Background Field Method, Batalin-Vilkovisky Formalism And Parametric Completeness Of Renormalization

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Abstract

We investigate the background field method with the Batalin-Vilkovisky formalism, to generalize known results, study the parametric completeness of general gauge theories and achieve a better understanding of several properties. In particular, we study renormalization and gauge dependence to all orders. Switching between the background field approach and the usual approach by means of canonical transformations, we prove parametric completeness without making use of cohomological theorems; namely we show that if the starting classical action is sufficiently general all divergences can be subtracted by means of parameter redefinitions and canonical transformations. Our approach applies to renormalizable and nonrenormalizable theories that are manifestly free of gauge anomalies and satisfy the following assumptions: the gauge algebra is irreducible and closes off shell, the gauge transformations are linear functions of the fields, and closure is field independent. Yang-Mills theories and quantum gravity in arbitrary dimensions are included, as well as effective and higher-derivative versions of them, but several other theories, such as supergravity, are left out.

1 Introduction

The background field method [1, 2] is a convenient tool to quantize gauge theories and make explicit calculations, particularly when it is used in combination with the dimensional-regularization technique. It amounts to choosing a nonstandard gauge fixing in the conventional approach and, among its virtues, it keeps the gauge transformations intact under renormalization. However, it takes advantage of properties that only particular classes of theories have.

The Batalin-Vilkovisky formalism [3] is also useful for quantizing general gauge theories, especially because it collects all ingredients of infinitesimal gauge symmetries in a single identity, the master equation, which remains intact through renormalization, at least in the absence of gauge anomalies.

Merging the background field method with the Batalin-Vilkovisky formalism is not only an interesting theoretical subject *per se*, but can also offer a better understanding of known results, make us appreciate aspects that have been overlooked, generalize the validity of crucial theorems about the quantization of gauge theories and renormalization, and help us address open problems. For example, an important issue concerns the generality of the background field method. It would be nice to formulate a unique treatment for all gauge theories, renormalizable and nonrenormalizable, unitary and higher derivative, with irreducible or reducible gauge algebras that close off shell or only on shell. However, we will see that at this stage it is not possible to achieve that goal, due to some intrinsic features of the background field method.

Another important issue that we want to emphasize more than has been done so far is the problem of *parametric completeness* in general gauge theories [4]. To ensure renormalization-group (RG) invariance, all divergences must be subtracted by redefining parameters and making canonical transformations. When a theory contains all independent parameters necessary to achieve this goal, we say that it is parametrically complete. The RG-invariant renormalization of divergences may require the introduction of missing Lagrangian terms, multiplied by new physical constants, or even deform the symmetry algebra in nontrivial ways. However, in nonrenormalizable theories such as quantum gravity and supergravity it is not obvious that the action can indeed be adjusted to achieve parametric completeness. One way to deal with this problem is to classify the whole cohomology of invariants and hope that the solution satisfies suitable properties. This method requires lengthy technical proofs that must be done case by case [5], and therefore lacks generality. Another way is to let renormalization build the new invariants automatically, as shown in ref. [4], with an algorithm that is able to iteratively extend the classical action converting divergences into finite counterterms. However, that procedure is mainly a theoretical tool, because although very general and conceptually minimal, it is practically unaffordable. Among the other things, it leaves the possibility that renormalization may dynamically deform the gauge symmetry in physically observable ways. A third possibility is the one we are going to treat here, taking advantage

of the background field method. Where it applies, it makes cohomological classifications unnecessary and excludes that renormalization may dynamically deform the symmetry in observable ways. Because of the intrinsic properties of the background field method, the approach of this paper, although general enough, is not exhaustive. It is general enough because it includes the gauge symmetries we need for physical applications, namely Abelian and non-Abelian Yang-Mills symmetries, local Lorentz symmetry and invariance under general changes of coordinates. At the same time, it is not exhaustive because it excludes other potentially interesting symmetries, such as local supersymmetry.

To be precise, our results hold for every gauge symmetry that satisfies the following properties: the algebra of gauge transformations (i) closes off shell and (ii) is irreducible; moreover (iii) there exists a choice of field variables where the gauge transformations $\delta_\Lambda \phi$ of the physical fields ϕ are linear functions of ϕ and the closure $[\delta_\Lambda, \delta_\Sigma] = \delta_{[\Lambda,\Sigma]}$ of the algebra is ϕ independent. We expect that with some technical work it will be possible to extend our results to theories that do not satisfy assumption (ii), but our impression is that removing assumptions (i) and (iii) will be much harder, if not impossible.

In this paper we also assume that the theory is manifestly free of gauge anomalies. Our results apply to renormalizable and nonrenormalizable theories that satisfy the assumptions listed so far, among which are QED, Yang-Mills theories, quantum gravity and Lorentz-violating gauge theories [6], as well as effective [7], higher-derivative [8] and nonlocal [9] versions of such theories, in arbitrary dimensions, and extensions obtained including any set of composite fields. We recall that Stelle's proof [8] that higher-derivative quantum gravity is renormalizable was incomplete, because it assumed without proof a generalization of the Kluberg-Stern–Zuber conjecture [10] for the cohomological problem satisfied by counterterms. Even the cohomological analysis of refs. [5] does not directly apply to higher-derivative quantum gravity, because the field equations of higher-derivative theories are not equal to perturbative corrections of the ordinary field equations. These remarks show that our results are quite powerful, because they overcome a number of difficulties that otherwise need to be addressed case by case.

Strictly speaking, our results, in their present form, do not apply to chiral theories, such as the Standard Model coupled to quantum gravity, where the cancellation of anomalies is not manifest. Nevertheless, since all other assumptions we have made concern just the forms of gauge symmetries, not the forms of classical actions, nor the limits around which perturbative expansions are defined, we expect that our results can be extended to all theories involving the Standard Model or Lorentz-violating extensions of it [11, 12]. However, to make derivations more easily understandable it is customary to first make proofs in the framework where gauge anomalies are manifestly absent, and later extend the results by means of the Adler-Bardeen theorem [13]. We follow the tradition on this, and plan to devote a separate investigation to anomaly cancellation.

Although some of our results are better understandings or generalizations of known properties,

we do include them for the sake of clarity and self-consistence. We think that our formalism offers insight on the issues mentioned above and gives a more satisfactory picture. In particular, the fact that background field method makes cohomological classifications unnecessary is something that apparently has not been appreciated enough so far. Moreover, our approach points out the limits of applicability of the background field method.

To achieve parametric completeness we proceed in four basic steps. First, we study renormalization to all orders subtracting divergences "as they come", which means without worrying whether the theory contains enough independent parameters for RG invariance or not. Second, we study how the renormalized action and the renormalized Γ functional depend on the gauge fixing, and work out how the renormalization algorithm maps a canonical transformation of the classical theory into a canonical transformation of the renormalized theory. Third, we renormalize the canonical transformation that continuously interpolates between the background field approach and the conventional approach. Fourth, comparing the two approaches we show that if the classical action $S_c(\phi, \lambda)$ contains all gauge invariant terms determined by the starting gauge symmetry, then there exists a canonical transformation $\Phi, K \to \hat{\Phi}, \hat{K}$ such that

$$S_{R\min}(\Phi, \underline{\phi}, K) = S_c(\hat{\phi} + \underline{\phi}, \tau(\lambda)) - \int R^\alpha(\hat{\phi} + \underline{\phi}, \hat{C})\hat{K}_\alpha, \qquad (1.1)$$

where $S_{R\min}$ is the renormalized action with the gauge-fixing sector switched off, $\Phi^\alpha = \{\phi, C\}$ are the fields (C being the ghosts), K_α are the sources for the Φ^α transformations $R^\alpha(\Phi)$, $\underline{\phi}$ are the background fields, λ are the physical couplings and $\tau(\lambda)$ are λ redefinitions. Identity (1.1) shows that all divergences can be renormalized by means of parameter redefinitions and canonical transformations, which proves parametric completeness. Power counting may or may not restrict the form of $S_c(\phi, \lambda)$.

Basically, under the assumptions we have made the background transformations do not renormalize, and the quantum fields ϕ can be switched off and then restored from their background partners $\underline{\phi}$. Nevertheless, the restoration works only up to a canonical transformation, which gives (1.1). The story is a bit more complicated than this, but this simplified version is enough to appreciate the main point. However, when the assumptions we have made do not hold, the argument fails, which shows how peculiar the background field method is. Besides giving explicit examples where the construction works, we address some problems that arise when the assumptions listed above are not satisfied.

A somewhat different approach to the background field method in the framework of the Batalin-Vilkovisky formalism exists in the literature. In refs. [14] Binosi and Quadri considered the most general variation $\delta \underline{A} = \Omega$ of the background gauge field $\underline{A}$ in Yang-Mills theory, and obtained a modified Batalin-Vilkovisky master equation that controls how the functional Γ depends on $\underline{A}$. Instead, here we introduce background copies of both physical fields and ghosts, which allows us to split the symmetry transformations into "quantum transformations" and "background

transformations". The master equation is split into the three identities (2.21), which control invariances under the two types of transformations.

The paper is organized as follows. In section 2 we formulate our approach and derive its basic properties, emphasizing the assumptions we make and why they are necessary. In section 3 we renormalize divergences to all orders, subtracting them "as they come". In section 4 we derive the basic differential equations of gauge dependence and integrate them, which allows us to show how a renormalized canonical transformation emerges from its tree-level limit. In section 5 we derive (1.1) and prove parametric completeness. In section 6 we give two examples, non-Abelian Yang-Mills theory and quantum gravity. In section 7 we make remarks about parametric completeness and recapitulate where we stand now on this issue. Section 8 contains our conclusions, while the appendix collects several theorems and identities that are used in the paper.

We use the dimensional-regularization technique and the minimal subtraction scheme. Recall that the functional integration measure is invariant with respect to perturbatively local changes of field variables. Averages $\langle \cdots \rangle$ always denote the sums of *connected* Feynman diagrams. We use the Euclidean notation in theoretical derivations and switch to Minkowski spacetime in the examples.

2 Background field method and Batalin-Vilkovisky formalism

In this section we formulate our approach to the background field method with the Batalin-Vilkovisky formalism. To better appreciate the arguments given below it may be useful to jump back and forth between this section and section 6, where explicit examples are given.

If the gauge algebra closes off shell, there exists a canonical transformation that makes the solution $S(\Phi, K)$ of the master equation $(S, S) = 0$ depend linearly on the sources K. We write

$$S(\Phi, K) = \mathcal{S}(\Phi) - \int R^\alpha(\Phi) K_\alpha. \tag{2.1}$$

The fields $\Phi^\alpha = \{\phi^i, C^I, \bar{C}^I, B^I\}$ are made of physical fields ϕ^i, ghosts C^I (possibly including ghosts of ghosts and so on), antighosts $\bar{C}^I$ and Lagrange multipliers B^I for the gauge fixing. Moreover, $K_\alpha = \{K^i_\phi, K^I_C, K^I_{\bar{C}}, K^I_B\}$ are the sources associated with the symmetry transformations $R^\alpha(\Phi)$ of the fields Φ^α, while

$$\mathcal{S}(\Phi) = S_c(\phi) + (S, \Psi)$$

is the sum of the classical action $S_c(\phi)$ plus the gauge fixing, which is expressed as the antiparenthesis of S with a K-independent gauge fermion $\Psi(\Phi)$. We recall that the antiparentheses are defined as

$$(X, Y) = \int \left\{ \frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K_\alpha} - \frac{\delta_r X}{\delta K_\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} \right\},$$

where the summation over the index α is understood. The integral is over spacetime points associated with repeated indices.

The non-gauge-fixed action

$$S_{\min}(\Phi, K) = S_c(\phi) - \int R_\phi^i(\phi, C) K_\phi^i - \int R_C^I(\phi, C) K_C^I, \tag{2.2}$$

obtained by dropping antighosts, Lagrange multipliers and their sources, also solves the master equation, and is called the minimal solution. Antighosts $\bar{C}$ and Lagrange multipliers B form trivial gauge systems, and typically enter (2.1) by means of the gauge fixing (S, Ψ) and a contribution

$$\Delta S_{\mathrm{nm}} = - \int B^I K_{\bar{C}}^I, \tag{2.3}$$

to $- \int R^\alpha K_\alpha$.

Let $\mathcal{R}^\alpha(\Phi, C)$ denote the transformations the fields Φ^α would have if they were matter fields. Each function $\mathcal{R}^\alpha(\Phi, C)$ is a bilinear form of Φ^α and C. Sometimes, to be more explicit, we also use the notation $\mathcal{R}_{\bar{C}}^I(\bar{C}, C)$ and $\mathcal{R}_B^I(B, C)$ for $\bar{C}$ and B, respectively. It is often convenient to replace (2.3) with the alternative nonminimal extension

$$\Delta S_{\mathrm{nm}}' = - \int \left(B^I + \mathcal{R}_{\bar{C}}^I(\bar{C}, C) \right) K_{\bar{C}}^I - \int \mathcal{R}_B^I(B, C) K_B^I. \tag{2.4}$$

For example, in Yang-Mills theories we have

$$\Delta S_{\mathrm{nm}}' = - \int \left(B^a - g f^{abc} C^b \bar{C}^c \right) K_{\bar{C}}^a + g \int f^{abc} C^b B^c K_B^a$$

and in quantum gravity

$$\Delta S_{\mathrm{nm}}' = - \int \left(B_\mu + \bar{C}_\rho \partial_\mu C^\rho - C^\rho \partial_\rho \bar{C}_\mu \right) K_{\bar{C}}^\mu + \int \left(B_\rho \partial_\mu C^\rho + C^\rho \partial_\rho B_\mu \right) K_B^\mu, \tag{2.5}$$

where C^μ are the ghosts of diffeomorphisms.

Observe that (2.4) can be obtained from (2.3) making the canonical transformation generated by

$$F_{\mathrm{nm}}(\Phi, K') = \int \Phi^\alpha K_\alpha' + \int \mathcal{R}_{\bar{C}}^I(\bar{C}, C) K_B^{I'}.$$

Requiring that F_{nm} indeed give (2.4) we get the identities

$$\mathcal{R}_B^I(B, C) = - \int B^J \frac{\delta_l}{\delta \bar{C}^J} \mathcal{R}_{\bar{C}}^I(\bar{C}, C), \qquad \int \left(R_C^J \frac{\delta_l}{\delta C^J} + \mathcal{R}_{\bar{C}}^J(\bar{C}, C) \frac{\delta_l}{\delta \bar{C}^J} \right) \mathcal{R}_{\bar{C}}^I(\bar{C}, C) = 0, \tag{2.6}$$

which can be easily checked both for Yang-Mills theories and gravity. In this paper the notation $R^\alpha(\Phi)$ refers to the field transformations of (2.2) plus those of the nonminimal extension (2.3), while $\bar{R}^\alpha(\Phi)$ refers to the transformations of (2.2) plus (2.4).

2.1 Background field action

To apply the background field method, we start from the gauge invariance of the classical action $S_c(\phi)$,

$$\int R_c^i(\phi, \Lambda) \frac{\delta_l S_c(\phi)}{\delta \phi^i} = 0, \tag{2.7}$$

where Λ are the arbitrary functions that parametrize the gauge transformations $\delta \phi^i = R_c^i$. Shifting the fields ϕ by background fields $\underline{\phi}$, and introducing arbitrary background functions $\underline{\Lambda}$ we can write the identity

$$\int \left[R_c^i(\phi + \underline{\phi}, \Lambda) + X^i \right] \frac{\delta_l S_c(\phi + \underline{\phi})}{\delta \phi^i} + \int \left[R_c^i(\phi + \underline{\phi}, \underline{\Lambda}) - X^i \right] \frac{\delta_l S_c(\phi + \underline{\phi})}{\delta \underline{\phi}^i} = 0,$$

which is true for arbitrary functions X^i. If we choose

$$X^i = R_c^i(\phi + \underline{\phi}, \underline{\Lambda}) - R_c^i(\underline{\phi}, \underline{\Lambda}),$$

the transformations of the background fields contain only background fields and coincide with $R_c^i(\underline{\phi}, \underline{\Lambda})$. We find

$$\int \left[R_c^i(\phi + \underline{\phi}, \Lambda + \underline{\Lambda}) - R_c^i(\underline{\phi}, \underline{\Lambda}) \right] \frac{\delta_l S_c(\phi + \underline{\phi})}{\delta \phi^i} + \int R_c^i(\underline{\phi}, \underline{\Lambda}) \frac{\delta_l S_c(\phi + \underline{\phi})}{\delta \underline{\phi}^i} = 0. \tag{2.8}$$

Thus, denoting background quantities by means of an underlining, we are led to consider the action

$$S(\Phi, \underline{\Phi}, K, \underline{K}) = S_c(\phi + \underline{\phi}) - \int R^\alpha(\Phi + \underline{\Phi}) K_\alpha - \int R^\alpha(\underline{\Phi})(\underline{K}_\alpha - K_\alpha), \tag{2.9}$$

which solves the master equation $[\![S, S]\!] = 0$, where the antiparentheses are defined as

$$[\![X, Y]\!] = \int \left\{ \frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K_\alpha} + \frac{\delta_r X}{\delta \underline{\Phi}^\alpha} \frac{\delta_l Y}{\delta \underline{K}_\alpha} - \frac{\delta_r X}{\delta K_\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} - \frac{\delta_r X}{\delta \underline{K}_\alpha} \frac{\delta_l Y}{\delta \underline{\Phi}^\alpha} \right\}.$$

More directly, if $S(\Phi, K) = S_c(\phi) - \int R^\alpha(\Phi) K_\alpha$ solves $(S, S) = 0$, the background field can be introduced with a canonical transformation. Start from the action

$$S(\Phi, \underline{\Phi}, K, \underline{K}) = S_c(\phi) - \int R^\alpha(\Phi) K_\alpha - \int R^\alpha(\underline{\Phi}) \underline{K}_\alpha, \tag{2.10}$$

which obviously satisfies two master equations, one in the variables Φ, K and the other one in the variables $\underline{\Phi}, \underline{K}$. *A fortiori*, it also satisfies $[\![S, S]\!] = 0$. Relabeling fields and sources with primes and making the canonical transformation generated by the functional

$$F_{\mathrm{b}}(\Phi, \underline{\Phi}, K', \underline{K}') = \int (\Phi^\alpha + \underline{\Phi}^\alpha) K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha, \tag{2.11}$$

we obtain (2.9), and clearly preserve $[\![S, S]\!] = 0$.

The shift $\underline{\Phi}$ is called background field, while Φ is called quantum field. We also have quantum sources K and background sources $\underline{K}$. Finally, we have background transformations, those described by the background ghosts $\underline{C}$ or the functions $\underline{\Lambda}$ in (2.8), and quantum transformations, those described by the quantum ghosts C and (2.3) or the functions Λ in (2.8).

The action (2.9) is not the most convenient one to study renormalization. It is fine in the minimal sector (the one with antighosts and Lagrange multipliers switched off), but not in the nonminimal one. Now we describe the improvements we need to make.

Non-minimal sector

So far we have introduced background copies of all fields. Nevertheless, strictly speaking we do not need to introduce copies of the antighosts $\bar{C}$ and the Lagrange multipliers B, since we do not need to gauge-fix the background. Thus we drop $\underline{\bar{C}}$, $\underline{B}$ and their sources from now on, and define $\underline{\Phi}^\alpha = \{\underline{\phi}^i, \underline{C}^I, 0, 0\}$, $\underline{K}_\alpha = \{\underline{K}^i_\phi, \underline{K}^I_C, 0, 0\}$. Observe that then we have $R^\alpha(\underline{\Phi}) = \bar{R}^\alpha(\underline{\Phi}) = \{R^i_\phi(\underline{\phi}, \underline{C}), R^I_C(\underline{\Phi}), 0, 0\}$.

Let us compare the nonminimal sectors (2.3) and (2.4). If we choose (2.3), $\bar{C}$ and B do not transform under background transformations. Since (2.3) are the only terms that contain $K_{\bar{C}}$, they do not contribute to one-particle irreducible diagrams and do not receive radiative corrections. Moreover, K_B does not appear in the action. Instead, if we choose the nonminimal sector (2.4), namely if we start from

$$S(\Phi, \underline{\Phi}, K, \underline{K}) = S_c(\phi) - \int \bar{R}^\alpha(\Phi) K_\alpha - \int R^\alpha(\underline{\Phi}) \underline{K}_\alpha \qquad (2.12)$$

instead of (2.10), the transformation (2.11) gives the action

$$S(\Phi, \underline{\Phi}, K, \underline{K}) = S_c(\phi + \underline{\phi}) - \int (\bar{R}^\alpha(\Phi + \underline{\Phi}) - \bar{R}^\alpha(\underline{\Phi})) K_\alpha - \int R^\alpha(\underline{\Phi}) \underline{K}_\alpha. \qquad (2.13)$$

In particular, using the linearity of $\mathcal{R}^I_{\bar{C}}$ and $\mathcal{R}^I_B$ in C, we see that (2.4) is turned into itself plus

$$- \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C}) K^I_{\bar{C}} - \int \mathcal{R}^I_B(B, \underline{C}) K^I_B. \qquad (2.14)$$

Because of these new terms, $\bar{C}$ and B now transform as ordinary matter fields under background transformations. This is the correct background transformation law we need for them. On the other hand, the nonminimal sector (2.4) also generates nontrivial quantum transformations for $\bar{C}$ and B, which are renormalized and complicate our derivations.

It would be better to have (2.4) in the background sector and (2.3) in the nonbackground sector. To achieve this goal, we make the canonical transformation generated by

$$F'_{\mathrm{nm}}(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C}) K^{I\prime}_B \qquad (2.15)$$

on (2.9). Using (2.6) again, the result is

$$S(\Phi, \underline{\Phi}, K, \underline{K}) = S_c(\phi + \underline{\phi}) - \int (R^\alpha(\Phi + \underline{\Phi}) - R^\alpha(\underline{\Phi}))K_\alpha$$
$$- \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C})K^I_{\bar{C}} - \int \mathcal{R}^I_B(B, \underline{C})K^I_B - \int R^\alpha(\underline{\Phi})\underline{K}_\alpha. \qquad (2.16)$$

This is the background field action we are going to work with. It is straightforward to check that (2.16) satisfies $[\![S, S]\!] = 0$.

Separating the background and quantum sectors

Now we separate the background sector from the quantum sector. To do this properly we need to make further assumptions.

First, we assume that there exists a choice of field variables where the functions $R^\alpha(\Phi)$ are at most quadratic in Φ. We call it *linearity assumption*. It is equivalent to assume that the gauge transformations $\delta_\Lambda \phi^i = R^i_c(\phi, \Lambda)$ of (2.7) are linear functions of the fields ϕ and closure is expressed by ϕ-independent identities $[\delta_\Lambda, \delta_\Sigma] = \delta_{[\Lambda, \Sigma]}$. The linearity assumption is satisfied by all gauge symmetries of physical interest, such as those of QED, non-Abelian Yang-Mills theory, quantum gravity and the Standard Model. On the other hand, it is not satisfied by other important symmetries, among which is supergravity, where the gauge transformations either close only on shell or are not linear in the fields.

Second, we assume that the gauge algebra is irreducible, which ensures that the set Φ contains only ghosts and not ghosts of ghosts.

Under these assumptions, we make the canonical transformation generated by

$$F_\tau(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + (\tau - 1) \int \underline{C}^I \underline{K}^{I\prime}_C \qquad (2.17)$$

on the action (2.16). This transformation amounts to rescaling the background ghosts $\underline{C}^I$ by a factor τ and their sources $\underline{K}^I_C$ by a factor $1/\tau$. Since we do not have background antighosts, (2.17) is the background-ghost-number transformation combined with a rescaling of the background sources.

The action (2.16) is not invariant under (2.17). Using the linearity assumption it is easy to check that the transformed action S_τ is linear in τ. Writing $S_\tau = \hat{S} + \tau\bar{S}$ we can split the total action S into the sum $\hat{S} + \bar{S}$ of a *quantum action* $\hat{S}$ and a *background action* $\bar{S}$.

Precisely, the quantum action $\hat{S}$ does not depend on the background sources $\underline{K}$ and the background ghosts $\underline{C}$, but only on the background copies $\underline{\phi}$ of the physical fields. We have

$$\hat{S} = \hat{S}(\Phi, \underline{\phi}, K) = S_c(\phi + \underline{\phi}) - \int R^\alpha(\phi + \underline{\phi}, C, \bar{C}, B)K_\alpha. \qquad (2.18)$$

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Note that, in spite of the notation, the functions $R^\alpha(\Phi)$ are actually $\bar{C}$ independent. Moreover, we find

$$\bar{S}(\Phi, \underline{\Phi}, K, \underline{K}) = -\int \mathcal{R}^\alpha(\Phi, \underline{C})K_\alpha - \int R^\alpha(\underline{\Phi})\underline{K}_\alpha, \qquad (2.19)$$

where, for ϕ and C,

$$\mathcal{R}^\alpha(\Phi, \underline{C}) = R^\alpha(\Phi + \underline{\Phi}) - R^\alpha(\underline{\Phi}) - R^\alpha(\phi + \underline{\phi}, C, \bar{C}, B). \qquad (2.20)$$

These functions transform ϕ and C as if they were matter fields and are of course linear in Φ and $\underline{C}$. Note that formula (2.20) does not hold for antighosts and Lagrange multipliers. In the end all quantum fields transform as matter fields under background transformations.

The master equation $[\![S, S]\!] = 0$ decomposes into the three identities

$$[\![\hat{S}, \hat{S}]\!] = [\![\hat{S}, \bar{S}]\!] = [\![\bar{S}, \bar{S}]\!] = 0, \qquad (2.21)$$

which we call *background field master equations*.

The quantum transformations are described by $\hat{S}$ and the background ones are described by $\bar{S}$. Background fields are inert under quantum transformations, because $[\![\hat{S}, \underline{\Phi}]\!] = 0$. Note that

$$[\![\hat{S}, [\![\bar{S}, X]\!]]\!] + [\![\bar{S}, [\![\hat{S}, X]\!]]\!] = 0, \qquad (2.22)$$

where X is an arbitrary local functional. This property follows from the Jacobi identity of the antiparentheses and $[\![\hat{S}, \bar{S}]\!] = 0$, and states that background and quantum transformations commute.

Gauge-fixing

Now we come to the gauge fixing. In the usual approach, the theory is typically gauge-fixed by means of a canonical transformation that amounts to replacing the action S by $S + (S, \Psi)$, where Ψ is a local functional of ghost number -1 and depends only on the fields Φ. Using the background field method it is convenient to search for a $\underline{C}$-independent gauge-fixing functional $\Psi(\Phi, \underline{\phi})$ that is also invariant under background transformations, namely such that

$$[\![\bar{S}, \Psi]\!] = 0. \qquad (2.23)$$

Then we fix the gauge with the usual procedure, namely we make a canonical transformation generated by

$$F_{\mathrm{gf}}(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + \Psi(\Phi, \underline{\phi}). \qquad (2.24)$$

Because of (2.23) the gauge-fixed action reads

$$S_{\mathrm{gf}} = \hat{S} + \bar{S} + [\![\hat{S}, \Psi]\!]. \qquad (2.25)$$

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Defining $\hat{S}_{\rm gf} = \hat{S} + [\![\hat{S}, \Psi]\!]$, identities (2.21), (2.22) and (2.23) give $[\![\hat{S}_{\rm gf}, \hat{S}_{\rm gf}]\!] = [\![\hat{S}_{\rm gf}, \bar{S}]\!] = 0$, so it is just like gauge-fixing $\hat{S}$. Since both $\hat{S}$ and Ψ are $\underline{K}$ and $\underline{C}$ independent, $\hat{S}_{\rm gf}$ is also $\underline{K}$ and $\underline{C}$ independent. Observe that the canonical transformations (2.17) and (2.24) commute; therefore we can safely apply the transformation (2.17) to the gauge-fixed action. A gauge fixing satisfying (2.23) is called *background-preserving gauge fixing*.

In some derivations of this paper the background field master equations (2.21) are violated in intermediate steps; therefore we need to prove properties that hold more generally. Specifically, consider an action

$$S(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}(\Phi, \underline{\phi}, K) + \bar{S}(\Phi, \underline{\Phi}, K, \underline{K}), \tag{2.26}$$

equal to the sum of a $\underline{K}$- and $\underline{C}$-independent "quantum action" $\hat{S}$, plus a "background action" $\bar{S}$ that satisfies the following requirements: (i) it is a linear function of the quantum fields Φ, (ii) it gets multiplied by τ when applying the canonical transformation (2.17), and (iii) $\delta_l \bar{S}/\delta \underline{K}_\alpha$ is Φ independent. In particular, requirement (ii) implies that $\bar{S}$ vanishes at $\underline{C} = 0$. Since $\bar{S}$ is a linear function of Φ, it does not contribute to one-particle irreducible diagrams. Since $\hat{S}$ does not depend on $\underline{C}$, while $\bar{S}$ vanishes at $\underline{C} = 0$, $\bar{S}$ receives no radiative corrections. Thus the Γ functional associated with the action (2.26) satisfies

$$\Gamma(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{\Gamma}(\Phi, \underline{\phi}, K) + \bar{S}(\Phi, \underline{\Phi}, K, \underline{K}). \tag{2.27}$$

Moreover, thanks to theorem 6 of the appendix we have the general identity

$$[\![\Gamma, \Gamma]\!] = \langle [\![S, S]\!] \rangle, \tag{2.28}$$

under the sole assumption that $\delta_l S/\delta \underline{K}_\alpha$ is Φ independent.

Applying the canonical transformation (2.17) to Γ we find $\Gamma_\tau = \hat{\Gamma} + \tau \bar{S}$, so (2.28) gives the identities

$$[\![\hat{\Gamma}, \hat{\Gamma}]\!] = \langle [\![\hat{S}, \hat{S}]\!] \rangle, \qquad [\![\bar{S}, \hat{\Gamma}]\!] = \langle [\![\bar{S}, \hat{S}]\!] \rangle. \tag{2.29}$$

When $[\![S, S]\!] = 0$ we have

$$[\![\Gamma, \Gamma]\!] = [\![\hat{\Gamma}, \hat{\Gamma}]\!] = [\![\bar{S}, \hat{\Gamma}]\!] = 0. \tag{2.30}$$

Observe that, thanks to the linearity assumption, an $\bar{S}$ equal to (2.19) satisfies the requirements of formula (2.26).

Now we give details about the background-preserving gauge fixing we pick for the action (2.16). It is convenient to choose gauge-fixing functions $G^{Ii}(\underline{\phi}, \partial)\phi^i$ that are linear in the quantum fields ϕ, where $G^{Ii}(\underline{\phi}, \partial)$ may contain derivative operators. Precisely, we choose the gauge fermion

$$\Psi(\Phi, \underline{\phi}) = \int \bar{C}^I G^{Ii}(\underline{\phi}, \partial)\phi^i, \tag{2.31}$$

and assume that it satisfies (2.23). A more common choice would be (see (6.2) for Yang-Mills theory)

$$\Psi(\Phi, \underline{\phi}) = \int \bar{C}^I \left(G^{Ii}(\underline{\phi}, \partial)\phi^i + \xi_{IJ}B^J \right),$$

where ξ_{IJ} are gauge-fixing parameters. In this case, when we integrate the B fields out the expressions $G^{Ii}(\underline{\phi}, \partial)\phi^i$ get squared. However, (2.31) is better for our purposes, because it makes the canonical transformations (2.15) and (2.24) commute with each other. We call the choice (2.31) *regular Landau gauge*. The gauge-field propagators coincide with the ones of the Landau gauge. Nevertheless, while the usual Landau gauge (with no B's around) is singular, here gauge fields are part of multiplets that include the B's, therefore (2.31) is regular.

In the regular Landau gauge, using (2.23) and applying (2.24) to (2.16) we find

$$S_{\text{gf}} = \hat{S}_{\text{gf}} + \bar{S} = S_c(\phi + \underline{\phi}) - \int R^\alpha(\phi + \underline{\phi}, C, \bar{C}, B)\tilde{K}_\alpha - \int \mathcal{R}^\alpha(\Phi, \underline{C})K_\alpha - \int R^\alpha(\underline{\Phi})\underline{K}_\alpha, \quad (2.32)$$

where the tilde sources $\tilde{K}_\alpha$ coincide with K_α apart from $\tilde{K}_\phi^i$ and $\tilde{K}_{\bar{C}}^I$, which are

$$\tilde{K}_\phi^i = K_\phi^i - \bar{C}^I G^{Ii}(\underline{\phi}, -\overleftarrow{\partial}), \qquad \tilde{K}_{\bar{C}}^I = K_{\bar{C}}^I - G^{Ii}(\underline{\phi}, \partial)\phi^i. \quad (2.33)$$

Recalling that the functions $R^\alpha(\Phi)$ are $\bar{C}$ independent, we see that $\hat{S}_{\text{gf}}$ does not depend on K_ϕ^i and $\bar{C}$ separately, but only through the combination $\tilde{K}_\phi^i$. Every one-particle irreducible diagram with $\bar{C}^I$ external legs actually factorizes a $-\bar{C}^I G^{Ii}(\underline{\phi}, -\overleftarrow{\partial})$ on those legs. Replacing one or more such objects with K_ϕ^is, we obtain other contributing diagrams. Conversely, replacing one or more K_ϕ^i-external legs with $-\bar{C}^I G^{Ii}(\underline{\phi}, -\overleftarrow{\partial})$ we also obtain contributing diagrams. Therefore, all radiative corrections, as well as the renormalized action $\hat{S}_R$ and the Γ functionals $\hat{\Gamma}$ and $\hat{\Gamma}_R$ associated with the action (2.32), do not depend on K_ϕ^i and $\bar{C}$ separately, but only through the combination $\tilde{K}_\phi^i$.

The only B-dependent terms of $\hat{S}_{\text{gf}}$, provided by $[\![S, \Psi]\!]$ and (2.3), are

$$\Delta S_B \equiv - \int B^I \tilde{K}_{\bar{C}}^I = \int B^I \left(G^{Ii}(\underline{\phi}, \partial)\phi^i - K_{\bar{C}}^I \right), \quad (2.34)$$

and are quadratic or linear in the quantum fields. For this reason, no one-particle irreducible diagrams can contain external B legs, therefore ΔS_B is nonrenormalized and goes into $\hat{S}_R$, $\hat{\Gamma}$ and $\hat{\Gamma}_R$ unmodified.

We thus learn that using linear gauge-fixing functions we can set $\bar{C} = B = 0$ and later restore the correct $\bar{C}$ and B dependencies in $\hat{S}_{\text{gf}}$, $\hat{S}_R$, $\hat{\Gamma}$ and $\hat{\Gamma}_R$ just by replacing K_ϕ^i with $\tilde{K}_\phi^i$ and adding ΔS_B.

From now on when no confusion can arise we drop the subscripts of S_{gf} and $\hat{S}_{\text{gf}}$ and assume that the background field theory is gauge-fixed in the way just explained.

2.2 Background-preserving canonical transformations

It is useful to characterize the most general canonical transformations $\Phi, \underline{\Phi}, K, \underline{K} \to \Phi', \underline{\Phi}', K', \underline{K}'$ that preserve the background field master equations (2.21) and the basic properties of $\hat{S}$ and $\bar{S}$.

By definition, all canonical transformations preserve the antiparentheses, so (2.21) are turned into

$$[\![\hat{S}', \hat{S}']\!]' = [\![\hat{S}', \bar{S}']\!]' = [\![\bar{S}', \bar{S}']\!]' = 0. \tag{2.35}$$

Moreover, $\hat{S}'$ should be $\underline{K}'$ and $\underline{C}'$ independent, while $\bar{S}$ should be invariant, because it encodes the background transformations. This means

$$\bar{S}'(\Phi, \underline{\Phi}, K, \underline{K}) = \bar{S}(\Phi, \underline{\Phi}, K, \underline{K}). \tag{2.36}$$

We prove that a canonical transformation defined by a generating functional of the form

$$F(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + Q(\Phi, \underline{\phi}, K'), \tag{2.37}$$

where Q is a $\underline{K}'$- and $\underline{C}$-independent local functional such that

$$[\![\bar{S}, Q(\Phi, \underline{\phi}, K)]\!] = 0, \tag{2.38}$$

satisfies our requirements.

Since Q is $\underline{K}'$ and $\underline{C}$ independent, the background fields and the sources $\underline{K}_C$ do not transform: $\underline{\Phi}' = \underline{\Phi}$, $\underline{K}'_C = \underline{K}_C$. Moreover, the action $\hat{S}'$ is clearly $\underline{K}'$ and $\underline{C}'$ independent, as desired, so we just need to prove (2.36). For convenience, multiply Q by a constant parameter ζ and consider the canonical transformations generated by

$$F_\zeta(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + \zeta Q(\Phi, \underline{\phi}, K'). \tag{2.39}$$

Given a functional $X(\Phi, \underline{\Phi}, K', \underline{K}')$ it is often useful to work with the tilde functional

$$\tilde{X}(\Phi, \underline{\Phi}, K, \underline{K}) = X(\Phi, \underline{\Phi}, K'(\Phi, \underline{\Phi}, K, \underline{K}), \underline{K}'(\Phi, \underline{\Phi}, K, \underline{K})). \tag{2.40}$$

obtained by expressing the primed sources in terms of unprimed fields and sources. Assumption (2.38) tells us that $Q(\Phi, \underline{\phi}, K)$ is invariant under background transformations. Since Φ^α and K_β transform as matter fields under such transformations, it is clear that $\delta Q / \delta K_\alpha$ and $\delta Q / \delta \Phi^\beta$ transform precisely like them, as well as $\Phi^{\alpha'}$ and K'_β. Moreover, we have $[\![\bar{S}, \tilde{Q}]\!] = 0$ for every ζ. Applying theorem 9 to $\chi = \bar{S}$ we obtain

$$\frac{\partial' \bar{S}'}{\partial \zeta} = \frac{\partial \bar{S}}{\delta \zeta} - [\![\bar{S}, \tilde{Q}]\!] = \frac{\partial \bar{S}}{\delta \zeta}, \tag{2.41}$$

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where $\partial'/\partial\zeta$ is taken at constant primed variables and $\partial/\partial\zeta$ is taken at constant unprimed variables. If we treat the unprimed variables as ζ independent, and the primed variables as functions of them and ζ, the right-hand side of (2.41) vanishes. Varying ζ from 0 to 1 we get

$$\bar{S}'(\Phi, \underline{\Phi}, K, \underline{K}) = \bar{S}'(\Phi', \underline{\Phi}', K', \underline{K}') = \bar{S}(\Phi, \underline{\Phi}, K, \underline{K}),$$

where now the relations among primed and unprimed variables are those specified by (2.37).

We call the canonical transformations just defined *background-preserving canonical transformations*. We stress once again that they do not just preserve the background field ($\underline{\Phi}' = \underline{\Phi}$), but also the background transformations ($\bar{S}' = \bar{S}$) and the $\underline{K}$ and $\underline{C}$ independence of $\hat{S}$. The gauge-fixing canonical transformation (2.24) is background preserving.

Canonical transformations may convert the sources K into functions of both fields and sources. However, the sources are external, while the fields are integrated over. Thus, canonical transformations must be applied at the level of the action S, not at the levels of generating functionals. In the functional integral they must be meant as mere replacements of integrands. Nevertheless, we recall that there exists a way [15, 16, 17] to upgrade the formalism of quantum field theory and overcome these problems. The upgraded formalism allows us to implement canonical transformations as true changes of field variables in the functional integral, and closely track their effects inside generating functionals, as well as throughout the renormalization algorithm.

3   Renormalization

In this section we give the basic algorithm to subtract divergences to all orders. As usual, we proceed by induction in the number of loops and use the dimensional-regularization technique and the minimal subtraction scheme. We assume that gauge anomalies are manifestly absent, i.e. that the background field master equations (2.21) hold exactly at the regularized level. We first work on the classical action $S = \hat{S} + \bar{S}$ of (2.32) and define a background-preserving subtraction algorithm. Then we generalize the results to non-background-preserving actions.

Call S_n and Γ_n the action and the Γ functional renormalized up to n loops included, with $S_0 = S$, and write the loop expansion as

$$\Gamma_n = \sum_{k=0}^{\infty} \hbar^n \Gamma_n^{(k)}.$$

The inductive assumptions are that S_n has the form (2.26), with $\bar{S}$ given by (2.19), and

$$S_n = S + \text{poles}, \qquad \Gamma_n^{(k)} < \infty \;\; \forall k \leqslant n, \tag{3.1}$$

$$[\![S_n, S_n]\!] = \mathcal{O}(\hbar^{n+1}), \qquad [\![\bar{S}, S_n]\!] = 0, \tag{3.2}$$

where "poles" refers to the divergences of the dimensional regularization. Clearly, the assumptions (3.1) and (3.2) are satisfied for $n = 0$.

Using formulas (2.29) and recalling that $[\![S_n, S_n]\!]$ is a local insertion of order $\mathcal{O}(\hbar^{n+1})$, we have

$$[\![\Gamma_n, \Gamma_n]\!] = \langle [\![S_n, S_n]\!] \rangle = [\![S_n, S_n]\!] + \mathcal{O}(\hbar^{n+2}), \qquad [\![\bar{S}, \Gamma_n]\!] = \langle [\![\bar{S}, S_n]\!] \rangle = 0. \tag{3.3}$$

By $[\![S, S]\!] = 0$ and the first of (3.1), $[\![S_n, S_n]\!]$ is made of pure poles.

Now, take the order $\hbar^{n+1}$ of equations (3.3) and then their divergent parts. The second of (3.1) tells us that all subdivergences are subtracted away, so the order-$\hbar^{n+1}$ divergent part $\Gamma^{(n+1)}_{n\,\mathrm{div}}$ of Γ_n is a local functional. We obtain

$$[\![S, \Gamma^{(n+1)}_{n\,\mathrm{div}}]\!] = \frac{1}{2}[\![S_n, S_n]\!] + \mathcal{O}(\hbar^{n+2}), \qquad [\![\bar{S}, \Gamma^{(n+1)}_{n\,\mathrm{div}}]\!] = 0. \tag{3.4}$$

Define

$$S_{n+1} = S_n - \Gamma^{(n+1)}_{n\,\mathrm{div}}. \tag{3.5}$$

Since S_n has the form (2.26), Γ_n has the form (2.27), therefore both $\hat{\Gamma}_n$ and $\Gamma^{(n+1)}_{n\,\mathrm{div}}$ are $\underline{K}$ and $\underline{C}$ independent, which ensures that S_{n+1} has the form (2.26) (with $\bar{S}$ given by (2.19)). Moreover, the first inductive assumption of (3.1) is promoted to S_{n+1}. The diagrams constructed with the vertices of S_{n+1} are the diagrams of S_n, plus new diagrams containing vertices of $-\Gamma^{(n+1)}_{n\,\mathrm{div}}$; therefore

$$\Gamma^{(k)}_{n+1} = \Gamma^{(k)}_n < \infty \;\; \forall k \leqslant n, \qquad \Gamma^{(n+1)}_{n+1} = \Gamma^{(n+1)}_n - \Gamma^{(n+1)}_{n\,\mathrm{div}} < \infty,$$

which promotes the second inductive assumption of (3.1) to $n + 1$ loops. Finally, formulas (3.4) and (3.5) give

$$[\![S_{n+1}, S_{n+1}]\!] = [\![S_n, S_n]\!] - 2[\![S, \Gamma^{(n+1)}_{n\,\mathrm{div}}]\!] + \mathcal{O}(\hbar^{n+2}) = \mathcal{O}(\hbar^{n+2}), \qquad [\![\bar{S}, S_{n+1}]\!] = 0,$$

so (3.2) are also promoted to $n + 1$ loops.

We conclude that the renormalized action $S_R = S_\infty$ and the renormalized generating functional $\Gamma_R = \Gamma_\infty$ satisfy the background field master equations

$$[\![S_R, S_R]\!] = [\![\bar{S}, S_R]\!] = 0, \qquad [\![\Gamma_R, \Gamma_R]\!] = [\![\bar{S}, \Gamma_R]\!] = 0. \tag{3.6}$$

For later convenience we write down the form of S_R, which is

$$S_R(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}_R(\Phi, \underline{\phi}, K) + \bar{S}(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}_R(\Phi, \underline{\phi}, K) - \int \mathcal{R}^\alpha(\Phi, \underline{C})K_\alpha - \int R^\alpha(\underline{\Phi})\underline{K}_\alpha. \tag{3.7}$$

In the usual (non-background field) approach the results just derived hold if we just ignore background fields and sources, as well as background transformations, and use the standard parentheses (X, Y) instead of $[\![X, Y]\!]$. Then the subtraction algorithm starts with a classical action

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$S(\Phi, K)$ that satisfies the usual master equation $(S, S) = 0$ exactly at the regularized level and ends with a renormalized action $S_R(\Phi, K) = S_\infty(\Phi, K)$ and a renormalized generating functional $\Gamma_R(\Phi, K) = \Gamma_\infty(\Phi, K)$ that satisfy the usual master equations $(S_R, S_R) = (\Gamma_R, \Gamma_R) = 0$.

In the presence of background fields $\underline{\Phi}$ and background sources $\underline{K}$, ignoring invariance under background transformations (encoded in the parentheses $[\![\bar{S}, S]\!]$, $[\![\bar{S}, S_n]\!]$, $[\![\bar{S}, S_R]\!]$ and similar ones for the Γ functionals), we can generalize the results found above to any classical action $S(\Phi, \underline{\Phi}, K, \underline{K})$ that satisfies $[\![S, S]\!] = 0$ at the regularized level and is such that $\delta_l S / \delta \underline{K}_\alpha$ is Φ independent. Indeed, these assumptions allow us to apply theorem 6, instead of formulas (2.29), which is enough to go through the subtraction algorithm ignoring the parentheses $[\![\bar{S}, X]\!]$. We have $\delta_l S / \delta \underline{K}_\alpha = \delta_l \Gamma / \delta \underline{K}_\alpha = \delta_l S_n / \delta \underline{K}_\alpha$ for every n. Thus, we conclude that a classical action $S(\Phi, \underline{\Phi}, K, \underline{K})$ that satisfies $[\![S, S]\!] = 0$ at the regularized level and is such that $\delta_l S / \delta \underline{K}_\alpha$ is Φ independent gives a renormalized action $S_R(\Phi, \underline{\Phi}, K, \underline{K})$ and a Γ functional $\Gamma_R(\Phi, \underline{\Phi}, K, \underline{K})$ that satisfy $[\![S_R, S_R]\!] = [\![\Gamma_R, \Gamma_R]\!] = 0$ and $\delta_l S_R / \delta \underline{K}_\alpha = \delta_l \Gamma_R / \delta \underline{K}_\alpha = \delta_l S / \delta \underline{K}_\alpha$.

The renormalization algorithm of this section is a generalization to the background field method of the procedure first given in ref. [18]. Since it subtracts divergences just as they come, as emphasized by formula (3.5), we use to call it "raw" subtraction [4], to distinguish it from algorithms where divergences are subtracted away at each step by means of parameter redefinitions and canonical transformations. The raw subtraction does not ensure RG invariance [4], because it subtracts divergent terms even when there is no (running) parameter associated with them. For the same reason, it tells us very little about parametric completeness.

In power-counting renormalizable theories the raw subtraction is satisfactory, since we can start from a classical action S_c that already contains all gauge-invariant terms that are generated back by renormalization. Nevertheless, in nonrenormalizable theories, such as quantum gravity, effective field theories and nonrenormalizable extensions of the Standard Model, in principle renormalization can modify the symmetry transformations in physically observable ways (see ref. [4] for a discussion about this possibility). In section 5 we prove that this actually does not happen under the assumptions we have made in this paper; namely when gauge anomalies are manifestly absent, the gauge algebra is irreducible and closes off shell, and $R^\alpha(\Phi)$ are quadratic functions of the fields Φ. Precisely, renormalization affects the symmetry only by means of canonical transformations and parameter redefinitions. Then, to achieve parametric completeness it is sufficient to include all gauge-invariant terms in the classical action $S_c(\phi)$, as classified by the starting gauge symmetry. The background field method is crucial to prove this result without advocating involved cohomological classifications.

4 Gauge dependence

In this section we study the dependence on the gauge fixing and the renormalization of canonical transformations. We first derive the differential equations that govern gauge dependence; then we integrate them and finally use the outcome to describe the renormalized canonical transformation that switches between the background field approach and the conventional approach. These results will be useful in the next section to prove parametric completeness.

The parameters of a canonical transformation are associated with changes of field variables and changes of gauge fixing. For brevity we call all of them "gauge-fixing parameters" and denote them with ξ. Let (2.37) be a tree-level canonical transformation satisfying (2.38). We write $Q(\Phi, \underline{\phi}, K', \xi)$ to emphasize the ξ dependence of Q. We prove that for every gauge-fixing parameter ξ there exists a local $\underline{K}$- and $\underline{C}$-independent functional $Q_{R,\xi}$ such that

$$Q_{R,\xi} = \widetilde{Q_\xi} + \mathcal{O}(\hbar)\text{-poles}, \qquad \langle Q_{R,\xi} \rangle < \infty, \tag{4.1}$$

and

$$\frac{\partial S_R}{\partial \xi} = [\![S_R, Q_{R,\xi}]\!], \qquad [\![\bar{S}, Q_{R,\xi}]\!] = 0, \qquad \frac{\partial \Gamma_R}{\partial \xi} = [\![\Gamma_R, \langle Q_{R,\xi} \rangle]\!], \tag{4.2}$$

where $Q_\xi = \partial Q / \partial \xi$, $\widetilde{Q_\xi}$ is defined as shown in (2.40) and the average is calculated with the action S_R. We call the first and last equations of the list (4.2) *differential equations of gauge dependence*. They ensure that renormalized functionals depend on gauge-fixing parameters in a cohomologically exact way. Later we integrate equations (4.2) and move every gauge dependence inside a (renormalized) canonical transformation. A consequence is that physical quantities are gauge independent.

We derive (4.2) proceeding inductively in the number of loops, as usual. The inductive assumption is that there exists a $\underline{K}$- and $\underline{C}$-independent local functional $Q_{n,\xi} = \widetilde{Q_\xi} + \mathcal{O}(\hbar)$-poles such that $\langle Q_{n,\xi} \rangle$ is convergent up to the nth loop included (the average being calculated with the action S_n) and

$$\frac{\partial S_n}{\partial \xi} = [\![S_n, Q_{n,\xi}]\!] + \mathcal{O}(\hbar^{n+1}), \qquad [\![\bar{S}, Q_{n,\xi}]\!] = 0. \tag{4.3}$$

Applying the identity (A.19), which here holds with the parentheses $[\![X, Y]\!]$, we easily see that $Q_{0,\xi} = \widetilde{Q_\xi}$ satisfies (4.3) for $n = 0$. Indeed, taking $\chi = S$ and noting that $\partial S'/\partial \xi|_{\Phi', K'} = 0$, since the parameter ξ is absent before the transformation (a situation that we describe using primed variables), we get the first relation of (4.3), without $\mathcal{O}(\hbar)$ corrections. Applying (A.19) to $\chi = \bar{S}$ and recalling that $\bar{S}$ is invariant, we get the second relation of (4.3).

Let $Q_{n,\xi\,\text{div}}^{(n+1)}$ denote the $\mathcal{O}(\hbar^{n+1})$ divergent part of $\langle Q_{n,\xi} \rangle$. The inductive assumption ensures that all subdivergences are subtracted away, so $Q_{n,\xi\,\text{div}}^{(n+1)}$ is local. Define

$$Q_{n+1,\xi} = Q_{n,\xi} - Q_{n,\xi\,\text{div}}^{(n+1)}. \tag{4.4}$$

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Clearly, $Q_{n+1,\xi}$ is $\underline{K}$ and $\underline{C}$ independent and equal to $\widetilde{Q}_{\xi} + \mathcal{O}(\hbar)$-poles. Moreover, by construction $\langle Q_{n+1,\xi} \rangle$ is convergent up to the $(n+1)$-th loop included, where the average is calculated with the action S_{n+1}.

Now, corollary 8 tells us that $[\![\bar{S}, Q_{n,\xi}]\!] = 0$ and $[\![\bar{S}, S_n]\!] = 0$ imply $[\![\bar{S}, \langle Q_{n,\xi} \rangle]\!] = 0$. Taking the $\mathcal{O}(\hbar^{n+1})$ divergent part of this formula we obtain $[\![\bar{S}, Q_{n,\xi\,\mathrm{div}}^{(n+1)}]\!] = 0$; therefore the second formula of (4.3) is promoted to $n+1$ loops.

Applying corollary 7 to Γ_n and S_n, with $X = Q_{n,\xi}$, we have the identity

$$\frac{\partial \Gamma_n}{\partial \xi} = [\![\Gamma_n, \langle Q_{n,\xi} \rangle]\!] + \left\langle \frac{\partial S_n}{\partial \xi} - [\![S_n, Q_{n,\xi}]\!] \right\rangle + \frac{1}{2} \langle [\![S_n, S_n]\!] Q_{n,\xi} \rangle_{\Gamma} , \tag{4.5}$$

where $\langle AB \rangle_{\Gamma}$ denotes the one-particle irreducible diagrams with one A insertion and one B insertion. Now, observe that if $A = \mathcal{O}(\hbar^{n_A})$ and $B = \mathcal{O}(\hbar^{n_B})$ then $\langle AB \rangle_{\Gamma} = \mathcal{O}(\hbar^{n_A + n_B + 1})$, since the A, B insertions can be connected only by loops. Let us take the $\mathcal{O}(\hbar^{n+1})$ divergent part of (4.5). By the inductive assumption (3.2), the last term of (4.5) can be neglected. By the inductive assumption (4.3) we can drop the average in the second-to-last term. We thus get

$$\frac{\partial \Gamma_{n\,\mathrm{div}}^{(n+1)}}{\partial \xi} = [\![\Gamma_{n\,\mathrm{div}}^{(n+1)}, Q_{0,\xi}]\!] + [\![S, Q_{n,\xi\,\mathrm{div}}^{(n+1)}]\!] + \frac{\partial S_n}{\partial \xi} - [\![S_n, Q_{n,\xi}]\!] + \mathcal{O}(\hbar^{n+2}).$$

Using this fact, (3.5) and (4.4) we obtain

$$\frac{\partial S_{n+1}}{\partial \xi} = [\![S_{n+1}, Q_{n+1,\xi}]\!] + \mathcal{O}(\hbar^{n+2}), \tag{4.6}$$

which promotes the first inductive hypothesis of (4.3) to order $\hbar^{n+1}$. When n is taken to infinity, the first two formulas of (4.2) follow, with $Q_{R,\xi} = Q_{\infty,\xi}$. The third identity of (4.2) follows from the first one, using (4.5) with $n = \infty$ and $[\![\hat{S}_R, \hat{S}_R]\!] = 0$. This concludes the derivation of (4.2).

4.1 Integrating the differential equations of gauge dependence

Now we integrate the first two equations of (4.2) and find the renormalized canonical transformation that corresponds to a tree-level transformation (2.37) satisfying (2.38). Specifically, we prove that

Theorem 1 *There exists a background-preserving canonical transformation*

$$F_R(\Phi, \underline{\Phi}, K', \underline{K}', \xi) = \int \Phi^A K_A' + \int \underline{\Phi}^A \underline{K}_A' + Q_R(\Phi, \phi, K', \xi), \tag{4.7}$$

where $Q_R(\Phi, \phi, K', \xi) = Q(\Phi, \phi, K', \xi) + \mathcal{O}(\hbar)$ is a $\underline{K}$- and $\underline{C}$-independent local functional, such that the transformed action $S_f(\Phi', \underline{\Phi}', K', \underline{K}') = S_R(\Phi, \underline{\Phi}, K, \underline{K}, \xi)$ is ξ independent and invariant under background transformations:

$$\frac{\partial S_f}{\partial \xi} = 0, \qquad [\![\bar{S}, S_f]\!] = 0. \tag{4.8}$$

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Proof. To prove this statement we introduce a new parameter ζ multiplying the whole functional Q of (2.37), as in (2.39). We know that $[\![\bar{S}, Q]\!] = 0$ implies $[\![\bar{S}, \tilde{Q}]\!] = 0$. If we prove that the ζ dependence can be reabsorbed into a background-preserving canonical transformation we also prove the same result for every gauge-fixing parameter ξ and also for all of them together. The differential equations of gauge dependence found above obviously apply with $\xi \to \zeta$.

Specifically, we show that the ζ dependence can be reabsorbed in a sequence of background-preserving canonical transformations $S_{Rn} \to S_{Rn+1}$ (with $S_{R0} = S_R$), generated by

$$F_n(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^A K'_A + \int \underline{\Phi}^A \underline{K}'_A + H_n(\Phi, \underline{\phi}, K', \zeta), \tag{4.9}$$

where $H_n = \mathcal{O}(\hbar^n)$, and such that

$$\frac{\partial S_{Rn}}{\partial \zeta} = [\![S_{Rn}, T_n]\!], \qquad T_n = \mathcal{O}(\hbar^n). \tag{4.10}$$

The functionals T_n and H_n are determined by the recursive relations

$$T_{n+1}(\Phi', \underline{\phi}, K', \zeta) = T_n(\Phi, \underline{\phi}, K, \zeta) - \widetilde{\frac{\partial H_n}{\partial \zeta}}, \tag{4.11}$$

$$H_n(\Phi, \underline{\phi}, K', \zeta) = \int_0^\zeta d\zeta' T_{n,n}(\Phi, \underline{\phi}, K', \zeta'), \tag{4.12}$$

with the initial conditions

$$T_0 = Q_{R,\zeta}, \qquad H_0 = \zeta Q.$$

In formula (4.11) the tilde operation (2.40) on $\partial H_n / \partial \zeta$ and the canonical transformation $\Phi, K \to \Phi', K'$ are the ones defined by F_n. In formula (4.12) $T_{n,n}(\Phi, \underline{\phi}, K')$ denotes the contributions of order $\hbar^n$ to $T_n(\Phi, \underline{\phi}, K(\Phi, \underline{\phi}, K'))$, the function $K(\Phi, \underline{\phi}, K')$ also being determined by F_n. Note that for $n > 0$ we have $T_n(\Phi, \underline{\phi}, K(\Phi, \underline{\phi}, K')) = T_n(\Phi, \underline{\phi}, K') + \mathcal{O}(\hbar^{n+1})$, therefore formula (4.12), which determines H_n (and so F_n), does not really need F_n on the right-hand side. Finally, (4.12) is self-consistent for $n = 0$.

Formula (A.19) of the appendix describes how the dependence on parameters is modified by a canonical transformation. Applying it to (4.10), we get

$$\frac{\partial S_{Rn+1}}{\partial \zeta} = \frac{\partial S_{Rn}}{\partial \zeta} - [\![S_{Rn}, \widetilde{\frac{\partial H_n}{\partial \zeta}}]\!] = [\![S_{Rn}, T_n - \widetilde{\frac{\partial H_n}{\partial \zeta}}]\!],$$

whence (4.11) follows. For $n = 0$ the first formula of (4.1) gives $T_0 = \tilde{Q} + \mathcal{O}(\hbar)$, therefore $T_1 = \mathcal{O}(\hbar)$. Then (4.12) gives $H_1 = \mathcal{O}(\hbar)$. For $n > 0$ the order $\hbar^n$ of T_{n+1} vanishes by formula (4.12); therefore $T_{n+1} = \mathcal{O}(\hbar^{n+1})$ and $H_{n+1} = \mathcal{O}(\hbar^{n+1})$, as desired.

Consequently, $S_f \equiv S_{R\infty}$ is ζ independent, since (4.10) implies $\partial S_{R\infty} / \partial \zeta = 0$. Observe that $\underline{K}$ and $\underline{C}$ independence is preserved at each step. Finally, all operations defined by (4.11) and

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(4.12) are background preserving. We conclude that the canonical transformation F_R obtained composing the F_ns solves the problem.

Using (4.8) and (2.29) we conclude that in the new variables

$$\frac{\partial \Gamma_f}{\partial \xi} = \left\langle \frac{\partial S_f}{\partial \xi} \right\rangle = 0 \qquad [\![\bar{S}, \Gamma_f]\!] = 0, \tag{4.13}$$

for all gauge-fixing parameters ξ.

4.2 Non-background-preserving canonical transformations

In the usual approach the results derived so far apply with straightforward modifications. It is sufficient to ignore the background fields and sources, as well as the background transformations, and use the standard parentheses (X, Y) instead of $[\![X, Y]\!]$. Thus, given a tree-level canonical transformation generated by

$$F(\Phi, K') = \int \Phi^\alpha K'_\alpha + Q(\Phi, K', \xi), \tag{4.14}$$

there exists a local functional $Q_{R,\xi}$ satisfying (4.1) such that

$$\frac{\partial S_R}{\partial \xi} = (S_R, Q_{R,\xi}), \qquad \frac{\partial \Gamma_R}{\partial \xi} = (\Gamma_R, \langle Q_{R,\xi} \rangle), \tag{4.15}$$

and there exists a renormalized canonical transformation

$$F_R(\Phi, K') = \int \Phi^A K'_A + Q_R(\Phi, K', \xi), \tag{4.16}$$

where $Q_R(\Phi, K', \xi) = Q(\Phi, K', \xi) + \mathcal{O}(\hbar)$ is a local functional, such that the transformed action $S_f(\Phi', K') = S_R(\Phi, K, \xi)$ is ξ independent. Said differently, the entire ξ dependence of S_R is reabsorbed into the transformation:

$$S_R(\Phi, K, \xi) = S_f(\Phi'(\Phi, K, \xi), K'(\Phi, K, \xi)).$$

In the presence of background fields $\underline{\Phi}$ and background sources $\underline{K}$, dropping assumption (2.38) and ignoring invariance under background transformations, encoded in the parentheses $[\![\bar{S}, X]\!]$, the results found above can be easily generalized to any classical action $S(\Phi, \underline{\Phi}, K, \underline{K})$ that solves $[\![S, S]\!] = 0$ and is such that $\delta_l S/\delta \underline{K}_\alpha$ is Φ independent, and to any $\underline{K}$-independent canonical transformation. Indeed, these assumptions are enough to apply theorem 6 and corollary 7, and go through the derivation ignoring the parentheses $[\![\bar{S}, X]\!]$. The tree-level canonical transformation is described by a generating functional of the form

$$F(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + Q(\Phi, \underline{\phi}, \underline{C}, K', \xi). \tag{4.17}$$

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We still find the differential equations

$$\frac{\partial S_R}{\partial \xi} = [\![S_R, Q_{R,\xi}]\!], \qquad \frac{\partial \Gamma_R}{\partial \xi} = [\![\Gamma_R, \langle Q_{R,\xi} \rangle]\!], \tag{4.18}$$

where $Q_{R,\xi}$ satisfies (4.1). When we integrate the first of these equations with the procedure defined above we build a renormalized canonical transformation

$$F_R(\Phi, \underline{\Phi}, K', \underline{K}', \xi) = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + Q_R(\Phi, \phi, \underline{C}, K', \xi), \tag{4.19}$$

where $Q_R = Q + \mathcal{O}(\hbar)$ is a local functional, such that the transformed action $S_f(\Phi', \underline{\Phi}', K', \underline{K}') = S_R(\Phi, \underline{\Phi}, K, \underline{K}, \xi)$ is ξindependent. The only difference is that now $Q_{R,\xi}$, $\langle Q_{R,\xi} \rangle$, T_n, H_n and Q_R can depend on $\underline{C}$, which does not disturb any of the arguments used in the derivation.

4.3   Canonical transformations in Γ

We have integrated the first equation of (4.18), and shown that the ξ dependence can be reabsorbed in the canonical transformation (4.19) on the renormalized action S_R, which gives the ξ-independent action with S_f. We know that the generating functional Γ_f of one-particle irreducible Green functions determined by S_f is ξ independent. We can also prove that Γ_f can be obtained applying a (non-local) canonical transformation directly on Γ_R.

To achieve this goal we integrate the second equation of (4.18). The integration algorithm is the same as the one of subsection 4.1, with the difference that $Q_{R,\xi}$ is replaced by $\langle Q_{R,\xi} \rangle$. The canonical transformation on Γ_R has a generating functional of the form

$$F_\Gamma(\Phi, \underline{\Phi}, K', \underline{K}', \xi) = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + Q_\Gamma(\Phi, \phi, \underline{C}, K', \xi), \tag{4.20}$$

where $Q_\Gamma = Q + \mathcal{O}(\hbar)$ (non-local) radiative corrections.

The result just obtained is actually more general, and proves that if S is any action that solves the master equation (it can be the classical action, the renormalized action, or any other action) canonical transformations on S correspond to canonical transformations on the Γ functional determined by S. See [14] for a different derivation of this result in Yang-Mills theory. Our line of reasoning can be recapitulated as follows: in the usual approach, (*i*) make a canonical transformation (4.14) on S; (*ii*) derive the equations of gauge dependence for the action, which are $\partial S/\partial \xi = (S, Q_\xi)$; (*iii*) derive the equations of gauge dependence for the Γ functional determined by S, which are $\partial \Gamma/\partial \xi = (\Gamma, \langle Q_\xi \rangle)$, and integrate them.

The property just mentioned may sound obvious, and is often taken for granted, but actually needed to be proved. The reason is that the canonical transformations we are talking about are not true changes of field variables inside functional integrals, but mere replacements of integrands [15]. Therefore, we cannot automatically infer how a transformation on the action S affects the

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generating functionals Z, $W = \ln Z$ and Γ, and need to make some additional effort to get where we want. We recall that to skip this kind of supplementary analysis we need to use the formalism of the master functional, explained in refs. [16, 17].

4.4 Application

An interesting application that illustrates the results of this section is the comparison between the renormalized action (3.7), which was obtained with the background field method and the raw subtraction procedure of section 3, and the renormalized action S'_R that can be obtained with the same raw subtraction in the usual non-background field approach.

The usual approach is retrieved by picking a gauge fermion Ψ' that depends on $\Phi + \underline{\Phi}$, such as

$$\Psi'(\Phi + \underline{\Phi}) = \int \bar{C}^I G^{Ii}(0, \partial)(\phi^i + \underline{\phi}^i). \tag{4.21}$$

Making the canonical transformation generated by

$$F'_{\mathrm{gf}}(\Phi, \underline{\Phi}, K', \underline{K}') = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + \Psi'(\Phi + \underline{\Phi}) \tag{4.22}$$

on (2.9) we find the classical action

$$S'(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}'(\Phi + \underline{\Phi}, K) + \bar{S}'(\Phi, \underline{\Phi}, K, \underline{K}), \tag{4.23}$$

where

$$\hat{S}'(\Phi + \underline{\Phi}, K) = S_c(\phi + \underline{\phi}) - \int R^\alpha(\Phi + \underline{\Phi})\bar{K}_\alpha, \qquad \bar{S}'(\Phi, \underline{\Phi}, K, \underline{K}) = -\int R^\alpha(\underline{\Phi})(\underline{K}_\alpha - K_\alpha), \tag{4.24}$$

and the barred sources $\bar{K}_\alpha$ coincide with K_α apart from $\bar{K}^i_\phi$ and $\bar{K}^I_{\bar{C}}$, which are

$$\bar{K}^i_\phi = K^i_\phi - \bar{C}^I G^{Ii}(0, -\overleftarrow{\partial}), \qquad \bar{K}^I_{\bar{C}} = K^I_{\bar{C}} - G^{Ii}(0, \partial)(\phi^i + \underline{\phi}^i). \tag{4.25}$$

Clearly, $\hat{S}'$ is the gauge-fixed classical action of the usual approach, apart from the shift $\Phi \to \Phi + \underline{\Phi}$.

The radiative corrections are generated only by $\hat{S}'$ and do not affect $\bar{S}'$. Indeed, $\hat{S}'$ as well as the radiative corrections are unaffected by setting $\underline{\Phi} = 0$ and then shifting Φ back to $\Phi + \underline{\Phi}$, while $\bar{S}'$ disappears doing this. Thus, $\bar{S}'$ is nonrenormalized, and the renormalized action S'_R has the form

$$S'_R(\Phi, \underline{\Phi}, K, \underline{K}) = \hat{S}'_R(\Phi + \underline{\Phi}, K) + \bar{S}'(\Phi, \underline{\Phi}, K, \underline{K}). \tag{4.26}$$

Now we compare the classical action (2.32) of the background field method with the classical action (4.23) of the usual approach. We recapitulate how they are obtained with the help of the following schemes:

$$(2.9) \quad \xrightarrow{(2.15)} \quad (2.16) \quad \xrightarrow{(2.24)} \quad S = (2.32) \qquad (2.9) \quad \xrightarrow{(4.22)} \quad S' = (4.24).$$

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Above the arrows we have put references to the corresponding canonical transformations, which are (2.15), (2.24) and (4.22) and commute with one another. We can interpolate between the classical actions (2.32) and (4.24) by means of a $\underline{K}$-independent non-background-preserving canonical transformation generated by

$$F_\xi(\Phi, \underline{\Phi}, K', \underline{K}', \xi) = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha \underline{K}'_\alpha + \xi \Delta \Psi + \xi \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C}) K^{I\prime}_B. \qquad (4.27)$$

where ξ is a gauge-fixing parameter that varies from 0 to 1, and

$$\Delta \Psi = \int \bar{C}^I \left(G^{Ii}(\underline{\phi}, \partial)\phi^i - G^{Ii}(0, \partial)(\phi^i + \underline{\phi}^i) \right).$$

Precisely, start from the non-background field theory (4.23), and take its variables to be primed ones. We know that $\hat{S}'$ depends on the combination

$$\tilde{K}^{i\prime}_\phi = K^{i\prime}_\phi - \bar{C}^I G^{Ii}(0, -\overleftarrow{\partial}), \qquad (4.28)$$

and we have $\bar{C}^I = \bar{C}^{I\prime}$. Expressing the primed fields and sources in terms of the unprimed ones and ξ, we find the interpolating classical action

$$S_\xi(\Phi, \underline{\Phi}, K, \underline{K}) = S_c(\phi + \underline{\phi}) - \int R^\alpha(\Phi + \underline{\Phi})\tilde{K}_\alpha(\xi) - \xi \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C})\tilde{K}^I_{\bar{C}}(\xi) - \int R^\alpha(\underline{\Phi})(\underline{\tilde{K}}_\alpha(\xi) - \tilde{K}_\alpha(\xi)), \qquad (4.29)$$

where $\tilde{K}^I_C(\xi) = K^I_C$,

$$\tilde{K}^i_\phi(\xi) = K^i_\phi - \xi\bar{C}^I G^{Ii}(\underline{\phi}, -\overleftarrow{\partial}) - (1 - \xi)\bar{C}^I G^{Ii}(0, -\overleftarrow{\partial}), \qquad (4.30)$$

while the other ξ-dependent tilde sources have expressions that we do not need to report here. It suffices to say that they are K^i_ϕ independent, such that $\delta_r S_\xi/\delta\underline{K}_\alpha = -R^\alpha(\underline{\Phi})$, and linear in the quantum fields Φ, apart from $\underline{\tilde{K}}^i_\phi(\xi)$, which is quadratic. Thus the action S_ξ and the transformation F_ξ satisfy the assumptions that allow us to apply theorem 6 and corollary 7. Actually, (4.27) is of type (4.17); therefore we have the differential equations (4.18) and the renormalized canonical transformation (4.19).

We want to better characterize the renormalized version F_R of F_ξ. We know that the derivative of the renormalized Γ functional with respect to ξ is governed by the renormalized version of the average

$$\left\langle \widetilde{\frac{\partial F_\xi}{\partial \xi}} \right\rangle = \langle \Delta \Psi \rangle + \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C}) K^I_B.$$

It is easy to see that $\langle \Delta \Psi \rangle$ is independent of $\underline{K}$, B, $K_{\bar{C}}$ and K_B, since no one-particle irreducible diagrams with such external legs can be constructed. In particular, $K^{I\prime}_B = K^I_B$. Moreover, using the explicit form (4.29) of the action S_ξ and arguments similar to the ones that lead to formulas

(2.33), we easily see that $\langle\Delta\Psi\rangle$ is equal to $\Delta\Psi$ plus a functional that does not depend on K_ϕ^i and $\bar{C}^I$ separately, but only on the convex combination (4.30). Indeed, the $\bar{C}$-dependent terms of (4.29) that do not fit into the combination (4.30) are K_ϕ^i independent and at most quadratic in the quantum fields, so they cannot generate one-particle irreducible diagrams that have either K_ϕ^i or $\bar{C}$ on the external legs. Clearly, the renormalization of $\langle\Delta\Psi\rangle$ also satisfies the properties just stated for $\langle\Delta\Psi\rangle$.

Following the steps of the previous section we can integrate the ξ derivative and reconstruct the full canonical transformation. However, formula (4.12) shows that the integration over ξ must be done by keeping fixed the unprimed fields Φ and the primed sources K'. When we do this for the zeroth canonical transformation F_0 of (4.9), the combination $\tilde{K}_\phi^i(\xi)$ is turned into (4.28), which is ξ independent. Every other transformation F_n of (4.9) preserves the combination (4.28), so the integrated canonical transformation does not depend on K_ϕ^i and $\bar{C}^I$ separately, but only on the combination $\tilde{K}_\phi^{i\prime}$, and the generating functional of the renormalized version F_R of F_ξ has the form

$$F_R(\Phi,\underline{\Phi},K',\underline{K}',\xi) = \int \Phi^\alpha K'_\alpha + \int \underline{\Phi}^\alpha\underline{K}'_\alpha + \xi\Delta\Psi + \xi\int\mathcal{R}_{\bar{C}}^I(\bar{C},\underline{C})K_B^{I\prime} + \Delta F_\xi(\phi,C,\underline{\phi},\underline{C},\tilde{K}'_\phi,K'_C,\xi).$$

$$(4.31)$$

Using this expression we can verify *a posteriori* that indeed $\tilde{K}_\phi^i(\xi)$ depends just on (4.28), not on $K_\phi^{i\prime}$ and $\bar{C}^I$ separately. Moreover, (4.31) implies

$$\underline{\Phi}^{\alpha\prime} = \underline{\Phi}^\alpha, \qquad B^{I\prime} = B^I + \xi\mathcal{R}_{\bar{C}}^I(\bar{C},\underline{C}), \qquad \bar{C}^{I\prime} = \bar{C}^I, \qquad K_B^{I\prime} = K_B^I. \qquad (4.32)$$

In the next section these results are used to achieve parametric completeness.

5 Renormalization and parametric completeness

The raw renormalization algorithm of section 3 subtracts away divergences just as they come. It does not ensure, *per se*, RG invariance, for which it is necessary to prove parametric completeness, namely that all divergences can be subtracted by redefining parameters and making canonical transformations. We must show that we can include in the classical action all invariants that are generated back by renormalization, and associate an independent parameter with each of them. The purpose of this section is to show that the background field method allows us to prove parametric completeness in a rather straightforward way, making cohomological classifications unnecessary.

We want to relate the renormalized actions (3.7) and (4.26). From the arguments of the previous section we know that these two actions are related by the canonical transformation generated by (4.31) at $\xi = 1$. We have

$$\hat{S}'_R(\Phi' + \underline{\Phi}, K') - \int R^\alpha(\underline{\Phi})(\underline{K}'_\alpha - K'_\alpha) = \hat{S}_R(\Phi,\underline{\phi},K) - \int\mathcal{R}^\alpha(\Phi,\underline{C})K_\alpha - \int R^\alpha(\underline{\Phi})\underline{K}_\alpha. \quad (5.1)$$

From (4.31) we find the transformation rules (4.32) at $\xi = 1$ and

$$\phi' = \phi + \frac{\delta \Delta F}{\delta \tilde{K}'_\phi}, \qquad K^I_{\bar{C}} = K^{I\prime}_{\bar{C}} + G^{Ii}(\underline{\phi}, \partial)\phi^i - G^{Ii}(0, \partial)(\phi^{i\prime} + \underline{\phi}^i) + \frac{\delta}{\delta \overline{C}^I} \int \mathcal{R}^J_{\bar{C}}(\bar{C}, \underline{C}) K^J_B, \ (5.2)$$

where ΔF is ΔF_ξ at $\xi = 1$. Here and below we sometimes understand indices when there is no loss of clarity. We want to express equation (5.1) in terms of unprimed fields and primed sources, and then set $\phi = C = \underline{K}' = 0$. We denote this operation with a subscript 0. Keeping $\underline{\phi}, \underline{C}, \bar{C}, B$ and K' as independent variables, we get

$$\hat{S}'_R(\Phi'_0 + \underline{\Phi}, K') = \hat{S}_R(0, \underline{\phi}, 0) - \int B^{I\prime}(K^{I\prime}_{\bar{C}} - G^{Ii}(0, \partial)(\phi^{i\prime}_0 + \underline{\phi}^i))$$

$$- \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C}) \frac{\delta}{\delta \overline{C}^I} \int \mathcal{R}^J_{\bar{C}}(\bar{C}, \underline{C}) K^{J\prime}_B - \int R^\alpha(\underline{\Phi})(\underline{K}_{\alpha 0} + K'_\alpha). \qquad (5.3)$$

To derive this formula we have used

$$\hat{S}_R(\{0, 0, \bar{C}, B\}, \underline{\phi}, K) = \hat{S}_R(0, \underline{\phi}, 0) - \int B^I K^I_{\bar{C}}, \qquad (5.4)$$

together with (2.6), (4.32) and (5.2). The reason why (5.4) holds is that at $C = 0$ there are no objects with positive ghost numbers inside the left-hand side of this equation; therefore we can drop every object that has a negative ghost number, which means $\bar{C}$ and all sources K but $K^I_{\bar{C}}$. Since (2.34) are the only B and $K^I_{\bar{C}}$-dependent terms, and they are not renormalized, at $\phi = 0$ we find (5.4).

Now, consider the canonical transformation $\{\underline{\phi}, \underline{C}, \bar{C}, B\}, \check{K} \to \Phi'', K'$ defined by the generating functional

$$F(\{\underline{\phi}, \underline{C}, \bar{C}, B\}, K') = \int \underline{\phi} K'_\phi + \int \underline{C} K'_C + F_R(\{0, 0, \bar{C}, B\}, \underline{\Phi}, K', 0, 1). \qquad (5.5)$$

It gives the transformation rules

$$\Phi'' = \underline{\Phi} + \Phi'_0, \qquad \check{K}_\phi = K'_\phi + \underline{K}_{\phi 0}, \qquad \check{K}_C = K'_C + \underline{K}_{C0},$$

$$\check{K}^I_{\bar{C}} = K^{I\prime}_{\bar{C}} - G^{Ii}(0, \partial)\phi^{i\prime\prime} + \frac{\delta}{\delta \overline{C}^I} \int \mathcal{R}^J_{\bar{C}}(\bar{C}, \underline{C}) K^{J\prime}_B, \qquad \check{K}_B = K'_B,$$

which turn formula (5.3) into

$$\hat{S}'_R(\Phi'', K') = \hat{S}_R(0, \underline{\phi}, 0) - \int R^i_\phi(\underline{\Phi}) \check{K}^i_\phi - \int R^I_C(\underline{\Phi}) \check{K}^I_C$$

$$- \int B^I \check{K}^I_{\bar{C}} - \int \mathcal{R}^I_{\bar{C}}(\bar{C}, \underline{C}) \check{K}^I_{\bar{C}} - \int \mathcal{R}^I_B(B, \underline{C}) \check{K}^I_B. \qquad (5.6)$$

Note that $(\hat{S}'_R, \hat{S}'_R) = 0$ is automatically satisfied by (5.6). Indeed, we know that $\hat{S}_R(\Phi, \underline{\phi}, K)$ is invariant under background transformations, and so is $\hat{S}_R(0, \underline{\phi}, 0)$, because Φ and K transform as matter fields.

We can classify $\hat{S}_R(0, \underline{\phi}, 0)$ using its gauge invariance. Let $\mathcal{G}_i(\phi)$ denote a basis of gauge-invariant local functionals constructed with the physical fields ϕ. Then

$$\hat{S}_R(0, \underline{\phi}, 0) = \sum_i \tau_i \mathcal{G}_i(\underline{\phi}), \tag{5.7}$$

for suitable constants τ_i.

Now we manipulate these results in several ways to make their consequences more explicit. To prepare the next discussion it is convenient to relabel $\{\underline{\phi}, \underline{C}, \bar{C}, B\}$ as $\check{\Phi}^\alpha$ and K' as K''. Then formulas (5.5) and (5.6) tell us that the canonical transformation

$$F_1(\check{\Phi}, K'') = \int \check{\phi} K''_\phi + \int \check{C} K''_C + F_R(\{0, 0, \check{\bar{C}}, \check{B}\}, \{\check{\phi}, \check{C}\}, K'', 0, 1) \tag{5.8}$$

is such that

$$\hat{S}'_R(\Phi'', K'') = \hat{S}_R(0, \check{\phi}, 0) - \int \bar{R}^\alpha(\check{\Phi}) \check{K}_\alpha. \tag{5.9}$$

Parametric completeness

Making the further canonical transformation $\check{\Phi}, \check{K} \to \Phi, K$ generated by

$$F_2(\Phi, \check{K}) = \int \Phi^\alpha \check{K}_\alpha + \int \bar{C}^I G^{Ii}(0, \partial) \phi^i - \int \mathcal{R}^I_{\bar{C}}(\bar{C}, C) \check{K}^I_B,$$

we get

$$\hat{S}'_R(\Phi'', K'') = \hat{S}_R(0, \phi, 0) - \int R^\alpha(\Phi) \bar{K}_\alpha, \tag{5.10}$$

where the barred sources are the ones of (4.25) at $\underline{\phi} = 0$. If we start from the most general gauge-invariant classical action,

$$S_c(\phi, \lambda) \equiv \sum_i \lambda_i \mathcal{G}_i(\phi), \tag{5.11}$$

where λ_i are physical couplings (apart from normalization constants), identities (5.7) and (5.10) give

$$\hat{S}'_R(\Phi'', K'') = S_c(\phi, \tau(\lambda)) - \int R^\alpha(\Phi) \bar{K}_\alpha. \tag{5.12}$$

This result proves parametric completeness in the usual approach, because it tells us that the renormalized action of the usual approach is equal to the classical action $\hat{S}'(\Phi, K)$ (check (4.23)-(4.24) at $\underline{\Phi} = \underline{K} = 0$), apart from parameter redefinitions $\lambda \to \tau$ and a canonical transformation. In this derivation the role of the background field method is just to provide the key tool to prove the statement.

We can also describe parametric completeness in the background field approach. Making the canonical transformation $\check{\Phi}, \check{K} \to \hat{\Phi}, \hat{K}$ generated by

$$F'_2(\hat{\Phi}, \check{K}) = \int \hat{\Phi}^\alpha \check{K}_\alpha + \int \underline{\phi}^i \check{K}^i_\phi + \int \hat{\bar{C}}^I G^{Ii}(\underline{\phi}, \partial) \hat{\phi}^i - \int \mathcal{R}^I_{\bar{C}}(\hat{\bar{C}}, \hat{C}) \check{K}^I_B, \tag{5.13}$$

formula (5.9) becomes

$$\hat{S}'_R(\Phi'', K'') = S_c(\hat{\phi} + \underline{\phi}, \tau) - \int R^\alpha(\hat{\phi} + \underline{\phi}, \hat{C}, \hat{\bar{C}}, \hat{B})\widetilde{\hat{K}}_\alpha, \tag{5.14}$$

where the relations between tilde and nontilde sources are the hat versions of (2.33). Next, we make a $\underline{\Phi}$ translation on the left-hand side of (5.14) applying the canonical transformation Φ'', K'' $\to \Phi', K'$ generated by

$$F_3(\Phi', K'') = \int (\Phi^{\alpha\prime} + \underline{\Phi}^\alpha) K''_\alpha. \tag{5.15}$$

Doing so, $\hat{S}'_R(\Phi'', K'')$ is turned into $\hat{S}'_R(\Phi' + \underline{\Phi}, K')$.

At this point, we want to compare the result we have obtained with (5.1). Recall that formula (5.1) involves the canonical transformation (4.31) at $\xi = 1$. If we set $\underline{K}' = 0$ we project that canonical transformation onto a canonical transformation $\Phi, K \to \Phi', K'$ generated by $F_R(\Phi, \underline{\Phi}, K', 0, 1)$, where $\underline{\Phi}$ is regarded as a spectator. Furthermore, it is convenient to set $\underline{C} = 0$, because then formula (5.1) turns into

$$\hat{S}'_R(\Phi' + \underline{\Phi}, K') = \hat{S}_R(\Phi, \underline{\phi}, K),$$

where now primed fields and sources are related to the unprimed ones by the canonical transformation generated by $F_R(\Phi, \{\underline{\phi}, 0\}, K', 0, 1)$. Finally, recalling that $\hat{S}'_R(\Phi'', K'') = \hat{S}'_R(\Phi' + \underline{\Phi}, K')$ and using (5.14) we get the key formula we wanted, namely

$$\hat{S}_R(\Phi, \underline{\phi}, K) = S_c(\hat{\phi} + \underline{\phi}, \tau(\lambda)) - \int R^\alpha(\hat{\phi} + \underline{\phi}, \hat{C}, \hat{\bar{C}}, \hat{B})\widetilde{\hat{K}}_\alpha. \tag{5.16}$$

Observe that formula (1.1) of the introduction is formula (5.16) with antighosts and Lagrange multipliers switched off. Checking (2.32), formula (5.16) tells us that the renormalized background field action $\hat{S}_R$ is equal to the classical background field action $\hat{S}_{\text{gf}}$ up to parameter redefinitions $\lambda \to \tau$ and a canonical transformation. This proves parametric completeness in the background field approach.

The canonical transformation $\Phi, K \to \hat{\Phi}, \hat{K}$ involved in formula (5.16) is generated by the functional $\hat{F}_R(\Phi, \underline{\phi}, \hat{K})$ obtained composing the transformations generated by $F_R(\Phi, \{\underline{\phi}, 0\}, K', 0, 1)$, $F_1(\check{\Phi}, K'')$, $F'_2(\hat{\Phi}, \check{K})$ and $F_3(\Phi', K'')$ of formulas (4.31), (5.8), (5.13) and (5.15) (at $\underline{C} = 0$). Working out the composition it is easy to prove that

$$\hat{\bar{C}} = \bar{C}, \qquad \hat{B} = B, \qquad \hat{K}_B = K_B, \qquad \hat{K}_{\bar{C}}^I - G^{Ii}(\underline{\phi}, \partial)\hat{\phi}^i = K_{\bar{C}}^I - G^{Ii}(\underline{\phi}, \partial)\phi^i,$$

and therefore $\hat{F}_R(\Phi, \underline{\phi}, \hat{K})$ has the form

$$\hat{F}_R(\Phi, \underline{\phi}, \hat{K}) = \int \Phi^\alpha \hat{K}_\alpha + \Delta\hat{F}(\phi, C, \underline{\phi}, \hat{K}_\phi^i - \hat{\bar{C}}^I G^{Ii}(\underline{\phi}, -\overleftarrow{\partial}), \hat{K}_C), \tag{5.17}$$

where $\Delta\hat{F} = \mathcal{O}(\hbar)$-poles.

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6 Examples

In this section we give two examples, non-Abelian gauge field theories and quantum gravity, which are also useful to familiarize oneself with the notation and the tools used in the paper. We switch to Minkowski spacetime. The dimensional-regularization technique is understood.

6.1 Yang-Mills theory

The first example is non-Abelian Yang-Mills theory with simple gauge group G and structure constants f^{abc}, coupled to fermions ψ^i in some representation described by anti-Hermitian matrices T^a_{ij}. The classical action $S_c(\phi)$ can be restricted by power counting, or enlarged to include all invariants of (5.11). The nonminimal non-gauge-fixed action S is the sum $\hat{S} + \bar{S}$ of (2.18) and (2.19). We find

$$\hat{S} = S_c(\phi + \underline{\phi}) + \int \left[g(\bar{\psi}^i + \underline{\bar{\psi}}^i) T^a_{ij} C^a K^j_\psi + g\bar{K}^i_\psi T^a_{ij} C^a (\psi^j + \underline{\psi}^j) \right]$$
$$- \int \left[(\underline{D}_\mu C^a + g f^{abc} A^b_\mu C^c) K^{\mu a} - \frac{1}{2} g f^{abc} C^b C^c K^a_C + B^a K^a_{\bar{C}} \right],$$

and

$$\bar{S} = g f^{abc} \int \underline{C}^b (A^c_\mu K^{\mu a} + C^c K^a_C + \bar{C}^c K^a_{\bar{C}} + B^c K^a_B) + g \int \left[\bar{\psi}^i T^a_{ij} \underline{C}^a K^j_\psi + \bar{K}^i_\psi T^a_{ij} \underline{C}^a \psi^j \right]$$
$$- \int \left[(\underline{D}_\mu \underline{C}^a) \underline{K}^{\mu a} - \frac{1}{2} g f^{abc} \underline{C}^b \underline{C}^c \underline{K}^a_C - g \underline{\bar{\psi}}^i T^a_{ij} \underline{C}^a \underline{K}^j_\psi - g \underline{\bar{K}}^i_\psi T^a_{ij} \underline{C}^a \underline{\psi}^j \right]. \tag{6.1}$$

The covariant derivative $\underline{D}_\mu$ is the background one; for example $\underline{D}_\mu \Lambda^a = \partial_\mu \Lambda^a + g f^{abc} \underline{A}^b_\mu \Lambda^c$. The first line of (6.1) shows that all quantum fields transform as matter fields under background transformations. It is easy to check that $\hat{S}$ and $\bar{S}$ satisfy $[\![\hat{S}, \hat{S}]\!] = [\![\hat{S}, \bar{S}]\!] = [\![\bar{S}, \bar{S}]\!] = 0$.

A common background-preserving gauge fermion is

$$\Psi = \int \bar{C}^a \left(-\frac{\lambda}{2} B^a + \underline{D}^\mu A^a_\mu \right), \tag{6.2}$$

and the gauge-fixed action $\hat{S}_{\text{gf}} = \hat{S} + [\![\hat{S}, \Psi]\!]$ reads

$$\hat{S}_{\text{gf}} = \hat{S} - \frac{\lambda}{2} \int (B^a)^2 + \int B^a \underline{D}^\mu A^a_\mu - \int \bar{C}^a \underline{D}^\mu (\underline{D}_\mu C^a + g f^{abc} A^b_\mu C^c).$$

Since the gauge fixing is linear in the quantum fields, the action $\hat{S}$ depends on the combination $K^a_\mu + \underline{D}_\mu \bar{C}^a$ and not on K^a_μ and $\bar{C}^a$ separately. From now on we switch matter fields off, for simplicity, and set $\lambda = 0$.

We describe renormalization using the approach of this paper. First we concentrate on the standard power-counting renormalizable case, where

$$S_c(A, g) = -\frac{1}{4} \int F_{\mu\nu}^a(A, g) F^{\mu\nu a}(A, g), \qquad F_{\mu\nu}^a(A, g) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c.$$

The key formula (5.16) gives

$$\hat{S}_R(\Phi, \underline{A}, K) = -\frac{Z}{4} \int F_{\mu\nu}^a(\hat{A} + \underline{A}, g) F^{\mu\nu a}(\hat{A} + \underline{A}, g) + \int \hat{B}^a \underline{D}^\mu \hat{A}_\mu^a - \int \hat{B}^a \hat{K}_{\bar{C}}^a$$
$$+ \int (\hat{K}^{\mu a} + \underline{D}^\mu \hat{\bar{C}}^a)(\underline{D}_\mu \hat{C}^a + g f^{abc} \hat{A}_\mu^b \hat{C}^c) + \frac{1}{2} g f^{abc} \int \hat{C}^b \hat{C}^c \hat{K}_C^a, \qquad (6.3)$$

where Z is a renormalization constant. The most general canonical transformation $\Phi, K \to \hat{\Phi}, \hat{K}$ that is compatible with power counting, global gauge invariance and ghost number conservation can be easily written down. Introducing unknown constants where necessary, we find that its generating functional has the form

$$\hat{F}_R(\Phi, \underline{A}, \hat{K}) = \int (Z_A^{1/2} A_\mu^a + \underline{Z}_A^{1/2} \underline{A}_\mu^a) \hat{K}^{\mu a} + \int Z_C^{1/2} C^a \hat{K}_C^a + \int Z_{\bar{C}}^{1/2} \bar{C}^a \hat{K}_{\bar{C}}^a$$
$$+ \int (Z_B^{1/2} B^a + \alpha \underline{D}^\mu A_\mu^a + \beta \partial^\mu \underline{A}_\mu^a + \gamma g f^{abc} \underline{A}^{\mu b} A_\mu^c + \delta g f^{abc} \bar{C}^b C^c) \hat{K}_B^a$$
$$+ \int \bar{C}^a (\zeta B^a + \xi \underline{D}^\mu A_\mu^a + \eta \partial^\mu \underline{A}_\mu^a + \theta g f^{abc} \underline{A}^{\mu b} A_\mu^c + \chi g f^{abc} \bar{C}^b C^c)$$
$$+ \int \sigma \hat{K}_{\bar{C}}^a \hat{K}_B^a + \int \tau g f^{abc} C^a \hat{K}_B^b \hat{K}_B^c.$$

Inserting it in (6.3) and using the nonrenormalization of the B and $K_{\bar{C}}$-dependent terms, we find $a = \beta = \gamma = \delta = \zeta = \theta = \chi = \sigma = \tau = 0$ and

$$\xi = 1 - Z_{\bar{C}}^{1/2} Z_A^{1/2}, \qquad \eta = -Z_{\bar{C}}^{1/2} \underline{Z}_A^{1/2}, \qquad Z_B = Z_{\bar{C}}. \qquad (6.4)$$

It is easy to check that $Z_{\bar{C}}$ disappears from the right-hand side of (6.3), so we can set $Z_{\bar{C}} = 1$. Furthermore, we know that $\hat{S}_R(\Phi, \underline{A}, K)$ is invariant under background transformations ($[\![\hat{S}_R, \bar{S}]\!] = 0$), which requires $\underline{Z}_A = 0$. Finally, the canonical transformation just reads

$$\hat{F}_R(\Phi, \underline{A}, \hat{K}) = \int Z_A^{1/2} A_\mu^a \hat{K}^{\mu a} + \int Z_C^{1/2} C^a \hat{K}_C^a + \int \bar{C}^a \hat{K}_{\bar{C}}^a + \int B^a \hat{K}_B^a + (1 - Z_A^{1/2}) \int \bar{C}^a (\underline{D}^\mu A_\mu^a),$$

which contains the right number of independent renormalization constants and is of the form (5.17).

Defining $Z_g = Z^{-1/2}$ and $Z_A' = Z Z_A$ we can describe renormalization in a more standard way. Writing

$$\hat{S}_R(0, \hat{A} + \underline{A}, 0) = -\frac{1}{4} \int F_{\mu\nu}^a(Z_A'^{1/2} A + Z_g^{-1} \underline{A}, g Z_g) F^{\mu\nu a}(Z_A'^{1/2} A + Z_g^{-1} \underline{A}, g Z_g),$$

we see that Z_g is the usual gauge-coupling renormalization constant, while Z'_A and Z_g^{-2} are the wave-function renormalization constants of the quantum gauge field A and the background gauge field $\underline{A}$, respectively. We remark that the local divergent canonical transformation $\Phi, K \to \hat{\Phi}, \hat{K}$ corresponds to a highly nontrivial, convergent but non-local canonical transformation at the level of the Γ functional.

If the theory is not power-counting renormalizable, then we need to consider the most general classical action, equal to the right-hand side of (5.11). Counterterms include vertices with arbitrary numbers of external Φ and K legs. Nevertheless, the key formula (5.16) ensures that the renormalized action $\hat{S}_R$ remains exactly the same, up to parameter redefinitions and a canonical transformation. The only difference is that now even the canonical transformation $\hat{F}_R(\Phi, \underline{\phi}, \hat{K})$ of (5.17) becomes nonpolynomial and highly nontrivial.

6.2 Quantum gravity

Having written detailed formulas for Yang-Mills theory, in the case of quantum gravity we can just outline the key ingredients. In particular, we stress that the linearity assumption is satisfied both in the first-order and second-order formalisms, both using the metric $g_{\mu\nu}$ and the vielbein e_μ^a. For example, using the second-order formalism and the vielbein, the symmetry transformations are encoded in the expressions

$$
-\int R^\alpha(\Phi)K_\alpha = \int (e_\rho^a \partial_\mu C^\rho + C^\rho \partial_\rho e_\mu^a + C^{ab}e_{\mu b})K_a^\mu + \int C^\rho(\partial_\rho C^\mu)K_\mu^C
$$
$$
+ \int (C^{ac}\eta_{cd}C^{db} + C^\rho \partial_\rho C^{ab})K_{ab}^C - \int B_\mu K_{\bar{C}}^\mu - \int B_{ab}K_{\bar{C}}^{ab},
$$
$$
-\int \bar{R}^\alpha(\Phi)K_\alpha = -\int R^\alpha(\Phi)K_\alpha - \int (\bar{C}_\rho \partial_\mu C^\rho - C^\rho \partial_\rho \bar{C}_\mu)K_{\bar{C}}^\mu + \int (B_\rho \partial_\mu C^\rho + C^\rho \partial_\rho B_\mu) K_B^\mu,
$$

in the minimal and nonminimal cases, respectively, where C^μ are the ghosts of diffeomorphisms, C^{ab} are the Lorentz ghosts and η_{ab} is the flat-space metric. We see that both $R^\alpha(\Phi)$ and $\bar{R}^\alpha(\Phi)$ are at most quadratic in Φ. Matter fields are also fine, since vectors A_μ, fermions ψ and scalars φ contribute with

$$
-\int (\partial_\mu C^a + gf^{abc}A_\mu^b C^c - C^\rho \partial_\rho A_\mu^a - A_\rho^a \partial_\mu C^\rho)K_A^{\mu a} + \int \left(C^\rho \partial_\mu C^a + \frac{1}{2}gf^{abc}C^b C^c\right)K_C^a
$$
$$
+ \int C^\rho(\partial_\rho\varphi)K_\varphi + \int C^\rho(\partial_\rho\bar{\psi})K_\psi - \frac{i}{4}\int \bar{\psi}\sigma^{ab}C_{ab}K_\psi + \int K_{\bar{\psi}}C^\rho(\partial_\rho\psi) - \frac{i}{4}\int K_{\bar{\psi}}\sigma^{ab}C_{ab}\psi,
$$

where $\sigma^{ab} = i[\gamma^a, \gamma^b]/2$. Expanding around flat space, common linear gauge-fixing conditions for diffeomorphisms and local Lorentz symmetry are $\eta^{\mu\nu}\partial_\mu e_\nu^a = \xi\eta^{a\mu}\partial_\mu e_\nu^b \delta_b^\nu$, $e_\mu^a = e_\nu^b \eta_{b\mu}\eta^{\nu a}$, respectively.

In the first-order formalism we just need to add the transformation of the spin connection ω_μ^{ab}, encoded in

$$\int (C^\rho \partial_\rho \omega_\mu^{ab} + \omega_\rho^{ab} \partial_\mu C^\rho - \partial_\mu C^{ab} + C^{ac} \eta_{cd} \omega_\mu^{db} - \omega_\mu^{ac} \eta_{cd} C^{db}) K_{ab}^\mu.$$

Moreover, in this case we can also gauge-fix local Lorentz symmetry with the linear gauge-fixing condition $\eta^{\mu\nu} \partial_\mu \omega_\nu^{ab} = 0$, instead of $e_\mu^a = e_\nu^b \eta_{b\mu} \eta^{\nu a}$.

We see that all gauge symmetries that are known to have physical interest satisfy the linearity assumption, together with irreducibility and off-shell closure. On the other hand, more speculative symmetries (such as local supersymmetry) do not satisfy those assumptions in an obvious way. When auxiliary fields are introduced to achieve off-shell closure, some symmetry transformations (typically, those of auxiliary fields) are nonlinear [19]. The relevance of this issue is already known in the literature. For example, in ref. [20] it is explained that in supersymmetric theories the standard background field method cannot be applied, precisely because the symmetry transformations are nonlinear. It is argued that the linearity assumption is tied to the linear splitting $\Phi \to \Phi + \underline{\Phi}$ between quantum fields Φ and background fields $\underline{\Phi}$, and that the problem of supersymmetric theories can be avoided with a nonlinear splitting of the form $\Phi \to \Phi + \underline{\Phi} +$ nonlinear corrections. Perhaps it is possible to generalize the results of this paper to supergravity following those guidelines. From our viewpoint, the crucial property is that the background transformations of the quantum fields are linear in the quantum fields themselves, because then they do not renormalize.

An alternative strategy, not bound to supersymmetric theories, is that of introducing (possibly infinitely many) auxiliary fields, replacing every nonlinear term appearing in the symmetry transformations with a new field N, and then proceeding similarly with the N transformations and the closure relations, till all functions R^α are at most quadratic. The natural framework for this kind of job is the one of refs. [15, 16, 17], where the fields N are dual to the sources L coupled to composite fields. Using that approach the Batalin-Vilkovisky formalism can be extended to the composite-field sector of the theory and all perturbative canonical transformations can be studied as true changes of field variables in the functional integral, instead of mere replacements of integrands. For reasons of space, though, we cannot pursue this strategy here.

7   The quest for parametric completeness: Where we stand now

In this section we make remarks about the problem of parametric completeness in general gauge theories and recapitulate where we stand now on this issue. To begin with, consider non-Abelian Yang-Mills theory as a deformation of its Abelian limit. The minimal solution $S(g)$ of the master equation $(S(g), S(g)) = 0$ reads

$$S(g) = -\frac{1}{4} \int F_{\mu\nu}^a F^{\mu\nu a} + \int K^{\mu a} \partial_\mu C^a + g f^{abc} \int \left(K^{\mu a} A_\mu^b + \frac{1}{2} K_C^a C^b \right) C^c.$$

Differentiating the master equation with respect to g and setting $g = 0$, we find

$$(S, S) = 0, \qquad (S, \omega) = 0, \qquad S = S(0), \qquad \omega = \left. \frac{\mathrm{d}S(g)}{\mathrm{d}g} \right|_{g=0}.$$

On the other hand, we can easily prove that there exists no local functional χ such that $\omega = (S, \chi)$. Thus, we can say that ω is a nontrivial solution of the cohomological problem associated with an Abelian Yang-Mills theory that contains a suitable number of photons [4]. Nevertheless, renormalization cannot turn ω on as a counterterm, because $S(0)$ is a free field theory. Even if we couple the theory to gravity and assume that massive fermions are present (which allows us to construct dimensionless parameters multiplying masses with the Newton constant), radiative corrections cannot dynamically "un-Abelian-ize" the theory, namely convert an Abelian theory into a non-Abelian one. One way to prove this fact is to note that the dependence on gauge fields is even at $g = 0$, but not at $g \neq 0$. The point is, however, that cohomology *per se* is unable to prove it. Other properties must be advocated, such as the discrete symmetry just mentioned. In general, we cannot rely on cohomology only, and the possibility that gauge symmetries may be dynamically deformed in nontrivial and observable ways remains open.

In ref. [4] the issue of parametric completeness was studied in general terms. In that approach, which applies to all theories that are manifestly free of gauge anomalies, renormalization triggers an automatic parametric extension till the classical action becomes parametrically complete. The results of ref. [4] leave the door open to dynamically induced nontrivial deformations of the gauge symmetry. Instead, the results found here close that door in all cases where they apply, which means manifestly nonanomalous irreducible gauge symmetries that close off shell and satisfy the linearity assumption. The reason is – we stress it again – that by formulas (5.12) and (5.16) all dynamically induced deformations can be organized into parameter redefinitions and canonical transformations. As far as we know now, gauge symmetries can still be dynamically deformed in observable ways in theories that do not satisfy the assumptions of this paper. Supergravities are natural candidates to provide explicit examples.

8 Conclusions

The background field method and the Batalin-Vilkovisky formalism are convenient tools to quantize general gauge field theories. In this paper we have merged the two to rephrase and generalize known results about renormalization, and to study parametric completeness. Our approach applies when gauge anomalies are manifestly absent, the gauge algebra is irreducible and closes off shell, the gauge transformations are linear functions of the fields, and closure is field independent. These assumptions are sufficient to include the gauge symmetries we need for physical applications, such as Abelian and non-Abelian Yang-Mills symmetries, local Lorentz symmetry and

general changes of coordinates, but exclude other potentially interesting symmetries, such as local supersymmetry. Both renormalizable and nonrenormalizable theories are covered, such as QED, non-Abelian Yang-Mills theories, quantum gravity and Lorentz-violating gauge theories, as well as effective and higher-derivative models, in arbitrary dimensions, and also extensions obtained adding any set of composite fields. At the same time, chiral theories, and therefore the Standard Model, possibly coupled with quantum gravity, require the analysis of anomaly cancellation and the Adler-Bardeen theorem, which we postpone to a future investigation. The fact that supergravities are left out from the start, on the other hand, suggests that there should exist either a no-go theorem or a more advanced framework. At any rate, we are convinced that our formalism is helpful to understand several properties better and address unsolved problems.

We have studied gauge dependence in detail, and renormalized the canonical transformation that continuously interpolates between the background field approach and the usual approach. Relating the two approaches, we have proved parametric completeness without making use of cohomological classifications. The outcome is that in all theories that satisfy our assumptions renormalization cannot hide any surprises; namely the gauge symmetry remains essentially the same throughout the quantization process. In the theories that do not satisfy our assumptions, instead, the gauge symmetry could be dynamically deformed in physically observable ways. It would be remarkable if we discovered explicit examples of theories where this sort of "dynamical creation" of gauge symmetries actually takes place.

Acknowledgments

A   Appendix

In this appendix we prove several theorems and identities that are used in the paper. We use the Euclidean notation and the dimensional-regularization technique, which guarantees, in particular, that the functional integration measure is invariant under perturbatively local changes of field variables. The generating functionals Z and W are defined from

$$Z(J, K, \underline{\Phi}, \underline{K}) = \int [\mathrm{d}\Phi] \exp(-S(\Phi, \underline{\Phi}, K, \underline{K}) + \int \Phi^\alpha J_\alpha) = \exp W(J, K, \underline{\Phi}, \underline{K}), \qquad (A.1)$$

and $\Gamma(\Phi, \underline{\Phi}, K, \underline{K}) = -W + \int \Phi^\alpha J_\alpha$ is the W Legendre transform. Averages denote the sums of connected diagrams (e.g. $\langle A(x)B(y) \rangle = \langle A(x)B(y) \rangle_{\mathrm{nc}} - \langle A(x) \rangle \langle B(y) \rangle$, where $\langle A(x)B(y) \rangle_{\mathrm{nc}}$ includes disconnected diagrams). Moreover, the average $\langle X \rangle$ of a local functional X can be viewed as a functional of $\Phi, \underline{\Phi}, K, \underline{K}$ (in which case it collects one-particle irreducible diagrams) or a functional of $J, \underline{\Phi}, K, \underline{K}$. When we need to distinguish the two options we specify whether

Φ or J are kept constant in functional derivatives. First we work in the usual (non-background field) framework; then we generalize the results to the background field method.

To begin with, we recall a property that is true even when the action $S(\Phi, K)$ does not satisfy the master equation.

Theorem 2 *The identity* $(\Gamma, \Gamma) = \langle (S, S) \rangle$ *holds.*

Proof. Applying the change of field variables

$$\Phi^\alpha \to \Phi^\alpha + \theta(S, \Phi^\alpha) \tag{A.2}$$

to (A.1), where θ is a constant anticommuting parameter, we obtain

$$\theta \int \left\langle \frac{\delta_r S}{\delta K_\alpha} \frac{\delta_l S}{\delta \Phi^\alpha} \right\rangle - \theta \int \left\langle \frac{\delta_r S}{\delta K_\alpha} \right\rangle J_\alpha = 0,$$

whence

$$\frac{1}{2} \langle (S, S) \rangle = - \int \left\langle \frac{\delta_r S}{\delta K_\alpha} \frac{\delta_l S}{\delta \Phi^\alpha} \right\rangle = - \int \left\langle \frac{\delta_r S}{\delta K_\alpha} \right\rangle J_\alpha = \int \frac{\delta_r W}{\delta K_\alpha} \frac{\delta_l \Gamma}{\delta \Phi^\alpha} = - \int \frac{\delta_r \Gamma}{\delta K_\alpha} \frac{\delta_l \Gamma}{\delta \Phi^\alpha} = \frac{1}{2} (\Gamma, \Gamma).$$

Now we prove results for an action S that satisfies the master equation $(S, S) = 0$.

Theorem 3 *If* $(S, S) = 0$ *then* $(\Gamma, \langle X \rangle) = \langle (S, X) \rangle$ *for every local functional X.*

Proof. Applying the change of field variables (A.2) to

$$\langle X \rangle = \frac{1}{Z(J, K)} \int [\mathrm{d}\Phi]\, X \exp(-S + \int \Phi^\alpha J_\alpha),$$

and using $(S, S) = 0$ we obtain

$$\int \left\langle \frac{\delta_r S}{\delta K_\alpha} \frac{\delta_l X}{\delta \Phi^\alpha} \right\rangle = (-1)^{\varepsilon_X + 1} \int \left\langle X \frac{\delta_r S}{\delta K_\alpha} \right\rangle \frac{\delta_l \Gamma}{\delta \Phi^\alpha}, \tag{A.3}$$

where ε_X denotes the statistics of the functional X (equal to 0 if X is bosonic, 1 if it is fermionic, modulo 2). Moreover, we also have

$$\int \left\langle \frac{\delta_r S}{\delta \Phi^\alpha} \frac{\delta_l X}{\delta K_\alpha} \right\rangle = \int \frac{\delta_r \Gamma}{\delta \Phi^\alpha} \left\langle \frac{\delta_l X}{\delta K_\alpha} \right\rangle, \tag{A.4}$$

which can be proved starting from the expression on the left-hand side and integrating by parts. In the derivation we use the fact that since X is local, $\delta_r \delta_l X / (\delta \Phi^\alpha \delta K_\alpha)$ is set to zero by the dimensional regularization, which kills the $\delta(0)$s and their derivatives.

Next, straightforward differentiations give

$$\left.\frac{\delta_l \langle X \rangle}{\delta K_\alpha}\right|_J = \left\langle \frac{\delta_l X}{\delta K_\alpha} \right\rangle - \left\langle \frac{\delta_l S}{\delta K_\alpha} X \right\rangle \tag{A.5}$$

$$= \left.\frac{\delta_l \langle X \rangle}{\delta K_\alpha}\right|_\Phi - \int \left.\frac{\delta_l J_\beta}{\delta K_\alpha}\right|_\Phi \left.\frac{\delta_l \langle X \rangle}{\delta J_\beta}\right|_K . \tag{A.6}$$

At this point, using (A.3)-(A.6) and $(J_\alpha, \Gamma) = 0$ (which can be proved by differentiating $(\Gamma, \Gamma) = 0$ with respect to Φ^α), we derive $(\Gamma, \langle X \rangle) = \langle (S, X) \rangle$.

Corollary 4 *If $(S, S) = 0$ and*

$$\frac{\partial S}{\partial \xi} = (S, X), \tag{A.7}$$

where X is a local functional and ξ is a parameter, then

$$\frac{\partial \Gamma}{\partial \xi} = (\Gamma, \langle X \rangle). \tag{A.8}$$

Proof. Using theorem 3 we have

$$\frac{\partial \Gamma}{\partial \xi} = -\frac{\partial W}{\partial \xi} = \langle \frac{\partial S}{\partial \xi} \rangle = \langle (S, X) \rangle = (\Gamma, \langle X \rangle).$$

Now we derive results that hold even when S does not satisfy the master equation.

Theorem 5 *The identity*

$$(\Gamma, \langle X \rangle) = \langle (S, X) \rangle - \frac{1}{2} \langle (S, S) X \rangle_\Gamma \tag{A.9}$$

holds, where X is a generic local functional and $\langle AB \cdots Z \rangle_\Gamma$ denotes the set of connected, one-particle irreducible diagrams with one insertion of A, B, ... Z.

This theorem is a generalization of theorem 3. It is proved by repeating the derivation without using $(S, S) = 0$. First, observe that formula (A.3) generalizes to

$$\int \left\langle \frac{\delta_r S}{\delta K_\alpha} \frac{\delta_l X}{\delta \Phi^\alpha} \right\rangle = (-1)^{\varepsilon_X + 1} \int \left\langle X \frac{\delta_r S}{\delta K_\alpha} \right\rangle \frac{\delta_l \Gamma}{\delta \Phi^\alpha} - \frac{1}{2} \langle (S, S) X \rangle. \tag{A.10}$$

On the other hand, formula (A.4) remains the same, as well as (A.5) and (A.6). We have

$$(\Gamma, \langle X \rangle) = \langle (S, X) \rangle - \frac{1}{2} \langle (S, S) X \rangle + \int \frac{\delta_r \Gamma}{\delta \Phi^\alpha} \left.\frac{\delta_l J_\beta}{\delta K_\alpha}\right|_\Phi \left.\frac{\delta_l \langle X \rangle}{\delta J_\beta}\right|_K - \int \frac{\delta_r \Gamma}{\delta K_\alpha} \left.\frac{\delta_l \langle X \rangle}{\delta \Phi^\alpha}\right|_K .$$

Differentiating (Γ, Γ) with respect to Φ^α we get

$$\frac{1}{2} \frac{\delta_r (\Gamma, \Gamma)}{\delta \Phi^\alpha} = \frac{1}{2} \frac{\delta_l (\Gamma, \Gamma)}{\delta \Phi^\alpha} = (J_\alpha, \Gamma) = (-1)^{\varepsilon_\alpha} (\Gamma, J_\alpha),$$

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where ε_α is the statistics of Φ^α. Using $(\Gamma, \Gamma) = \langle(S, S)\rangle$ we finally obtain

$$(\Gamma, \langle X\rangle) = \langle(S, X)\rangle - \frac{1}{2}\langle(S, S)X\rangle + \frac{1}{2}\int (-1)^{\varepsilon_\alpha} \frac{\delta_r \langle(S, S)\rangle}{\delta\Phi^\alpha} \left.\frac{\delta_l \langle X\rangle}{\delta J_\alpha}\right|_K . \tag{A.11}$$

The set of irreducible diagrams contained in $\langle AB\rangle$, where A and B are local functionals, is given by the formula

$$\langle AB\rangle_\Gamma = \langle AB\rangle - \{\langle A\rangle, \langle B\rangle\}, \tag{A.12}$$

where $\{X, Y\}$ are the "mixed brackets" [22]

$$\{X, Y\} \equiv \int \frac{\delta_r X}{\delta\Phi^\alpha} \langle \Phi^\alpha \Phi^\beta\rangle \frac{\delta_l Y}{\delta\Phi^\beta} = \int \frac{\delta_r X}{\delta\Phi^\alpha} \frac{\delta_r \delta_r W}{\delta J_\beta \delta J_\alpha} \frac{\delta_l Y}{\delta\Phi^\beta} = \int \left.\frac{\delta_r X}{\delta J_\alpha}\right|_K \frac{\delta_l Y}{\delta\Phi^\alpha}, \tag{A.13}$$

X and Y being functionals of Φ and K. Indeed, $\{\langle A\rangle, \langle B\rangle\}$ is precisely the set of diagrams in which the A and B insertions are connected in a one-particle reducible way. Thus, formula (A.11) coincides with (A.9).

Using (A.9) we also have the identity

$$\frac{\partial\Gamma}{\partial\xi} - (\Gamma, \langle X\rangle) = \left\langle \frac{\partial S}{\partial\xi} - (S, X)\right\rangle + \frac{1}{2}\langle(S, S)\, X\rangle_\Gamma, \tag{A.14}$$

which generalizes corollary 4.

Now we switch to the background field method. We begin by generalizing theorem 2.

Theorem 6 *If the action $S(\Phi, \underline{\Phi}, K, \underline{K})$ is such that $\delta_l S/\delta\underline{K}_\alpha$ is Φ independent, the identity*

$$[\![\Gamma, \Gamma]\!] = \langle[\![S, S]\!]\rangle$$

holds.

Proof. Since $\delta_l S/\delta\underline{K}_\alpha$ is Φ independent we have $\delta_l\Gamma/\delta\underline{K}_\alpha = \delta_l S/\delta\underline{K}_\alpha$. Using theorem 2 we find

$$[\![\Gamma, \Gamma]\!] = (\Gamma, \Gamma) + 2\int \frac{\delta_r\Gamma}{\delta\underline{\Phi}^\alpha} \frac{\delta_l\Gamma}{\delta\underline{K}_\alpha} = \langle(S, S)\rangle + 2\int \langle\frac{\delta_r S}{\delta\underline{\Phi}^\alpha}\rangle\frac{\delta_l S}{\delta\underline{K}_\alpha} = \langle(S, S)\rangle + 2\int \langle\frac{\delta_r S}{\delta\underline{\Phi}^\alpha}\frac{\delta_l S}{\delta\underline{K}_\alpha}\rangle = \langle[\![S, S]\!]\rangle.$$

Next, we mention the useful identity

$$\left.\frac{\delta_l\langle X\rangle}{\delta\underline{\Phi}^\alpha}\right|_\Phi = \left\langle\frac{\delta_l X}{\delta\underline{\Phi}^\alpha}\right\rangle - \left\langle\frac{\delta_l S}{\delta\underline{\Phi}^\alpha}X\right\rangle_\Gamma, \tag{A.15}$$

which holds for every local functional X. It can be proved by taking (A.5)–(A.6) with $K \to \underline{\Phi}$ and using (A.12)–(A.13).

Mimicking the proof of theorem 6 and using (A.15), it is easy to prove that theorem 5 implies the identity

$$[\![\Gamma, \langle X\rangle]\!] = \langle[\![S, X]\!]\rangle - \frac{1}{2}\langle[\![S, S]\!]X\rangle_\Gamma, \tag{A.16}$$

for every $\underline{K}$-independent local functional X. Thus we have the following property.

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Corollary 7 *The identity*

$$\frac{\partial \Gamma}{\partial \xi} - [\![\Gamma, \langle X \rangle]\!] = \left\langle \frac{\partial S}{\partial \xi} - [\![S, X]\!] \right\rangle + \frac{1}{2} \langle [\![S, S]\!] X \rangle_\Gamma \tag{A.17}$$

holds for every action S such that $\delta_l S / \delta \underline{K}_\alpha$ is Φ independent, for every $\underline{K}$-independent local functional X and for every parameter ξ.

If the action S has the form (2.26) and X is also $\underline{C}$ independent, applying (2.17) to (A.16) we obtain

$$[\![\hat{\Gamma}, \langle X \rangle]\!] = \langle [\![\hat{S}, X]\!] \rangle - \frac{1}{2} \langle [\![\hat{S}, \hat{S}]\!] X \rangle_\Gamma, \qquad [\![\bar{S}, \langle X \rangle]\!] = \langle [\![\bar{S}, X]\!] \rangle - \langle [\![\bar{S}, \hat{S}]\!] X \rangle_\Gamma, \qquad \langle [\![\bar{S}, \bar{S}]\!] X \rangle_\Gamma = 0, \tag{A.18}$$

which imply the following statement.

Corollary 8 *If S satisfies the assumptions of (2.26), $[\![\bar{S}, X]\!] = 0$ and $[\![\bar{S}, \hat{S}]\!] = 0$, where X is a $\underline{C}$- and $\underline{K}$-independent local functional, then $[\![\bar{S}, \langle X \rangle]\!] = 0$.*

Finally, we recall a result derived in ref. [23].

Theorem 9 *If $\Phi, K \to \Phi', K'$ is a canonical transformation generated by $F(\Phi, K')$, and $\chi(\Phi, K)$ is a functional behaving as a scalar (that is to say $\chi'(\Phi', K') = \chi(\Phi, K)$), then*

$$\frac{\partial \chi'}{\partial \varsigma} = \frac{\partial \chi}{\partial \varsigma} - (\chi, \tilde{F}_\varsigma) \tag{A.19}$$

for every parameter ς, where $\tilde{F}_\varsigma(\Phi, K) \equiv F_\varsigma(\Phi, K'(\Phi, K))$ and $F_\varsigma(\Phi, K') = \partial F / \partial \varsigma$.

Proof. When we do not specify the variables that are kept constant in partial derivatives, it is understood that they are the natural variables. Thus F, Φ' and K are functions of Φ, K', while χ and $\tilde{F}_\varsigma$ are functions of Φ, K and χ' is a function of Φ', K'. It is useful to write down the differentials of Φ' and K, which are [24]

$$d\Phi'^\alpha = \int \frac{\delta_l \delta F}{\delta K'_\alpha \delta \Phi^\beta} d\Phi^\beta + \int \frac{\delta_l \delta F}{\delta K'_\alpha \delta K'_\beta} dK'_\beta + \frac{\partial \Phi'^\alpha}{\partial \varsigma} d\varsigma,$$

$$dK_\alpha = \int d\Phi^\beta \frac{\delta_l \delta F}{\delta \Phi^\beta \delta \Phi^\alpha} + \int dK'_\beta \frac{\delta_l \delta F}{\delta K'_\beta \delta \Phi^\alpha} + \frac{\partial K_\alpha}{\partial \varsigma} d\varsigma. \tag{A.20}$$

Differentiating $\chi'(\Phi', K') = \chi(\Phi, K)$ with respect to ς at constant Φ' and K', we get

$$\frac{\partial \chi'}{\partial \varsigma} = \frac{\partial \chi}{\partial \varsigma} + \int \frac{\delta_r \chi}{\delta \Phi^\alpha} \left. \frac{\partial \Phi^\alpha}{\partial \varsigma} \right|_{\Phi', K'} + \int \frac{\delta_r \chi}{\delta K_\alpha} \left. \frac{\partial K_\alpha}{\partial \varsigma} \right|_{\Phi', K'}. \tag{A.21}$$

Formulas (A.20) allow us to write

$$\frac{\partial \Phi'^{\alpha}}{\partial \varsigma} = -\int \frac{\delta_l \delta F}{\delta K'_{\alpha} \delta \Phi^{\beta}} \left. \frac{\partial \Phi^{\beta}}{\partial \varsigma} \right|_{\Phi',K'}, \qquad \frac{\delta_l \delta F}{\delta K'_{\alpha} \delta \Phi^{\beta}} = \left. \frac{\delta_l K_{\beta}}{\delta K'_{\alpha}} \right|_{\Phi,\varsigma},$$

and therefore we have

$$\frac{\delta \tilde{F}_{\varsigma}}{\delta K_{\alpha}} = \int \left. \frac{\delta_l K'_{\beta}}{\delta K_{\alpha}} \right|_{\Phi,\varsigma} \frac{\partial \Phi'^{\beta}}{\partial \varsigma} = -\int \left. \frac{\delta_l K'_{\beta}}{\delta K_{\alpha}} \right|_{\Phi,\varsigma} \left. \frac{\delta_l K_{\gamma}}{\delta K'_{\beta}} \right|_{\Phi,\varsigma} \left. \frac{\partial \Phi^{\gamma}}{\partial \varsigma} \right|_{\Phi',K'} = -\left. \frac{\partial \Phi^{\alpha}}{\partial \varsigma} \right|_{\Phi',K'}. \qquad (A.22)$$

Following analogous steps, we also find

$$\frac{\delta \tilde{F}_{\varsigma}}{\delta \Phi^{\alpha}} = \frac{\partial K_{\alpha}}{\partial \varsigma} + \int \left. \frac{\delta_l K'_{\beta}}{\delta \Phi^{\alpha}} \right|_{K,\varsigma} \frac{\partial \Phi'^{\beta}}{\partial \varsigma}, \qquad \frac{\partial K_{\alpha}}{\partial \varsigma} = \left. \frac{\partial K_{\alpha}}{\partial \varsigma} \right|_{\Phi',K'} - \int \frac{\delta_l \delta F}{\delta \Phi^{\alpha} \delta \Phi^{\beta}} \left. \frac{\partial \Phi^{\beta}}{\partial \varsigma} \right|_{\Phi',K'},$$

whence

$$\left. \frac{\partial K_{\alpha}}{\partial \varsigma} \right|_{\Phi',K'} = \frac{\delta \tilde{F}_{\varsigma}}{\delta \Phi^{\alpha}} + \int \left(\frac{\delta_l K_{\gamma}}{\delta \Phi^{\alpha}} + \left. \frac{\delta_l K'_{\beta}}{\delta \Phi^{\alpha}} \right|_{K,\varsigma} \frac{\delta_l K_{\gamma}}{\delta K'_{\beta}} \right) \left. \frac{\partial \Phi^{\gamma}}{\partial \varsigma} \right|_{\Phi',K'} = \frac{\delta \tilde{F}_{\varsigma}}{\delta \Phi^{\alpha}}. \qquad (A.23)$$

This formula, together with (A.22), allows us to rewrite (A.21) in the form (A.19).

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