derivations [24] based on the Wess-Zumino consistency conditions [25] and the quantization of the Wess-Zumino-Witten action. Another method to prove the Adler-Bardeen theorem in renormalizable theories is obtained by extending the coupling constants to spacetime-dependent fields [26]. A proof that covers all power-counting renormalizable gauge theories was given in ref. [6]. It was obtained by elaborating on a previous proof [12] given for quantum field theories that violate Lorentz symmetry at high energies (in particular, Lorentz violating extensions of the standard model) and are renormalizable by weighted power counting [27]. Recently, the proof of [6] was further extended in ref. [7], to include a large class of nonrenormalizable theories, such as the standard model coupled to quantum gravity. We emphasize that a byproduct of our investigation is that the standard model, coupled to quantum gravity or not, is perturbatively unitary, and so are most of its extensions.

The paper is organized as follows. In section 2 we compare the Ward identities of chiral and nonchiral gauge theories, and illustrate the crucial new term that appears when the theory is potentially anomalous. In section 3 we prove the main theorem of this paper, by deriving and integrating the equations of gauge dependence in AB nonanomalous theories. We show that every canonical transformation on the classical action is mapped into a canonical transformation on the renormalized Γ functional, provided that the finite local counterterms are appropriately re-fine-tuned. We also integrate the equations of gauge dependence. In section 4 we derive the equations of gauge dependence of the renormalized action. In section 5 we prove that the canonical transformation preserves RG invariance and discuss two applications of the main theorem. In section 6 we study the gauge dependence of the beta functions in detail. In section 7 we explain how to switch off the ghosts, the antighosts, the Lagrange multipliers for the gauge fixing, and the sources for the symmetry transformations, and get to the physical quantities, collected into a "physical" Γ functional Γ_{ph}. We derive the (nonlocal) gauge symmetry of Γ_{ph}, and prove that it closes off shell. Finally, we prove that Γ_{ph} is gauge independent, up to field redefinitions, and perturbatively unitary. In section 8 we investigate the checks of high-order calculations provided by gauge independence and estimate how powerful they are. Section 9 contains our conclusions. In the appendices we prove some properties used in the paper, recall earlier results and collect some reference formulas for the standard model coupled to quantum gravity. Moreover, we revisit the gauge dependence of manifestly nonanomalous theories in the light of the new results.

2 Generalized Ward identities

In this section we fix some notation, recall the main properties of the Batalin-Vilkovisky formalism for general gauge theories [9] and derive the generalized Ward identities.

Let $D = d - \varepsilon$ denote the continued, complex dimension of spacetime, and d the physical spacetime dimension. The D-dimensional spacetime manifold $\mathbb{R}^D$ is split into the product $\mathbb{R}^d \times \mathbb{R}^{-\varepsilon}$ of the ordinary d-dimensional spacetime $\mathbb{R}^d$ times a residual $(-\varepsilon)$-dimensional evanescent space, $\mathbb{R}^{-\varepsilon}$. The spacetime indices $\mu, \nu, \ldots$ of vectors and tensors are split into the bar indices $\bar{\mu}, \bar{\nu}, \ldots$, which take the values of $0, 1, \cdots, d-1$, and the formal hat indices $\hat{\mu}, \hat{\nu}, \ldots$, which denote the $\mathbb{R}^{-\varepsilon}$ components. For example, the momenta p^{μ} are split into the pairs $p^{\bar{\mu}}$, $p^{\hat{\mu}}$, also written as $\bar{p}^{\mu}$, $\hat{p}^{\mu}$, and the coordinates x^{μ} are split into $\bar{x}^{\mu}$, $\hat{x}^{\mu}$. The formal flat-space metric $\eta_{\mu\nu}$ is split into the usual $d \times d$ flat-space metric $\eta_{\bar{\mu}\bar{\nu}} =\text{diag}(1, -1, \cdots, -1)$ and the formal evanescent metric $\eta_{\hat{\mu}\hat{\nu}} = -\delta_{\hat{\mu}\hat{\nu}}$. The off-diagonal components $\eta_{\bar{\mu}\hat{\nu}}$ vanish. The evanescent components are contracted among themselves by means of the metric $\eta_{\hat{\mu}\hat{\nu}}$, so for example $\hat{p}^2 = p^{\hat{\mu}}\eta_{\hat{\mu}\hat{\nu}}p^{\hat{\nu}}$. Full $SO(1, D-1)$ invariance is lost in most expressions, replaced by $SO(1, d-1) \times SO(-\varepsilon)$ invariance.

We recall that in the CD regularization the fields Φ have strictly d-dimensional components. The metric tensor $g_{\mu\nu}$ is block-diagonal: the diagonal blocks are $g_{\bar{\mu}\bar{\nu}}(x)$ and $\eta_{\hat{\mu}\hat{\nu}}$, while $g_{\bar{\mu}\hat{\nu}} = 0$. Moreover, the γ matrices are strictly d dimensional, and satisfy the usual Dirac algebra $\{\gamma^{\bar{a}}, \gamma^{\bar{b}}\} = 2\eta^{\bar{a}\bar{b}}$, where the indices $\bar{a}, \bar{b}, \ldots$ refer to the Lorentz group. If $d = 2k$ is even, the d-dimensional generalization of γ_5 is defined as

$$\tilde{\gamma} = -i^{k+1}\gamma^0\gamma^1 \cdots \gamma^{2k-1},$$

and satisfies $\tilde{\gamma}^{\dagger} = \tilde{\gamma}$, $\tilde{\gamma}^2 = 1$. The left and right projectors $P_L = (1 - \tilde{\gamma})/2$, $P_R = (1 + \tilde{\gamma})/2$ are defined as usual. The tensor $\varepsilon^{\bar{a}_1 \cdots \bar{a}_d}$ and the charge-conjugation matrix $\mathcal{C}$ also coincide with the usual ones.

The set of fields $\Phi^\alpha = \{\phi^i, C, \bar{C}, B\}$ contains the classical fields ϕ, the Fadeev-Popov ghosts C, the antighosts $\bar{C}$ and the Lagrange multipliers B for the gauge fixing. An external source K_α with opposite statistics is associated with each Φ^α, and coupled to the Φ^α transformations $R^\alpha(\Phi)$. If X and Y are functionals of Φ and K, their *antiparentheses* are defined as

$$(X,Y) \equiv \int \left(\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K_\alpha} - \frac{\delta_r X}{\delta K_\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} \right), \tag{2.1}$$

where the integral is over spacetime points associated with repeated indices and the subscripts l and r in δ_l and δ_r denote the left and right functional derivatives, respectively.

The action S should solve the *master equation* $(S,S) = 0$ in D dimensions, with the "boundary condition" $S(\Phi, K) = S_c(\phi)$ at $C = \bar{C} = B = K = 0$, where $S_c(\phi)$ is the classical action.

If the gauge algebra closes off shell, there exists a choice of field/source variables such that the non-gauge-fixed solution $\bar{S}_d(\Phi, K)$ of the master equation has the form

$$\bar{S}_d(\Phi, K) = S_c(\phi) + S_K, \qquad S_K(\Phi, K) = -\int R^\alpha(\Phi) K_\alpha. \tag{2.2}$$

In this case, $(\bar{S}_d, \bar{S}_d) = 0$ splits into the two identities

$$\int R^i(\phi) \frac{\delta_l S_c(\phi)}{\delta \phi^i} = 0, \qquad \int R^\beta(\Phi) \frac{\delta_l R^\alpha(\Phi)}{\delta \Phi^\beta} = 0,$$

which express the gauge invariance of the classical action and the closure of the algebra, respectively. The gauge-fixed solution $S_d(\Phi, K)$ of the master equation reads

$$S_d(\Phi, K) = S_c(\phi) + (S_K, \Psi) + S_K = \bar{S}_d + (S_K, \Psi), \tag{2.3}$$

where $\Psi(\Phi)$ is the *gauge fermion*, that is to say a local functional of ghost number -1 that encodes the gauge fixing. Reference formulas for S_c, S_K and Ψ in the case of the standard model coupled to quantum gravity can be found in appendix D. Typically, Ψ has the form

$$\Psi(\Phi) = \int \bar{C} \left(G(\phi, \xi) + \frac{1}{2} P(\phi, \xi', \partial) B \right), \tag{2.4}$$

where $G(\phi, \xi)$ is the gauge-fixing function, P is an operator that may contain derivatives acting on B, and ξ, ξ' are gauge-fixing parameters. For example, $G(\phi) = \partial^\mu A_\mu$ for the Lorenz gauge in Yang-Mills theories. Clearly, S_d also solves the master equation $(S_d, S_d) = 0$ in D dimensions.

If the gauge algebra does not close off shell, $\bar{S}_d(\Phi, K)$ is not linear in K and S_d is obtained from $\bar{S}_d$ by applying the canonical transformation generated by

$$F(\Phi, K') = \int \Phi^\alpha K'_\alpha + \Psi(\Phi). \tag{2.5}$$

In manifestly nonanomalous theories we can solve $(S,S) = 0$ in D dimensions at the regularized level. Typically, the solution coincides with (2.3). In potentially anomalous theories, instead, we cannot achieve this goal. There, the functional $S_d(\Phi, K)$ does solve $(S_d, S_d) = 0$ in D dimensions, but is not well regularized. The most common reason is the presence of chiral fermions. We can deform S_d into a well-regularized action

$$S(\Phi, K) = S_d + S_{\text{ev}} \tag{2.6}$$

by adding an evanescent part S_{ev} that collects suitable regularizing terms [11]. The deformed action S does not solve $(S, S) = 0$ in D dimensions. Instead, it solves the *deformed master equation*

$$(S, S) = \mathcal{O}(\varepsilon), \tag{2.7}$$

where the right-hand side denotes terms that vanish for $D \to d$.

Given a generic action $S(\Phi, K)$, the generating functionals Z and W of the (connected) correlation functions are defined by the formulas

$$Z(J, K) = \int [\mathrm{d}\Phi] \exp\left(iS(\Phi, K) + i \int \Phi^\alpha J_\alpha \right) = \exp iW(J, K), \tag{2.8}$$

and the generating functional $\Gamma(\Phi, K) = W(J, K) - \int \Phi^\alpha J_\alpha$ of the one-particle irreducible diagrams is the Legendre transform of $W(J, K)$ with respect to J. The anomaly functional is defined as

$$\mathcal{A} = (\Gamma, \Gamma) = \langle (S, S) \rangle \tag{2.9}$$

and collects the set of one-particle irreducible correlation functions that contain one insertion of (S, S), where $\langle \cdots \rangle$ denotes the average defined by S at arbitrary J . The last equality of (2.9) can be proved by making the change of variables

$$\Phi^\alpha \to \Phi^\alpha + \varpi(S, \Phi^\alpha) = \Phi^\alpha - \varpi \frac{\delta_r S}{\delta K_\alpha}, \tag{2.10}$$

in the functional integral (2.8), where ϖ is a constant anticommuting parameter. For the detailed proof, see for example the appendices of refs. [6, 8]. See also appendix A.

Let us explain the meaning of formula (2.9). The functional (S, S) represents the symmetry violation, so it is basically the integral of the divergence of the gauge current J^μ multiplied by the ghosts:

$$(S, S) \sim 2 \int \mathrm{d}^D x\, C(x) \partial_\mu J^\mu(x),$$

where the sign "$\sim$" means that the right-hand side is written up to terms proportional to the field equations and other terms that we can neglect in the present discussion. As said, formula (2.9) collects the one-particle irreducible diagrams that contain one insertion of (S, S) and arbitrary external Φ and K legs. The key diagram of this type in four dimensions is the one-loop triangle diagram that is responsible for the well-known ABJ anomaly [28], which arises by considering one (S,S) insertion and two external gauge field legs. Amputating those legs, we get

$$\frac{1}{2}\left\langle (S,S)\,J_{\mu}(x)\,J_{\nu}(y)\right\rangle \approx \int \mathrm{d}^{D}z\,C(z)\left\langle \partial_{\rho}J^{\rho}(z)\,J_{\mu}(x)\,J_{\nu}(y)\right\rangle . \tag{2.11}$$

The sign "≈" comes from the leg amputation and the fact that we have taken the ghosts out of the average, because this is the only way to get nontrivial contributions to anomalies at one loop. See ref. [11] for the calculation of the one-loop triangle anomaly in chiral Yang-Mills theories with formula (2.9) and the CD regularization technique.

The Adler-Bardeen theorem is the statement that if the gauge anomalies are trivial at one loop, there exists a class of subtraction schemes where they vanish to all orders, that is to say

$$\mathcal{A}_{R}=(\Gamma_{R},\Gamma_{R})=\langle (S_{R},S_{R})\rangle =\mathcal{O}(\varepsilon), \tag{2.12}$$

S_R and Γ_R being the renormalized action and the renormalized Γ functional, respectively. The right-hand side of (2.12) vanishes for $D\rightarrow d$, which ensures that the renormalized Γ functional is gauge invariant in the physical limit. The AB nonanomalous theories are those that admit subtraction schemes where (2.12) holds.

While the AB identity (2.12) ensures gauge invariance, it does not say much about gauge independence, which is a different statement, namely the property that a certain class of correlation functions (that we call "physical") do not depend on the gauge fixing.

One way to study the gauge independence is through Ward identities. We begin by recalling how those identities work in manifestly nonanomalous theories, where the master equation $(S,S)=0$ is satisfied exactly at the regularized level. Let $\Upsilon(\Phi)$ denote a K-independent, but otherwise completely arbitrary, product of elementary and local composite fields at distinct points. By making the change of field variables (2.10) in the functional integral

$$\int [\mathrm{d}\Phi]\,\Upsilon\,\mathrm{e}^{iS},$$

we find

$$\int [\mathrm{d}\Phi]\;(S,\Upsilon)\;\mathrm{e}^{iS}=0. \tag{2.13}$$

We omit details of the derivation, because the proof of this formula is a particular case of the more general proof given below. We just stress that it is crucial to use the master equation $(S,S)=0$, which implies that S is invariant under the field redefinition (2.10).

Equation (2.13) is the usual Ward identity. For example, if we take $\Upsilon=\bar{C}(x)\,\partial^{\mu}A_{\mu}(y)$ and $\Upsilon=\bar{C}(x)\bar{\psi}(y)\psi(z)$ in QED, we can derive the well-known formula $Z_{e}Z_{A}^{1/2}=1$ that relates the renormalization constants Z_e and Z_A of the electric charge and the gauge field [29].

In this paper, the average $\langle\cdots\rangle$ denotes the sum of connected diagrams. For example, if X and Y are local functionals, we have $\langle XY\rangle=\langle XY\rangle_{\text{nc}}-\langle X\rangle\langle Y\rangle$, where $\langle XY\rangle_{\text{nc}}$ includes disconnected diagrams. The subscript 0 in $\langle \cdots \rangle_0$ means that the correlation functions are evaluated at $J=0$. An equivalent form of the identity (2.13) is

$$\langle (S, \Upsilon) \rangle_0 = 0. \tag{2.14}$$

If we repeat the argument leading to (2.13) without assuming $(S,S)=0$, we get the generalized Ward identity that we consider in this paper, which reads

$$\langle (S, \Upsilon) \rangle_0 + \frac{i}{2} \langle (S,S) \Upsilon \rangle_0 = 0. \tag{2.15}$$

The extra term on the left-hand side of this formula is going to appear in many other contexts and is responsible for the new effects anticipated in the introduction.

To prove (2.15), express Υ as the product $\prod_i X_i$ of K-independent elementary and local composite fields X_i. Then, consider the functional integral

$$\int [\mathrm{d}\Phi] \mathrm{e}^{iS + \sum_i X_i \sigma_i}, \tag{2.16}$$

where σ_i are arbitrary constants. Under the field redefinition (2.10), the action S and the functionals X_i transform as follows:

$$S \to S - \varpi \int \frac{\delta_r S}{\delta K_\alpha} \frac{\delta_l S}{\delta \Phi^\alpha} = S + \frac{\varpi}{2}(S,S), \qquad X_i \to X_i - \varpi \int \frac{\delta_r S}{\delta K_\alpha} \frac{\delta_l X_i}{\delta \Phi^\alpha} = X_i + \varpi (S, X_i).$$

In the last step we have used the assumption that X_i depends only on the fields Φ. When we make the change of variables (2.10) inside (2.16) and divide by (2.16), we get

$$\frac{\int [\mathrm{d}\Phi] \left(\sum_j \varpi (S, X_j) \sigma_j + \frac{i}{2} \varpi (S,S) \right) \mathrm{e}^{iS + \sum_i X_i \sigma_i}}{\int [\mathrm{d}\Phi] \mathrm{e}^{iS + \sum_k X_k \sigma_k}} = 0.$$

The left-hand side of this formula is a sum of connected diagrams. Differentiating it once to the right with respect to each $\sigma_1, \ldots, \sigma_n$ and setting $\sigma_i = 0$ at the end, we project onto the diagrams that have one external σ_i leg for each i. So doing, we get precisely formula (2.15).

When the local functionals X_i of the product $\Upsilon = \prod_i X_i$ depend on both Φ and K, and the sources J are not set to zero, the generalized Ward identities can be worked out from formula (2.9), by deforming the action S into $S + \sum_i X_i \sigma_i$, where σ_i are constants, and taking the first order in all σ_is.

In particular, if Υ is equal to a local functional X, it is easy to show that when the action S is deformed into $S + X\sigma$, where σ is a constant, the Γ functional deforms into $\Gamma + \langle X \rangle \sigma + \mathcal{O}(\sigma^2)$, while the average $\langle Y \rangle$ of a local functional Y deforms into $\langle Y \rangle + i \langle YX \rangle_\Gamma \sigma + \mathcal{O}(\sigma^2)$, where $\langle \prod_i A_i \rangle_\Gamma$ denotes the set of one-particle irreducible diagrams that contain one A_i insertion for each i, A_i

being local functionals (details are given in appendix A). Expanding $(\Gamma,\Gamma)=\langle (S,S)\rangle$ in powers of σ and taking the first order of the expansion, we obtain the identity [8]

$$\langle (S,X)\rangle+\frac{i}{2}\langle (S,S)X\rangle_{\Gamma}=(\Gamma,\langle X\rangle). \tag{2.17}$$

Both sides of (2.17) are viewed as functionals of Φ and K (rather than functionals of J and K). Note that, in particular, $\langle X\rangle=\langle X\rangle_{\Gamma}$.

Repeating the derivation for $\Upsilon=XY$, where X and Y are both local functionals, we get the identity

$$\langle (S,XY)\rangle_{\Gamma}+\frac{i}{2}\langle (S,S)XY\rangle_{\Gamma}=(\Gamma,\langle XY\rangle_{\Gamma})-i(-1)^{\varepsilon_X}(\langle X\rangle,\langle Y\rangle)+i(-1)^{\varepsilon_X}\langle (X,Y)\rangle, \tag{2.18}$$

where ε_X denotes the statistics of the functional X (which is 0 if X is bosonic, 1 if it is fermionic). When Υ is the product of more local functionals, we can proceed similarly.

An important application of the generalized Ward identities is the derivation of the *equations of gauge dependence*, which tell us how the generating functional Γ depends on the gauge parameters. We first recall such equations in manifestly nonanomalous theories and then switch to AB nonanomalous theories.

In manifestly nonanomalous theories $(S,S)=0$ in D dimensions and $S_{\text{ev}}=0$, $S=S_d$. The functional Γ satisfies the equation

$$\frac{\partial\Gamma}{\partial\xi}=\left\langle\frac{\partial S}{\partial\xi}\right\rangle=\langle (S,\Psi_{\xi})\rangle=(\Gamma,\langle\Psi_{\xi}\rangle), \tag{2.19}$$

where ξ is any gauge-fixing parameter and $\Psi_{\xi}=\partial\Psi/\partial\xi$ is the ξ-derivative of the gauge fermion Ψ. The first equality is obvious. The second equality follows from formula (2.3). Indeed, recalling that the parameters ξ are contained only in Ψ, we have $\partial S/\partial\xi=(S_K,\Psi_{\xi})=(S,\Psi_{\xi})$. The third equality follows from formula (2.17).

More generally, if θ denotes any gauge parameter, introduced by a canonical transformation generated by (1.1), we find

$$\frac{\partial\Gamma}{\partial\theta}=\left\langle\frac{\partial S}{\partial\theta}\right\rangle=\langle (S,\tilde{Q}_{\theta})\rangle=(\Gamma,\langle\tilde{Q}_{\theta}\rangle), \tag{2.20}$$

where $\tilde{Q}_{\theta}$ is the derivative $F(\Phi,K',\theta)$ with respect to θ, reexpressed as a functional of Φ and K.

Equations (2.19) can be renormalized and integrated (see [8] and appendix C). The result is that the ξ dependence can be absorbed into a canonical transformation on Γ. Therefore, the contributions due to the right-hand side of (2.19), which are in general nonvanishing, do not affect the physical quantities, for example the S-matrix elements. See subsection 7.3 for details.

In AB nonanomalous theories the equations of gauge dependence are corrected by an extra term, which corresponds to the extra term of (2.15). Formula (2.20) turns into [8]

$$\frac{\partial\Gamma}{\partial\theta}=\left\langle\frac{\partial S}{\partial\theta}\right\rangle=\langle (S,\tilde{Q}_{\theta})\rangle=(\Gamma,\langle\tilde{Q}_{\theta}\rangle)-\frac{i}{2}\langle (S,S)\tilde{Q}_{\theta}\rangle_{\Gamma}. \tag{2.21}$$

Assuming that the primes denote the θ-independent quantities, the second equality of (2.21) follows from formula (A.6) recalled in appendix A, since $\partial S'/\partial\theta = 0$. The last equality of (2.21) follows from formula (2.17).

The identities (2.15), (2.17), (2.18) and (2.21) are so general that they also hold in truly anomalous theories. However, their most interesting applications are to AB nonanomalous theories, which are the main focus of this paper.

In the next sections we are going to renormalize the equations (2.21) and integrate their renormalized versions. The nontrivial part of this task is to work out the effects of the last term of formula (2.21). The result is that the θ dependence can be absorbed into a canonical transformation on the renormalized Γ functional Γ_R, provided that the finite local counterterms are appropriately fine-tuned.

We stress again that gauge invariance, which is expressed by formula (2.12), does not imply gauge independence in an obvious way. However, in this paper we prove that ultimately it does. Gauge independence allows us to prove the perturbative unitarity of the theory (see subsection 7.4).

Before concluding this section, we make some remarks to emphasize the role played by the evanescent terms $\mathcal{O}(\varepsilon)$ in our discussion. With respect to the limit $D \rightarrow d$ we can distinguish divergent, nonevanescent and evanescent terms. A contribution is called "nonevanescent" if it has a regular limit for $D \rightarrow d$ and coincides with the value of that limit. In the (ordinary, as well as chiral) dimensional regularization the evanescences can be of two types: *formal* or *analytic*. Analytically evanescent terms are those that factorize at least one ε, such as $\varepsilon F_{\bar{\mu}\bar{\nu}}F^{\bar{\mu}\bar{\nu}}$, $\varepsilon\bar{\psi}_L ie_{\bar{a}}^{\bar{\mu}}\gamma^{\bar{a}}D_{\bar{\mu}}\psi_L$, etc., where ψ_L is a left-handed fermion. Formally evanescent terms are those that formally disappear when $D \rightarrow d$, although they do not factorize powers of ε, such as $\psi_L^T\hat{\partial}^2\psi_L$. The divergences are poles in ε, and can multiply either nonevanescent terms or formally evanescent terms. In the latter case they are called *divergent evanescences*. An example is $\psi_L^T\hat{\partial}^2\psi_L/\varepsilon$. It is convenient to subtract away the divergent evanescences like any other divergences.

In most derivations it is necessary to extract the divergent parts of functionals and antiparentheses of functionals. We have to take some precautions to ensure that this operation can safely cross the antiparentheses, so that for example $(S, X)_{\text{div}} = (S, X_{\text{div}})$. The first thing to do is define the classical action (2.6) so that it does not contain analytically evanescent terms, but only nonevanescent and formally evanescent terms, multiplied by ε-independent coefficients. In this way, S does not contain dangerous ε factors that could simplify the divergences of X inside (S, X). For the same reason, it is convenient to use the chiral dimensional regularization of [11], instead of the ordinary dimensional regularization. In particular, we must use the CD regularization when the theory in not power-counting renormalizable. So doing, we avoid a number of ambiguities that would complicate our operations. For details on this subject, see refs. [6, 11].

3 The theorem of gauge dependence

Consider a general gauge theory with action $S(\Phi, K, \omega)$, where ω denotes its parameters. Let S_R denote the renormalized action and Γ_R the renormalized Γ functional. Assume that the theory is AB nonanomalous, i.e.

$$(\Gamma_R, \Gamma_R) = \mathcal{O}(\varepsilon). \tag{3.1}$$

For the purposes of this section, we do not need to make other assumptions. The gauge algebra may be irreducible or reducible, and close off shell or on shell. The theory may be renormalizable or nonrenormalizable, perturbatively unitary or not. In particular, it may contain higher-derivative fields. The action S does not need to satisfy special cohomological properties. We can also include local composite fields $\mathcal{O}^I(x)$, in renormalizable and nonrenormalizable theories, by coupling them to external sources $L_I(x)$ and appropriately extending the actions S_c, $\bar{S}_d$, S_d and S. In the arguments that follow, the dependence on such types of external sources is not made explicit. However, we understand that it may be there, whenever necessary.

Consider a canonical transformation $\Phi, K \rightarrow \Phi', K'$ with generating functional

$$F(\Phi, K', \theta) = \int \Phi^\alpha K'_\alpha + Q(\Phi, K', \theta), \tag{3.2}$$

where $Q = \mathcal{O}(\theta)$ is a local functional. Let S_θ denote the action obtained by applying (3.2) to S, $S_{R\theta}$ the renormalized version of S_θ and $\Gamma_{R\theta}$ the renormalized Γ functional associated with $S_{R\theta}$. We assume, for simplicity, that Q does not contain analytically evanescent contributions.

We work out how Γ_R and the identity (3.1) change when we make the transformation (3.2) on S. To reach S_θ from S, it is useful to embed the theory into a more general theory, by considering the extended action

$$\Sigma(\Phi, K, \omega, \hbar\tau) \equiv S(\Phi, K, \omega) + \sum_i \hbar\tau_i \mathcal{H}_i(\Phi, K), \tag{3.3}$$

where τ_i are arbitrary parameters and $\{\mathcal{H}_i\}$ is a basis of local functionals of Φ and K. Specifically, the $\mathcal{H}_i$ are integrals of local monomials constructed with the fields, the sources and their derivatives. They can be restricted by demanding that they be invariant under the nonanomalous symmetries of the theory. However, they are not restricted by gauge invariance, or power counting. To simplify a number of formulas, we include duplicates of the terms that are already present in Σ, multiplied by new independent parameters $\hbar\tau_i$. The difference $\Sigma - S$ is made of $\mathcal{O}(\hbar)$-terms and is also assumed to contain evanescent terms (including those that are already present in S). Basically, $\Sigma - S$ parametrizes the arbitrariness of the subtraction scheme. We denote the Γ functional calculated with the action Σ by $\Omega(\Phi, K, \omega, \hbar\tau)$.

Now, we renormalize Σ. We denote its renormalized action by Σ_R and the Γ functional associated with Σ_R by Ω_R. We can imagine, for a moment, that we replace each $\hbar\tau_i$ with an ordinary parameter ρ_i of order zero in $\hbar$. In that case, the construction of Σ_R is straightforward, since every divergence can be subtracted by means of ρ_i redefinitions. At a second stage, we raise the order of the parameters ρ_i by restoring $\hbar\tau_i$ in their places. The consistency of this operation is justified by the arguments that follow.

We organize the renormalization of Σ so that Σ_R coincides with S_R when the parameters τ_i are equal to suitable finite functions $\tau_i^*(\omega)$, which identify the subtraction scheme where formula (3.1) holds:

$$\Sigma_R(\Phi, K, \omega, \hbar\tau^*) = S_R(\Phi, K, \omega), \qquad \Omega_R(\Phi, K, \omega, \hbar\tau^*) = \Gamma_R(\Phi, K, \omega). \tag{3.4}$$

At arbitrary τ, the action Σ_R can be viewed as an extended renormalization of S, which includes the most general subtraction scheme. We say that Σ_R is the *arbitrary renormalization* of S. When we set $\tau_i = \tau_i^*$ we specialize the subtraction scheme to the one used for S_R, which, by assumption (3.1), preserves gauge invariance to all orders.

Since it is consistent to set $\tau_i \equiv \tau_i^*$, it is also consistent to set $\tau_i = \tau_i^* + \hbar^n \tilde{\nu}_{n+1i}$, $n \geqslant 0$, for arbitrary new parameters $\tilde{\nu}_{n+1i}$. By this we mean that the renormalization of each $\tilde{\nu}_{n+1i}$ remains analytic in $\hbar$. We can better explain this fact by noting that the renormalizations of the differences $\delta_i \equiv \tau_i - \tau_i^*$ vanish at $\delta_j = 0$, so they must be proportional to δ_j. Thus, if we replace δ_i by $\hbar^n \tilde{\nu}_{n+1i}$, $n > 1$, the renormalizations of $\tilde{\nu}_{n+1i}$ remain analytic in $\hbar$. These remarks illustrate a trick that we use in the recursive proof given below. Precisely, at each step we raise the $\hbar$ order of certain residual parameters by one unit, till we make those parameters disappear, and show that we can do this while preserving the analyticity in $\hbar$.

The definition (3.3) understands that the difference $\Sigma - S$ starts from $\mathcal{O}(\hbar)$. Indeed, we do not want to modify the classical action, but just parametrize the arbitrariness of the subtraction scheme. The reason why we move to the more general theory Σ is that if we want to cancel the anomalies after the canonical transformation, we generically need to re-fine-tune all sorts of finite, local terms, including the gauge noninvariant ones.

As said, $S_\theta(\Phi, K, \omega, \theta)$ denotes the action obtained by applying (3.2) to $S(\Phi, K, \omega)$. Let $\Sigma_\theta(\Phi, K, \omega, \hbar\tau, \theta)$ denote the action obtained by applying (3.2) to Σ. We obviously have $\Sigma_\theta = S_\theta + \mathcal{O}(\hbar)$. We denote the renormalized version of Σ_θ by $\Sigma_{R\theta}(\Phi, K, \omega, \hbar\tau, \theta)$. Since $\Sigma_{R\theta} = \Sigma_\theta + \mathcal{O}(\hbar) = S_\theta + \mathcal{O}(\hbar)$, $\Sigma_{R\theta}$ can be viewed as the arbitrary renormalization of S_θ. Note that Σ_θ is not gauge invariant, so its renormalization is not subject to particular restrictions, aside from the continuity condition $\Sigma_{R\theta}(\Phi, K, \omega, \hbar\tau, \theta) = \Sigma_R(\Phi, K, \omega, \hbar\tau) + \mathcal{O}(\theta)$. We denote the Γ functional associated with $\Sigma_{R\theta}$ by $\Omega_{R\theta}$.

Finally, consider the local functional $Q(\Phi, K')$ defined by the canonical transformation (3.2), and define $Q_\theta(\Phi, K') = \partial Q(\Phi, K')/\partial\theta$ and $\tilde{Q}_\theta(\Phi, K) = Q_\theta(\Phi, K'(\Phi, K))$. Let $\tilde{\mathcal{Q}}_{R\theta}$ denote the renormalized version of $\tilde{Q}_\theta(\Phi, K)$ at generic τ.

We prove that

Theorem 1 *there exist finite functions $\tau_j^{*\prime}(\omega,\theta)=\mathcal{O}(\theta)$, such that, defining $\tilde{\tau}_j^*(\omega,\theta)=\tau_j^*(\omega)+\tau_j^{\prime *}(\omega,\theta)$, the action*

$$S_{R\theta}(\Phi,K,\omega,\theta)\equiv\Sigma_{R\theta}(\Phi,K,\omega,\hbar\tilde{\tau}_j^*(\omega,\theta),\theta) \tag{3.5}$$

gives a Γ functional $\Gamma_{R\theta}$ that satisfies the identities

$$(\Gamma_{R\theta},\Gamma_{R\theta})=\mathcal{O}(\varepsilon), \tag{3.6}$$

$$\frac{\partial\Gamma_{R\theta}}{\partial\theta}-(\Gamma_{R\theta},\langle\tilde{Q}_{R\theta}\rangle)=\mathcal{O}(\varepsilon), \tag{3.7}$$

for arbitrary θ, where $\tilde{Q}_{R\theta}$ denotes the functional $\tilde{\mathcal{Q}}_{R\theta}$ calculated at $\hbar\tau_i=\hbar\tilde{\tau}_i^*$.

Note that formula (3.5) ensures that $S_{R\theta}$ also satisfies the continuity condition $S_{R\theta}(\Phi,K,\omega,\theta)=S_R(\Phi,K,\omega)+\mathcal{O}(\theta)$. In fact, all the operations we make preserve the continuity in θ.

For clarity, it is useful to summarize the definitions given so far in a table:

$$\begin{array}{ccccccc}
 & & & & S_R & \stackrel{\Gamma}{\longrightarrow} & \Gamma_R \\
 & & & & \uparrow_{\hbar\tau^*} & & \uparrow_{\hbar\tau^*} \\
S & \stackrel{\hbar\tau}{\longrightarrow} & \Sigma & \stackrel{R}{\longrightarrow} & \Sigma_R & \stackrel{\Gamma}{\longrightarrow} & \Omega_R \\
\downarrow^{\theta} & & \downarrow^{\theta} & & \downarrow^{\theta} & & \downarrow^{\theta} \\
S_\theta & \stackrel{\hbar\tau}{\longrightarrow} & \Sigma_\theta & \stackrel{R}{\longrightarrow} & \Sigma_{R\theta} & \stackrel{\Gamma}{\longrightarrow} & \Omega_{R\theta} \\
 & & & & \downarrow^{\hbar\tilde{\tau}^*} & & \downarrow^{\hbar\tilde{\tau}^*} \\
 & & & & S_{R\theta} & \stackrel{\Gamma}{\longrightarrow} & \Gamma_{R\theta}
\end{array}$$

3.1 The equations of gauge dependence

If we apply the identity (A.5) of appendix A to the renormalized action $\Sigma_{R\theta}$ and the renormalized Γ functional $\Omega_{R\theta}$, with $X=\tilde{\mathcal{Q}}_{R\theta}$, we obtain

$$\frac{\partial\Omega_{R\theta}}{\partial\theta}-(\Omega_{R\theta},\langle\tilde{\mathcal{Q}}_{R\theta}\rangle)=\left\langle\frac{\partial\Sigma_{R\theta}}{\partial\theta}-(\Sigma_{R\theta},\tilde{\mathcal{Q}}_{R\theta})-\frac{i}{2}(\Sigma_{R\theta},\Sigma_{R\theta})\tilde{\mathcal{Q}}_{R\theta}\right\rangle_\Gamma. \tag{3.8}$$

It is convenient to organize this formula in the form

$$\frac{\partial\Omega_{R\theta}}{\partial\theta}=(\Omega_{R\theta},\langle\tilde{\mathcal{Q}}_{R\theta}\rangle)+\langle\mathcal{Y}_{R\theta}\rangle_\Gamma, \tag{3.9}$$

where

$$\mathcal{Y}_{R\theta}\equiv-\frac{i}{2}(\Sigma_{R\theta},\Sigma_{R\theta})\tilde{\mathcal{Q}}_{R\theta}+\frac{\partial\Sigma_{R\theta}}{\partial\theta}-(\Sigma_{R\theta},\tilde{\mathcal{Q}}_{R\theta}). \tag{3.10}$$

If the right-hand side of formula (3.9) contained no $\langle\mathcal{Y}_{R\theta}\rangle_\Gamma$ (which happens, for example, in manifestly nonanomalous theories) or we knew that $\langle\mathcal{Y}_{R\theta}\rangle_\Gamma$ is for some reason equal to $\mathcal{O}(\varepsilon)$, the solution of our problem would be straightforward. Formula (3.9) would turn into a much simpler equation, which is integrated in ref. [8] and in appendix C. The result would be that the entire θ dependence of $\Omega_{R\theta}$ can be absorbed into a convergent canonical transformation acting on Ω_R, up to $\mathcal{O}(\varepsilon)$. Moreover, there would be no reason to keep τ generic. More simply, we could just work with $\tau = \tau^*$ from the start. Then, formula (3.9) would give (3.7). Integrating (3.7) with the procedure of appendix C, we would find a convergent canonical transformation that turns Γ_R into $\Gamma_{R\theta}$, again up to $\mathcal{O}(\varepsilon)$. That canonical transformation would also turn formula (3.1) directly into (3.6), since the right-hand side would remain evanescent.

Unfortunately, $\langle \mathcal{Y}_{R\theta} \rangle_\Gamma$ is there, because the theory we are considering is potentially anomalous, so we must study the effects of such an extra term. To achieve this goal, a few facts need to be noticed.

(i) By construction, $\Omega_{R\theta}$ and $\langle \tilde{\mathcal{Q}}_{R\theta} \rangle$ are convergent.

(ii) The local functional $(\Sigma_{R\theta}, \Sigma_{R\theta})$ is already renormalized. Indeed, formula (2.9) tells us that $\langle (\Sigma_{R\theta}, \Sigma_{R\theta}) \rangle = (\Omega_{R\theta}, \Omega_{R\theta})$, which is convergent. Since $\Sigma_{R\theta} = S_\theta + \mathcal{O}(\hbar)$, we can say that $(\Sigma_{R\theta}, \Sigma_{R\theta})$ is the arbitrary renormalization of (S_θ, S_θ).

(iii) By points (i) and (ii), all the subdiagrams of the diagrams that contribute to the average $\langle (\Sigma_{R\theta}, \Sigma_{R\theta}) \tilde{\mathcal{Q}}_{R\theta} \rangle_\Gamma$ are already renormalized, except those that contain *both* insertions of $(\Sigma_{R\theta}, \Sigma_{R\theta})$ and $\tilde{\mathcal{Q}}_{R\theta}$.

(iv) The object $\mathcal{Y}_{R\theta}$ is a bit peculiar, because at the tree level it is equal to

$$Y_\theta \equiv -\frac{i}{2}(S_\theta, S_\theta)\,\tilde{Q}_\theta. \tag{3.11}$$

The reason why the last two terms of (3.10) do not contribute at $\hbar = 0$ is that

$$\frac{\partial S_\theta}{\partial \theta} - (S_\theta, \tilde{Q}_\theta) = \frac{\partial S}{\partial \theta} = 0, \tag{3.12}$$

which follows from formula (A.6), if we understand that the primes denote the fields and the sources before the transformation, i.e. write $S = S(\Phi', K')$ and $S_\theta = S_\theta(\Phi, K)$. We see that Y_θ is the product of two local functionals. We call Y_θ a local *bifunctional*. We extend the definition of local bifunctional to any expression of the form

$$\mathcal{B} = \sum_i A_i B_i + C \tag{3.13}$$

where A_i, B_i and C are local functionals. An evanescent local bifunctional is a local bifunctional (3.13) where C and A_i (or B_i) are evanescent.

Now, (S, S) is an evanescent local functional, by formula (2.7), and S_θ is obtained from S by means of a finite canonical transformation, which preserves the antiparentheses and maps $\mathcal{O}(\varepsilon)$ into $\mathcal{O}(\varepsilon)$. Thus, (S_θ, S_θ) is also evanescent, and $\mathcal{Y}_{R\theta}$ is an evanescent local bifunctional. Actually,

(v) $\mathcal{Y}_{R\theta}$ is a renormalized evanescent local bifunctional, since formula (3.9) implies that $\langle \mathcal{Y}_{R\theta} \rangle_\Gamma$ is convergent.

The procedure to renormalize a local bifunctional is explained in appendix B. There, it is also shown how to renormalize an evanescent local bifunctional $\mathcal{E}$ in such a way that $\langle \mathcal{E}_R \rangle_\Gamma = \mathcal{O}(\varepsilon)$. To describe what happens order by order in the perturbative expansion, consider for simplicity an evanescent local bifunctional of the form $\mathcal{E} = EB + F$ where E and F are evanescent local functionals. Let E_n and B_n denote the functionals E and B renormalized up to and including n loops, and inductively assume that E_n satisfies $\langle E_n \rangle = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+1})$. Also assume that F_n is a local functional such that $\langle \mathcal{E}_n \rangle_\Gamma = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+1})$, where $\mathcal{E}_n = E_n B_n + F_n$. Then, the $\mathcal{O}(\hbar^{n+1})$ contributions to $\langle \mathcal{E}_n \rangle_\Gamma$ are the sum of a local divergent part, a local nonevanescent part and a generically nonlocal evanescent part. If B_{n+1} is the functional B renormalized up to and including $n+1$ loops, there exist local functionals E_{n+1} and F_{n+1} such that $\langle E_{n+1} \rangle = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+2})$ and $\langle \mathcal{E}_{n+1} \rangle_\Gamma = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+2})$, where $\mathcal{E}_{n+1} = E_{n+1} B_{n+1} + F_{n+1}$. The subtraction can be iterated in n to obtain $\langle E_R \rangle = \mathcal{O}(\varepsilon)$ and $\langle \mathcal{E}_R \rangle_\Gamma = \mathcal{O}(\varepsilon)$, where $E_R = E_\infty$ and $\mathcal{E}_R = \mathcal{E}_\infty$.

Although $\mathcal{Y}_{R\theta}$ is renormalized, it does not satisfy $\langle \mathcal{Y}_{R\theta} \rangle_\Gamma = \mathcal{O}(\varepsilon)$, as far as we know. However, we will obtain $\langle \mathcal{Y}_{R\theta} \rangle_\Gamma = \mathcal{O}(\varepsilon)$ by identifying the functions $\tau_j^{*\prime}(\omega, \theta)$ and setting $\tau_i = \tau_i^* + \tau_i^{\prime *}$.

To prove (3.5), (3.6) and (3.7), we proceed by induction. Let ν_{nj} denote free parameters of order $\hbar^n$. The first inductive assumption is that

(a_n) there exist finite functions $\mu_{nj}(\omega, \nu_{n+1k}, \theta) = \mathcal{O}(\theta)\mathcal{O}(\hbar)$, such that the action

$$\Sigma_n(\Phi, K, \omega, \nu_{n+1j}, \theta) \equiv \Sigma_{R\theta}(\Phi, K, \omega, \hbar\tau_j^* + \mu_{nj} + \nu_{n+1j}, \theta) \tag{3.14}$$

gives a Γ functional Ω_n that satisfies

$$(\Omega_n, \Omega_n) = \langle (\Sigma_n, \Sigma_n) \rangle_n = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+1}), \tag{3.15}$$

where $\langle \cdots \rangle_n$ denotes the average calculated with the action Σ_n.

Now, define

$$\begin{aligned} \tilde{Q}_n &\equiv \tilde{\mathcal{Q}}_{R\theta}(\Phi, K, \omega, \hbar\tau_j^* + \mu_{nj} + \nu_{n+1j}, \theta), \\ Y_n &\equiv -\frac{i}{2}(\Sigma_n, \Sigma_n)\tilde{Q}_n + \frac{\partial \Sigma_n}{\partial \theta} - (\Sigma_n, \tilde{Q}_n). \end{aligned} \tag{3.16}$$

Applying formula (A.5) to the action Σ_n and its Γ functional Ω_n, with $X = \tilde{Q}_n$, we obtain

$$\frac{\partial \Omega_n}{\partial \theta} = (\Omega_n, \langle \tilde{Q}_n \rangle_n) + \langle Y_n \rangle_{n\Gamma}, \tag{3.17}$$

where $\langle \cdots \rangle_{n\Gamma}$ denotes the one-particle irreducible diagrams of the average $\langle \cdots \rangle_n$. The second inductive assumption is that

(b_n)

$$\langle Y_n \rangle_{n\Gamma} = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+1}). \tag{3.18}$$

Statement (a_0) is true with $\mu_{0j} = 0$, because $\Sigma_0 = S_\theta + \mathcal{O}(\hbar)$ and (S_θ, S_θ) is evanescent, so $\langle (\Sigma_0, \Sigma_0) \rangle_0 = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar)$. Statement ($b_0$) is also true, because $\tilde{Q}_0 = \tilde{Q}_\theta + \mathcal{O}(\hbar)$, so $Y_0 = Y_\theta + \mathcal{O}(\hbar)$.

3.2 Inductive proof

Assume that (a_n) and (b_n) hold. Then, the averages $\langle(\Sigma_n,\Sigma_n)\rangle_n$ and $\langle Y_n\rangle_{n\Gamma}$ are evanescent up to and including n loops. The arguments of appendix B ensure that the $(n+1)$-loop contributions $Y_n^{(n+1)}$ to $\langle Y_n\rangle_{n\Gamma}$, which are convergent by formula (3.17), are the sum of a local nonevanescent part $Y_{n\text{nonev}}^{(n+1)}$ plus a generically nonlocal evanescent part. We have

$$\langle Y_n\rangle_{n\Gamma}=\mathcal{O}(\varepsilon)+Y_{n\text{nonev}}^{(n+1)}+\mathcal{O}(\hbar^{n+2}). \tag{3.19}$$

We can write an explicit expression for $Y_{n\text{nonev}}^{(n+1)}$. Recall, from formula (3.3), that the derivatives $\partial\Sigma/\partial(\hbar\tau_j)$ form a basis for the local functionals of Φ and K. Obviously, so do the derivatives $\partial\Sigma_\theta/\partial(\hbar\tau_j)\equiv\mathcal{H}_{j\theta}$. Up to higher orders in $\hbar$, the derivatives $\partial\Sigma_{R\theta}/\partial(\hbar\tau_j)=\mathcal{H}_{j\theta}+\mathcal{O}(\hbar)$ are also a basis, as well as the derivatives $\partial\Sigma_n/\partial\nu_{n+1j}$. Thus, there exist finite order-$\hbar^{n+1}$ functions $\sigma_j^{(n)}$, which depend analytically on ω, ν_{n+1k} and θ, such that

$$Y_{n\text{nonev}}^{(n+1)}=\sum_j\sigma_j^{(n)}\frac{\partial\Sigma_n}{\partial\nu_{n+1j}}+\mathcal{O}(\hbar^{n+2}). \tag{3.20}$$

Now, define

$$\mathcal{Y}_{n+1}=Y_n-\sum_j\sigma_j^{(n)}\frac{\partial\Sigma_n}{\partial\nu_{n+1j}}. \tag{3.21}$$

Taking the average of both sides, and using (A.3), we get

$$\langle\mathcal{Y}_{n+1}\rangle_{n\Gamma}=\langle Y_n\rangle_{n\Gamma}-\sum_j\sigma_j^{(n)}\frac{\partial\Omega_n}{\partial\nu_{n+1j}}. \tag{3.22}$$

Using (3.19) and (3.20), we obtain

$$\langle\mathcal{Y}_{n+1}\rangle_{n\Gamma}=Y_{n\text{nonev}}^{(n+1)}-Y_{n\text{nonev}}^{(n+1)}+\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2})=\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2}). \tag{3.23}$$

Using (3.22) inside (3.17), we also find

$$\frac{\partial\Omega_n}{\partial\theta}=(\Omega_n,\langle\tilde{Q}_n\rangle_n)+\sum_j\sigma_j^{(n)}\frac{\partial\Omega_n}{\partial\nu_{n+1j}}+\langle\mathcal{Y}_{n+1}\rangle_{n\Gamma}\,. \tag{3.24}$$

Define finite functions $\nu_{n+1j}(\omega,\bar{\nu}_{n+1k},\theta)$ as the solutions of the evolution equations

$$\frac{\partial\nu_{n+1j}}{\partial\theta}=-\sigma_j^{(n)}(\omega,\nu_{n+1k},\theta), \tag{3.25}$$

with the initial conditions $\nu_{n+1j}(\omega,\bar{\nu}_{n+1k},0)=\bar{\nu}_{n+1j}$. Clearly, $\nu_{n+1j}=\bar{\nu}_{n+1j}+\mathcal{O}(\theta)$. Given a functional $X(\Phi,K,\omega,\nu_{n+1j},\theta)$, define

$$\bar{X}(\Phi,K,\omega,\bar{\nu}_{n+1j},\theta)=X(\Phi,K,\omega,\nu_{n+1j}(\omega,\bar{\nu}_{n+1k},\theta),\theta). \tag{3.26}$$

Then,

$$\frac{\partial \bar{X}}{\partial \theta}=\overline{\frac{\partial X}{\partial \theta}}-\sum_{i}\bar{\sigma}_{j}^{(n)}\overline{\frac{\partial X}{\partial \nu_{n+1j}}}, \tag{3.27}$$

where $\bar{\sigma}_{j}^{(n)}$ are the functions obtained by applying the redefinitions $\nu_{n+1j}(\omega,\bar{\nu}_{n+1k},\theta)$ to $\sigma_{j}^{(n)}$. Choosing $X=\Omega_{n}$, we can turn equation (3.24) into

$$\frac{\partial \bar{\Omega}_{n}}{\partial \theta}=(\bar{\Omega}_{n},\overline{\langle \tilde{Q}_{n}\rangle_{n}})+\overline{\langle \mathcal{Y}_{n+1}\rangle_{n\Gamma}}, \tag{3.28}$$

Applying the redefinitions $\nu_{n+1j}(\omega,\bar{\nu}_{n+1k},\theta)$ to the functions $\mu_{nj}(\omega,\nu_{n+1k},\theta)$ of assumption (a_{n}), and including the contributions coming from $\nu_{n+1j}-\bar{\nu}_{n+1j}$, which are proportional to θ, we can define new $\mathcal{O}(\theta)\mathcal{O}(\hbar)$ functions $\bar{\mu}_{nj}(\omega,\bar{\nu}_{n+1k},\theta)$ by the formula

$$\bar{\mu}_{nj}(\omega,\bar{\nu}_{n+1k},\theta)\equiv\mu_{nj}(\omega,\nu_{n+1k}(\omega,\bar{\nu}_{n+1l},\theta),\theta)+\nu_{n+1j}(\omega,\bar{\nu}_{n+1k},\theta)-\bar{\nu}_{n+1j}.$$

Then, using (3.26) and (3.14), we have

$$\bar{\Sigma}_{n}(\Phi,K,\omega,\bar{\nu}_{n+1j},\theta)=\Sigma_{R\theta}(\Phi,K,\omega,\hbar\tau_{j}^{*}+\bar{\mu}_{nj}+\bar{\nu}_{n+1j},\theta).$$

At this point, the independent parameters are ω, $\bar{\nu}_{n+1j}$ and θ. The formulas we have written so far hold for every value of $\bar{\nu}_{n+1j}$, as long as it is $\mathcal{O}(\hbar^{n+1})$. Now we want to raise the $\hbar$ order of $\bar{\nu}_{n+1j}$ by one unit. The validity of this choice will be self-evident. By this we mean that it allows us to iterate all the arguments of the proof without difficulties till the very end and preserve the analyticity in $\hbar$.

Define

$$\bar{\nu}_{n+1j}=\nu_{n+2j},\qquad \mu_{n+1j}(\omega,\nu_{n+2k},\theta)=\bar{\mu}_{nj}(\omega,\bar{\nu}_{n+1k},\theta)|_{\bar{\nu}_{n+1k}\rightarrow\nu_{n+2k}}. \tag{3.29}$$

So doing, we obtain the action Σ_{n+1}, given by formula (3.14) with the replacement $n\rightarrow n+1$:

$$\Sigma_{n+1}(\Phi,K,\omega,\nu_{n+2j},\theta)=\Sigma_{R\theta}(\Phi,K,\omega,\hbar\tau_{j}^{*}+\mu_{n+1j}+\nu_{n+2j},\theta)=\bar{\Sigma}_{n}(\Phi,K,\omega,\nu_{n+2j},\theta). \tag{3.30}$$

Recalling that $\Sigma_{R\theta}=\Sigma_{R}+\mathcal{O}(\theta)$ and $\mu_{n+1j}=\mathcal{O}(\theta)\mathcal{O}(\hbar)$, formula (3.30) tells us that, at $\theta=0$,

$$\Sigma_{n+1}|_{\theta=0}=\Sigma_{R}(\Phi,K,\omega,\hbar\tau_{j}^{*}+\nu_{n+2j})=\Sigma_{R}(\Phi,K,\omega,\hbar\tau_{j}^{*})+\mathcal{O}(\hbar^{n+2})=S_{R}(\Phi,K,\omega)+\mathcal{O}(\hbar^{n+2}),$$

where the last equality follows from the first equation of (3.4). Finally, the second equation of (3.4) and formula (3.1) give

$$(\Omega_{n+1},\Omega_{n+1})|_{\theta=0}=(\Gamma_{R},\Gamma_{R})+\mathcal{O}(\hbar^{n+2})=\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2}). \tag{3.31}$$

This is a check that the new action Σ_{n+1} is AB nonanomalous at $\theta=0$, up to $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\hbar^{n+2})$. Now we show that Σ_{n+1} satisfies the same property for every θ.

Using formula (3.28), we get

$$\frac{\partial\Omega_{n+1}}{\partial\theta}=(\Omega_{n+1},\langle\tilde{Q}_{n+1}\rangle_{n+1})+\langle Y_{n+1}\rangle_{n+1\Gamma}\,, \tag{3.32}$$

where the functionals $\tilde{Q}_{n+1}$ and Y_{n+1} are obtained from $\tilde{Q}_n$ and $\mathcal{Y}_{n+1}$ by applying the redefinitions $\nu_{n+1j}(\omega,\bar{\nu}_{n+1k},\theta)$ and (3.29). Using (3.26), (3.27) and (3.21), it is easy to see that formulas (3.16) hold with $n\rightarrow n+1$.

Moreover, formula (3.23) ensures that

$$\langle Y_{n+1}\rangle_{n+1\Gamma}=\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2}),$$

that is to say formulas (3.17) and (3.18) hold with $n\rightarrow n+1$.

Taking the antiparentheses of (3.32) with Ω_{n+1} and using the Jacobi identity, we also find

$$\frac{\partial}{\partial\theta}(\Omega_{n+1},\Omega_{n+1})=((\Omega_{n+1},\Omega_{n+1}),\langle\tilde{Q}_{n+1}\rangle_{n+1})+2(\Omega_{n+1},\langle Y_{n+1}\rangle_{n+1\Gamma}). \tag{3.33}$$

The last term of the right-hand side is $\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2})$. In appendix C we show how to integrate equation (3.33) and prove that the θ dependence of $(\Omega_{n+1},\Omega_{n+1})$ is encoded into a canonical transformation, up to $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\hbar^{n+2})$. By formula (3.31), the value of $(\Omega_{n+1},\Omega_{n+1})$ at $\theta=0$ is also of such orders. Moreover, the canonical transformation is convergent, because it is uniquely determined by $\langle\tilde{Q}_{n+1}\rangle_{n+1}$, which is convergent. Therefore, we find

$$(\Omega_{n+1},\Omega_{n+1})=\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2})$$

for arbitrary θ, which is formula (3.15) with $n\rightarrow n+1$. As promised, the action Σ_{n+1} is AB nonanomalous for arbitrary θ, up to $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\hbar^{n+2})$. We have thus proved statements (a_{n+1}) and (b_{n+1}).

Finally, formulas (3.6) and (3.7) follow by taking n to infinity, with $\nu_{\infty j}=0$ and $\hbar\tau_j^{*\prime}(\omega,\theta)=\mu_{\infty j}(\omega,0,\theta)$. Indeed, because of (3.14), if we define $S_{R\theta}$ according to (3.5), we have $\Sigma_\infty=S_{R\theta}$, so $\Omega_\infty=\Gamma_{R\theta}$. Then, formula (3.15) becomes (3.6) at $n=\infty$. By (3.18), formula (3.17) turns into (3.7) at $n=\infty$, with $\tilde{Q}_{R\theta}=\tilde{Q}_\infty$.

3.3 Integrating the equations of gauge dependence

Equation (3.7) can be integrated with the method of appendix C (see also [8]). There, it is shown that we can consistently ignore the terms $\mathcal{O}(\varepsilon)$ appearing on the right-hand side, in the sense that the solution we find by ignoring those terms is correct up to $\mathcal{O}(\varepsilon)$. The basic reason is that the equations involve only convergent functionals. Alternatively, we can just remove the cutoff by taking the physical limit $\varepsilon\rightarrow 0$ in (3.7) and then work in the physical dimension d. The result is that every θ dependence of $\Gamma_{R\theta}$ can be absorbed into a convergent canonical transformation, up to $\mathcal{O}(\varepsilon)$.

More precisely, the theorem of appendix C ensures that there exists a canonical transformation $\Phi, K \rightarrow \Phi', K'$ such that the Γ functional Γ'_R defined by

$$\Gamma'_R(\Phi', K', \omega) = \Gamma_{R\theta}(\Phi(\Phi', K', \omega, \theta), K(\Phi', K', \omega, \theta), \omega, \theta) \tag{3.34}$$

is θ independent, up to $\mathcal{O}(\varepsilon)$. Setting $\theta = 0$, we find $\Gamma'_R = \Gamma_R$, since

$$\Gamma'_R(\Phi', K', \omega) = \Gamma_{R\theta}(\Phi', K', \omega, 0) = \Gamma_R(\Phi', K', \omega).$$

Finally, inverting the transformations, we get

$$\Gamma_{R\theta}(\Phi, K, \omega, \theta) = \Gamma_R(\Phi'(\Phi, K, \omega, \theta), K'(\Phi, K, \omega, \theta), \omega). \tag{3.35}$$

As promised, the dependence of $\Gamma_{R\theta}$ on the gauge parameter θ can be fully absorbed inside a canonical transformation.

We recall that the canonical transformations we are talking about, which are convergent, nonlocal and act on the renormalized Γ functional, originate from a local canonical transformation of the form (3.2) that acts on the tree-level action. The connection between the two is a procedure of re-renormalization and a re-fine-tuning of the finite local counterterms. We call such canonical transformations on Γ_R *special*. Clearly, the composition of special canonical transformations is a special canonical transformation. If we repeat the argument of this subsection for any other gauge parameter θ that satisfies (3.7), taking one at a time, we can prove that the entire dependence of the Γ functional on the gauge parameters can be absorbed into a special canonical transformation.

In subsection 7.3 the equations of gauge dependence are used to prove that the physical quantities are gauge independent.

4 Gauge dependence of the renormalized action

In this section we study the counterparts of equations (3.6) and (3.7) at the level of the renormalized action. Using the identity (2.9), formula (3.6) gives $(\Gamma_{R\theta}, \Gamma_{R\theta}) = \langle (S_{R\theta}, S_{R\theta}) \rangle = \mathcal{O}(\varepsilon)$, which implies that $(S_{R\theta}, S_{R\theta})$ is a "truly evanescent" local functional, i.e. a local functional such that its average is evanescent. We use the symbol $\mathcal{E}$ to denote such type of functionals. Thus, we have the formula

$$(S_{R\theta}, S_{R\theta}) = \mathcal{E}, \tag{4.1}$$

where $\langle \mathcal{E} \rangle = \mathcal{O}(\varepsilon)$. Equation (4.1) expresses the cancellation of the gauge anomalies to all orders at the level of the renormalized action.

Next, if we apply formulas (A.3) and (A.4) to (3.7), we obtain

$$\left\langle \frac{\partial S_{R\theta}}{\partial \theta} - (S_{R\theta}, \tilde{Q}_{R\theta}) \right\rangle - \frac{i}{2}\langle (S_{R\theta}, S_{R\theta})\tilde{Q}_{R\theta}\rangle_{\Gamma} = \mathcal{O}(\varepsilon). \tag{4.2}$$

By formula (4.1), $(S_{R\theta}, S_{R\theta})$ is a renormalized local functional such that its average is evanescent. In appendix B we prove that there exists an $\mathcal{O}(\hbar)$ local functional F_R, such that the local bifunctional $Y_R \equiv -(i/2)(S_{R\theta}, S_{R\theta})\tilde{Q}_{R\theta} + F_R$ is renormalized and the average $\langle Y_R\rangle_{\Gamma}$ is evanescent to all orders. We denote such F_R by the symbolic expression $(S_{R\theta}, S_{R\theta}; \tilde{Q}_{R\theta})$. Thus, formula (4.2) gives

$$\left\langle \frac{\partial S_{R\theta}}{\partial \theta} - (S_{R\theta}, \tilde{Q}_{R\theta}) - (S_{R\theta}, S_{R\theta}; \tilde{Q}_{R\theta}) \right\rangle = \mathcal{O}(\varepsilon).$$

In turn, this equation implies

$$\frac{\partial S_{R\theta}}{\partial \theta} = (S_{R\theta}, \tilde{Q}_{R\theta}) + (S_{R\theta}, S_{R\theta}; \tilde{Q}_{R\theta}) + \mathcal{E}. \tag{4.3}$$

Formula (4.3) is the equation of gauge dependence for the renormalized action $S_{R\theta}$. Note that $(S_{R\theta}, S_{R\theta}; \tilde{Q}_{R\theta})$ encodes the re-fine-tuning of the finite local counterterms.

Equation (4.3) can be integrated with the method explained in appendix C. Although the term $(S_{R\theta}, S_{R\theta}; \tilde{Q}_{R\theta})$ depends on $S_{R\theta}$, a recursive procedure allows us to treat it as a known functional at every step.

5 RG invariance and other applications

In this section we give a few applications of the theorem proved in section 3. The first application is the proof that RG invariance is preserved by the canonical transformation. The second application is the proof that renormalizable chiral gauge theories gauge-fixed by means of a nonrenormalizable gauge fixing remain renormalizable, although in a nonmanifest way. The third application is a step of the proof of the Adler-Bardeen theorem in nonrenormalizable theories [7].

RG invariance is expressed by the Callan-Symanzik equation (which is derived at the end of this section)

$$\mu\frac{\partial \Gamma_R}{\partial \mu} + \hat{\beta}^i \frac{\partial \Gamma_R}{\partial \omega_i} - (\Gamma_R, \langle U_R\rangle) = \mathcal{O}(\varepsilon), \tag{5.1}$$

where $\hat{\beta}^i$ are the ω_i beta functions (at $\varepsilon \neq 0$) and U_R is a local functional. At the level of the renormalized action S_R, the Callan-Symanzik equation reads

$$\mu\frac{\partial S_R}{\partial \mu} + \hat{\beta}^i \frac{\partial S_R}{\partial \omega_i} - (S_R, U_R) - (S_R, S_R; U_R) = \mathcal{E}. \tag{5.2}$$

Let $\tilde{\Gamma}_R$ denote the renormalized Γ functional where the parameters ω_i are written in terms of their running versions $\tilde{\omega}_i(\mu)$ and μ, where $\tilde{\omega}_i$ are the solutions of $\mu \mathrm{d}\tilde{\omega}_i/\mathrm{d}\mu = -\hat{\beta}^i(\tilde{\omega})$ with initial conditions ω_i. We have

$$\tilde{\Gamma}_R(\Phi, K, \tilde{\omega}, \mu) = \Gamma_R(\Phi, K, \omega, \mu).$$

Let $\tilde{U}_R$ denote the functional $\langle U_R \rangle$ reparametrized in a similar way. Then the Callan-Symanzik equation becomes

$$\mu \frac{\tilde{\partial}\tilde{\Gamma}_R}{\tilde{\partial}\mu} - (\tilde{\Gamma}_R, \tilde{U}_R) = \mathcal{O}(\varepsilon),$$

where $\tilde{\partial}$ denotes the derivative at fixed $\tilde{\lambda}$. The new equation has the same form as (3.7), so it is solved by making a canonical transformation. From formula (3.34), we learn that there exists a canonical transformation that takes us to new fields and sources $\tilde{\Phi}$, $\tilde{K}$ and a reference value of μ, which we denote by $\bar{\mu}$ and leave implicit, such that

$$\tilde{\Gamma}_R(\Phi, K, \tilde{\omega}, \mu) = \bar{\Gamma}_R(\tilde{\Phi}(\Phi, K, \tilde{\omega}, \mu), \tilde{K}(\Phi, K, \tilde{\omega}, \mu), \tilde{\omega}),$$

for a certain other functional $\bar{\Gamma}_R$.

Now, if we make the canonical transformation (3.2) on the tree-level action, we get, by formula (3.35),

$$\begin{aligned}\tilde{\Gamma}_{R\theta}(\Phi, K, \tilde{\omega}, \mu, \theta) &= \tilde{\Gamma}_R(\Phi'(\Phi, K, \tilde{\omega}, \mu, \theta), K'(\Phi, K, \tilde{\omega}, \mu, \theta), \tilde{\omega}, \mu) = \\ &= \bar{\Gamma}_R(\tilde{\Phi}(\Phi', K', \tilde{\omega}, \mu), \tilde{K}(\Phi', K', \tilde{\omega}, \mu), \tilde{\omega}).\end{aligned}$$

Going back to the parameters ω, we also have

$$\Gamma_{R\theta}(\Phi, K, \omega, \mu, \theta) \equiv \tilde{\Gamma}_{R\theta}(\Phi, K, \tilde{\omega}, \mu, \theta) = \bar{\Gamma}_R(\bar{\Phi}'(\Phi, K, \omega, \mu, \theta), \bar{K}'(\Phi, K, \omega, \mu, \theta), \tilde{\omega}(\omega, \mu)).$$

having defined $\tilde{\Phi}(\Phi', K', \tilde{\omega}, \mu) = \bar{\Phi}'(\Phi, K, \omega, \mu, \theta)$ and similarly for $\tilde{K}$. Differentiating with respect to $\ln \mu$, and recalling that our canonical transformations are special, we get

$$\mu \frac{\partial \Gamma_{R\theta}}{\partial \mu} + \hat{\beta}^i \frac{\partial \Gamma_{R\theta}}{\partial \omega_i} - (\Gamma_{R\theta}, \langle U_{R\theta} \rangle) = \mathcal{O}(\varepsilon), \qquad (5.3)$$

for some new local functional $U_{R\theta}$. Formula (5.3) is the transformed RG equation. Note that the beta functions do not depend on θ in this approach.

Another application that we mention is to power-counting renormalizable chiral gauge theories gauge-fixed by means of a nonrenormalizable gauge fixing. If a renormalizable theory is nonchiral, it is rather straightforward to prove that it remains renormalizable when a nonrenormalizable gauge fixing is used. When the theory is chiral, on the other hand, the matter is more complicated. In principle, the simplifications between divergences and evanescences can make the parameters of negative dimensions, introduced by the gauge fixing, propagate into the physical sector and turn the theory into a truly nonrenormalizable one. The theorem of section 3, combined with RG invariance, ensures that this cannot happen.

Consider for example the standard model in flat space and gauge fix a non-Abelian gauge symmetry by means of a gauge-fixing function such as

$$\bar{G}^a(\phi) = \partial^\mu A_\mu^a + \kappa A_\mu^b A_\nu^a F^{b\mu\nu}.$$

Since the constant κ has dimension -2 in units of mass, power counting alone is not sufficient to classify the counterterms in a convenient way at $\kappa \neq 0$. However, the change of gauge fixing that turns $G^a(\phi) = \partial^\mu A_\mu^a$ into $\bar{G}^a(\phi)$ is a canonical transformation, so we can apply the theorem of section 3. Formula (3.3) teaches us that infinitely many terms $\mathcal{H}_i$ of arbitrary dimensions are switched on, including the gauge noninvariant ones. Nevertheless, the theorem ensures that once we have done that, it is possible to express the coefficients of all of those terms as functions of the other parameters of the theory, and fine-tune those functions to enforce again the cancellation of gauge anomalies to all orders. Moreover, the argument given above ensures that RG invariance is preserved. We conclude that no new independent parameters are necessary to subtract the divergences and cancel the gauge anomalies in a RG invariant way: the physical couplings are still finitely many. Thus, when a power counting renormalizable chiral gauge theory, such as the standard model in flat space, is gauge-fixed by means of a nonrenormalizable gauge fixing, it remains a renormalizable theory, although its renormalizability is not manifest anymore. A similar conclusion holds when the theory is renormalizable by weighted power counting [27] or any other criterion.

The third application we mention is the proof of the Adler-Bardeen theorem in nonrenormalizable theories, recently obtained in ref. [7] by upgrading the arguments of [6]. It applies to the theories whose gauge symmetries are general covariance, local Lorentz symmetry and Abelian and non-Abelian Yang-Mills symmetries, and satisfy a variant of the Kluberg-Stern–Zuber conjecture. Quantum gravity coupled to the standard model satisfies all the assumptions and so is free of gauge anomalies to all orders. In the approach of [7], the CD regularization is combined with a higher-derivative regularization. If the scale Λ associated with the higher-derivative terms is kept fixed, we obtain a super-renormalizable higher-derivative (HD) theory, which satisfies the Adler-Bardeen theorem by simple power-counting arguments. When the scale Λ is sent to infinity, the Λ divergences are renormalized inductively. At each step, the theorem of section 3 allows us to resubtract the divergences in ε and re-fine-tune the finite local terms, in order to enforce the cancellation of gauge anomalies to all orders at Λ fixed. In the end, thanks to this, the cancellation of gauge anomalies survives the renormalization of both types of divergences. Moreover, the approach of ref. [7] identifies a special subtraction scheme where the cancellation of gauge anomalies is manifest from two loops onwards, within any given truncation. We stress that it is not possible to achieve a similar goal by means of regularization-independent methods.

The Callan-Symanzik equation (5.1) can be proved from the results of ref. [7] as follows, under the assumptions specified there. At Λ fixed the HD theory is renormalized by redefinitions of parameters, while the trivially anomalous terms are canceled by adding a finite local functional $-\chi/2$ to the action. The renormalized action coincides with its bare version apart from χ itself, which satisfies $\chi = \mu^{-\varepsilon}\chi_{\mathrm{B}}$, where χ_{B} is RG invariant. Then, the HD theory satisfies formulas (5.1) and (5.2) with $U_R = 0$, the right-hand side of (5.2) being equal $\varepsilon\chi/2$. The average $\langle\chi\rangle$ is convergent in the HD theory (since its divergences, which would start from two loops, are excluded by the arguments of [7]), so the product $\varepsilon\langle\chi\rangle/2$ is truly evanescent at Λ fixed. At a second stage, the renormalization is completed by removing the Λ divergences. This is done by means of special canonical transformations and redefinitions of parameters. In this section we have proved that those operations preserve the Callan-Symanzik equation, although they can affect the beta functions and the functional U_R. In the end, we obtain equations of the forms (5.1) and (5.2).

6 Gauge dependence of the beta functions

Often, we can prove that a theory is AB nonanomalous in a family of gauges, parametrized by certain gauge-fixing parameters ξ. In various common situations we can achieve this goal by applying the results of ref. [6], where the Adler-Bardeen theorem was proved for arbitrary values of the gauge-fixing parameter ξ of the Lorenz gauge, in power counting renormalizable gauge theories that have unitary free-field limits. More generally, if the theory is coupled to quantum gravity, we can apply the results of [7]. Then, when we study the dependence of the correlation functions on ξ, we can proceed more straightforwardly than in section 3, since we already know that $(\Gamma_R, \Gamma_R) = \mathcal{O}(\varepsilon)$ for arbitrary ξ. It is worth recalling that in section 3 we had to derive this result from just knowing that (Γ_R, Γ_R) was $\mathcal{O}(\varepsilon)$ for ξ equal to some initial value ξ^*.

In this section we study the equations of gauge dependence in theories that are AB nonanomalous for arbitrary values of some gauge parameter θ and satisfy some additional assumptions. Those assumptions are not very restrictive, since they are fulfilled quite commonly. When θ varies, we do not need to readjust the subtraction scheme by fine-tuning the finite local counterterms. Then, however, the beta functions of the couplings are in general gauge dependent. Their gauge dependence can be removed by redefining the couplings themselves.

We begin by listing the assumptions we need.

(I) We assume that the gauge algebra is irreducible and closes off shell. This assumption is satisfied by the theories whose gauge symmetries are general covariance, local Lorentz symmetry and Abelian and non-Abelian Yang-Mills symmetries, such as the standard model coupled to quantum gravity. It allows us to make a number of simplifications. For example, we can choose the fields Φ and the sources K so that the gauge-fixed tree-level solution S_d of the D-dimensional master equation $(S_d, S_d) = 0$ is linear in K and has the very simple structure (2.3).

We have already remarked that in various cases, for example when the theory is chiral or parity violating, the action S_d, embedded in D dimensions using the standard rules of the dimensional regularization technique, is in general not well regularized, due to the key role played by the d-dimensional analogue $\tilde{\gamma}$ of the matrix γ_5, or the tensor $\varepsilon^{\bar{a}_1 \cdots \bar{a}_d}$. Using the chiral dimensional regularization, a well-regularized classical action $S(\Phi, K)$ is obtained by adding a number of evanescent corrections S_{ev} to S_d [11], as shown in formula (2.6). We denote the parameters contained in S_{ev} by η_I. For convenience, we assume that S_{ev} depends linearly on the parameters η, and vanishes for $\eta = 0$.

Let $\{\mathcal{G}_i(\phi)\}$ denote a basis of local gauge invariant functionals of the classical fields ϕ. Expand the classical action as

$$S_c(\phi) = \sum_i \lambda_i \mathcal{G}_i(\phi), \tag{6.1}$$

where λ_i are independent parameters. We call the constants λ_i "physical parameters", since they contain or are related to the gauge coupling constants, the masses, the Yukawa couplings, etc. In our notation some parameters λ_i may be actually redundant. Nevertheless, to simplify some derivations we prefer to keep an independent λ_i for every $\mathcal{G}_i$. For example, it is often useful to restrict S_c by dropping the terms that are proportional to the S_c field equations, because those terms can be renormalized by means of canonical transformations, rather than λ_i redefinitions. We do not implement this restriction right now, to make some arguments of the derivations that follow more transparent. We can always remove that class of redundant terms at the end by means of a convergent canonical transformation, by applying either the procedure of section 3, which is more general, or the one of this section, which holds under specific assumptions. Both procedures preserve the cancellation of gauge anomalies and the equations of gauge dependence.

In total, we have physical parameters λ, gauge-fixing parameters ξ, contained in Ψ, and regularizing parameters η. The classical action is written as $S(\Phi, K, \lambda, \xi, \eta)$.

The action S_c may contain *accidental* symmetries, which are the global symmetries unrelated to the gauge transformations. Some accidental symmetries are dynamically lost, because they are anomalous, others are nonanomalous. Let G_{nas} denote the group of nonanomalous accidental symmetries, or the identity group, depending on whether the gauge group contains $U(1)$ factors or not. By definition, the set $\{\mathcal{G}_i(\phi)\}$ includes the invariants that explicitly break the anomalous accidental symmetries, but excludes the invariants, denoted by $\check{\mathcal{G}}_i(\phi)$, that explicitly break G_{nas}. Then the actions S_c and S_d do not contain the invariants $\check{\mathcal{G}}_i$, so we define extended actions $\check{S}_c$ and $\check{S}_d = \check{S}_c + (S_K, \Psi) + S_K$ that do include them, multiplied by independent parameters $\check{\lambda}_i$. Both choices of including and excluding the invariants $\check{\mathcal{G}}_i$, are consistent, from the point of view of renormalization.

We say that the action S_d satisfies the Kluberg-Stern–Zuber assumption [19], if every nonevanescent local functional X of ghost number zero that solves the equation $(S_d, X) = 0$ has the form

$$X = \sum_i a_i \mathcal{G}_i + (S_d, Y),$$

where a_i are constants depending on the parameters of the theory, and Y is a local functional of ghost number -1.

We say that the action S_d is *cohomologically complete* if its extension $\check{S}_d$ satisfies the extended Kluberg-Stern–Zuber assumption, that is to say every nonevanescent local functional X of ghost number zero that solves $(\check{S}_d, X) = 0$ has the form

$$X = \sum_i a_i \mathcal{G}_i + \sum_i b_i \check{\mathcal{G}}_i + (\check{S}_d, Y), \tag{6.2}$$

where b_i are other constants, and Y is a local functional.

(II) We assume that the action S_d of (2.3) is cohomologically complete and the group G_{nas} is compact.

The Kluberg-Stern–Zuber assumption is satisfied when the Yang-Mills gauge group is semisimple and the action S_d satisfies generic properties [30]. It is not satisfied when the gauge group has $U(1)$ factors and accidental symmetries are present. In particular, it is not satisfied by the standard model. However, it can be proved, using the Ward identities that hold in the Lorenz gauge, that the standard model is cohomologically complete [6]. So are the Lorentz violating extensions of the standard model of refs. [12, 31], which are renormalizable by weighted power counting [27]. Starting from the cohomological theorems proved in ref. [30], it can be proved that the standard model coupled to quantum gravity is also cohomologically complete [7], and so are most of its extensions.

The condition $(S_d, X) = 0$ is the one typically satisfied by the counterterms. In this section we show that the contributions of the extra term contained in the generalized Ward identity (2.15) satisfy the same condition. Thus, assumption (II) will give us control on the effects of the new term.

We can imagine that θ is one of the parameters ξ, or another parameter introduced by a field redefinition. We keep it distinct from the other parameters λ, ξ, η contained in the action S and assume that $S(\Phi, K, \lambda, \xi, \eta)$ denotes the action at some specific value θ^* of θ. With no loss of generality, we take $\theta^* = 0$. By definition of gauge parameter, when we vary θ, we make a canonical transformation generated by a functional of the form (3.2) on the action S, and this operation gives the action S_θ. As before, let $S_{R\theta}$ denote the renormalized action and $\Gamma_{R\theta}$ the Γ functional associated with it.

(III) We assume that the theory is AB nonanomalous for arbitrary values of some gauge parameter θ. Precisely, we assume that there exists a class of subtraction schemes where the renormalized Γ functional $\Gamma_{R\theta}$ satisfies the identity

$$(\Gamma_{R\theta}, \Gamma_{R\theta}) = \mathcal{O}(\varepsilon), \tag{6.3}$$

where θ takes values in some continuous range that includes $\theta = 0$. From now on we understand that we work in that class of subtraction schemes.

Assumption (III) has been proved, for common families of gauge conditions, in the power-counting renormalizable gauge theories that have unitary free-field limits [6], in the Lorentz violating extensions of the standard model that are renormalizable by weighted power counting [12, 31], in the standard model coupled to quantum gravity and a large class of other nonrenormalizable theories [7].

We prove that there exist finite functions ρ_j of λ, ξ, η and θ, which start from $\mathcal{O}(\hbar)$, and a renormalized local functional $H_{R\theta}(\Phi, K) = \tilde{Q}_\theta(\Phi, K) + \mathcal{O}(\hbar)$, where $\tilde{Q}_\theta(\Phi, K) = Q_\theta(\Phi, K'(\Phi, K))$ and $Q_\theta = \partial Q/\partial\theta$, such that $\Gamma_{R\theta}$ satisfies the equation

$$\frac{\partial \Gamma_{R\theta}}{\partial\theta} = \sum_j \rho_j \frac{\partial \Gamma_{R\theta}}{\partial\lambda_j} + (\Gamma_{R\theta}, \langle H_{R\theta}\rangle) + \mathcal{O}(\varepsilon). \tag{6.4}$$

The first term on the right-hand side of (6.4) can be absorbed by means of finite redefinitions of the parameters λ (which correspond to the re-fine-tuning of the previous section). The second term is the one that can be absorbed into a canonical transformation.

In the rest of this section we derive the equations of gauge dependence (6.4) under the assumptions listed above, and integrate them. Before beginning the derivation, a few preliminary remarks are in order. If we differentiate (6.3) with respect to any parameter ζ, we find

$$\left(\Gamma_{R\theta}, \frac{\partial \Gamma_{R\theta}}{\partial\zeta}\right) = \mathcal{O}(\varepsilon). \tag{6.5}$$

Now we take the antiparentheses of both sides of formula (3.7) or (6.4) with $\Gamma_{R\theta}$, and use (6.5) for $\zeta = \theta$ and $\zeta = \lambda_j$, the Jacobi identity satisfied by the antiparentheses and formula (6.3) again. At the end, we find a consistent relation of the form $\mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon)$. Thus, we can view formulas (3.7) and (6.4) as the solutions to the condition (6.5) for $\zeta = \theta$.

To explain this issue more clearly, let us define an operator δ_Γ that acts on a (generically nonlocal) functional Y by taking its antiparentheses with $\Gamma_{R\theta}$: $\delta_\Gamma Y = (\Gamma_{R\theta}, Y)$. Formula (6.3) ensures that δ_Γ is nilpotent up to $\mathcal{O}(\varepsilon)$, because the Jacobi identity gives

$$\delta_\Gamma^2 Y = (\Gamma_{R\theta}, (\Gamma_{R\theta}, Y)) = \frac{1}{2}((\Gamma_{R\theta}, \Gamma_{R\theta}), Y) = \mathcal{O}(\varepsilon). \tag{6.6}$$

Therefore, it is meaningful to study the cohomology of δ_Γ. Consider the problem $\delta_\Gamma Y = 0$, of which the $\varepsilon \to 0$ limit of (6.5) is an example. It is a nonlocal upgrade of the more standard cohomological problem $(S_d, X) = 0$, where X is local. Formula (6.5) tells us that $\partial\Gamma_{R\theta}/\partial\theta$ is closed, in the sense of the δ_Γ cohomology, up to $\mathcal{O}(\varepsilon)$. On the other hand, formula (6.4) ensures that there exist finite linear combinations of $\partial\Gamma_{R\theta}/\partial\zeta$ that are δ_Γ-exact, up to $\mathcal{O}(\varepsilon)$.

However, nonlocal cohomological problems are difficult to solve and must be treated with care, because if we do not specify which nonlocalities are allowed and which are not, any closed functional can in principle be exact. In other words, we cannot derive (6.4) immediately from (6.5), which is why gauge dependence deserves a separate investigation.

6.1 The equations of gauge dependence

We apply formula (A.5) of appendix A to the renormalized action $S_{R\theta}$ and the renormalized Γ functional $\Gamma_{R\theta}$, with $X = \tilde{Q}_{R\theta}$, where $\tilde{Q}_{R\theta}$ denotes the renormalized version of the functional $\tilde{Q}_\theta(\Phi, K)$. We obtain

$$\frac{\partial\Gamma_{R\theta}}{\partial\theta} = (\Gamma_{R\theta}, \langle\tilde{Q}_{R\theta}\rangle) + \langle U_{R\theta}\rangle_\Gamma\,, \tag{6.7}$$

where

$$U_{R\theta} = -\frac{i}{2}(S_{R\theta}, S_{R\theta})\tilde{Q}_{R\theta} + \frac{\partial S_{R\theta}}{\partial\theta} - (S_{R\theta}, \tilde{Q}_{R\theta}). \tag{6.8}$$

Taking the antiparentheses of both sides of (6.7) with $\Gamma_{R\theta}$ and using (6.5) and (6.6), we obtain

$$(\Gamma_{R\theta}, \langle U_{R\theta}\rangle_\Gamma) = \mathcal{O}(\varepsilon). \tag{6.9}$$

Differently from (6.5), this nonlocal cohomological problem can be reduced to a local one, and solved. The reason is that $U_{R\theta}$ is originated by an evanescent local bifunctional. We prove that there exist *finite* functions $\rho_j = \mathcal{O}(\hbar)$ of λ, ξ, η and θ, and a renormalized local functional $W_{R\theta} = \mathcal{O}(\hbar)$, such that

$$\langle U_{R\theta}\rangle_\Gamma = \sum_j \rho_j \frac{\partial\Gamma_{R\theta}}{\partial\lambda_j} + (\Gamma_{R\theta}, \langle W_{R\theta}\rangle) + \mathcal{O}(\varepsilon). \tag{6.10}$$

We proceed by induction. Assume that there exist finite functions $\rho_{nj} = \mathcal{O}(\hbar)$ of λ, ξ, η and θ, and a renormalized local functional $W_n = \mathcal{O}(\hbar)$, such that the partially subtracted functional

$$U_n \equiv U_{R\theta} - \sum_j \rho_{nj}\frac{\partial S_{R\theta}}{\partial\lambda_j} - (S_{R\theta}, W_n) - \frac{i}{2}(S_{R\theta}, S_{R\theta})W_n, \tag{6.11}$$

satisfies

$$\langle U_n\rangle_\Gamma = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+1}). \tag{6.12}$$

This assumption is clearly satisfied at the zeroth order, where $\rho_{0j} = 0$ and $W_0 = 0$, because by formula (3.12) we have $U_{R\theta} = Y_\theta + \mathcal{O}(\hbar)$, where Y_θ is evanescent and given by (3.11).

Using formulas (A.3) and (A.4), we obtain the average

$$\langle U_n\rangle_\Gamma = \langle U_{R\theta}\rangle_\Gamma - \sum_j \rho_{nj}\frac{\partial\Gamma_{R\theta}}{\partial\lambda_j} - (\Gamma_{R\theta}, \langle W_n\rangle), \tag{6.13}$$

which is clearly convergent. Consider the $(n+1)$-loop contributions $U_n^{(n+1)}$ to $\langle U_n \rangle_\Gamma$. They are convergent, because so is the right-hand side of (6.13). Moreover, the inductive assumption (6.12) states that the average $\langle U_n \rangle_\Gamma$ is evanescent up to and including n loops, while (6.3) ensures that $\langle (S_{R\theta}, S_{R\theta}) \rangle = (\Gamma_{R\theta}, \Gamma_{R\theta})$ is evanescent to all orders. The arguments of appendix B ensure that the functional $U_n^{(n+1)}$ is the sum of a local nonevanescent part $U_{n\text{nonev}}^{(n+1)}$ plus a generically nonlocal evanescent part:

$$\langle U_n \rangle_\Gamma = U_{n\text{nonev}}^{(n+1)} + \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+2}). \tag{6.14}$$

Thus, using (6.13) and (6.14), we have

$$\langle U_{R\theta} \rangle_\Gamma = \sum_j \rho_{nj} \frac{\partial \Gamma_{R\theta}}{\partial \lambda_j} + (\Gamma_{R\theta}, \langle W_n \rangle) + U_{n\text{nonev}}^{(n+1)} + \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+2}). \tag{6.15}$$

Inserting this expression inside (6.9) and using (6.5), (6.6) and (6.3), we obtain

$$(\Gamma_{R\theta}, U_{n\text{nonev}}^{(n+1)}) = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+2}).$$

Taking the $(n+1)$-loop nonevanescent contributions to this formula, we find

$$(S_{d\theta}, U_{n\text{nonev}}^{(n+1)}) = 0, \tag{6.16}$$

where $S_{d\theta}$ is the action obtained by applying the canonical transformation (3.2) to S_d. In deriving the result (6.16), it is important to recall that the tree-level action (2.6) and the canonical transformation (3.2) do not contain analytically evanescent terms. In turn, S_θ and $S_{d\theta}$ satisfy the same property, and $S_{d\theta}$ is the full nonevanescent part of S_θ.

Applying the inverse of the transformation (3.2) to equation (6.16) and letting $\tilde{U}_{n\text{nonev}}^{(n+1)}$denote the functional obtained from $U_{n\text{nonev}}^{(n+1)}$, we get

$$(S_d, \tilde{U}_{n\text{nonev}}^{(n+1)}) = 0. \tag{6.17}$$

At this point, we apply assumption (II). Let us imagine that instead of working with the classical action S_c we work with its extension $\check{S}_c$, which includes the invariants $\check{\mathcal{G}}_i$ that break the nonanomalous accidental symmetries belonging to the group G_{nas}. Similarly, we extend S_d to $\check{S}_d$, S_{ev} to $\check{S}_{\text{ev}}$ and $S = S_d + S_{\text{ev}}$ to $\check{S}$. Every extended functional reduces to the nonextended one when we set $\check{\lambda} = \check{\eta} = 0$, where $\check{\lambda}$ and $\check{\eta}$ are the extra parameters of $\check{S}_c$ and $\check{S}_{\text{ev}}$, respectively. If we repeat the operations that lead to (6.17), we obtain an extended, nonevanescent local functional $\check{U}_{n\text{nonev}}^{(n+1)}$ that satisfies $(\check{S}_d, \check{U}_{n\text{nonev}}^{(n+1)}) = 0$. By assumption (II), the action $\check{S}_d$ satisfies the extended Kluberg-Stern–Zuber assumption. Therefore, there exist finite order-$\hbar^{n+1}$ constants $\check{\sigma}_i^{(n+1)}$, $\check{\tau}_i^{(n+1)}$, depending on the parameters, and a finite nonevanescent local functional $\check{V}_\theta^{(n+1)}$ of order $\hbar^{n+1}$ such that

$$\check{U}_{n\text{nonev}}^{(n+1)} = \sum_i \check{\sigma}_i^{(n+1)} \mathcal{G}_i + \sum_i \check{\tau}_i^{(n+1)} \check{\mathcal{G}}_i + (\check{S}_d, \check{V}_\theta^{(n+1)}).$$

If we set $\check{\lambda}=\check{\eta}=0$ in this equation, we obtain

$$\tilde{U}_{\text{nnonev}}^{(n+1)}=\sum_i \bar{\sigma}_i^{(n+1)}\mathcal{G}_i+\sum_i \bar{\tau}_i^{(n+1)}\check{\mathcal{G}}_i+(S_d,\bar{V}_\theta^{(n+1)}), \tag{6.18}$$

where $\bar{\sigma}_i^{(n+1)}$, $\bar{\tau}_i^{(n+1)}$ and $\bar{V}_\theta^{(n+1)}$ are equal to $\check{\sigma}_i^{(n+1)}$, $\check{\tau}_i^{(n+1)}$ and $\check{V}_\theta^{(n+1)}$ at $\check{\lambda}=\check{\eta}=0$. However, $\tilde{U}_{\text{nnonev}}^{(n+1)}$ and S_d are invariant under G_{nas}, while the functionals $\check{\mathcal{G}}_i$ are not. If we average on G_{nas} (which we can do, since G_{nas} is assumed to be compact), the $\check{\mathcal{G}}_i$ disappear or give linear combinations of the invariants $\mathcal{G}_i$, and $\bar{V}_\theta^{(n+1)}$ turns into some $\tilde{V}_\theta^{(n+1)}$. We obtain[1]

$$\tilde{U}_{\text{nnonev}}^{(n+1)}=\sum_j \sigma_j^{(n+1)}\frac{\partial S_d}{\partial\lambda_j}+(S_d,\tilde{V}_\theta^{(n+1)}), \tag{6.19}$$

for some new constants $\sigma_j^{(n+1)}$. We have used $\mathcal{G}_i=\partial S_d/\partial\lambda_j$. At this point, we apply the canonical transformation (3.2) again, and note that, by formula (A.6) the difference between the transformed $\partial S_d/\partial\lambda_j$ and $\partial S_{d\theta}/\partial\lambda_j$ is equal to $(S_{d\theta},X_\theta)$ for some local functional X_θ. In the end, we get

$$U_{\text{nnonev}}^{(n+1)}=\sum_j \sigma_j^{(n+1)}\frac{\partial S_{d\theta}}{\partial\lambda_j}+(S_{d\theta},V_\theta^{(n+1)}) \tag{6.20}$$

for some new local functional $V_\theta^{(n+1)}$ of order $\hbar^{n+1}$. Now, define

$$U_{n+1}=U_{R\theta}-\sum_j \rho_{n+1\,j}\frac{\partial S_{R\theta}}{\partial\lambda_j}-(S_{R\theta},W_{n+1})-\frac{i}{2}(S_{R\theta},S_{R\theta})W_{n+1},$$
$$\rho_{n+1\,j}=\rho_{nj}+\sigma_j^{(n+1)},\qquad W_{n+1}=W_n+V_{R\theta}^{(n+1)},$$

where $V_{R\theta}^{(n+1)}$ are the renormalized versions of the functionals $V_\theta^{(n+1)}$. Using (6.11), we also have

$$U_{n+1}=U_n-\sum_j \sigma_j^{(n+1)}\frac{\partial S_{R\theta}}{\partial\lambda_j}-(S_{R\theta},V_{R\theta}^{(n+1)})-\frac{i}{2}(S_{R\theta},S_{R\theta})V_{R\theta}^{(n+1)}. \tag{6.21}$$

Recall that $S=S_d+S_{\text{ev}}$, which implies $S_\theta=S_{d\theta}+\mathcal{O}(\varepsilon)$ and $S_{R\theta}=S_{d\theta}+\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar)$. Taking the average of both sides of (6.21), and using (A.3), (A.4), (6.20) and then (6.14), we find

$$\langle U_{n+1}\rangle_\Gamma=\langle U_n\rangle_\Gamma-U_{\text{nnonev}}^{(n+1)}+\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2})=\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^{n+2}),$$

which extends the inductive assumption (6.12) to the order $n+1$. Formula (6.10) follows from formula (6.15) for $n=\infty$, with $\rho_j=\rho_{\infty j}$ and $W_{R\theta}=W_\infty=\mathcal{O}(\hbar)$.

[1]If the terms proportional to the S_c field equations are dropped from S_c, the average on G_{nas} may generate them back. In the case of general covariance, local Lorentz symmetry and Yang-Mills symmetries, the average of $\check{\mathcal{G}}_i$ may also affect $\tilde{V}_\theta^{(n+1)}$, besides the coefficients $\sigma_i^{(n+1)}$, but the final result is still of the form (6.19).

Finally, using (6.10) inside (6.7), we get

$$\frac{\partial \Gamma_{R\theta}}{\partial \theta} = \sum_j \rho_j \frac{\partial \Gamma_{R\theta}}{\partial \lambda_j} + (\Gamma_{R\theta}, \langle \tilde{Q}_{R\theta} + W_{R\theta} \rangle) + \mathcal{O}(\varepsilon).$$

This formula is equivalent to (6.4) with the identification $H_{R\theta} = \tilde{Q}_{R\theta} + W_{R\theta}$. Observe that $H_{R\theta}$ is another renormalized version of the functional $\tilde{Q}_\theta(\Phi, K)$, and just differs from $\tilde{Q}_{R\theta}$ by a choice of subtraction scheme.

6.2 Integrating the new equations and RG invariance

Now we integrate the equations (6.4). We can easily absorb away the first term on the right-hand side by making finite redefinitions $\lambda(\lambda', \theta)$ of the parameters λ. We choose functions $\lambda_i(\lambda', \xi, \eta, \theta)$ that solve the evolution equations

$$\frac{\partial \lambda_i}{\partial \theta} = -\rho_i(\lambda, \xi, \eta, \theta), \tag{6.22}$$

with the initial conditions $\lambda_i(\lambda', \xi, \eta, 0) = \lambda'_i$. Using formulas (6.4) and (3.27), we obtain

$$\frac{\partial \bar{\Gamma}_{R\theta}}{\partial \theta} = (\bar{\Gamma}_{R\theta}, \overline{\langle H_{R\theta} \rangle}) + \mathcal{O}(\varepsilon), \tag{6.23}$$

where $\bar{\Gamma}_{R\theta}$ is related to $\Gamma_{R\theta}$ according to the definition $\lambda(\lambda', \xi, \eta, \theta)$ [see (3.26) and the arguments given right after that formula].

Observe that equation (6.23) is equivalent to formula (3.7) of section 3. This means the redefinitions $\lambda_i(\lambda', \xi, \eta, \theta)$ perform the re-fine-tuning of finite local counterterms (automatically incorporated in the approach of section 3) that was missing so far in the approach of the present section. As in subsection 3.3, equation (6.23) can be integrated with the method of appendix C. We find that there exists a canonical transformation $\Phi, K \rightarrow \Phi', K'$ such that the Γ functional Γ'_R defined by

$$\Gamma'_R(\Phi', K', \lambda', \xi, \eta) = \Gamma_{R\theta}(\Phi(\Phi', K', \lambda', \xi, \eta, \theta), K(\Phi', K', \lambda', \xi, \eta, \theta), \lambda(\lambda', \xi, \eta, \theta), \xi, \eta, \theta) \tag{6.24}$$

is θ independent, up to $\mathcal{O}(\varepsilon)$. Since $\theta = 0$ gives $\Gamma'_R = \Gamma_R$, we also have

$$\Gamma'_R(\Phi', K', \lambda', \xi, \eta) = \Gamma_{R\theta}(\Phi', K', \lambda', \xi, \eta, 0) = \Gamma_R(\Phi', K', \lambda', \xi, \eta).$$

Inverting the transformations, we obtain the formula

$$\Gamma_{R\theta}(\Phi, K, \lambda, \xi, \eta, \theta) = \Gamma_R(\Phi'(\Phi, K, \lambda, \xi, \eta, \theta), K'(\Phi, K, \lambda, \xi, \eta, \theta), \lambda'(\lambda, \xi, \eta, \theta), \xi, \eta), \tag{6.25}$$

which shows that the dependence of $\Gamma_{R\theta}$ on the gauge parameter θ can be fully absorbed inside a finite redefinition of the parameters λ and a canonical transformation.

According to formulas (6.24) and (6.25), the beta functions $\beta'_{\lambda'}$ of the parameters λ' (in the framework where the fields and the sources have primes) are θ independent. That means, however, that the beta functions β_λ of the couplings λ do depend on θ. However, their θ dependence is not arbitrary, because it disappears by making the redefinitions $\lambda(\lambda', \xi, \eta, \theta)$.

We can repeat the argument for any other gauge parameter θ for which formula (6.3) is known to hold, taking one at a time. Since the composition of special canonical transformations and redefinitions of parameters is a special canonical transformation combined with a redefinition of parameters, we reach the conclusion that the entire dependence on the gauge parameters can be absorbed into such operations, which do not affect the physical quantities (see subsection 7.3).

We can also repeat the arguments of section 5 and prove that RG invariance is preserved. The difference is that now instead of (5.3) we get a transformed Callan-Symanzik equation that contains θ-dependent beta functions.

7 Gauge independence and unitarity

In general gauge theories we need to introduce extra fields, such as the Fedeev-Popov ghosts C, the antighosts $\bar{C}$ and the Lagrange multipliers B, and choose gauge-fixing conditions to make the functional integral perturbatively well defined. In addition, to implement the renormalization of divergences to all orders, study the gauge dependence and prove the Adler-Bardeen theorem, it is also convenient to introduce the sources K and use the Batalin-Vilkovisky formalism. The extra fields and the sources must be switched off at some point. In this section we explain how to define the physical quantities and show that they are gauge independent, under the sole assumption that the theory is AB nonanomalous, as in section 3. We work with convergent functionals, so we can set $\varepsilon = 0$. We denote the $\varepsilon \to 0$ limits of Γ_R and the other functionals involved in our arguments by the same symbols used so far, since no confusion is expected to arise.

First, we need to "un-gauge-fix" the theory, by switching off $\bar{C}$, B and their sources $K_{\bar{C}}$, K_B. This operation is regular inside the Γ functionals, once Feynman diagrams have been evaluated, but not inside the actions S and S_R, in the sense that if we un-gauge-fix the action, Feynman diagrams obviously become ill defined. For this reason, some gauge dependence survives the un-gauge-fixing procedure. Besides un-gauge-fixing, we must switch off the sources K. The combined switch-off procedure allows us to define a physical Γ functional, identify its gauge symmetries, check that they close on-shell, and prove that no gauge dependence affects the physical quantities.

Since the gauge fixing is introduced by means of a canonical transformation, such as (2.5), when we vary the gauge-fixing parameters $\theta = \xi$ we make a canonical transformation. Therefore, the equations (3.7) and (6.4) can be used to study the dependence of the physical quantities on the parameters ξ.

The information gathered so far is encoded in the key formulas

$$(\Gamma_{R\theta},\Gamma_{R\theta})=0, \tag{7.1}$$

$$\frac{\partial\Gamma_{R\theta}}{\partial\theta}-\sum_j\rho_j\frac{\partial\Gamma_{R\theta}}{\partial\lambda_j}-(\Gamma_{R\theta},\langle H_{R\theta}\rangle)=0, \tag{7.2}$$

and is sufficient to achieve the goals of this section. We work on the $\varepsilon\rightarrow 0$ limit of (6.4), rather than the one of (3.7), because everything we say starting from the former can be easily generalized to the other case.

7.1 Quantum gauge algebra

Formula (7.1) gives

$$0=-\int\frac{\delta_r\Gamma_{R\theta}}{\delta K_\alpha}\frac{\delta_l\Gamma_{R\theta}}{\delta\Phi^\alpha}=-\int\left\langle\frac{\delta_r S_{R\theta}}{\delta K_\alpha}\right\rangle\frac{\delta_l\Gamma_{R\theta}}{\delta\Phi^\alpha}=\int\langle(S_{R\theta},\Phi^\alpha)\rangle\frac{\delta_l\Gamma_{R\theta}}{\delta\Phi^\alpha}, \tag{7.3}$$

and tells us that $\Gamma_{R\theta}$ is invariant under the infinitesimal (nonlocal) transformations

$$\Phi^\alpha\rightarrow\Phi^\alpha+\delta\Phi^\alpha,\qquad \delta\Phi^\alpha\equiv\varpi\,\langle(S_{R\theta},\Phi^\alpha)\rangle=-\varpi\frac{\delta_r\Gamma_{R\theta}}{\delta K_\alpha}.$$

Here and below ϖ, ϖ', etc., denote constant anticommuting parameters. Write $\Phi^\alpha=\{\phi^i,C^a,\bar{C}^a,B^a\}$ and $K_\alpha=\{K_\phi^i,K_C^a,K_{\bar{C}}^a,K_B^a\}$, to separate the classical fields ϕ^i and their sources K_ϕ^i from the extra fields and their sources.

Observe that S is independent of K_B and contains $K_{\bar{C}}$ only through the term $-\int B^aK_{\bar{C}}^a$. This is also true after the canonical transformation (3.2), if we assume, for simplicity, that the functional $Q(\Phi,K')$ appearing in (3.2) is independent of $K_{\bar{C}}$ and K_B. Then S_θ also satisfies $(S_\theta,\bar{C})=B$ and $(S_\theta,B)=0$. Moreover, the sources $K_{\bar{C}}$ and K_B cannot contribute to any nontrivial one-particle irreducible diagrams. Thus, after renormalization we still have $(S_{R\theta},\bar{C})=B$ and $(S_{R\theta},B)=0$, i.e. $\delta\bar{C}^a=\varpi B^a$ and $\delta B^a=0$.

Define

$$\hat{\Gamma}_R(\phi)\equiv\Gamma_{R\theta}(\Phi,K)|_{\bar{C}=B=K=0}\,,\qquad \hat{\delta}\Phi^\alpha=\delta\Phi^\alpha|_{\bar{C}=B=K=0}\,.$$

Observe that $\hat{\Gamma}_R(\phi)$ is independent of the ghosts C, because it has ghost number zero and after suppressing $\bar{C}$ and K no fields and/or sources of negative ghost numbers survive. For the same reason, $\hat{\delta}\phi^i$, which has ghost number equal to one, is linear in C. Clearly, $\hat{\delta}\bar{C}=\hat{\delta}B=0$. Thus, when $\bar{C}$, B and K are switched off, formula (7.3) turns into

$$0=\int\hat{\delta}\phi^i\frac{\delta_l\hat{\Gamma}_R(\phi)}{\delta\phi^i}. \tag{7.4}$$

The terms proportional to $\delta_l \Gamma_{R\theta}/\delta C$ do not contribute to (7.4) because $\hat{\Gamma}_R(\phi)$ is C independent. The terms proportional to $\delta_l \Gamma_{R\theta}/\delta \bar{C}$ and $\delta_l \Gamma_{R\theta}/\delta B$ disappear, because they multiply $\hat{\delta}\bar{C}$ and $\hat{\delta}B$, respectively.

We call $\hat{\Gamma}_R(\phi)$ the "physical" Γ functional. The transformations $\hat{\delta}\phi^i$ encode the gauge symmetry of $\hat{\Gamma}_R$. Indeed, recall that $\hat{\delta}\phi^i$ is linear in C and of course ϖ. Replacing each ghost C with $\varpi'\Lambda$, where $\Lambda(x)$ is a function having statistics opposite to the one of C, and dropping the products $\varpi\varpi'$ after moving them to the left, we can define a symmetry transformation $\delta_\Lambda \phi^i$ by the formula

$$\varpi\varpi'\delta_\Lambda \phi^i = \hat{\delta}\phi^i\Big|_{C\rightarrow\varpi'\Lambda}$$

and prove, using equation (7.4), that $\hat{\Gamma}_R(\phi)$ is invariant under this symmetry:

$$\delta_\Lambda \hat{\Gamma}_R(\phi) = \int \delta_\Lambda \phi^i \frac{\delta_l \hat{\Gamma}_R(\phi)}{\delta \phi^i} = 0.$$

We call $\delta_\Lambda \phi^i$ the quantum gauge transformations. To the lowest order in $\hbar$ they coincide with the starting gauge transformations, but at higher orders they are in general nonlocal functionals. We call the algebra of the transformations δ_Λ *quantum gauge algebra*.

7.2 Closure of the quantum gauge algebra

Now we study the closure of the quantum gauge algebra. If we differentiate (7.1) with respect to K, we obtain

$$(\Gamma_{R\theta}, \delta\Phi^\alpha) = 0.$$

Consider this equation in the case $\delta\Phi^\alpha \rightarrow \delta\phi^i$, then switch off $\bar{C}$ and B, and set $K = 0$ at the end. Recalling that $\delta\bar{C} = \varpi B$ and $\delta B = 0$, and observing that $\delta\phi^i$ does not depend on $K_{\bar{C}}$ and K_B, we obtain

$$\int \hat{\delta}'\phi^j \frac{\delta_l(\hat{\delta}\phi^i)}{\delta\phi^j} + \int \hat{\delta}'C^a \frac{\delta_l(\hat{\delta}\phi^i)}{\delta C^a} = \int \left.\frac{\delta_r(\delta\phi^i\varpi')}{\delta K^j_\phi}\right|_{\bar{C}=B=K=0} \frac{\delta_l \hat{\Gamma}_R(\phi)}{\delta\phi^j}, \tag{7.5}$$

having multiplied to the left by ϖ' and having defined $\hat{\delta}'\Phi^\alpha = \hat{\delta}\Phi^\alpha\Big|_{\varpi\rightarrow\varpi'}$. The right-hand side of (7.5) is proportional to the ϕ^j "Γ field equations", which means that closure is achieved on shell. The left-hand side of (7.5) can be handled as follows. Since $\hat{\delta}\phi^i$ and $\hat{\delta}C^a$ are linearly and quadratically proportional to the ghosts, respectively, we can write them in the form

$$\hat{\delta}\phi^i = \varpi \int C^{\bar{a}} T^i_{\bar{a}}(\phi), \qquad \hat{\delta}C^a = -\frac{1}{2}\int C^{\bar{b}}\varpi C^{\bar{c}} T^a_{\bar{b}\bar{c}}(\phi),$$

where $T^i_{\bar{a}}$ and $T^a_{\bar{b}\bar{c}}$ are nonlocal functionals. Here the bar indices include the spacetime points where the corresponding fields are located and the summation over repeated bar indices understands the integration over those spacetime points. Now, take formula (7.5) and replace C^a with $\varpi''\Lambda^a + \varpi'''\Sigma^a$, Λ^a and Σ^a being functions of the coordinates. The left-hand side of (7.5) is turned into $\varpi\varpi'\varpi''\varpi'''$ times

$$\int \delta_\Lambda \phi^j \frac{\delta_l(\delta_\Sigma \phi^i)}{\delta \phi^j} - \int \delta_\Sigma \phi^j \frac{\delta_l(\delta_\Lambda \phi^i)}{\delta \phi^j} - \int \Lambda^{\bar{a}} \Sigma^{\bar{b}} T^{\bar{c}}_{\bar{a}\bar{b}}(\phi) T^i_{\bar{c}}(\phi).$$

Finally, the whole formula (7.5) is equivalent

$$[\delta_\Lambda, \delta_\Sigma]\phi^i = \delta_{[\Lambda,\Sigma]}\phi^i + \int v^{ij}(\phi, \Lambda, \Sigma) \frac{\delta_l \hat{\Gamma}_R(\phi)}{\delta \phi^j}, \tag{7.6}$$

where

$$[\Lambda, \Sigma]^a = \int \Lambda^{\bar{b}} \Sigma^{\bar{c}} T^a_{\bar{b}\bar{c}}(\phi)$$

and $v^{ij}(\phi, \Lambda, \Sigma)$ are suitable functions. Formula (7.6) expresses the on shell closure of the quantum gauge algebra.

The field transformations and the closure relations become clearer if we switch to a more explicit notation, where they read

$$\delta_\Lambda \phi^i(x) = \int \mathrm{d}^d y \Lambda^a(y) T^i_a[\phi](x, y), \qquad [\Lambda, \Sigma]^a(x) = \int \mathrm{d}^d y \mathrm{d}^d z \Lambda^b(y) \Sigma^c(z) T^a_{bc}[\phi](x, y, z),$$

$T^i_a[\phi]$ and $T^a_{bc}[\phi]$ being (nonlocal) functionals that depend on two and three spacetime points, respectively.

7.3 Gauge dependence of the physical Γ functional

The last goal is to study the gauge dependence of $\hat{\Gamma}_R(\phi)$. Observe that the functional $\langle H_{R\theta} \rangle$ that appears in formula (7.2) has ghost number equal to -1. Therefore, it must be proportional to the antighosts $\bar{C}$ and/or some sources K. This fact implies that the derivatives $\delta_l \langle H_{R\theta} \rangle / \delta \phi^i$ and $\delta_l \langle H_{R\theta} \rangle / \delta C^a$ are zero at $\bar{C} = K = 0$. Moreover, $\langle H_{R\theta} \rangle$ does not depend on $K_{\bar{C}}$ and K_B, if the functional $Q(\Phi, K')$ of (3.2) satisfies the same property, as we are assuming here. Setting $\bar{C} = B = K = 0$ in (7.2) we obtain

$$\frac{\partial \hat{\Gamma}_R(\phi)}{\partial \theta} = \sum_j \rho_j \frac{\partial \hat{\Gamma}_R(\phi)}{\partial \lambda_j} + \int u^i(\phi) \frac{\delta_l \hat{\Gamma}_R(\phi)}{\delta \phi^i}, \tag{7.7}$$

where

$$u^i(\phi) = \left. \frac{\delta_r \langle H_{R\theta} \rangle}{\delta K^i_\phi} \right|_{\bar{C}=B=K=0}.$$

Formula (7.7) is the equation of gauge dependence satisfied by the physical functional $\hat{\Gamma}_R(\phi)$. We can integrate it with the procedure described in subsection 6.2. The first term on the right-hand side of (7.7) can be absorbed into redefinitions of the parameters λ, while the second term can be absorbed into a change of field variables. We can do this for each gauge parameter θ, taking one at a time. We obtain that there exists redefinitions $\lambda(\lambda', \theta)$ and a change of field variables $\phi(\phi', \lambda', \theta)$ such that the transformed physical functional

$$\hat{\Gamma}'_R(\phi', \lambda') = \hat{\Gamma}_R(\phi(\phi', \lambda', \theta), \lambda(\lambda', \theta), \theta)$$

is θ independent. Setting $\theta = 0$ we get $\hat{\Gamma}'_R(\phi', \lambda') = \hat{\Gamma}_R(\phi', \lambda', 0)$, which in the end allows us to write

$$\hat{\Gamma}_R(\phi, \lambda, \theta) = \hat{\Gamma}_R(\phi'(\phi, \lambda, \theta), \lambda'(\lambda, \theta), 0).$$

Since the entire gauge dependence is encoded into changes of field variables and redefinitions of parameters, it cannot affect the physical quantities contained in $\hat{\Gamma}_R(\phi)$.

7.4 Unitarity

In this subsection we prove (perturbative) unitarity, to emphasize why gauge independence is so crucial. For definiteness, we illustrate our arguments in Yang-Mills theories, but everything we say can be applied to quantum gravity, as well as any general gauge theory. We recall that perturbative unitarity is the statement that the identity $SS^\dagger = 1$ holds diagrammatically, order by order in the perturbative expansion [32]. A necessary condition is that the free-field theory we perturb around propagates only physical degrees of freedom. A necessary and sufficient condition is that when the identity $SS^\dagger = 1$ is written as a cutting equation no unphysical degrees of freedom contribute to the cut propagators.

There exists no gauge-fixing conditions where both unitarity and the locality of counterterms are manifest. If we want manifest unitarity, propagators must have only physical poles. This happens when we choose gauge-fixing functions of the Coulomb type, such as $G(\phi) = \partial_i A_i$, where $i, j, \ldots$ are space indices, inside the gauge fermion $\Psi(\Phi)$ of (2.4). However, the locality of counterterms is not manifest in that gauge, since the Coulomb propagators contain denominators whose dominant terms (those that determine their ultraviolet behavior) do not depend on the energy (or do not depend on it in the correct way). Then, when we differentiate a Feynman diagram with respect to the energies of its external legs, the overall degree of divergence is not guaranteed to decrease, so we cannot prove the locality of counterterms in this way. Besides having a bad power-counting behavior at high energies, the propagators of the Coulomb gauge generate spurious divergences that are difficult to handle.

To have a good power-counting behavior we need to equip the propagators with extra poles, some of which are unphysical. This is achieved for example by choosing the Lorenz gauge-fixing function $G(\phi) = \partial^\mu A_\mu$ in (2.4). The Fadeev-Popov ghosts then also have poles. The locality of counterterms is manifest, but unitarity is not.

The extra poles must cancel somehow, but their mutual compensation is not evident. The best way to prove this compensation is to use the gauge independence of the physical amplitudes, which allows us to switch back and forth between gauge-fixing conditions of the Lorentz type and gauge-fixing conditions of the Coulomb type. The former make the locality of counterterms manifest and hide unitarity, while the latter make unitarity manifest and hide the locality of counterterms.

For example, choose the gauge fermion

$$\Psi(\Phi)=\int \bar{C}^a\left(\zeta\partial_0 A_0^a-\partial_i A_i^a+\frac{\xi}{2}B^a\right), \tag{7.8}$$

which contains two gauge-fixing parameters, ξ and ζ. This functional interpolates between the Lorenz gauge ($\zeta=1$) and the Coulomb gauge ($\zeta=0$). After integrating B out, the propagators of the gauge fields are

$$\begin{aligned}\langle A_0(k)A_0(-k)\rangle_0 &= -\frac{i\xi^2}{P(k)}(\xi E^2-\bar{k}^2), \qquad \langle A_i(k)A_0(-k)\rangle_0=\frac{i\xi^2}{P(k)}(\zeta-\xi)Ek_i,\\ \langle A_i(k)A_j(-k)\rangle_0 &= \frac{i}{E^2-\bar{k}^2}\left(\delta_{ij}-\frac{k_ik_j}{\bar{k}^2}\right)+i(\zeta^2E^2-\xi^2\bar{k}^2)\frac{\xi k^ik^j}{\bar{k}^2P(k)},\end{aligned} \tag{7.9}$$

where $\bar{k}^2=k_ik_i$ and

$$P(k)=\xi(\zeta E^2-\xi\bar{k}^2)^2-\bar{k}^2(1-\xi)(\zeta^2E^2-\xi^2\bar{k}^2),$$

while the ghost propagator is

$$\langle C(k)\bar{C}(-k)\rangle=\frac{i}{\zeta E^2-\bar{k}^2}. \tag{7.10}$$

We see that the propagators are well behaved, from the point of view of power counting, whenever $\zeta\neq 0$. They are not well behaved for $\zeta=0$, which is the Coulomb limit. The parameter ζ is a sort of cutoff that regulates the spurious divergences of the Coulomb gauge. Moreover, at $\zeta=0$ $P(k)$ is equal to $\xi^2(\bar{k}^2)^2$ and only the physical poles survive. Instead, unphysical poles are present whenever $\zeta\neq 0$.

In the previous sections we have proved that the physical quantities are gauge independent. In particular, they are independent of ξ and ζ. Thus, they are also unitary, and obey the locality of counterterms. We see that they are unitary by taking $\zeta=0$. We see that they obey the locality of counterterms by taking $\zeta\neq 0$.

In the case of the standard model in flat space, we can easily generalize the proof of the Adler-Bardeen theorem given in ref. [6] to the family of gauge fermions (7.8), because they are all renormalizable. Then, the remarks of this subsection allow us to infer that the standard model in flat space is perturbatively unitary.

In ref. [7] a more general proof of the Adler-Bardeen theorem was given. It holds in a large class of nonrenormalizable theories, which includes the standard model coupled to quantum gravity. Combining the results of [7] with those of section 3, we can extend the validity of the Adler-Bardeen theorem to the most general local gauge fermions. In particular, using an analogue of (7.8), to switch between the Lorenz and Coulomb gauges of diffeomorphisms and Yang-Mills symmetries, we infer that the standard model coupled to quantum gravity is unitary as a perturbative quantum field theory. So are its extensions, as long as they satisfy the assumptions we have made.

We stress again that gauge independence is crucial to reach these conclusions, since the Adler-Bardeen theorem *per se* ensures gauge invariance, but not gauge independence.

8 Checks of high-order calculations based on gauge independence

In this section we discuss how to use the results of this paper to check high-order calculations, under the assumptions of section 6. We have proved that

Proposition 1 *The beta functions of the physical parameters λ may depend on the gauge parameters ξ, but that dependence can always be reabsorbed into finite λ redefinitions.*

This proposition also reminds us that there exists a class of subtraction schemes where the beta functions are gauge independent, in agreement with the general theorem proved in section 3. If we are extremely lucky, the framework we choose to simplify high-order calculations might belong to that class. In ordinary situations, we may expect to be lucky only to the lowest orders, which may mean till three or four loops, or for special choices of the gauge fixing. However, we may not be able to identify the right framework in advance. Therefore, contrary to the usual lore, in general we cannot make checks of high-order calculations based on the assumption that λ beta functions are completely gauge independent.

Nevertheless, the beta functions cannot be gauge dependent in an arbitrary way, precisely because their gauge dependence must disappear in a suitable class of subtraction schemes. Thanks to this, a criterion to make checks of high-order calculations, based on gauge independence, still exists. It amounts to verify that every ξ dependence contained in the λ beta functions can be cancelled by means of finite λ redefinitions. In this section we show that the correct criterion, although less powerful than expected, is nontrivial and powerful enough.

For definiteness, consider the standard model in flat space, and let λ_i collect the φ^4 coupling, the squared gauge couplings, and the squared Yukawa couplings. The most general λ beta functions have the form

$$\beta_i = \sum_{n=2}^{\infty} \hbar^{n-1} \chi_{i_1 \cdots i_n i} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{8.1}$$

where $\chi_{ii_1\cdots i_n}$ are constants and the powers of $\hbar$ are inserted to emphasize the order of the loop expansion. The most general perturbative λ redefinitions can be parametrized as

$$\lambda_i' = \lambda_i + \sum_{n=2}^{\infty} \hbar^{n-1} \vartheta_{ii_1\cdots i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{8.2}$$

where $\vartheta_{ii_1\cdots i_n}$ are other constants. We have

$$\begin{aligned} \beta_i' &= \sum_{n=2}^{\infty} \hbar^{n-1} \chi_{i_1\cdots i_n i} \lambda_{i_1} \cdots \lambda_{i_n} + \sum_{n=2}^{\infty} n\hbar^{n-1} \vartheta_{ii_1\cdots i_n} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \sum_{m=2}^{\infty} \hbar^{m-1} \chi_{k_1\cdots k_m i_n} \lambda_{k_1} \cdots \lambda_{k_m} \\ &\equiv \sum_{n=2}^{\infty} \hbar^{n-1} \chi_{i_1\cdots i_n i}' \lambda_{i_1}' \cdots \lambda_{i_n}'. \end{aligned} \tag{8.3}$$

Proposition 1 ensures that the gauge dependence contained in the beta functions β_i can be absorbed inside the redefinitions (8.2), that is to say there exist constants $\vartheta_{ii_1\cdots i_n}$ such that the couplings λ_i' have gauge independent beta functions β_i'. Using this piece of information, we can determine which nontrivial checks of high-order calculations are available.

The one-loop coefficients $\chi_{i_1 i_2 i}$ cannot be changed, because they are scheme independent ($\chi_{i_1 i_2 i}' = \chi_{i_1 i_2 i}$). Therefore, they are also gauge independent. Comparing (8.1) and (8.3), we find that the other coefficients are related by the formula

$$\chi_{i_1\cdots i_n i}' = \chi_{i_1\cdots i_n i} + (n-1)\vartheta_{ij\{i_1\cdots i_{n-2}} \chi_{i_{n-1} i_n\} j} - 2\vartheta_{j\{i_1\cdots i_{n-1}} \chi_{i_n\} ji} + \cdots, \tag{8.4}$$

where the dots stand for contributions involving $\vartheta_{i_1\cdots i_k}$ with $k < n$. We can define an iterative procedure to determine $\vartheta_{i_1\cdots i_n}$ by assuming that the constants $\vartheta_{i_1\cdots i_k}$ with $k < n$ are known, and requiring that $\chi_{i_1\cdots i_n i}'$ be gauge independent.

Now, if the number of couplings λ is N, the tensors $\chi_{i_1\cdots i_{\ell+1} i}$ have $c_{N,\ell} \equiv N\binom{N+\ell}{\ell+1}$ independent components [33], while the tensors $\vartheta_{i_1\cdots i_\ell j}$ have $c_{N,\ell-1}$ components, where ℓ is the number of loops. For $N = 1$ (that is to say a single coupling λ) and $\ell > 2$ it is always possible to absorb the gauge dependence into λ redefinitions (as long as the one-loop coefficient χ of the beta function does not vanish), because $c_{1,\ell} = c_{1,\ell-1} = 1$. For $\ell = 2$ it is not possible, because the second and third terms on the right-hand side of formula (8.4) cancel each other. Thus, two nontrivial checks are available for $N = 1$, due to the gauge independence of the one-loop and two-loop coefficients of the beta function.

For $N > 1$ more nontrivial checks of high-order calculations based on gauge independence are available, because $c_{N,\ell} > c_{N,\ell-1}$. Proposition 1 implies that the number of ξ-independent components of the tensors $\chi_{i_1\cdots i_{\ell+1} i}$ is obtained by modding out the redefinitions (8.2). Generically, this operation leaves

$$c_{N,\ell} - c_{N,\ell-1} = N\binom{N+\ell-1}{\ell+1}$$

independent checks at ℓ loops. This number is $(\ell+N)/(N-1)$ times less than the number we would obtain if the beta functions were completely gauge independent. Indeed, in that case we would have $c_{N,\ell}$ independent checks at ℓ loops, which is equal to the number of constants $\chi_{i_1\cdots i_{\ell+1}i}$.

So far, the beta functions of the standard model have been calculated to three loops [34] and the results are fully independent of the gauge-fixing parameters. Presumably, the convenient gauge-fixing functions and the clever treatments of the matrix γ_5 used in refs. [34] project onto the class of subtraction schemes where the beta functions are already gauge independent, at least to the lowest orders. However, we may expect that this coincidence will stop, sooner or later. When that happens, we must be aware of the facts pointed out in this section. Moreover, we stress that in the proofs of properties to all orders, such as the proof of the Adler-Bardeen theorem in nonrenormalizable theories [7], it is often more convenient to use subtraction schemes that are less practical from the calculational point of view, but more convenient from the theoretical side. There, it is also important to keep in mind that the beta functions do not need to be gauge independent.

9 Conclusions

In this paper we have derived generalized Ward identities for potentially anomalous theories, and used them to study the problem of gauge independence. The new equations contain an extra term that is responsible for a number of interesting effects. We have renormalized the equations of gauge dependence and integrated them. The result is that every gauge dependence can be absorbed into a canonical transformation acting on the renormalized Γ functional, provided that the finite local counterterms are appropriately fine-tuned. RG invariance is preserved and, as expected, the physical quantities are gauge independent. Nevertheless, the beta functions of the couplings may in general depend on the gauge choice. Gauge independence is useful to switch back and forth between gauge conditions that exhibit perturbative unitarity and gauge conditions that exhibit a correct power-counting behavior and the locality of counterterms.

In several cases, the Adler-Bardeen theorem ensures that the gauge anomalies cancel to all orders, when they are trivial at one loop. However, it is not sufficient, *per se*, to ensure that the physical quantities are independent of the gauge fixing. In this paper we have proved that, in the end, gauge invariance does imply the gauge independence of the physical quantities. Precisely, we have shown that it is possible to renormalize the theory and fine-tune its finite local counterterms so that the cancellation of gauge anomalies ensured by the Adler-Bardeen theorem is preserved for arbitrary values of the gauge parameters.

Said differently, assume that the gauge anomalies vanish for some specific choices of the gauge parameters. Varying or turning on a gauge parameter is equivalent to making a canonical transformation. After the canonical transformation, it is always possible to re-renormalize the theory and re-fine-tune its finite local counterterms to enforce the cancellation of gauge anomalies again. Moreover, the gauge dependence of the renormalized Γ functional is encoded into a convergent canonical transformation. The theorem proved in section 3 is very general, to the extent that we did not need to make particular assumptions about the gauge algebra or the properties of the theory under renormalization. In particular, it holds for renormalizable and nonrenormalizable, chiral and nonchiral, theories and for arbitrary composite fields. Once we know that the cancellation of gauge anomalies holds in the framework we prefer, we know that it holds in every other framework.

One application of the theorem is to power-counting renormalizable chiral gauge theories gauge-fixed by means of a nonrenormalizable gauge fixing. It allows us to show that the parameters of negative dimensions introduced by the gauge fixing do not propagate into the physical quantities. In other words, the theory remains renormalizable, although in a nonmanifest form. A second application is a crucial step in the proof of the Adler-Bardeen theorem for nonrenormalizable theories elaborated in ref. [7].

It is often possible to prove the cancellation of gauge anomalies in a family of gauges. In that case, if the assumptions listed in section 6 hold, we do not need a new fine-tuning to enforce the cancellation of gauge anomalies after the variation of a gauge parameter. Then, the gauge dependence of the theory is encoded into a convergent canonical transformation on the renormalized Γ functional, combined with a finite redefinition of the parameters. This fact makes it apparent that in general the beta functions of the couplings may depend on the gauge fixing. We expect that high-order calculations of the beta functions in the standard model will exhibit, sooner or later, dependences of the type mentioned here.

The gauge dependences of the beta functions can be eliminated by redefining the couplings in *ad hoc* ways. Thanks to this fact, gauge independence can still be used to make nontrivial checks of the calculations.

Appendices

A Useful formulas

In this appendix we collect a few identities that are used in the paper. First, we recall that

$$(\Gamma,\Gamma)=\langle (S,S)\rangle\,, \tag{A.1}$$

where S is any action (renormalized or not), Γ denotes the Γ functional associated with S and

$$\langle X\rangle=\frac{1}{Z(J,K)}\int[\mathrm{d}\Phi]\,X\exp\left(iS(\Phi,K)+i\int\Phi^{\alpha}J_{\alpha}\right) \tag{A.2}$$

is the average defined by S, X being a local functional. Formula (A.1) can be proved by making the change of field variables (2.10) in the functional integral (2.8), and recalling that in any dimensional regularization the local perturbative changes of field variables have Jacobian determinants identically equal to one. For details on the derivation, see the appendices of refs. [6, 8].

If ζ is any parameter, we also have the formulas

$$\frac{\partial \Gamma}{\partial \zeta} = \left\langle \frac{\partial S}{\partial \zeta} \right\rangle, \tag{A.3}$$

$$(\Gamma, \langle X \rangle) = \langle (S, X) \rangle + \frac{i}{2} \langle (S, S) X \rangle_\Gamma, \tag{A.4}$$

where X is an arbitrary local functional and $\langle XY \rangle_\Gamma$ denotes the set of one-particle irreducible diagrams that have one X insertion, one Y insertion, and arbitrary Φ and K external legs, Y being another local functional. Formula (A.3) follows from the definition of Γ as the Legendre transform of W. Formula (A.4) can be proved by making the change of field variables (2.10) in the average (A.2), and expressing the final result in terms of Φ and K. For details on this method, see the appendix of ref. [8][2].

A simpler method to derive formula (A.4) is to deform the action S into $S + X\sigma$, where σ is a constant, consider the deformed version of formula (A.1) and take the first order of its expansion in powers of σ. By (A.3), Γ is deformed into $\Gamma + \langle X \rangle \sigma + \mathcal{O}(\sigma^2)$. Instead, the average $\langle Y \rangle$ of a local functional Y is deformed into $\langle Y \rangle + i \langle YX \rangle_\Gamma \sigma + \mathcal{O}(\sigma^2)$. Indeed, the factor e^{iS} appearing in the integrands of $Z(J, K)$ and $Z(J, K)\langle Y \rangle$ [check (2.8) and (A.2)] is deformed into $\mathrm{e}^{iS}(1 + iX\sigma + \mathcal{O}(\sigma^2))$. Moreover, the deformed average, considered as a functional of Φ and K, is still a collection of one-particle irreducible diagrams. Thus, the first correction to $\langle Y \rangle$ is precisely $i\langle YX \rangle_\Gamma \sigma$. Taking $Y = (S, S)$, we obtain $\langle (S, S) \rangle \rightarrow \langle (S, S) \rangle + i \langle (S, S) X \rangle_\Gamma \sigma + \mathcal{O}(\sigma^2)$, wherefrom (A.4) follows.

If we subtract the equations (A.3) and (A.4) we also get

$$\frac{\partial \Gamma}{\partial \zeta} - (\Gamma, \langle X \rangle) = \left\langle \frac{\partial S}{\partial \zeta} - (S, X) - \frac{i}{2}(S, S) X \right\rangle_\Gamma, \tag{A.5}$$

which is the starting point to derive the equations of gauge dependence.

Another useful identity tells us that [16, 8], if $\Phi, K \rightarrow \Phi', K'$ is a canonical transformation with generating functional $F(\Phi, K')$, and $Y(\Phi, K)$ is a functional behaving as a scalar, i.e. such that $Y'(\Phi', K') = Y(\Phi, K)$, then

$$\frac{\partial Y'}{\partial \zeta} = \frac{\partial Y}{\partial \zeta} - (Y, \tilde{F}_\zeta), \tag{A.6}$$

where $\tilde{F}_\zeta(\Phi, K) = F_\zeta(\Phi, K'(\Phi, K))$ and $F_\zeta(\Phi, K') = \partial F / \partial \zeta$. The field and source variables that are kept constant in the ζ derivative of a functional are the natural field and source variables of that functional (that is to say Φ' and K' for Y', Φ and K for Y, Φ and K' for F).

[2] Note that we have switched from the Euclidean notation used in [8] to the Minkowskian notation used here.

B Renormalization of local bifunctionals

In this appendix we show how to renormalize a generic local bifunctional, and then specialize to evanescent local bifunctionals. Given a theory with action S, assume that a local bifunctional $\mathcal{F}$ has the form AB, where A and B are local functionals. Couple A and B to external (constant) sources h_A and h_B, by deforming the action S into $\breve{S} = S - ih_A A - ih_B B$. Then, renormalize the extended action $\breve{S}$. The renormalized version of $\breve{S}$ has the form

$$\breve{S}_R = S_R - i\breve{h}_A A_R - i\breve{h}_B B_R - i\breve{h}_B \breve{h}_A C_R + \mathcal{O}(\breve{h}_A^2) + \mathcal{O}(\breve{h}_B^2),$$

where A_R and B_R are the renormalized functionals A and B, respectively, $\breve{h}_A$, $\breve{h}_B$ are the renormalized sources, and C_R is a local functional. Consider the Γ functional $\breve{\Gamma}_R$ associated with $\breve{S}_R$. Differentiating it from the left-hand side with respect to $\breve{h}_B$ and then $\breve{h}_A$, and later setting $\breve{h}_A = \breve{h}_B = 0$, we find that the renormalized $\mathcal{F}$ is equal to $\mathcal{F}_R = A_R B_R + C_R$.

It is a known fact (see for example [21], chapter 13, or [6], section 6) that an evanescent local functional E can be renormalized so that its renormalized version E_R satisfies $\langle E_R \rangle = \mathcal{O}(\varepsilon)$. This property extends to evanescent local bifunctionals in a straightforward way. However, we have to pay attention to some details.

By writing $\hat{\partial}^\mu = \hat{\eta}^{\mu\nu}\partial_\nu$ and $\hat{p}^\mu = \hat{\eta}^{\mu\nu}p_\nu$ everywhere inside E, we can express each vertex of E in a factorized form $\mathcal{T}_k \hat{\delta}_k$, where $\hat{\delta}_k$ denotes the evanescent part, made of tensors $\eta^{\hat{\mu}\hat{\nu}}$, possibly ε factors and other structures that stay outside of the diagrams, while $\mathcal{T}_k$ is a nonevanescent local functional and collects all the momenta. We then have $E = \sum_k \mathcal{T}_k \hat{\delta}_k$. Instead of considering the average $\langle E \rangle$, consider first the diagrams $\langle \mathcal{T}_k \rangle$ that contain one insertion of $\mathcal{T}_k$. Iterating in $n = 0, 1, \ldots$, let $\mathcal{T}_{k\text{div}}^{(n+1)}$ denote the $(n+1)$-loop divergent part of $\langle \mathcal{T}_{nk} \rangle$, where

$$\mathcal{T}_{nk} = \mathcal{T}_k - \sum_{p=1}^{n} \mathcal{T}_{k\text{div}}^{(p)}$$

are the functionals $\mathcal{T}_k$ renormalized up to and including n loops. By the locality of counterterms, each $\mathcal{T}_{k\text{div}}^{(p)}$ is local. Then, the functional $E_n = \sum_k \mathcal{T}_{nk} \hat{\delta}_k$ is renormalized up to and including n loops, and satisfies

$$\langle E_n \rangle = \sum_k \langle \mathcal{T}_{nk} \rangle \, \hat{\delta}_k = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+1}), \tag{B.1}$$

because each $\langle \mathcal{T}_{nk} \rangle$ is convergent up to $\mathcal{O}(\hbar^{n+1})$. Finally, the functional $E_R \equiv E_\infty$ satisfies $\langle E_R \rangle = \mathcal{O}(\varepsilon)$.

In the procedure just outlined we have subtracted away all sorts of contributions $\mathcal{T}_{k\text{div}}^{(p)}$, order by order. More generally, we do not need to subtract those that, once multiplied by $\hat{\delta}_k$, give evanescent results. Indeed, collecting those evanescent local parts inside a local functional ΔE, anything we have said so far for E can be repeated for ΔE. We reach the conclusion that $\langle E_R \rangle = \mathcal{O}(\varepsilon)$ even if we "forget" to subtract any evanescent local parts.

Once we have renormalized E so that $\langle E_R \rangle$ is evanescent to all orders, we can apply the same procedure to the bifunctional $Y = EB$, where B is an arbitrary local functional. The outcome is that we can find a $\mathcal{O}(\hbar)$ local functional F_R, such that the local bifunctional $Y_R = E_R B_R + F_R$ is renormalized and the average $\langle Y_R \rangle_\Gamma$ is evanescent to all orders.

More precisely, we can iterate the renormalization of Y as follows. Write $Y = EB = \sum_k \hat{\delta}_k \mathcal{U}_k$, where $\mathcal{U}_k = \mathcal{T}_k B$. Let B_n denote the functional B renormalized up to and including n loops. Inductively assume that the n-loop renormalized $\mathcal{U}_k$ have the form $\mathcal{U}_{nk} = \mathcal{T}_{nk} B_n + \mathcal{C}_{nk}$, where $\mathcal{C}_{nk}$ are local functionals. Define $Y_n = \sum_k \hat{\delta}_k \mathcal{U}_{nk} = E_n B_n + F_n$, where $F_n = \sum_k \hat{\delta}_k \mathcal{C}_{nk}$. Clearly, $\langle Y_n \rangle_\Gamma = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+1})$, because each $\langle \mathcal{U}_{nk} \rangle_\Gamma$ is convergent up to $\mathcal{O}(\hbar^{n+1})$. By the locality of counterterms, the $(n+1)$-loop contributions $\mathcal{U}_{nk}^{(n+1)}$ to $\langle \mathcal{U}_{nk} \rangle_\Gamma$ are made of a local divergent part $U_{nk\text{div}}^{(n+1)}$, plus a generically nonlocal convergent part. Consequently, the $(n+1)$-loop contributions to $\langle Y_n \rangle_\Gamma$ are the sum of a local divergent part, a local nonevanescent part, plus a generically nonlocal evanescent part. If we define $\mathcal{U}_{n+1k} = \mathcal{T}_{n+1k} B_{n+1} + \mathcal{C}_{n+1k}$, where B_{n+1} is the functional B renormalized up to and including $n+1$ loops, and $\mathcal{C}_{n+1k} = \mathcal{C}_{nk} - \mathcal{U}_{nk\text{div}}^{(n+1)}$, we see that $\langle \mathcal{U}_{n+1k} \rangle_\Gamma$ is convergent up to $\mathcal{O}(\hbar^{n+2})$, and so $\langle Y_{n+1} \rangle_\Gamma = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^{n+2})$, where $Y_{n+1} = \sum_k \hat{\delta}_k \mathcal{U}_{n+1k} = E_{n+1} B_{n+1} + F_{n+1}$, and $F_{n+1} = \sum_k \hat{\delta}_k \mathcal{C}_{n+1k}$. The conclusion also holds if we "forget" to subtract any evanescent local parts of E_{n+1} and/or F_{n+1}. The subtraction can be iterated in n so that in the end $\langle Y_R \rangle_\Gamma$ is evanescent to all orders in $\hbar$, where $Y_R = Y_\infty$.

C Integrating equation (3.33)

In this appendix we integrate the equations (3.33) and (3.32). First, we recall how to integrate the simpler equation

$$\frac{\partial X}{\partial \theta} = (X, V), \tag{C.1}$$

for the functional $X(\Phi, K, \theta)$, given the functional $V(\Phi, K, \theta)$. Expanding in powers of θ, write

$$V(\Phi, K, \theta) = \sum_{n=0}^{\infty} \theta^n V_n(\Phi, K).$$

We want to show that there exists a canonical transformation $\Phi, K \to \Phi', K'$, with generating functional

$$\mathcal{F}(\Phi, K', \theta) = \int \Phi^\alpha K'_\alpha + \sum_{n=1}^{\infty} \theta^n \mathcal{F}_n(\Phi, K'), \tag{C.2}$$

such that

$$X'(\Phi', K') \equiv X(\Phi(\Phi', K', \theta), K(\Phi', K', \theta), \theta)$$

is independent of θ.

We can derive conditions on the unknown functionals $\mathcal{F}_n$ by applying formula (A.6), which relates the functional V of (C.1) to the canonical transformation $\mathcal{F}$. A sufficient condition to have $\partial X'/\partial\theta = 0$ is $V = \tilde{\mathcal{F}}_\theta$, where $\mathcal{F}_\theta = \partial\mathcal{F}/\partial\theta$. In other words,

$$0 = \sum_{n=0}^{\infty} \theta^n \left[V_n(\Phi, K) - (n+1)\mathcal{F}_{n+1}(\Phi, K')\right], \qquad K_\alpha = K'_\alpha + \sum_{n=1}^{\infty} \theta^n \frac{\delta\mathcal{F}_n(\Phi, K')}{\delta\Phi^\alpha}.$$

The first equation can be solved for $\mathcal{F}_{n+1}$ by working recursively in n. It is sufficient to express each $V_k(\Phi, K)$ as a functional of Φ and K', by using the second equation, and then set the coefficient of θ^n to zero. This proves that the desired canonical transformation (C.2) does exist. Clearly, $X'(\Phi', K')$ coincides with $X(\Phi', K', 0)$. Therefore, expressing everything by means of fields and sources without primes, we get

$$X(\Phi, K, \theta) = X(\Phi'(\Phi, K, \theta), K'(\Phi, K, \theta), 0).$$

Now, assume that a functional $Y(\Phi, K, \theta)$ satisfies

$$\frac{\partial Y}{\partial\theta} = (Y, V) + G, \tag{C.3}$$

where $V(\Phi, K, \theta)$ and $G(\Phi, K, \theta)$ are two other functionals. Define a new functional $\tilde{G}$ and a map $\mathcal{L}_\theta : Z \to \mathcal{L}_\theta Z$, where Z is a functional, as

$$\tilde{G}(\Phi, K, \theta) = \int_0^\theta \mathrm{d}\bar{\theta} G(\Phi, K, \bar{\theta}), \qquad \mathcal{L}_\theta Z(\Phi, K, \theta) = \int_0^\theta \mathrm{d}\bar{\theta}\, (Z_{\bar{\theta}}, V_{\bar{\theta}}),$$

where $Z_{\bar{\theta}} = Z(\Phi, K, \bar{\theta})$ and $V_{\bar{\theta}} = V(\Phi, K, \bar{\theta})$. Observe that

$$\frac{\partial}{\partial\theta}\mathcal{L}_\theta Z = (Z, V).$$

Then, equation (C.3) turns into equation

$$\frac{\partial\tilde{Y}}{\partial\theta} = (\tilde{Y}, V), \qquad \text{for } \tilde{Y} = Y - \sum_{n=0}^{\infty} \mathcal{L}_\theta^n \tilde{G}.$$

Note that the terms $\mathcal{L}_\theta^n \tilde{G}$ are at least $\mathcal{O}(\theta^{n+1})$. Using the result found above, the canonical transformation $\Phi, K \to \Phi', K'$ given by formula (C.2) is such that the transformed functional

$$\tilde{Y}'(\Phi', K') \equiv \tilde{Y}(\Phi(\Phi', K', \theta), K(\Phi', K', \theta), \theta)$$

is θ independent. Finally, if $G = \mathcal{O}(u^n)$ for some expansion parameter u (which is ε or $\hbar$, when we apply this theorem in subsection 3.2) and V is regular in u, then the canonical transformation $\Phi, K \to \Phi', K'$ is also regular in u, which implies

$$Y(\Phi(\Phi', K', \theta), K(\Phi', K', \theta), \theta) = \tilde{Y}'(\Phi', K') + \mathcal{O}(u^n).$$

Setting $\theta = 0$, we get

$$Y(\Phi', K', 0) = \tilde{Y}'(\Phi', K') + \mathcal{O}(u^n).$$

Hence, expressing everything by means of fields and sources without primes,

$$Y(\Phi, K, \theta) = Y(\Phi'(\Phi, K, \theta), K'(\Phi, K, \theta), 0) + \mathcal{O}(u^n).$$

In other words, the functional $Y(\Phi, K, \theta)$ still evolves by means of a canonical transformation, but only up to $\mathcal{O}(u^n)$.

In most applications, the functionals V and G of equation (C.3) may intrinsically depend on Y. For example, this happens when Y is some renormalized action (or the Γ functional associated with it) and V, G are (the averages of) some renormalized local functionals, calculated with that action. We can disentangle this difficulty by expanding each functional in powers of $\hbar$ and proceeding inductively in this expansion. Writing

$$Y = \sum_{n=0}^{\infty} \hbar^n Y_n, \qquad V = \sum_{n=0}^{\infty} \hbar^n V_n, \qquad G = \sum_{n=0}^{\infty} \hbar^n G_n,$$

we obtain the equations

$$\frac{\partial Y_n}{\partial \theta} - (Y_n, V_0) = \sum_{k=0}^{n-1} (Y_k, V_{n-k}) + G_n, \tag{C.4}$$

which have the same form as (C.3). The contributions V_k and G_k to V and G with $k \leqslant n$ do not depend on Y_n. For $k = 0$ this is obvious. For $k > 0$ it is sufficient to observe that the vertices Y_n of order $\hbar^n$ of the renormalized action Y can only contribute to the one-particle irreducible diagrams associated with V and G that have $n + 1$ or more loops. Indeed, at least one additional loop must be closed to connect a vertex Y_n with the insertions provided by V or G. When Y is the Γ functional and V, G are averages of local functionals, we can argue similarly.

Now, assume that we have solved the equations (C.4) for $n < \bar{n}$, and consider the equations (C.4) for $n = \bar{n}$. The unknown is $Y_{\bar{n}}$, while V_k and G_k with $k \leqslant \bar{n}$ are independent of it. Thus, equations (C.4) can be solved with the method explained above. We conclude that the procedure we have given to solve the equations (C.3) is well defined.

D Standard model coupled to quantum gravity

In this appendix we report some reference formulas for the standard model coupled to quantum gravity. The classical fields ϕ contain the vielbein $e_{\mu}^{\bar{a}}$, the Yang-Mills gauge fields A_{μ}^{a} and the matter fields, where the indices $a, b, \ldots$ refer to the Yang-Mills gauge group (within which we include the Abelian subgroup) and $\bar{a}, \bar{b}, \ldots$ refer to the Lorentz group. The classical action $S_c(\phi)$

is equal to the sum $S_{c\text{SM}}+\Delta S_c$, where

$$S_{c\text{SM}}=\int\sqrt{|g|}\left[-\frac{1}{2\kappa^2}(R+2\Lambda_{\text{c}})-\frac{1}{4}F_{\bar{\mu}\bar{\nu}}^{a}F^{a\bar{\mu}\bar{\nu}}+\mathcal{L}_m\right]$$

and ΔS_c collects the invariants generated by renormalization as counterterms, multiplied by independent parameters. Here, R is the Ricci curvature, g is the determinant of the metric tensor, $F_{\bar{\mu}\bar{\nu}}^{a}$ is the Yang-Mills field strength, $\mathcal{L}_m$ is the matter Lagrangian coupled to gravity, Λ_{c} is the cosmological constant, and $\kappa^2=8\pi G$, where G is Newton's constant.

The functional S_K of formula (2.2) reads

$$\begin{aligned}S_K=&\int(C^{\bar{\rho}}\partial_{\bar{\rho}}A_{\bar{\mu}}^{a}+A_{\bar{\rho}}^{a}\partial_{\bar{\mu}}C^{\bar{\rho}}-\partial_{\bar{\mu}}C^{a}-gf^{abc}A_{\bar{\mu}}^{b}C^{c})K_A^{\bar{\mu}a}+\int\left(C^{\bar{\rho}}\partial_{\bar{\rho}}C^{a}+\frac{g}{2}f^{abc}C^{b}C^{c}\right)K_C^{a}\\&+\int(C^{\bar{\rho}}\partial_{\bar{\rho}}e_{\bar{\mu}}^{\bar{a}}+e_{\bar{\rho}}^{\bar{a}}\partial_{\bar{\mu}}C^{\bar{\rho}}+C^{\bar{a}\bar{b}}e_{\bar{\mu}\bar{b}})K_{\bar{a}}^{\bar{\mu}}+\int C^{\bar{\rho}}(\partial_{\bar{\rho}}C^{\bar{\mu}})K_{\bar{\mu}}^{C}+\int(C^{\bar{a}\bar{c}}\eta_{\bar{c}\bar{d}}C^{\bar{d}\bar{b}}+C^{\bar{\rho}}\partial_{\bar{\rho}}C^{\bar{a}\bar{b}})K_{\bar{a}\bar{b}}^{C}\\&+\int\left(C^{\bar{\rho}}\partial_{\bar{\rho}}\bar{\psi}_L-\frac{i}{4}\bar{\psi}_L\sigma^{\bar{a}\bar{b}}C_{\bar{a}\bar{b}}+g\bar{\psi}_LT^{a}C^{a}\right)K_\psi+\int K_{\bar{\psi}}\left(C^{\bar{\rho}}\partial_{\bar{\rho}}\psi_L-\frac{i}{4}\sigma^{\bar{a}\bar{b}}C_{\bar{a}\bar{b}}\psi_L+gT^{a}C^{a}\psi_L\right)\\&+\int\left(C^{\bar{\rho}}(\partial_{\bar{\rho}}\varphi)+g\mathfrak{T}^{a}C^{a}\varphi\right)K_\varphi-\int B^{a}K_{\bar{C}}^{a}-\int B_{\bar{\mu}}K_{\bar{C}}^{\bar{\mu}}-\int B_{\bar{a}\bar{b}}K_{\bar{C}}^{\bar{a}\bar{b}},\end{aligned}$$

where ψ_L are left-handed fermions, φ are scalars, while T^a and $\mathfrak{T}^a$ are the anti-Hermitian matrices associated with their representations. The triplets C^a-$\bar{C}^a$-B^a, $C^{\bar{a}\bar{b}}$-$\bar{C}_{\bar{a}\bar{b}}$-$B^{\bar{a}\bar{b}}$ and $C^{\bar{\mu}}$-$\bar{C}_{\bar{\mu}}$-$B_{\bar{\mu}}$ collect the ghosts, the antighosts and the Lagrange multipliers of Yang-Mills symmetry, local Lorentz symmetry and diffeomorphisms, respectively. It is easy to check that $(S_K,S_K)=0$ in arbitrary D dimensions.

Finally, the gauge fermion of formula (2.4) reads

$$\begin{aligned}\Psi(\Phi)=&\int\sqrt{|g|}\bar{C}^{a}\left(g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}A_{\bar{\nu}}^{a}+\frac{\xi}{2}B^{a}\right)+\int e\bar{C}_{\bar{a}\bar{b}}\left(\frac{1}{\kappa}e^{\bar{\rho}\bar{a}}g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\partial_{\bar{\nu}}e_{\bar{\rho}}^{\bar{b}}+\frac{\xi_L}{2}B^{\bar{a}\bar{b}}+\frac{\xi_L'}{2}g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\partial_{\bar{\nu}}B^{\bar{a}\bar{b}}\right)\\&-\int\sqrt{|g|}\bar{C}_{\bar{\mu}}\left(\frac{1}{\kappa}\partial_{\bar{\nu}}g^{\bar{\mu}\bar{\nu}}+\frac{\xi_G}{\kappa}g^{\bar{\mu}\bar{\nu}}g_{\bar{\rho}\bar{\sigma}}\partial_{\bar{\nu}}g^{\bar{\rho}\bar{\sigma}}-\frac{\xi_G'}{2}g^{\bar{\mu}\bar{\nu}}B_{\bar{\nu}}\right),\end{aligned}$$

where ξ, ξ_L, ξ_L', ξ_G and ξ_G' are gauge-fixing parameters.

E Comparison with manifestly nonanomalous theories

We have mentioned that an unexpected consequence of our results is that in AB nonanomalous theories the beta functions of the couplings can depend on the gauge-fixing parameters. It is interesting to better understand why this does not happen in manifestly nonanomalous theories.

We actually begin with nongauge theories, that is to say theories that have no gauge symmetries. There the action $S(\Phi,K)$ does not even depend on the sources K and the canonical transformations are just arbitrary changes of field variables.

Denote the classical action by $S(\phi)$, the renormalized action by $S_R(\phi)$ and the renormalized Γ functional by $\Gamma_R(\phi)$. We assume that S_R and Γ_R are defined by subtracting away the divergences just as they come, in the minimal subtraction scheme.

Consider a local, perturbative change of field variables

$$\psi^i(\phi,\theta) = \phi^i + \mathcal{O}(\theta) \tag{E.1}$$

for the classical action S. Let $S_\theta(\phi,\theta)$ denote the transformed classical action,

$$S_\theta(\phi,\theta) = S(\psi(\phi,\theta)),$$

which obviously satisfies

$$\frac{\partial S_\theta}{\partial\theta} = \int \Delta\phi^i \frac{\delta_l S_\theta}{\delta\phi^i},$$

where

$$\Delta\phi^i = \int \frac{\delta\psi^j}{\delta\theta}\frac{\delta_l\phi^i}{\delta\psi^j}. \tag{E.2}$$

Denote the renormalized S_θ by $S_{R\theta}$ and the Γ functional associated with it by $\Gamma_{R\theta}$.

We want to show that the change of field variables (E.1) on S is mapped onto a renormalized change of field variables on S_R and a nonlocal, convergent change of field variables on Γ_R. This property is encoded into the equations of gauge dependence, which now read

$$\frac{\partial S_{R\theta}}{\partial\theta} = \int \Delta_R\phi^i \frac{\delta_l S_{R\theta}}{\delta\phi^i}, \qquad \frac{\partial\Gamma_{R\theta}}{\partial\theta} = \int \left\langle \Delta\phi_R^i \right\rangle \frac{\delta_l \Gamma_{R\theta}}{\delta\phi^i}, \tag{E.3}$$

where $\Delta_R\phi^i$ is the renormalized version of the composite field (E.2). Equations (E.3) are just particular cases of equation (C.1), and can be integrated with the method explained in appendix C. So doing, it is straightforward to prove that the θ dependences of both $S_{R\theta}$ and $\Gamma_{R\theta}$ are encoded into pure changes of field variables, with no redefinitions of parameters.

We point out that the first equations of formula (E.3) are highly nonlinear in $S_{R\theta}$, because $\Delta_R\phi^i$, being a renormalized composite field, intrinsically depends on $S_{R\theta}$. Nevertheless, with the inductive procedure explained in appendix C we can disentangle this dependence. Similarly, the equations satisfied by $\Gamma_{R\theta}$ contain the average $\left\langle \Delta_R\phi^i \right\rangle$ on the right-hand side, which is also determined by $S_{R\theta}$. The procedure to integrate the equations of $\Gamma_{R\theta}$ is basically the same as the one for $S_{R\theta}$ and is again given in appendix C.

Formulas (E.3) can be proved by induction, using the minimal subtraction scheme. Let $S_n = S_\theta + \mathcal{O}(\hbar)\times$poles and $\Delta_n\phi^i = \Delta\phi^i + \mathcal{O}(\hbar)\times$poles denote the action and the composite field (E.2) renormalized up to and including n loops. Assume that

$$\mathcal{R}_n \equiv \frac{\partial S_n}{\partial\theta} - \int \Delta_n\phi^i \frac{\delta_l S_n}{\delta\phi^i} = \mathcal{O}(\hbar^{n+1}). \tag{E.4}$$

Clearly, this assumption is satisfied for $n=0$. Moreover, in the minimal subtraction scheme $\mathcal{R}_n$ is made of pure poles.

Differentiating the Γ functional Γ_n, associated with S_n, with respect to θ, we get

$$\frac{\partial \Gamma_n}{\partial \theta} = \left\langle \frac{\partial S_n}{\partial \theta} \right\rangle_n = \int \mathrm{d}^D x \left\langle \Delta_n \phi^i(x) \frac{\delta_l S_n}{\delta \phi^i(x)} \right\rangle_n + \langle \mathcal{R}_n \rangle_n. \tag{E.5}$$

Now,

$$\frac{\delta_l S_n}{\delta \phi^i(x)} \exp\left(iS_n + i\int \phi^j J_j\right) = -J_i(x) - i\frac{\delta_l}{\delta \phi^i(x)} \exp\left(iS_n + i\int \phi^j J_j\right).$$

Using this formula inside (E.5) we can drop the last term by integrating by parts, because when the derivative $\delta_l/\delta\phi^i(x)$ acts on $\Delta_n\phi^i(x)$ it gives zero in dimensional regularization. Finally, we obtain

$$\frac{\partial \Gamma_n}{\partial \theta} = -\int \left\langle \Delta_n \phi^i \right\rangle_n J_i + \langle \mathcal{R}_n \rangle_n = \int \left\langle \Delta_n \phi^i \right\rangle_n \frac{\delta_l \Gamma_n}{\delta \phi^i} + \langle \mathcal{R}_n \rangle_n. \tag{E.6}$$

Since S_n and $\Delta_n\phi^i$ are renormalized up to and including n loops, the $(n+1)$-loop divergent parts $\Gamma_{n\mathrm{div}}^{(n+1)}$ and $\Delta_{n\mathrm{div}}^{(n+1)}\phi^i$ of Γ_n and $\left\langle \Delta_n\phi^i\right\rangle_n$ are local. Moreover, the $\mathcal{O}(\hbar^{n+1})$ divergent part of $\langle \mathcal{R}_n\rangle_n$ coincides with the $\mathcal{O}(\hbar^{n+1})$ part of $\mathcal{R}_n$, because $\mathcal{R}_n$ starts from $\mathcal{O}(\hbar^{n+1})$ and it is just made of poles. Thus, taking the $\mathcal{O}(\hbar^{n+1})$ divergent parts of formula (E.6) we get

$$\frac{\partial \Gamma_{n\mathrm{div}}^{(n+1)}}{\partial \theta} = \int \Delta_{n\mathrm{div}}^{(n+1)} \phi^i \frac{\delta_l S_\theta}{\delta \phi^i} + \int \Delta \phi^i \frac{\delta_l \Gamma_{n\mathrm{div}}^{(n+1)}}{\delta \phi^i} + \frac{\partial S_n}{\partial \theta} - \int \Delta_n \phi^i \frac{\delta_l S_n}{\delta \phi^i} + \mathcal{O}(\hbar^{n+2}). \tag{E.7}$$

Subtracting the divergences just as they come, we define

$$S_{n+1} = S_n - \Gamma_{n\mathrm{div}}^{(n+1)}, \qquad \Delta_{n+1}\phi^i = \Delta_n\phi^i - \Delta_{n\mathrm{div}}^{(n+1)}\phi^i.$$

Clearly, the Γ functional Γ_{n+1} associated with S_{n+1} is renormalized up to and including $n+1$ loops. Using (E.7), we find

$$\mathcal{R}_{n+1} \equiv \frac{\partial S_{n+1}}{\partial \theta} - \int \Delta_{n+1}\phi^i \frac{\delta_l S_{n+1}}{\delta \phi^i} = \mathcal{O}(\hbar^{n+2}).$$

Thus, the inductive assumption (E.4) is promoted to the next order. The equations (E.3) follow by taking $n=\infty$ in (E.4) and (E.6).

We see that in theories with no gauge symmetries a change of field variables on the classical action does not generate redefinitions of parameters in the renormalized Γ functional: the parameters θ introduced by the field redefinition do not propagate into the beta functions of the couplings. Moreover, we do not need to re-fine-tune the finite local counterterms.

Another approach to these issues was given in refs. [35, 36], where the changes of field variables were mapped from the classical action to the renormalized action and the (renormalized) generating functionals Z, W and Γ, as well as a more general type of Γ functional, called master functional. That approach also shows that a change of field variables does not affect the beta functions of the couplings, in the theories that have no gauge symmetries.

Similar properties hold in manifestly nonanomalous gauge theories, where the equations

$$\frac{\partial S_{R\theta}}{\partial\theta} = (S_{R\theta}, \tilde{Q}_{R\theta}), \qquad \frac{\partial \Gamma_{R\theta}}{\partial\theta} = (\Gamma_{R\theta}, \langle\tilde{Q}_{R\theta}\rangle) \tag{E.8}$$

hold and can be integrated [8]. Again, the conclusion is that a canonical transformation acting on the classical action is converted into a renormalized canonical transformation acting on the renormalized action, and a nonlocal, convergent canonical transformation acting on the renormalized Γ functional, with no effect on the beta functions of the couplings. Equations (E.3) can also be obtained by switching off the sources K in formulas (E.8).

What "goes wrong" in AB nonanomalous theories, is that "small things", that is to say evanescent terms $\mathcal{O}(\varepsilon)$, are around all the time, and can generate unexpected finite corrections by simplifying some divergences. For this reason, they force us to re-fine-tune the subtraction scheme at every, even minor, modification of the framework in which we formulate the theory. Yet, we have shown in the paper that we can put their effects under control and preserve the correct physical properties.

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