IFUP-TH 2013/07

Properties Of The Classical Action Of Quantum Gravity

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13A2 Renorm

Abstract

The classical action of quantum gravity, determined by renormalization, contains infinitely many independent couplings and can be expressed in different perturbatively equivalent ways. We organize it in a convenient form, which is based on invariants constructed with the Weyl tensor. We show that the FLRW metrics are exact solutions of the field equations in arbitrary dimensions, and so are all locally conformally flat solutions of the Einstein equations. Moreover, expanding the metric tensor around locally conformally flat backgrounds the quadratic part of the action is free of higher derivatives. Black-hole solutions of Schwarzschild and Kerr type are modified in a non-trivial way. We work out the first corrections to their metrics and study their properties.

1 Introduction

Having control on the classical action of quantum gravity, its properties and the solutions of its field equations can be useful to address the search for detectable effects that may single out some significant departure from Einstein gravity. Since quantum gravity is not power-counting renormalizable, its classical action contains infinitely many independent couplings. Nevertheless, some interesting solutions of the field equations may depend only on a finite subset of parameters and allow us to make physical predictions. Moreover, even if each correction is small, the presence of infinitely many of them opens the door to effects that might be detectable in particular experimental arrangements or astrophysical observations, situated beyond the domains tested so far and before radiative corrections become important. Finally, in extreme situations, such as inside black holes, or close to the event horizon, or in the primordial phases of the universe, classical corrections of quantum origin may play a relevant role.

Using field redefinitions, the classical action of quantum gravity can be written in different, perturbatively equivalent expansions around the Einstein action. In particular, we can rearrange the terms proportional to the Einstein vacuum field equations. Equivalent actions can be useful to uncover different classes of exact solutions of the field equations, or reduce the effort to study approximate solutions.

In this paper we single out a form S_{QG} that we deem convenient for several purposes. Besides the Hilbert term, the cosmological term and a peculiar combination that is non-trivial in higher dimensions, the action S_{QG} contains invariants constructed with the Weyl tensor $C_{\mu\nu\rho\sigma}$, rather than the Riemann tensor $R_{\mu\nu\rho\sigma}$. Precisely, we write, in arbitrary dimensions $d > 2$,

$$S_{\mathrm{QG}} = -\frac{1}{2\kappa^{d-2}} \int \sqrt{-g} \left(R + 2\Lambda + \lambda_0 \kappa^2 \hat{\mathrm{G}} + \lambda_1 \kappa^4 C_3 + \lambda_1' \kappa^4 C_3' + \sum_{n=2}^{\infty} \lambda_n \kappa^{2n+2} \mathcal{J}_n(\nabla, C) \right) + S_m, \tag{1.1}$$

where κ has dimension -1 in units of mass, λ_n are dimensionless constants, S_m are the contributions of matter fields and other gauge fields, $\hat{\mathrm{G}}$ is the special combination [1]

$$\hat{\mathrm{G}} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 + \frac{4(d-3)(d-4)}{(d-1)(d-2)} \Lambda \left(R + \Lambda \right),$$

and $\mathcal{J}_n(\nabla, C)$ collectively denotes the local scalars of dimension $2n+4$ that can be constructed with three or more Weyl tensors $C_{\mu\nu\rho\sigma}$ and covariant derivatives ∇_μ, up to covariant divergences of vectors. Each such scalar must be multiplied by an independent λ_n. For future use, we explicitly write the terms $\mathcal{J}_1(\nabla, C)$, which are two contractions of three Weyl tensors:

$$C_3 = C_{\mu\nu\rho\sigma} C^{\rho\sigma\alpha\beta} C_{\alpha\beta}{}^{\mu\nu}, \qquad C_3' = C_{\mu\rho\nu\sigma} C^{\alpha\mu\beta\nu} C^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta}.$$

For simplicity in this paper we assume parity invariance. Parity-violating terms may be treated along the same guidelines.

In four dimensions $\sqrt{-g}\hat{\mathrm{G}}$ is the Gauss-Bonnet integrand, which does not contribute to the field equations. Thanks to this fact, quantum gravity is finite at one loop in the absence of matter [2]. Moreover, C_3 and C_3' are proportional to each other, so we can set $\lambda_1' = 0$ in $d = 4$. Goroff and Sagnotti showed [3] that C_3 is switched on as a two-loop counterterm in pure gravity. The result of their calculation can be interpreted as the running of the coupling constant λ_1, therefore allows us to infer that quantum gravity predicts $\lambda_1 \neq 0$. In principle, the presence of matter can modify this conclusion, but only if matter fields exactly cancel the C_3-counterterm generated by pure gravity, and the cancellation is consistent with renormalization-group invariance. As far as we know today, this happens only in supergravity. Similarly, C_3 is turned on at one loop in six dimensional pure gravity [4].

The action (1.1) is preserved by renormalization. It is perturbatively equivalent to actions written previously and to the most general local perturbative extension of the Einstein action [5]. The form (1.1) is convenient in various respects, for example it allows us to find interesting classes of exact solutions of the field equations, which include all locally conformally flat metrics (which we just call "conformally flat" from now on) that solve the Einstein equations. In particular, the Friedmann-Lemaître-Robertson-Walker (FLRW) metrics are exact solutions of the S_{QG}-field equations in arbitrary dimensions $d > 2$ with a homogeneous and isotropic matter distribution. In four dimensions such solutions coincide with the usual ones, while in higher dimensions they coincide with the usual solutions once the energy density ρ and the pressure p are replaced by suitable functions of ρ and p. Metric-independent maps also relate conformally flat solutions of the S_{QG}-equations to conformally flat solutions of the Einstein equations. On the other hand, solutions that are not conformally flat are deformed in a nontrivial way by the couplings λ_n. In the paper we study the first modifications to the Schwarzschild and Kerr metrics in four dimensions.

Another property of S_{QG} is that expanding the metric tensor around conformally flat backgrounds the quadratic part of the action is free of higher derivatives. Vertices, instead, as well as quadratic terms obtained expanding around more general backgrounds, do not have this feature. Working perturbatively in the couplings λ_n, every term of the field equations that contains higher derivatives can be converted into a linear combination of terms that contain at most two derivatives. Then the solutions of the S_{QG}-field equations are uniquely determined by their $\lambda_n \to 0$-limits.

The paper is organized as follows. In section 2 we study the action (1.1) and its field equations. We compare S_{QG} with other local perturbative extensions of the Einstein action. In section 3 we study exact solutions of the S_{QG}-field equations in arbitrary dimensions, in particular metrics of FLRW type and conformally flat metrics. In section 4 we work out the first corrections to the Schwarzschild and Kerr black-hole solutions in four dimensions and discuss their properties. In section 5 we study the perturbative equivalence of actions in detail. Section 6 collects our conclusions. In the appendix we show how to truncate the actions to finite numbers of terms, consistently with the diagrammatic expansion of quantum gravity.

2 The action and its field equations

In this section we study S_{QG} and compare its properties with those of two other actions: the most general local perturbative extension S_{loc} of the Einstein action [5] and an action written in ref. [1], which inspires the simplification proposed here. It is convenient to parametrize S_{loc} as

$$S_{\text{loc}}=-\frac{1}{2\kappa^{d-2}}\int\sqrt{-g}\left(R+2\Lambda+\sum_{n=0}^{\infty}\bar{\lambda}_n\kappa^{2n+2}\bar{\mathcal{I}}_n^{(\Lambda)}(\nabla,\hat{R})\right)+S_m, \tag{2.1}$$

where $\bar{\mathcal{I}}_n^{(\Lambda)}(\nabla,\hat{R})$ denotes the scalars of dimensions $2n+4$ that can be constructed with two or more tensors

$$\hat{R}_{\mu\nu\rho\sigma}=R_{\mu\nu\rho\sigma}+\frac{2\Lambda}{(d-1)\,(d-2)}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right), \tag{2.2}$$

as well as their contractions $\hat{R}_{\mu\nu}=\hat{R}^{\rho}{}_{\mu\rho\nu}$ and $\hat{R}=\hat{R}_{\mu}^{\mu}$, and covariant derivatives ∇_μ, up to covariant divergences of vectors.

In ref. [1] the properties of renormalization were used to write a different action, namely[1]

$$\tilde{S}_{\text{QG}}=-\frac{1}{2\kappa^{d-2}}\int\sqrt{-g}\left(R+2\Lambda+\tilde{\lambda}_0\kappa^2\hat{\mathrm{G}}+\sum_{n=1}^{\infty}\tilde{\lambda}_n\kappa^{2n+2}\mathcal{I}_n^{(\Lambda)}(\nabla,\hat{R})\right)+S_m, \tag{2.3}$$

where $\mathcal{I}_n^{(\Lambda)}(\nabla,\hat{R})$ denotes the local scalars of dimensions $2n+4$ that can be constructed with three or more tensors $\hat{R}_{\mu\nu\rho\sigma}$ and covariant derivatives ∇_μ, up to covariant divergences of vectors. The contractions $\hat{R}_{\mu\nu}$ and $\hat{R}$ can appear inside the scalars $\mathcal{I}_n^{(\Lambda)}$ or not, the resulting different actions being perturbatively equivalent (see section 5 for more details).

The actions S_{QG} and $\tilde{S}_{\text{QG}}$ look like restrictions on S_{loc}, but they are actually perturbatively equivalent to each other and to S_{loc}. Precisely, these actions can be mapped into one another by means of local field redefinitions and parameter-redefinitions, the parameters λ_n, $\tilde{\lambda}_n$ and $\bar{\lambda}_n$ being treated perturbatively. Consequently, S_{QG} and $\tilde{S}_{\text{QG}}$ are preserved by renormalization, namely all divergences generated by Feynman diagrams can be subtracted redefining the metric tensor and the parameters λ_n, or $\tilde{\lambda}_n$, the matter fields and the parameters contained inside S_m.

Specifically, the renormalizability of S_{loc} is obvious, since it is the most general local action. Instead, the actions S_{QG} and $\tilde{S}_{\text{QG}}$ are renormalizable, since using Bianchi identities, commuting covariant derivatives and integrating by parts every (counter)term that does not appear in those actions can be reabsorbed away redefining fields and parameters [1]. In particular, the scalar $\hat{\mathrm{G}}$

[1] Up to notational changes.

is used to write $\hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma}$ as a linear combination of terms present in S_{QG} ($\tilde{S}_{\mathrm{QG}}$), plus terms quadratically proportional to $\hat{R}_{\mu\nu}$. In turn, these can be converted into terms of S_{QG} ($\tilde{S}_{\mathrm{QG}}$) redefining the metric tensor.

Spaces of constant curvature play a peculiar role, since in the absence of matter they are exact solutions of the field equations of the most general action. Indeed, once $R_{\mu\nu\rho\sigma} = K(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ is used, with K =constant, any covariant gravitational field equations must reduce to a simple condition

$$f(\kappa^2 K, \kappa^2\Lambda, \lambda)g_{\mu\nu} = 0, \tag{2.4}$$

where f is some function of the parameters of the theory, which can be solved to obtain K. The parametrizations of (1.1), (2.1) and (2.3), which use hatted tensors or Weyl tensors, are such that the solution of (2.4) simply reads

$$K = -\frac{2\Lambda}{(d-1)\,(d-2)}. \tag{2.5}$$

An important fact is that S_{QG} and $\tilde{S}_{\mathrm{QG}}$, differently from S_{loc}, do not contain terms that are quadratic in the curvature tensors, with the exception of those appearing in the peculiar combination $\hat{\mathrm{G}}$. The special scalar $\hat{\mathrm{G}}$ is a generalization of the Gauss-Bonnet integrand. Its main property is that expanding the metric around a background $\bar{g}_{\mu\nu}$ of constant curvature K equal to (2.5), the integral $\int\sqrt{-g}\hat{\mathrm{G}}$ does not contain terms that are linear or quadratic in the quantum fluctuations. Precisely, writing $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and using $\hat{R}_{\mu\nu\rho\sigma}(\bar{g}) = 0$ it is straightforward to check that

$$\int\sqrt{-g}\hat{\mathrm{G}} = \frac{32(d-3)\Lambda^2}{(d-1)(d-2)^2}\int\sqrt{-\bar{g}} + \mathcal{O}\left(h^3\right). \tag{2.6}$$

The invariants $\int\sqrt{-g}\mathcal{I}_n$ and $\int\sqrt{-g}\mathcal{I}_n^{(\Lambda)}$, $n \geqslant 1$, clearly have the same property. Thus, in this expansion the quadratic parts of the actions S_{QG} and $\tilde{S}_{\mathrm{QG}}$ do not contain higher derivatives and coincide with the quadratic part obtained from Einstein gravity. The absence of higher derivatives in propagators is important to prevent the propagation of unphysical degrees of freedom, such as those of higher-derivative quantum gravity [6].

In every even dimensions $d = 2k$ we can drop one term $\sim \int\sqrt{-g}C^k$ containing k Weyl tensors and no derivatives and add the topological invariant

$$\int\sqrt{-g}\mathrm{G}_k \equiv \int\sqrt{-g}\,\delta^{\alpha_1\beta_1\cdots\alpha_k\beta_k}_{\mu_1\nu_1\cdots\mu_k\nu_k}R^{\mu_1\nu_1}{}_{\alpha_1\beta_1}\cdots R^{\mu_k\nu_k}{}_{\alpha_k\beta_k} \tag{2.7}$$

instead, which does not contribute to the field equations. The difference between two such actions is a linear combination of other terms $\sim \int\sqrt{-g}C^k$ plus terms containing the Ricci tensor [7]. Writing the Ricci tensor as a linear combination of $\hat{R}_{\mu\nu}$ and $\Lambda g_{\mu\nu}$, we can reabsorb the difference into a perturbative local field redefinition and parameter-redefinitions (see section 5). For example, in six dimensions we can set $\lambda_1' = 0$ and add $\int\sqrt{-g}\mathrm{G}_3$.

The invariants (2.7) with $k < d/2$ are not topological. Nevertheless, their variations with respect to the metric tensor are free of higher derivatives [8]. The action of Lovelock gravity [8] in d dimensions contains only the invariants (2.7) with $k \leqslant d/2$, therefore its field equations are completely free of higher derivatives. Nevertheless, that kind of action is not preserved by renormalization. For example, in four dimensions Lovelock gravity is just Einstein gravity with the Gauss-Bonnet term.

In our approach, which is based on renormalization, the mentioned property of G_k is used only for $k = 2$ [9, 1], to build the special invariant $\int \sqrt{-g}\hat{\mathrm{G}}$ and ensure that when we expand the metric around backgrounds of special classes, the quadratic parts of actions such as S_{QG} and $\tilde{S}_{\mathrm{QG}}$ are free of higher derivatives. However, we cannot guarantee similar results for vertices, or for the quadratic parts obtained expanding around more general backgrounds.

The matter action S_m is the most general local one, as long as it has correct unitary propagators. If the classical action has correct propagators, the renormalized one also has. Indeed, in a quantum field theory of matter fields of spins $\leqslant 1/2$ and gauge fields of spins $\leqslant 2$, higher-derivative quadratic terms are not turned on by renormalization if they are absent at the tree level [1]. This fact ensures that a unitary propagator is not driven by renormalization into a non-unitary one.

The new form S_{QG} of the classical action improves $\tilde{S}_{\mathrm{QG}}$ in various respects. First, the $\tilde{S}_{\mathrm{QG}}$-scalars $\mathcal{I}_n^{(\Lambda)}$ are intrinsically Λ-dependent, being constructed with hatted curvature tensors. This gives the impression that the action $\tilde{S}_{\mathrm{QG}}$ is chosen ad hoc. It is better to have independent terms multiplied by independent couplings, as in S_{QG}. Moreover, (1.1) allows us to easily find other, more interesting exact solutions of the field equations, besides spaces of constant curvature, such as the FLRW metrics. More generally, all conformally flat solutions of the Einstein equations solve the S_{QG}-field equations (in four dimensions) or can be easily mapped into solutions of the S_{QG}-field equations (in higher dimensions). Finally, expanding the metric tensor $g_{\mu\nu}$ around any conformally flat background $\bar{g}_{\mu\nu}$ the quadratic part of the action S_{QG} is free of higher derivatives. The action $\tilde{S}_{\mathrm{QG}}$ satisfies this property only for the expansion around spaces of constant curvature.

Field equations

Writing

$$S_{\mathrm{QG}} = -\frac{1}{2\kappa^{d-2}}\int \sqrt{-g}\,(R + 2\Lambda) + S_m + S^{(g)},$$

the S_{QG}-field equations read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^{d-2}T_{\mu\nu} + \kappa^{d-2}T_{\mu\nu}^{(g)}, \tag{2.8}$$

where

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}, \qquad T_{\mu\nu}^{(g)} = \frac{2}{\sqrt{-g}}\frac{\delta S^{(g)}}{\delta g^{\mu\nu}},$$

are the matter energy-momentum tensor and the gravitational self-energy-momentum tensor, respectively. Varying $\int\sqrt{-g}\hat{\mathrm{G}}$ explicitly, we find

$$\kappa^{d-2}T_{\mu\nu}^{(g)}=-\lambda_0\kappa^2\left[2C_{\mu\rho\sigma\alpha}C_{\nu}{}^{\rho\sigma\alpha}-\frac{1}{2}g_{\mu\nu}C_{\rho\sigma\alpha\beta}C^{\rho\sigma\alpha\beta}-\frac{4(d-4)}{d-2}C_{\mu\rho\nu\sigma}\hat{R}^{\rho\sigma}\right.$$
$$\left.-\frac{(d-3)(d-4)}{(d-2)^2}\left(4\hat{R}_{\mu\rho}\hat{R}_{\nu}^{\rho}-2g_{\mu\nu}\hat{R}_{\rho\sigma}\hat{R}^{\rho\sigma}-\frac{2d}{d-1}\hat{R}_{\mu\nu}\hat{R}+\frac{d+2}{2(d-1)}g_{\mu\nu}\hat{R}^2\right)\right]$$
$$+\mathcal{O}(\nabla^2C^2)+\mathcal{O}(RC^2)+\mathcal{O}(C^3). \qquad (2.9)$$

The field equations of $\tilde{S}_{\mathrm{QG}}$ are very similar, the only difference being that in the third line of (2.9) the Weyl tensors are replaced by hatted curvature tensors. The notation $\mathcal{O}(\hat{R}^n)$ means terms containing at least n powers of $\hat{R}_{\mu\nu\rho\sigma}$ and its contractions.

Observe that the variation of $\int\sqrt{-g}\hat{\mathrm{G}}$ with respect to the metric is $\mathcal{O}(\hat{R}^2)$, in agreement with (2.6). Clearly, the tensor $T_{\mu\nu}^{(g)}$ of (2.9) is identically zero in three dimensions. In four dimensions, instead, it reduces to the last line of (2.9). For future use we explicitly work out the first non-trivial contributions to $T_{\mu\nu}^{(g)}$ in $d=4$, which are the ones proportional to the *Goroff-Sagnotti constant* $\Lambda_{\mathrm{GS}}\equiv 3\lambda_1\kappa^4$. Setting $\lambda_1'=0$ and dropping the Gauss-Bonnet term, we write the four dimensional action as

$$S_{\mathrm{QG}}^{(d=4)}=-\frac{1}{2\kappa^2}\int\sqrt{-g}\left(R+2\Lambda+\frac{\Lambda_{\mathrm{GS}}}{3}C_3+\sum_{n=2}^{\infty}\lambda_n\kappa^{2n+2}\mathcal{I}_n(\nabla,C)\right)+S_m.$$

Then we find

$$\kappa^2T_{\mu\nu}^{(g)}=\Lambda_{\mathrm{GS}}\left(\nabla^{\rho}\nabla^{\sigma}C_{\mu\rho\sigma\nu}^{(2)}+\nabla^{\rho}\nabla^{\sigma}C_{\nu\rho\sigma\mu}^{(2)}-\frac{1}{2}C_{\mu\alpha\rho\sigma}^{(2)}R_{\nu}{}^{\alpha\rho\sigma}-\frac{1}{2}C_{\nu\alpha\rho\sigma}^{(2)}R_{\mu}{}^{\alpha\rho\sigma}+\frac{1}{6}g_{\mu\nu}C_3\right.$$
$$\left.-\frac{1}{6}\nabla_{\mu}\nabla_{\nu}C_2+\frac{1}{6}g_{\mu\nu}\nabla^2C_2+\frac{1}{6}R_{\mu\nu}C_2\right)+\mathcal{O}(\nabla^4C^2)+\mathcal{O}(\nabla^2C^3)+\mathcal{O}(C^4), \qquad (2.10)$$

where

$$C_{\mu\nu\rho\sigma}^{(2)}=C_{\mu\nu\alpha\beta}C^{\alpha\beta}{}_{\rho\sigma}, \qquad C_2=C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}.$$

In the list of higher orders that appears in the second line of (2.10) it is understood that pairs of covariant derivatives can be replaced by curvature tensors, so $\mathcal{O}(\nabla^4C^2)=\mathcal{O}(\nabla^2RC^2)$, etc.

As promised, when the metric tensor is expanded around the metric $\overline{g}_{\mu\nu}$ of a space of constant curvature, an FLRW metric, or more generally a conformally flat metric, then the quadratic part of the expanded action S_{QG} does not contain higher derivatives. We can prove this fact considering the variation of $T_{\mu\nu}^{(g)}$ with respect to the metric. The first two lines of (2.9) give contributions that contain at most two derivatives of the fluctuation. The third line of (2.9) gives contributions that are proportional to the Weyl tensor, therefore vanish on conformally flat metrics. If $\overline{g}_{\mu\nu}$ does not belong to these classes of backgrounds then the quadratic part of the action may contain higher derivatives. In general vertices do contain higher derivatives of $g_{\mu\nu}$, multiplied by the couplings λ_n.

To understant how to deal with such higher derivatives, recall that renormalization, which is responsible for turning on the couplings λ_n, is purely perturbative. To be consistent, the action S_{QG} must be treated perturbatively in the λ_ns. In particular, we must search for solutions of the field equations that are analytic in the λ_ns, at least away from singularities. Such solutions exist and are uniquely determined by their limits $\lambda_n \to 0$. Indeed, the field equations contain at most two time derivatives at $\lambda_n = 0$. Therefore, working perturbatively in λ_n we can convert every terms that contain higher time derivatives into terms that contain at most two time derivatives. In this way we obtain new field equations that are perturbatively equivalent to (2.8). Explicit examples of this procedure are illustrated in section 4, when we study solutions of black-hole type.

Similar methods are commonly used to eliminate runaway solutions caused by higher-time derivatives, as in the case of the Abraham-Lorentz force in classical electrodynamics [10]. For applications to gravity see refs. [11, 12, 13]. The elimination of unphysical solutions has a price, because it generates violations of microcausality [10]. We discuss these issues in detail at the end of section 4.

These facts, together with the presence of infinitely many independent couplings, are there to remind us that S_{QG} is not the action of a fundamental theory, but must be viewed as an effective action that can be obtained from a more complete theory in a particular limit or integrating out some massive fields. In the same way as the Fermi theory of weak interactions helped building the Standard Model, studying the properties of S_{QG} can be useful to identify the missing ultimate theory of quantum gravity, which should be unitary, causal and renormalizable with a finite number of independent couplings.

3 Exact solutions of the field equations

In this section we study exact solutions of the S_{QG}-field equations and relate them to known solutions of the Einstein field equations. Because of the theorems proved in section 5 any solution of S_{QG} can be perturbatively mapped into a solution of the field equations of any action that is perturbatively equivalent to S_{QG}, for example S_{loc} and $\tilde{S}_{\mathrm{QG}}$.

We begin observing that in four dimensions all conformally flat metrics that solve the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{3.1}$$

also solve the S_{QG}-field equations (2.8), and vice versa. The reason is that when $d = 4$ and $C_{\mu\rho\nu\sigma} = 0$ formulas (2.9) and (2.10), ensure that the gravitational self-energy-momentum tensor

$T^{(g)}_{\mu\nu}$ identically vanishes. Moreover, the variation of $T^{(g)}_{\mu\nu}$ with respect to the metric is proportional to the Weyl tensor, therefore it also vanishes on conformally flat metrics. If we expand the metric tensor around conformally flat backgrounds that solve (2.8) in four dimensions the propagator coincides with the one of Einstein gravity (if the same gauge-fixing is used).

Now, if $d\Omega^2_{d-2}$ denotes the standard metric of the $(d-2)$-dimensional sphere, the metrics

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{d-2}\right) \tag{3.2}$$

of homogeneous and isotropic spaces are conformally flat in arbitrary dimensions. Indeed, it is easy to prove that the Weyl tensor vanishes everywhere. The FLRW metrics have the form (3.2) and solve (3.1) with a homogeneous and isotropic distribution of matter, described by an energy-momentum tensor T_μ^ν equal to

$$T_\mu^\nu(\rho, p) = \rho\delta^0_\mu\delta^\nu_0 - p\sum_{i=1}^{d-1}\delta^i_\mu\delta^\nu_i, \tag{3.3}$$

where the energy density ρ and the pressure p can be time-dependent.

Thus, the FLRW metrics are exact solutions of the $S_{\rm QG}$-field equations (2.8) in four dimensions.

In higher dimensions we have to take the term $\int\sqrt{-g}\hat{\rm G}$ into account. Nevertheless, in the classes of FLRW metrics and conformally flat metrics we can find metric-independent maps that convert solutions of the Einstein equations into solutions of the $S_{\rm QG}$-field equations, and vice versa.

FLRW solutions in arbitrary dimensions

Consider the $S_{\rm QG}$-field equations (2.8) with matter energy-momentum tensor given by (3.3). We want to show that the FLRW metrics (3.2) that solve

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^{d-2}T_{\mu\nu}(\tilde{\rho}, \tilde{p}) \tag{3.4}$$

also solve (2.8), and vive versa, where $\tilde{\rho}$ and $\tilde{p}$ are suitable functions of ρ and p. Inserting (3.4) into (2.8) we find that this statement is true if and only if

$$T_{\mu\nu}(\tilde{\rho}, \tilde{p}) = T_{\mu\nu}(\rho, p) + T^{(g)}_{\mu\nu}. \tag{3.5}$$

Using (3.4) inside (2.9) (and recalling that $C_{\mu\nu\rho\sigma} = 0$) we easily get

$$T^{(g)\nu}_{\ \ \mu} = \Lambda_0\tilde{\rho}\left(\tilde{\rho}\delta^0_\mu\delta^\nu_0 - (\tilde{\rho} + 2\tilde{p})\sum_{i=1}^{d-1}\delta^i_\mu\delta^\nu_i\right),$$

where

$$\Lambda_0 = 2\lambda_0 \kappa^d \frac{(d-3)(d-4)}{(d-2)(d-1)},$$

therefore equation (3.5) is equivalent to the pair of metric-independent quadratic equations

$$\rho = \tilde{\rho} - \Lambda_0 \tilde{\rho}^2, \qquad p = \tilde{p} - \Lambda_0 \tilde{\rho}(\tilde{\rho} + 2\tilde{p}), \tag{3.6}$$

for $\tilde{\rho}$ and $\tilde{p}$.

Given ρ and p, we determine $\tilde{\rho}$ and $\tilde{p}$ solving the equations (3.6). Then the usual FLRW solution with energy density $\tilde{\rho}$ and pressure $\tilde{p}$ solves the $S_{\rm QG}$-field equations with energy density ρ and pressure p. Assuming $\rho\Lambda_0, p\Lambda_0 \ll 1$ the solution can be worked out perturbatively. For convenience, we report here the differential equations satisfied by a, ρ and p in arbitrary dimensions:

$$\frac{\ddot{a}}{a} = \frac{2\Lambda - (d-1)\tilde{p}\kappa^{d-2} - (d-3)\tilde{\rho}\kappa^{d-2}}{(d-1)(d-2)}, \qquad \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -(d-1)(\tilde{p}+\tilde{\rho})\left(\frac{\dot{a}}{a}\right).$$

The cases $d = 3, 4$ can be seen as particular cases of the more general solution.

Observe that in higher dimensions when we expand the metric around FLRW backgrounds the propagator does not coincide with the one obtained in Einstein gravity (even if we use the same gauge-fixing). Nevertheless, formula (2.9) shows that the quadratic part of the expanded action $S_{\rm QG}$ does not contain higher derivatives. Indeed, it is just affected by terms $\sim \tilde{\rho}\bigtriangledown^2$ and $\sim \tilde{p}\bigtriangledown^2$, and terms with fewer derivatives.

Conformally flat solutions in arbitrary dimensions

More generally, if $\tilde{T}_\mu^\nu$ and T_μ^ν are related by the metric-independent polynomial equation

$$T_\mu^\nu = \tilde{T}_\mu^\nu - \Lambda_0 \frac{d-1}{d-2}\left(2\tilde{T}_\mu^\rho \tilde{T}_\rho^\nu - \delta_\mu^\nu \tilde{T}_2 - \frac{2}{d-1}\tilde{T}_\mu^\nu \tilde{T} + \frac{1}{d-1}\delta_\mu^\nu \tilde{T}^2\right). \tag{3.7}$$

where $\tilde{T} = \tilde{T}_\rho^\rho$ and $\tilde{T}_2 = \tilde{T}_\rho^\sigma \tilde{T}_\sigma^\rho$, then the conformally flat metrics that solve

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa^{d-2}\tilde{T}_{\mu\nu} \tag{3.8}$$

also solve the $S_{\rm QG}$-field equations, and vice versa. The condition (3.7) is obtained inserting (3.8) into (2.9) and (2.8), and using $C_{\mu\nu\rho\sigma} = 0$. Expanding the metric tensor around a conformally flat solution the quadratic part of the action $S_{\rm QG}$ is free of higher derivatives.

4 Approximate black-hole solutions

From the observational point of view, deformed black-hole solutions can offer interesting possibilities to test modifications of general relativity. Deviations from the Kerr metric, in particular, are the easiest to detect [14]. Since black-hole solutions are not conformally flat, they are affected in a non-trivial way by the corrections to Einstein gravity contained in S_{QG}. In this section we study deformations of the metrics of Schwarzschild and Kerr types.

We work in four dimensions and in the absence of matter, and keep only the Goroff-Sagnotti constant Λ_{GS}, besides the Newton constant $G = \kappa^2/8\pi$ and the cosmological constant Λ. The action reads

$$S_{\text{QG}}' = -\frac{1}{2\kappa^2}\int \sqrt{-g}\left(R + 2\Lambda + \frac{\Lambda_{\text{GS}}}{3}C_3\right). \tag{4.1}$$

We begin looking for spherically symmetric solutions of the form

$$ds^2 = \mathrm{e}^{\nu(r)+\omega(r)}dt^2 - \mathrm{e}^{-\nu(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.2}$$

It is worth mentioning that metrics of this type satisfy the peculiar identity

$$C^{(2)}_{\mu\nu\rho\sigma} = -\frac{\Omega}{2\sqrt{3}}C_{\mu\nu\rho\sigma} + \frac{\Omega^2}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \qquad \Omega^2 = C_2, \tag{4.3}$$

where the sign of Ω is determined to have $\Omega > 0$ for the Schwarzschild metric. This identity is useful to simplify various expressions. Inserting the ansatz (4.2) into the field equations (2.8) and using (2.10) we find differential equations for $\nu(r)$ and $\omega(r)$. The Λ_{GS}-dependent contributions involve up to four derivatives of these functions. Clearly, higher-derivatives do not appear at $\Lambda_{\text{GS}} = 0$ and, as explained in section 2, we must search for solutions that are analytic in Λ_{GS}, at least away from singularities. Thus we can work iteratively in Λ_{GS}, which allows us to convert the higher-derivative terms into terms that have at most two derivatives. After this conversion we find two (involved) equations of the form

$$\nu' = F_1(\nu, \omega, r), \qquad \omega' = F_2(\nu, \omega, r), \tag{4.4}$$

for certain functions F_1 and F_2 that are analytic in Λ_{GS}, and two other equations that are automatically satisfied when (4.4) hold. We see that the solutions certainly exist and are uniquely determined by their limits $\Lambda_{\text{GS}} \to 0$. However, we do not have closed expressions for the functions F_1 and F_2, therefore both the search for exact solutions and the numerical analysis appear to be challenging tasks, also considering that the higher-derivative form of the equations does not make numerical integration easy. Here we content ourselves with the first perturbative corrections in Λ_{GS}.

Defining

$$\chi(r) = r\left(1 - \mathrm{e}^{\nu(r)} - \frac{\Lambda}{3}r^2\right),$$

we find

$$\chi' = -\frac{2\Lambda_{\text{GS}}}{r^7}\chi^2(16\chi - 15r + 4\Lambda r^3) + \mathcal{O}(\Lambda_{\text{GS}}^2), \qquad \omega' = \frac{24\Lambda_{\text{GS}}}{r^7}\chi^2 + \mathcal{O}(\Lambda_{\text{GS}}^2).$$

To the lowest order of approximation, the solutions can be found replacing χ with a constant on the right-hand sides of these equations. We obtain

$$\begin{aligned} e^{\nu(r)} &= 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2 + \frac{6\Lambda_{\mathrm{GS}} r_s^2}{r^6}\left(1 - \frac{8r_s}{9r} - \frac{4}{9}\Lambda r^2\right) + \mathcal{O}(\Lambda_{\mathrm{GS}}^2), \\ \omega(r) &= -\frac{4\Lambda_{\mathrm{GS}} r_s^2}{r^6} + \mathcal{O}(\Lambda_{\mathrm{GS}}^2), \end{aligned} \tag{4.5}$$

$r_s = 2Gm$ being the usual Schwarzschild radius.

Using a computer program we worked out the metric up to the order Λ_{GS}^4 included. Higher-order corrections show that the solution has the form

$$-g_{rr}^{-1} = e^{\nu(r)} = 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2 + \frac{r_s}{r}\sum_{n=1}^{\infty}\xi^n P_n, \qquad \omega(r) = \frac{r_s}{r}\sum_{n=1}^{\infty}\xi^n Q_{n-1}, \tag{4.6}$$

where

$$\xi(r) = \frac{\Lambda_{\mathrm{GS}} r_s}{r^5}$$

and P_n, Q_n are polynomials of degree n in r_s/r and Λr^2. It is easy to verify that the expansion of g_{tt} has the same form as the one of $-g_{rr}^{-1}$. Thus the approximation obtained expanding in powers of Λ_{GS} is valid for $\xi \ll 1$, with r_s/r and Λr^2 bounded.

At $\Lambda = 0$ the metric has an event horizon at a modified radius equal to

$$\bar{r}_s = r_s\left(1 - \frac{2}{3}\xi_s + \mathcal{O}(\xi_s^2)\right), \tag{4.7}$$

where $\xi_s = \xi(r_s)$. The form (4.6) of the solution shows that both g_{tt} and g_{rr}^{-1} vanish at $r = \bar{r}_s$.

The informations we have gathered so far do not allow us to study the curvature singularity at $r = 0$. We just mention that once the action is written in the form S_{QG} it makes more sense to consider curvature scalars such as C_2, C_3, etc., instead of the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (which coincides with C_2 for Ricci flat metrics). Because of the identity (4.3) we have

$$C_3 = -\frac{1}{2\sqrt{3}}C_2^{3/2}.$$

We find (at $\Lambda = 0$)

$$C_2 = \frac{12r_s^2}{r^6}\left(1 - 4\xi(r)\left(12 - 13\frac{r_s}{r}\right)\right) + \mathcal{O}(\xi^2).$$

To this order C_2 is equal to $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, because the difference is quadratic in $R_{\mu\nu}$, therefore at least $\mathcal{O}(\Lambda_{\mathrm{GS}}^2)$.

Now we switch to the modified Kerr metric. We study it at $\Lambda = 0$ in two limiting situations. We first consider slowly rotating black holes. To the first order in $a = J/m$ at $\Lambda = 0$, where J is the angular momentum, we find

$$ds^2 = e^{\bar{\nu}(r)+\bar{\omega}(r)}dt^2 - e^{-\bar{\nu}(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + 2a\frac{r_s}{r}\left(1 + \frac{4\Lambda_{\mathrm{GS}} r_s^2}{3r^6}\right)\sin^2\theta dt d\phi,$$

plus $\mathcal{O}(\Lambda_{\mathrm{GS}}^2)$ and $\mathcal{O}(a^2)$, where $\bar{\nu}$ and $\bar{\omega}$ are the same functions as before calculated at $\Lambda = 0$. The location of the event horizon is unmodified to this order of approximation.

Moving one step forward, we study the large-distance expansion of the deformed Kerr metric. Precisely, we take $r_s, a \sim \varepsilon$ and $\Lambda_{\mathrm{GS}} \sim \varepsilon^4$, $\varepsilon \ll 1$ (i.e. we assume that the constants r_s, a and Λ_{GS} are of orders equal to their dimensions in units of coordinates), and calculate the metric to the order ε^8. Doing so, we automatically exclude orders of Λ_{GS} higher than the first. Indeed, Λ_{GS} must always be multiplied by r_s, because $r_s = 0$ gives flat space. In Boyer-Lindquist coordinates we write

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi$$

and obtain

$$\begin{aligned}
g_{tt} &= 1 - \frac{rr_s}{\rho^2} + \frac{2\Lambda_{\mathrm{GS}}r_s^2}{3\rho^8}(3\rho^2 - 2rr_s - 54a^2\cos^2\theta), \qquad g_{\theta\theta} = -\rho^2 + \frac{6a^2r_s^2}{\rho^6}\Lambda_{\mathrm{GS}}\sin^2\theta, \\
g_{rr} &= -\frac{\rho^2}{\Delta} + \frac{2\Lambda_{\mathrm{GS}}r_s^2}{3\rho^6\Delta}\left(9a^2 + 9\rho^2 + rr_s + r_s^2 - 297a^2\cos^2\theta\right), \\
g_{t\phi} &= \frac{arr_s}{\rho^2}\sin^2\theta\left(1 + \frac{4\Lambda_{\mathrm{GS}}r_s^2}{3\rho^6}\right), \qquad g_{\phi\phi} = -\sin^2\theta\left(a^2 + r^2 + \frac{a^2rr_s}{\rho^2}\sin^2\theta\right),
\end{aligned} \tag{4.8}$$

where, as usual,

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - rr_s + a^2.$$

Observe that the modified Kerr metric (4.8) is more general than the deformed metrics considered in ref. [15], where deviations from Kerr are parametrized by one function h of r and θ. Because of this, calculations are rather involved. Using a computer program, five independent functions of r and θ have been used to work out the approximate solution given above. Note that at the end there is no deformation of $g_{\phi\phi}$.

We stress again that renormalization predicts $\Lambda_{\mathrm{GS}} \neq 0$, therefore the deviations worked out in this section can be viewed as predictions of quantum gravity. Their practical detectability depends on the actual value of the constant Λ_{GS}. Theoretically, we cannot predict the value of Λ_{GS}, but only the Λ_{GS}-running, which gives us an estimate of the minimum value of $|\Lambda_{\mathrm{GS}}|$. Using the two-loop result of [3] we find

$$\Delta\Lambda_{\mathrm{GS}}(\ell, \ell') = \Lambda_{\mathrm{GS}}(\ell) - \Lambda_{\mathrm{GS}}(\ell') = \frac{209 l_P^4}{30(4\pi)^2}\ln\frac{\ell}{\ell'},$$

where $l_P = \sqrt{G}$ is the Planck length and $\Lambda_{\mathrm{GS}}(x)$ is the running coupling at the scale x. If we take ℓ equal to the diameter of the observable universe and ℓ' equal to the Planck length itself, we obtain

$$|\Delta\Lambda_{\mathrm{GS}}| \sim 6 l_P^4.$$

If the value of $|\Lambda_{\rm GS}|$ were around $6l_P^4$ there would be no chance to detect the deviations we have worked out so far. We can only hope that $|\Lambda_{\rm GS}|$ has a much larger value in nature. Light black holes are the ones that are affected more sensibly. Taking a mass equal to 5 solar masses, we need at least

$$|\Lambda_{\rm GS}| \sim 10^{156} l_P^4 = 10^{44} ({\rm eV})^{-4} \tag{4.9}$$

to get $\xi_s \sim 1$. In this case the deviations would be appreciable right outside the black hole. The Schwarzschild radius (4.7) would be modified in a sensible way and effects on the deflection of light, for example, could be detected. Depending on the precision of our instruments, smaller values of ξ_s could suffice. In case no deviations are observed it is possible to put experimental bounds on $\Lambda_{\rm GS}$. Observe that as long as $|\Lambda_{\rm GS}|$ is much larger than $6l_P^4$, for all practical purposes $\Lambda_{\rm GS}$ does not run throughout the universe.

So far we have studied static and stationary solutions, but if we are interested in metrics that depend on time, as well as the motion of light and particles in the metrics we have found, we must discuss the violations of causality induced by the presence of higher time derivatives.

To understand the problem it is useful to briefly recall the case of the Abraham-Lorentz force [10] in classical electrodynamics, where the radiation emitted by an accelerated charged particle of mass m is described by one of the equations

$$m\left(1-\tau\frac{\rm d}{{\rm d}t}\right)a(t) = F(t), \qquad ma(t) = \langle F(t)\rangle \equiv \frac{1}{\tau}\int_t^\infty {\rm d}t' {\rm e}^{(t-t')/\tau} F(t'), \tag{4.10}$$

where $\tau = 2e^2/(3mc^3)$, a is the acceleration and F is an external force. The first equation is the standard, higher-derivative one. The second equation is obtained from the first one with the same procedure used to obtain (4.4), i.e. demanding analyticity in τ. This requirement eliminates the runaway solution, but generates a violation of *micro*causality. Indeed, to determine the motion at a given time t we must know the external force at future times t' such that $t \leqslant t' \lesssim t+\tau$. On the other hand, if $F(t') \neq 0$ only for $0 \leqslant t' \leqslant T$ all events appear to be causal at any time $t > T$.

Let us now turn to the case of gravity. Even if the metric deviations predicted here were detected, they would not necessarily provide an indirect evidence that microcausality is violated. The reason is that the action $S_{\rm QG}$ is most probably the effective theory of a more complete, causal theory. It could be obtained, for example, integrating out some degrees of freedom. That said, to detect violations of microcausality we should catch acausal events in the act, compare a sufficient number of different situations, and prove that no causal equations can explain the data.

Considering a fluctuation δg around the metric given by (4.2) and (4.5), higher-time derivative terms provided by $\delta T_{\mu\nu}^{(g)}$ are mutiplied by the Weyl tensor $C \sim r_s/r^3$ or by $\nabla C \sim r_s/r^4$:

$$\kappa^2 \delta T_{\mu\nu}^{(g)} \sim \Lambda_{\rm GS} \nabla C \nabla^3 \delta g + \Lambda_{\rm GS} C \nabla^4 \delta g$$

Comparing these terms with the ones contained in the Einstein field equations and assuming that the derivatives of δg are time ones, for $\xi(r) < 1$ causality violations last for a typical time equal to

$$\tau(r) = r\sqrt{\xi(r)}.$$

In the case of gravitational lensing by a light black hole, taking r around a few times r_s and assuming $\xi(r) \sim 1$ it is necessary to resolve time intervals of about 10^{-4} seconds.

5 Perturbative equivalence of actions and solutions of the field equations

Renormalization cannot determine the action unambiguously. It only determines the *perturbative equivalence* class to which the action belongs. We say that two actions S_1 and S_2 are perturbatively equivalent if

i) they are perturbative expansions around the same unperturbed action $\bar{S}$ and

ii) they can be mapped into each other by means of perturbative field redefinitions and parameter redefinitions.

A perturbative field redefinition is a field redefinition that can be expressed as the identity map plus a perturbative series of local monomials of the fields and their derivatives. Using an appropriate field-covariant formalism [16] perturbative field redefinitions can be implemented in functional integrals as true changes of integration variables, instead of mere replacements of integrands. Generating functionals, suitably generalized [17, 18], behave as scalars.

The actions S_{QG}, $\tilde{S}_{\text{QG}}$ and S_{loc} are perturbatively equivalent. They are mapped into one another by perturbative redefinitions of the metric tensor and redefinitions of the parameters λ and ζ, where ζ denote the parameters of the matter action S_m. As a consequence, the solutions of their field equations can also be perturbatively mapped into one another. In this section we study the map in detail.

We begin with the perturbative equivalence of S_{QG} and $\tilde{S}_{\text{QG}}$. There exists a redefinition of the metric tensor of the form

$$g = g' + \mathcal{O}(\hat{R}^2), \tag{5.1}$$

where $\hat{R}$ denotes the tensor (2.2) and its contractions, and parameter redefinitions λ',ζ' such that

$$S_{\text{QG}}(g, \varphi, \Lambda, \lambda, \zeta) = \tilde{S}_{\text{QG}}(g', \varphi, \Lambda, \lambda', \zeta'). \tag{5.2}$$

We work inductively in the power n_R of Weyl or hatted curvature tensors. Specifically, we assume

$$S_{\text{QG}}(g, \varphi, \Lambda, \lambda, \zeta) = \tilde{S}_{\text{QG}}(g', \varphi, \Lambda, \lambda', \zeta') + Y_{\bar{n}_R+1}, \tag{5.3}$$

where g and g' are related by a field redefinition of the form (5.1), $\bar{n}_R \geqslant 2$ and $Y_{\bar{n}_R+1}$ is matter-independent and $\mathcal{O}(\hat{R}^{\bar{n}_R+1})$. The identity (5.3) is obviously satisfied for $\bar{n}_R = 2$. It is sufficient to show that formula (5.3) with arbitrary $\bar{n}_R \geqslant 2$ implies a similar relation with $\bar{n}_R \to \bar{n}_R + 1$.

Consider the terms of $Y_{\bar{n}_R+1}$ that have precisely $\bar{n}_R+1$ hatted curvature tensors. Express $\hat{R}_{\mu\nu\rho\sigma}$ in terms of the Weyl tensor, $\hat{R}_{\mu\nu}$ and $\hat{R}$. The terms containing only Weyl tensors can be mapped defining relations between the appropriate parameters λ and λ'. Once we have done this, we obtain

$$S_{\mathrm{QG}}(g,\varphi,\Lambda,\lambda,\zeta)=\tilde{S}_{\mathrm{QG}}(g',\varphi,\Lambda,\lambda'',\zeta')+\tilde{Y}_{\bar{n}_R+1},$$

where $\tilde{Y}_{\bar{n}_R+1}=\mathcal{O}(\hat{R}^{\bar{n}_R+1})$ is still matter-independent, but now it is also proportional to $\hat{R}_{\mu\nu}$ or $\hat{R}$. We can write

$$\tilde{Y}_{\bar{n}_R+1}=\int\left(\hat{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\hat{R}\right)X^{\mu\nu}_{\bar{n}_R},\qquad X^{\mu\nu}_{\bar{n}_R}=\mathcal{O}(\hat{R}^{\bar{n}_R}).$$

Using (2.8) and (2.9), adapted to $\tilde{S}_{\mathrm{QG}}$, the variation $\tilde{E}^{\mu\nu}_{\mathrm{QG}}$ of $\tilde{S}_{\mathrm{QG}}$ with respect to the metric tensor can be written in the form

$$\tilde{E}^{\mu\nu}_{\mathrm{QG}}=\hat{R}^{\mu\nu}-\frac{1}{2}g^{\mu\nu}\hat{R}+E^{\mu\nu}_m+\mathcal{O}(\hat{R}^2),$$

where $E^{\mu\nu}_m$ is the analogous variation of S_m, therefore

$$\tilde{Y}_{\bar{n}_R+1}=\int\tilde{E}_{\mathrm{QG}}X_{\bar{n}_R}+Y_{m,\bar{n}_R}+\mathcal{O}(\hat{R}^{\bar{n}_R+2}),$$

$Y_{m,\bar{n}_R}$ denoting $\mathcal{O}(\hat{R}^{\bar{n}_R})$-terms proportional to the matter fields φ. At this point, we have

$$S_{\mathrm{QG}}(g,\varphi,\Lambda,\lambda,\zeta)=\tilde{S}_{\mathrm{QG}}(g',\varphi,\Lambda,\lambda'',\zeta')+\int\tilde{E}'_{\mathrm{QG}}X'_{\bar{n}_R}+Y'_{m,\bar{n}_R}+\mathcal{O}(\hat{R}^{\bar{n}_R+2}),$$

where $\tilde{E}'_{\mathrm{QG}}$, $X'_{\bar{n}_R}$ and $Y'_{m,\bar{n}_R}$ are $\tilde{E}_{\mathrm{QG}}$, $X_{\bar{n}_R}$ and $Y_{m,\bar{n}_R}$ once the metric tensor g is expressed in terms of g'. Consider the redefinition

$$g''=g'+X'_{\bar{n}_R}$$

of the metric tensor. We have

$$S_{\mathrm{QG}}(g,\varphi,\Lambda,\lambda,\zeta)=\tilde{S}_{\mathrm{QG}}(g'',\varphi,\Lambda,\lambda'',\zeta')+Y''_{m,\bar{n}_R}+\mathcal{O}(\hat{R}^{\bar{n}_R+2}),$$

where we have used $\bar{n}_R\geqslant 2$. Finally, the terms $Y_{m,\bar{n}_R}$ can be reabsorbed redefining the parameters ζ'. Therefore there exist ζ'' such that

$$S_{\mathrm{QG}}(g,\varphi,\Lambda,\lambda,\zeta)=\tilde{S}_{\mathrm{QG}}(g'',\varphi,\Lambda,\lambda'',\zeta'')+\mathcal{O}(\hat{R}^{\bar{n}_R+2}).$$

This relation promotes the inductive hypothesis (5.3) from $\bar{n}_R$ to $\bar{n}_R+1$, which proves the theorem.

The same procedure can be used to modify the $\mathcal{O}(\hat{R}^3)$-sector of the action $\tilde{S}_{\mathrm{QG}}$ adding any $\mathcal{O}(\hat{R}^3)$-terms proportional to the hatted Ricci tensor.

Now we show the perturbative equivalence of S_{QG} and S_{loc}. These actions differ by terms quadratically proportional to the hatted Ricci tensor and $\mathcal{O}(\hat{R}^3)$-terms proportional to the hatted Ricci tensor. To quickly prove their equivalence we use a theorem derived in ref. [13], stating that any terms quadratically proportional to the field equations can be reabsorbed into a perturbative field redefinition. In particular, there exists a field redefinition

$$\tilde{g} = g + \mathcal{O}(\hat{R}_{\mu\nu}) \tag{5.4}$$

such that

$$-\frac{1}{2\kappa^{d-2}}\int\sqrt{-g}(R+2\Lambda)+\frac{1}{2}\int\sqrt{-g}\hat{R}_{\mu\nu}Q^{\mu\nu\rho\sigma}\hat{R}_{\rho\sigma}=-\frac{1}{2\kappa^{d-2}}\int\sqrt{-\tilde{g}}\left(R(\tilde{g})+2\Lambda\right),$$

where Q is any perturbatively local derivative operator. Using this map and the properties of $\int\sqrt{-g}\hat{\mathrm{G}}$, in particular its variation with respect to the metric, encoded in (2.9), we can convert S_{loc} into an action $\tilde{S}_{\mathrm{QG}}$ with unrestricted scalars $\mathcal{J}_n^{(\Lambda)}$. Then using the map (5.1) and parameter-redefinitions we can convert the action to S_{QG}.

Finally, recall that when maps such as (5.1) and (5.4) lower the number of time derivatives, they also generate violations of microcausality [13, 19], to which the arguments of the previous section apply.

6 Conclusions

The action of quantum gravity is determined by renormalization. It can be simplified dropping terms proportional to the hatted Ricci tensor, because those terms can be reabsorbed into perturbative field redefinitions and parameter redefinitions. Doing so, we can arrange the action in different perturbatively equivalent ways, which may help us uncover different properties, identify different classes of exact solutions, or reduce the effort to study approximate solutions. We singled out a convenient form S_{QG} that allows us, to some extent, to have control on the infinitely many couplings of the theory. Among the other things, we can show that some well known metrics, such as the FLRW metrics, are exact solutions of the field equations or can be mapped into exact solutions. Precisely, in four dimensions the solutions coincide with the usual ones, while in dimensions greater than four they coincide with the usual ones once the density and the pressure are mapped into simple functions of themselves. More generally, all conformally flat solutions of Einstein gravity can be mapped in a metric-independent way into conformally flat solutions of S_{QG}, and vice versa. The quadratic terms of the action, generated expanding the metric around these solutions, are free of higher derivatives. Solutions that are not conformally flat are instead modified in a nontrivial way. We have studied the first corrections to the metrics of Schwarzschild and Kerr types, expanding in powers of the Goroff-Sagnotti constant.

Vertices can contain arbitrarily high derivatives of the metric tensor. The solutions of the field equations that are analytic in the couplings λ_n, at least away from singularities, are uniquely determined by initial conditions of Cauchy type. However, those solutions violate microcausality. These features and the presence of infinitely many independent couplings point towards a missing, more fundamental theory, which should be unitary, causal and renormalizable with a finite number of independent couplings.

Most of the properties we have studied originate from high-energy physics, specifically renormalization. However, they may have effects detectable in astrophysical observations. For example, it would be desirable to compare predictions and observational data to put constraints on the magnitude of the Goroff-Sagnotti constant. Renormalization only tells us that this constant is non-vanishing. A further reason to motivate investigations of the properties of S_{QG} is that they could help us identify the ultimate theory of quantum gravity, in the same way as the Fermi theory of weak interactions was helpful to build the Standard Model.

Acknowledgments

I thank C. Bambi, G. Cella, P. Menotti and L. Modesto for inspiring discussions and correspondence. I also thank the Physics Department of Fudan University, Shanghai, for hospitality during the initial stage of this work.

A Appendix: Equivalent truncations of the action

Since the number of parameters is infinite, it is useful to define appropriate truncations to classify the invariants and expand perturbatively, consistently with the diagrammatic expansion.

We study Feynman diagrams expanding the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa^{(d-2)/2} h_{\mu\nu}, \tag{A.1}$$

where the background $\bar{g}_{\mu\nu}$ is a conformally flat solution of the field equations, in the case of S_{QG}, or a space of constant curvature, in the cases of $\tilde{S}_{\mathrm{QG}}$ and S_{loc}. Invariance of the functional integral under translations ensures that the results we obtain do not depend on the choice of background $\bar{g}_{\mu\nu}$. The graviton propagator is determined by the Hilbert term and the cosmological term. It depends on $\bar{g}_{\mu\nu}$ and Λ, but not on κ and the λ_ns. The propagators of matter fields can of course depend on masses m. Let E denote the overall energy scale of correlation functions. We assume $\kappa E, \kappa m, \kappa|\Lambda|^{1/2} \ll 1$, and that the values of λ_n are bounded from above (namely there exists a constant M such that $|\lambda_n| < M$ for every n). We do not assume particular inequalities among E, m and Λ, in the same way as we normally do not expand Feynman diagrams in powers of m or $1/m$. Thus for our purposes E, m and Λ can be assumed to be of the same order. Vertices are multiplied by powers of κ, Λ and λ_n. Apart from this kind of factors, Feynman diagrams give integrals that depend only on $\bar{g}_{\mu\nu}$, m and Λ. Therefore, by the Wick theorem and power-counting we can write $h_{\mu\nu} \sim E$ (or $h_{\mu\nu} \sim |\Lambda|^{1/2}, m$).

The diagrammatic expansion is an expansion in powers of κE, $\kappa|\Lambda|^{1/2}$ and κm. A truncation of the diagrammatic expansion amounts to discard powers of these quantities larger than some T. Below we concentrate on the gravitational sector, since the matter sector can be treated similarly. Moreover, we identify Λ and m^2. Precisely, we classify the contributions as

$$\kappa^{-d}(\kappa^2\Lambda)^a(\kappa\bar{\nabla})^b(\kappa^{(d-2)/2}h)^c, \tag{A.2}$$

where $\bar{\nabla}$ denotes the covariant derivative in the background $\bar{g}_{\mu\nu}$, and pairs of $\bar{\nabla}$s can also stand for curvature tensors $\bar{R}$. The number c is the number of external legs of the diagram (or vertex, at the tree level), while b is the power of (external) momenta and a is the power of Λ. Higher powers of $\kappa^2\Lambda$ can be generated by radiative corrections and renormalize the parameters λ_n. The truncation to order T is obtained discarding the contributions that have

$$2a + b + \frac{d-2}{2}c > T. \tag{A.3}$$

This kind of truncation preserves general covariance only within the truncation, namely up to powers $T' > T$ of κE, $\kappa|\Lambda|^{1/2}$ and κm. Clearly, the Feynman diagrams that contribute within the truncation are constructed with a finite number of vertices. Moreover, they are themselves finitely many, since every loop raises the power of κ. We call this truncation *diagrammatic truncation*.

There is actually an alternative truncation [1], which simply amounts to truncate the sums appearing in (1.1), (2.1) and (2.3) to finite numbers of terms. Its advantage is that it is manifestly general covariant, although its connection with Feynman diagrams is less apparent. Precisely, we discard, according to the case (S_{QG} or $\tilde{S}_{\mathrm{QG}}$-S_{loc}), the terms

$$\sim \kappa^{-d}(\kappa^2\Lambda)^{n_\Lambda}(\kappa\nabla)^{n_\nabla}(\kappa^2 C)^{n_R}, \qquad \text{or} \qquad \sim \kappa^{-d}(\kappa^2\Lambda)^{n_\Lambda}(\kappa\nabla)^{n_\nabla}(\kappa^2\hat{R})^{n_R}, \tag{A.4}$$

with

$$2n_\Lambda + n_\nabla + 2n_R > N, \tag{A.5}$$

for some N. Expanding the structures (A.4) according to (A.1) we get terms (A.2) with

$$a = n_\Lambda, \qquad b = n_\nabla + 2n_R, \qquad c = n_R + q,$$

where $q \geqslant 0$ is integer. We can choose a basis such that each invariant $\int\sqrt{-g}\,\mathfrak{I}_n$, $\int\sqrt{-g}\,\bar{\mathfrak{I}}_n^{(\Lambda)}$ and $\int\sqrt{-g}\,\mathfrak{I}_n^{(\Lambda)}$ is uniquely determined by its $q = 0$-contribution. The other contributions are then fixed by general covariance.

The two truncations are actually equivalent, in the sense that a diagrammatic truncation covers a certain general covariant truncation, and vice versa. Let us describe how to switch back and forth between the two. Since (A.5) implies (A.3) with $T = N$, the general covariant truncation to order N covers the diagrammatic truncation to order N. To study the converse implication, we recall that, by general covariance, it is enough to determine the $q = 0$-contribution to an invariant to determine the full invariant. Consider the terms (A.2) and analyze them for increasing number of external legs c. Doing so, $q = 0$-contributions coming from new invariants can be disentangled from $q > 0$-contributions coming from invariants determined for smaller values of c. This procedure allows us to determine the structures (A.4) with

$$n_\Lambda = a, \qquad n_\nabla = b - 2c, \qquad n_R = c.$$

Because of (A.3), the terms we cannot determine satisfy

$$2n_\Lambda + n_\nabla + 2n_R > \frac{4}{d+2}\left(2n_\Lambda + n_\nabla + \frac{d+2}{2}n_R\right) > \frac{4T}{d+2}. \tag{A.6}$$

We conclude that the diagrammatic truncation to order T covers the general covariant truncation to order

$$N = \frac{4T}{d+2}.$$

References

[1] D. Anselmi, Absence of higher derivatives in the renormalization of propagators in quantum field theories with infinitely many couplings, Class. Quant. Grav. 20 (2003) 2355, 02A4 Renormalization.com and arXiv:hep-th/0212013.

[2] G. 't Hooft and M. Veltman, One-loop divergences in the theory of gravitation, Ann. Inst. Poincarè, 20 (1974) 69.

[3] M.H. Goroff and A. Sagnotti, The ultraviolet behavior of Einstein gravity, Nucl. Phys. B 266 (1986) 709.

[4] P. van Nieuwenhuizen, On the renormalization of quantum gravitation without matter, Annals Phys. 104 (1977) 197.

[5] S. Weinberg, Ultraviolet divergences in quantum theories of gravitation, in *An Einstein centenary survey*, Edited by S. Hawking and W. Israel, Cambridge University Press, Cambridge 1979, p. 790.

[6] K.S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D 16 (1977) 953.

[7] P. van Nieuwenhuizen and C.C. Wu, On integral relations for invariants constructed from three Riemann tensors and their applications to quantum gravity, J. Math. Phys. 18 (1977) 182.

[8] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys. 12 (1971) 498.

[9] B. Zwiebach, Curvature squared terms and string theories, Phys. Lett. 156B (1985) 315.

[10] J.D. Jackson, *Classical electrodynamics*, John Wiley and Sons, Inc. (1975), chap. 17.

[11] L. Bel and H. Sirouss Zia, Regular reduction of relativistic theories of gravitation with a quadratic Lagrangian, Phys. Rev. D 32 (1985) 3128.

[12] L. Parker and J.Z. Simon, Einstein equation with quantum corrections reduced to second order, Phys. Rev. D 47 (1993) 1339 and arXiv:gr-qc/9211002.

[13] D. Anselmi, Renormalization and causality violations in classical gravity coupled with quantum matter, JHEP 0701 (2007) 062, 06A1 Renormalization.com and arXiv:hep-th/0605205.

[14] C. Bambi, Testing the Kerr black hole hypothesis, Mod. Phys. Lett. A26 (2011) 2453 and arXiv:1109.4256 [gr-qc].

[15] T. Johannsen and D. Psaltis, A metric for rapidly spinning black holes suitable for strong-field tests of the no-hair theorem, Phys. Rev. D 83 (2011) 124015 and arXiv:1105.3191 [gr-qc].

[16] D. Anselmi, A general field-covariant approach to quantum field theory, Eur. Phys. J. C73 (2013) 2338, 12A1 Renormalization.com and arXiv:1205.3279 [hep-th].

[17] D. Anselmi, A master functional for quantum field theory, Eur. Phys. J. C73 (2013) 2385, 12A2 Renormalization.com and arXiv:1205.3584 [hep-th].

[18] D. Anselmi, Master functional and proper formalism for quantum gauge field theory, Eur. Phys. J. C73 (2013) 2363, 12A3 Renormalization.com and arXiv:1205.3862 [hep-th].

[19] D. Anselmi and M. Halat, Renormalizable acausal theories of classical gravity coupled with interacting quantum fields, Class. Quant. Grav. 24 (2007) 1927 and arXiv:hep-th/0611131.