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Renormalization Of Gauge Theories Without Cohomology

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Abstract

We investigate the renormalization of gauge theories without assuming cohomological properties. We define a renormalization algorithm that preserves the Batalin-Vilkovisky master equation at each step and automatically extends the classical action till it contains sufficiently many independent parameters to reabsorb all divergences into parameter-redefinitions and canonical transformations. The construction is then generalized to the master functional and the field-covariant proper formalism for gauge theories. Our results hold in all manifestly anomaly-free gauge theories, power-counting renormalizable or not. The extension algorithm allows us to solve a quadratic problem, such as finding a sufficiently general solution of the master equation, even when it is not possible to reduce it to a linear (cohomological) problem.

1 Introduction

When a gauge theory is power-counting renormalizable, studying its renormalization is not a difficult task, because the terms contained in the classical action and the counterterms are finitely many, and usually a small number, and it is possible to write down all of them and work out their transformation properties explicitly. When, however, composite fields of higher dimensions are turned on, infinitely many others must be included. If the theory is not power-counting renormalizable, like quantum gravity, we have to deal with infinitely many terms in any case. In these situations, cohomological properties can simplify several tasks, because they allow us to classify terms and counterterms into gauge-invariant ones, gauge-trivial ones, and gauge non-invariant ones. This classification is useful to prove that divergences can be subtracted redefining the ingredients of the classical action, that is to say parameters, fields and sources, or prove that it is possible to extend the classical action so that divergences can be subtracted that way. A natural question arises whether cohomological properties are essential for the renormalization of gauge theories or not. In this paper we prove that they are not.

The only assumption we make is that the theory is manifestly free of gauge anomalies (global symmetries are instead allowed to be anomalous), which means that there must exist a regularization such that the classical action S_c satisfies the Batalin-Vilkovisky master equation $(S_c, S_c) = 0$ [1] exactly at the regularized level. Without using or assuming cohomological properties we show that it is always possible to extend S_c, preserving the master equation, till the extended S_c contains enough independent parameters to subtract all divergences by means of parameter-redefinitions and canonical transformations. A classical action with these properties is called *parameter-complete*. In our approach it is renormalization itself that guides us through the appropriate S_c-extensions, till parameter-completion is achieved. Among the other things, parameter-completion is necessary to have renormalization-group (RG) invariance.

We do not assume that the gauge algerba closes off shell, nor that the number of independent parameters necessary to renormalize divergences is finite. The search for theories that are renormalizable with a finite number of independent parameters, and do not obey known power-counting criteria, is out of the purposes of this paper. Nevertheless, we do believe that the formalism developed here will help organize that search in a more convenient way.

We can illustrate the basic idea of our extension algorithm in a simple case that involves no gauge symmetries. Consider the massless φ^6-theory

$$S_{\rm scal}(\varphi, \lambda_6\mu^{2\varepsilon}) = \frac{1}{2}\int(\partial_\mu\varphi)(\partial^\mu\varphi) - \lambda_6\mu^{2\varepsilon}\int\frac{\varphi^6}{6!},$$

in four dimensions, using the dimensional-regularization technique. The coupling λ_6 has dimension -2 and $\varepsilon = 4 - D$, D being the continued dimension. RG invariance is apparent if we hide the parameter μ inside the bare coupling $\lambda_{6\rm B} = \lambda_6\mu^{2\varepsilon}$. Now we calculate the one-loop divergences and subtract them just as they come. In the minimal subtraction scheme the one-loop renormalized action reads

$$S_{\text{scal}}^{\text{1-loop}}(\varphi, \lambda_6\mu^{2\varepsilon}, \mu^{-\varepsilon}\hbar/\varepsilon) = S_{\text{scal}} - \frac{35\hbar\lambda_6^2\mu^{3\varepsilon}}{(4\pi)^2\varepsilon}\int \frac{\varphi^8}{8!}, \tag{1.1}$$

which is clearly not RG invariant. The reason is that it misses an independent parameter for the new vertex φ^8. However, it does depend on a new quantity, which is $\mu^{-\varepsilon}\hbar/\varepsilon$. Therefore it is sufficient to replace $\hbar/\varepsilon$ with a new dimensionless parameter λ_8', and define the extended RG invariant classical action

$$S_{\text{scal}}^{(1)}(\varphi, m, \lambda_6\mu^{2\varepsilon}, \lambda_8'\mu^{-\varepsilon}) = \frac{1}{2}\int(\partial_\mu\varphi)(\partial^\mu\varphi) - \lambda_6\mu^{2\varepsilon}\int\frac{\varphi^6}{6!} - \frac{35\lambda_6^2\lambda_8'\mu^{3\varepsilon}}{(4\pi)^2}\int\frac{\varphi^8}{8!}, \tag{1.2}$$

with $\lambda_{8\text{B}}' = \lambda_8'\mu^{-\varepsilon}$ at the tree level. The divergence contained in (1.1) can now be reabsorbed into a renormalization of λ_8', which reads at one loop

$$\lambda_{8\text{B}}' = \mu^{-\varepsilon}\left(\lambda_8' + \frac{\hbar}{\varepsilon}\right). \tag{1.3}$$

This is not the end of the story, however, not even at one loop. For example, the classical action $S_{\text{scal}}^{(1)}$ generates one-loop counterterms $\sim \varphi^{10}$. Thus we need to iterate the procedure, introduce a new parameter λ_{10}', and proceed like this indefinitely.

In this simple example what we have done is redundant. On the other hand, when gauge symmetries are present and cohomological theorems do not help us, extending the classical action preserving the master equation is not so straightforward. Nevertheless, we can use the strategy just sketched and let renormalization build the extended classical action by itself. Observe that the parametrization we obtain from this kind of procedure is a bit unusual. Indeed, the vertex φ^8 in (1.2) in not just multiplied by some parameter λ_8, but by a complicated product of parameters. Here we can just replace the coefficient of $\varphi^8/8!$ with $-\lambda_8\mu^{3\varepsilon}$, but in the most general case we cannot assume that a nice parametrization exists. This forces us to work with the unusual one. The perturbative expansion is also organized in an unusual, but consistent way. If take $\lambda_6 \sim g^4$ and $\lambda_8' \sim 1/g^2$, with $g \ll 1$, the parameter λ_8' is large, but it always appears inside combinations that are altogether small. The running predicted by (1.3) is also consistent, since $\beta_8' = \hbar$ is turned into $\beta_g \sim \hbar g^3$.

Switching to gauge theories, the results of this paper prove that all divergences generated by renormalization fit into suitable gauge-invariant extensions of the action. The price is that the gauge symmetry itself may be extended, or modified in a non-trivial way, because there is no guarantee that after the extension procedure the final gauge symmetry will be equivalent to the starting one. At the same time, our results do not prove that all gauge-invariant terms we can construct fit into extensions of the action. In other words, if we can construct some gauge-invariant terms that do not fit into an extended action, renormalization will never be able to generate them as counterterms. Furthermore, the parameter-complete action we obtain may not be the most general extended action. It is just the minimally extended action that can reabsorb all divergences into parameter-redefinitions and canonical transformations.

We emphasize the main issue, here. The master equation $(S_c, S_c) = 0$ is quadratic in the action S_c, thus the compatibility of gauge symmetry and renormalization is encoded in a quadratic problem. Instead, cohomological problems are linear in S_c. When we assume cohomological properties we in practice assume that the quadratic problem can be reduced to a linear one, which makes life much easier. It is nice to know that if cohomological properties do not hold, or are not assumed to hold, we can still build the action we need, even if the problem cannot be linearized and at every step the subtraction algorithm becomes more and more involved. The procedure we outline is conceptually simple, but rather involved at the practical level. At this stage, its most important applications appear to be theoretical.

Cohomological properties provide a purely algebraic classification and have no strict relation with the renormalization algorithm. Typically, they ensure that all gauge-invariant terms we can construct, even those that renormalization cannot generate as counterterms, can be included extending the classical action. From the algebraic point of view, cohomological theorems can be more general than our results. That kind of generality, however, may be unnecessary for the purposes of renormalization. At the same time, our construction is more general in a different direction, because it also works when cohomological theorems do not hold or are unavailable.

The classical action S_c we start from can be any particular local solution of the master equation. Then we use the properties of renormalization to build the parameter-complete local extension $S_{\subset}$. The extension map

$$S_c \rightarrow S_{\subset} \tag{1.4}$$

is also a powerful machine to prove the existence of new solutions of the master equation, even when we are unable to write them down explicitly.

In the last part of the paper we generalize our results to the master functional defined in ref. [2] and the field-covariant *proper formalism* for gauge theories. The master functional Ω satisfies the proper master equation $\lfloor \Omega, \Omega \rfloor = 0$, where the squared antiparentheses are obtained generalizing the Batalin-Vilkovisky antiparentheses to the sector made of composite fields and their gauge transformations. The proper formalism allows us to express all local perturbative field redefinitions and changes of gauge-fixing as "proper" canonical transformations (see section 6 for details), and interpret them as true changes of variables in the functional integral, instead of simple replacements of integrands.

The generalization of the results of this paper to theories that are not manifestly free of gauge anomalies is left to a separate investigation.

Throughout this paper we use the dimensional-regularization technique and the minimal subtraction scheme. Nevertheless, once the action is extended to contain enough independent parameters, it is possible to switch to an arbitrary scheme making finite redefinitions of those parameters. From now on we switch to the Euclidean notation.

The paper is organized as follows. In section 2 we briefly recall how the renormalization algorithm works when cohomological properties hold or are assumed to hold. In section 3 we formulate the completion algorithm induced by renormalization. At this stage, we introduce redundant parameters so that all divergences can be subtracted by means of parameter-redefinitions, without involving canonical transformations. In section 4 we study the perturbative expansion and discuss consistent truncations that allow us to work with the desired precision with a finite number of terms and a finite number of operations. In section 5 we extend the completion algorithm to include both canonical transformations and parameter-redefinitions. In section 6 we generalize the construction to the master functional and the field-covariant proper formalism for gauge theories. In section 7 we collect a few remarks to point out some interesting features of our construction. In section 8 we comment on the search for the most general solution of the master equation, obtained extending the starting classical action. We show that in general this strategy does not allow us to achieve our goals. Section 9 contains our conclusions.

Before starting our investigation we recall a few definitions and facts that will be useful throughout the paper. The antiparentheses of two functionals X and Y of the fields Φ and the sources K coupled to the Φ-gauge transformations are

$$(X,Y)=\int\left(\frac{\delta_r X}{\delta\Phi^A}\frac{\delta_l Y}{\delta K_A}-\frac{\delta_r X}{\delta K_A}\frac{\delta_l Y}{\delta\Phi^A}\right).$$

If S denotes the classical action, the generating functionals Z and W are defined by

$$Z(J,K)=\int[\mathrm{d}\Phi]\exp\left(-S(\Phi,K)+\int\Phi^A J_A\right)=\exp W(J,K), \tag{1.5}$$

while $\Gamma(\Phi,K)$ is the Legendre transform of W with respect to J. A general theorem says that Γ satisfies the identity

$$(\Gamma,\Gamma)=\langle(S,S)\rangle. \tag{1.6}$$

This formula can be proved making a change of variables $\Phi\rightarrow\Phi+\xi(S,\Phi)$ in the functional integral (1.5), where ξ is a constant anticommuting parameter. In particular, $(S,S)=0$ implies $(\Gamma,\Gamma)=0$. Finally, in dimensional regularization the functional integration measure $[\mathrm{d}\Phi]$ is invariant under arbitrary perturbative changes of field variables.

2 Renormalization with cohomology

In this section we briefly review how renormalization proceeds when suitable cohomological properties hold, specified in formulas (2.2) and (2.3) below. To make the presentation simpler, here we also assume that the gauge algebra closes off shell, so there exists a choice of variables such that S_c has the form

$$S_c(\Phi, K) = \mathcal{S}(\Phi) - \int R^A(\Phi) K_A, \tag{2.1}$$

where $R^A(\Phi)$ is the symmetry transformation of the field Φ^A . As said, we use the dimensional-regularization technique and the minimal subtraction scheme, and the starting classical action S_c must satisfy the master equation $(S_c, S_c) = 0$ exactly at the regularized level. Among the other things, minimal subtraction scheme means that S_c does not contain evanescent terms equal to the product of finite local terms times some powers of $\varepsilon = 4 - D$. Indeed, if terms of this type were present we would not be able to extract the divergent parts (of master equations, see below) in an efficient way, since finite local contributions could originate from products between divergent and evanescent terms.

For example, in pure non-Abelian Yang-Mills theory we have $\Phi^A = (A_\mu^a, C^a, \bar{C}^a, B^a)$, where A_μ^a are the gauge fields, C^a and $\bar{C}^a$ are the Fadeev-Popov ghosts and antighosts, respectively, and B^a are the Lagrange multipliers for the gauge-fixing. We write the sources as $K_A = (K_a^\mu, K_C^a, K_{\bar{C}}^a, K_B^a)$. The classical action is

$$\begin{aligned} S_c(\Phi, K) = &\int \left(\frac{1}{4} F_{\mu\nu}^{a\ 2} - \frac{\lambda}{2}(B^a)^2 + B^a \partial \cdot A^a - \bar{C}^a \partial_\mu D_\mu C^a \right) \\ &- \int D_\mu C^a K_\mu^a + \frac{g}{2} \int f^{abc} C^b C^c K_C^a - \int B^a K_{\bar{C}}^a, \end{aligned}$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ is the field strength, $D_\mu C^a = \partial_\mu C^a + g f^{abc} A_\mu^b C^c$ is the covariant derivative of the ghosts and f^{abc} are the structure constants of the Lie algebra. The theory is power-counting renormalizable, hence its renormalization is straightforward. However, we can imagine to consider a more general class of theories, obtained adding gauge invariant composite fields of arbitrary dimensions, such as $(F_{\mu\nu}^{a\ 2})^n$ with $n > 1$, multiplied by new coupling costants. Then to renormalize the theory we can either apply the method recalled in this section, which uses cohomological properties, or follow the strategy developed in the next sections.

Call S_n the action renormalized up to n loops, with $S_0 = S_c$. By assumption, S_n has the form S_c+poles in ε. Assume, by induction, that S_n also satisfies the master equation $(S_n, S_n) = 0$ exactly at the regularized level. Then (1.6) tells us that the n-loop renormalized Γ-functional Γ_n satisfies the master equation $(\Gamma_n, \Gamma_n) = 0$. Call $\Gamma_{n\,\mathrm{div}}^{(n+1)}$ the $(n+1)$-loop divergent part of Γ_n. By the theorem of locality of counterterms, $\Gamma_{n\,\mathrm{div}}^{(n+1)}$ is a local functional. Then the $(n+1)$-loop divergent part of the master equation $(\Gamma_n, \Gamma_n) = 0$ gives the cohomological problem

$$(S_c, \Gamma_{n\,\mathrm{div}}^{(n+1)}) = 0. \tag{2.2}$$

The cohomological assumption we make now is that the most general solution of this problem has the form

$$\Gamma_{n\,\mathrm{div}}^{(n+1)}(\Phi, K) = \mathcal{G}(\Phi) + (S_c, \chi), \tag{2.3}$$

where $\mathcal{G}(\Phi)$ and $\chi(\Phi, K)$ are local functionals. The main meaning of (2.3) is that the possibly non-trivial part $\mathcal{G}$ depends only on the fields Φ. Theorems that ensure (2.3) have been proved both for Yang-Mills theory and gravity, for local composite fields and local functionals of arbitrary ghost numbers [3].

Often we can characterize $\mathcal{G}(\Phi)$ even more precisely. Assuming that the set of fields Φ^A is made of the physical fields ϕ, the ghosts C, plus the gauge-trivial subsystem $\bar{C}$-B made of antighosts $\bar{C}$ and Lagrange multipliers B, then it is possible to further decompose $\mathcal{G}(\Phi)$ as

$$\mathcal{G}(\Phi) = \mathcal{G}'(\phi) + (S_c, \chi'), \tag{2.4}$$

where $\mathcal{G}'(\phi)$ and $\chi'(\Phi, K)$ are also local functionals. This formula shows that the cohomologically non-trivial solutions $\mathcal{G}'(\phi)$ are just the gauge-invariant terms constructed with the physical fields ϕ and their derivatives.

Assumption (2.3), instead of (2.4), is actually sufficient for the arguments that follow. Let $\{\mathcal{G}_i(\Phi)\}$ denote a basis for the non-trivial solutions $\mathcal{G}(\Phi)$ appearing in (2.3). Extend the action S_c of (2.1) replacing $\mathcal{S}(\Phi)$ with a linear combination $\mathcal{S}'(\Phi)$ of all $\mathcal{G}_i(\Phi)$s, multiplied by independent parameters λ_i:

$$S_c'(\Phi, K) = \mathcal{S}'(\Phi) - \int R^A(\Phi) K_A, \qquad \mathcal{S}'(\Phi) = \sum_i \lambda_i \mathcal{G}_i(\Phi). \tag{2.5}$$

Since $(S_c, \mathcal{G}_i) = 0$ and $(\mathcal{G}_i, \mathcal{G}_j) = 0$ the extended action $S_c'(\Phi, K)$ still solves the master equation $(S_c', S_c') = 0$. From now on we drop the primes in S_c' and $\mathcal{S}'$ and assume that S_c is the extended action.

Now, by assumption (2.3) we can decompose $\Gamma^{(n+1)}_{n\,\mathrm{div}}$ as

$$\Gamma^{(n+1)}_{n\,\mathrm{div}} = \sum_i (\Delta_{n+1}\lambda_i) \mathcal{G}_i(\Phi) + (S_c, \chi_{n+1}),$$

where $\Delta_{n+1}\lambda_i$ are constants and χ_{n+1} is a local functional. The divergences $\mathcal{G}_i(\Phi)$ are subtracted redefining the parameters λ_i as $\lambda_i' = \lambda_i - \Delta_{n+1}\lambda_i$, while the cohomologically trivial divergences (S_c, χ_{n+1}) are subtracted by means of the canonical transformation generated by

$$F_{n+1}(\Phi, K') = \int \Phi^A K_A' - \chi_{n+1}(\Phi, K').$$

Indeed,

$$\Phi'^A = \Phi^A - \frac{\delta \chi_{n+1}}{\delta K_A}, \qquad K_A' = K_A + \frac{\delta \chi_{n+1}}{\delta \Phi^A}, \qquad S_n(\Phi', K') = S_n(\Phi, K) - (S_c, \chi_{n+1}),$$

plus higher orders, therefore the action

$$S_{n+1}(\Phi, K, \lambda) \equiv S_n(\Phi', K', \lambda')$$

is such that

$$S_{n+1}(\Phi,K,\lambda)=S_n(\Phi,K,\lambda)-\sum_i(\Delta_{n+1}\lambda_i)\mathcal{G}_i(\Phi)-(S_c,\chi_{n+1})=S_n(\Phi,K,\lambda)-\Gamma^{(n+1)}_{n\,\mathrm{div}}$$

plus higher orders. Clearly, S_{n+1} is the $(n+1)$-loop renormalized action, because the functional Γ_{n+1} is equal to $\Gamma_n-\Gamma^{(n+1)}_{n\,\mathrm{div}}$ plus higher orders. Finally, S_{n+1} also satisfies the master equation $(S_{n+1},S_{n+1})=0$, since canonical transformations and parameter-redefinitions preserve the antiparentheses. Thus, the inductive hypotheses are promoted to the order $n+1$. Iterating the argument, we find that S_∞ is the renormalized action to all orders and satisfies the master equation $(S_\infty,S_\infty)=0$.

In ref. [2] the derivation just recalled was extended to the master functional and the field-covariant proper formalism for gauge theories. It was also shown that when the cohomological assumption (2.3) holds, then it generalizes to an analogous cohomological property for the proper formalism.

3 Parameter-completion without cohomology

From now on we do not assume that the gauge algebra closes off shell, nor that the symmetry satisfies particular cohomological properties, such as (2.3) and its generalizations. The only assumption we retain is that the theory is manifestly free of gauge anomalies, namely S_c satisfies the master equation

$$(S_c,S_c)=0 \tag{3.1}$$

exactly at the regularized level. We show that renormalization itself allows us to extend the classical action S_c preserving the master equation till the extended S_c becomes parameter-complete.

We present our arguments in two steps. In this section, *i*) we introduce enough redundant parameters so that all redefinitions of fields and sources can actually be traded for redefinitions of the redundant parameters. In section 5, *ii*) we remove those ad hoc parameters and take full advantage of the possibility to make canonical transformations. Option *i*) is a formal trick for intermediate derivations. Option *ii*) is the right way to go to determine if our theory belongs to some special class with respect to its renormalizability properties, for example it is finite, or renormalizable with a finite number of parameters.

Raw renormalization algorithm

Before deriving our main results, we need to recall a "raw" renormalization algorithm [4], where divergences are subtracted just as they come, without checking whether they can be reabsorbed into parameter- and/or field-source-redefinitions. This construction allows us to define a map that is crucial to build the extension map (1.4).

As before, call S_n and Γ_n the action and the Γ-functional renormalized up to n loops, with $S_0 = S_c$. We allow the starting action S_c to be an expansion in $\hbar$, although we still call it "classical action". Later we will appreciate why it is useful to have an $\hbar$-dependent S_c. We denote the $\hbar \to 0$ limit of $S_c(\lambda, \hbar)$ with $\bar{S}_0$. Since we use the minimal subtraction scheme, $S_n = S_c$+poles and S_c does not contain evanescent terms equal to powers of ε times finite local terms.

We inductively assume that S_n satisfies the master equation up to higher orders, namely

$$(S_n, S_n) = \mathcal{O}(\hbar^{n+1}). \tag{3.2}$$

Applying the theorem (1.6) we get the identity

$$(\Gamma_n, \Gamma_n) = \langle (S_n, S_n) \rangle. \tag{3.3}$$

Using (3.2), formula (3.3) gives $(\Gamma_n, \Gamma_n) = \mathcal{O}(\hbar^{n+1})$. Now, (S_n, S_n) is a local functional, and $\langle (S_n, S_n) \rangle$ is the functional that collects the one-particle irreducible correlations functions containing one insertion of (S_n, S_n). Because of (3.2), the $\mathcal{O}(\hbar^{n+1})$-contributions to $\langle (S_n, S_n) \rangle$ coincide with the $\mathcal{O}(\hbar^{n+1})$-contributions to (S_n, S_n). Moreover, since $S_n = S_c$+poles and $(S_c, S_c) = 0$, we know that (S_n, S_n) =poles.

Call $\Gamma^{(n+1)}_{n\,\mathrm{div}}$ the order-$(n+1)$ divergent part of Γ_n. By the theorem of locality of counterterms, $\Gamma^{(n+1)}_{n\,\mathrm{div}}$ is a local functional. By the observations just made, if we take the order-$(n+1)$ divergent part of (3.3), we get

$$(\bar{S}_0, \Gamma^{(n+1)}_{n\,\mathrm{div}}) = \frac{1}{2}(S_n, S_n) + \mathcal{O}(\hbar^{n+2}). \tag{3.4}$$

Now we define

$$S_{n+1} = S_n - \Gamma^{(n+1)}_{n\,\mathrm{div}}. \tag{3.5}$$

Clearly, S_{n+1} is the $(n+1)$-loop renormalized action, since $\Gamma_{n+1} = \Gamma_n - \Gamma^{(n+1)}_{n\,\mathrm{div}} + \mathcal{O}(\hbar^{n+2})$. Moreover, we still have $S_{n+1} = S_c$+poles, and, using (3.4) and (3.5),

$$(S_{n+1}, S_{n+1}) = \mathcal{O}(\hbar^{n+2}),$$

which promotes the inductive assumption to $n+1$ loops. Iterating the argument, we can construct the renormalized action S_∞ and the renormalized functional Γ_∞, and prove that both satisfy their master equations exactly.

Let us study S_∞ more closely. In dimensional regularization the L-loop divergences are multiplied by

$$\frac{\hbar^L}{\varepsilon^n}, \qquad 1 \leqslant n \leqslant L. \tag{3.6}$$

Thus, while S_c depends on λ and $\hbar$, S_∞ depends on one additional quantity, which is $\hbar/\varepsilon$, and the new dependence is (order-by-order) polynomial. Given a solution S_c of the master equation, the map

$$S_c(\lambda, \hbar) \to S_\infty(\lambda, \hbar/\varepsilon, \hbar) \tag{3.7}$$

builds an *extended solution* S_∞ of the master equation, such that the functional Γ_∞ associated with S_∞ is convergent. Since $S_\infty = S_c$+poles, we have $S_\infty(\lambda, 0, \hbar) = S_c(\lambda, \hbar)$. We discover that renormalization knows how to automatically extend the solutions of the master equation. This piece of information is crucial for the arguments of this paper.

From the physical point of view, the raw subtraction algorithm is not the final answer to the problem of renormalization, because when divergences are subtracted just as they come, instead of by means of field-, parameter- and source-redefinitions, renormalization-group invariance is lost. We cannot define a bare action, because the renormalized action S_∞ does not contain enough independent constants to define all bare parameters we need. To have RG invariance we must extend the classical action introducing new independent parameters where appropriate.

Parameter-extension maps

For the moment we adopt the option *i*) mentioned above and view renormalization as a redefinition of parameters only, with no field/source redefinitions. We prove that the classical action always admits an extension that satisfies the master equation and contains enough independent parameters to subtract all divergences by means of parameter-redefinitions. The basic argument is that in case it is not so, we can use renormalization to build an extended solution containing at least one additional independent parameter. Iterating this procedure indefinitely, we end up with a parameter-complete action, namely an action that can reabsorb its own divergences redefining its own parameters.

We learned from (3.7) that renormalization generates a new perturbatively local solution S_∞ of the master equation that depends on one quantity $(\hbar/\varepsilon)$ not contained in S_c. Turning $\hbar/\varepsilon$ into an independent parameter λ' we obtain the *parameter-extension map* (often abbreviated to *extension map*)

$$S_c(\lambda, \hbar) \rightarrow S_\infty(\lambda, \lambda', \hbar), \tag{3.8}$$

from a classical solution $S_c(\lambda, \hbar)$ to the master equation, depending on certain parameters λ, to an extended classical solution $S_\infty(\lambda, \lambda', \hbar)$, which can (polynomially) depend on one additional parameter λ'. We say that $S_\infty(\lambda, \lambda', \hbar)$ is the parameter extension of $S_c(\lambda, \hbar)$.

Now, construct an *extension chain* $\{S^{(0)}, S^{(1)}, S^{(2)} \cdots\}$ of classical actions, all of which are solutions of the master equation, where $S^{(0)}$ is the starting classical action $S_c(\lambda, \hbar)$, and $S^{(i)}$, $i > 0$, is the parameter extension of $S^{(i-1)}$. Denote the parameters contained in $S^{(i)}$ with $\lambda^{(i)}$. We have $\{\lambda^{(i)}\} \subset \{\lambda^{(i+1)}\}$. Writing $\{\lambda^{(i+1)}\} \equiv \{\lambda^{(i)}, \lambda'^{(i)}\}$, we also have $S^{(i+1)}(\lambda^{(i+1)}, \hbar)\big|_{\lambda'^{(i)}=0} = S^{(i)}(\lambda^{(i)}, \hbar)$.

If there exists an $i = I$ such that the parameter extension of $S^{(I)}$ is stable up to parameter-redefinitions $\tilde{\lambda}$, namely such that

$$S^{(I+1)}(\lambda^{(I+1)}, \hbar) = S^{(I)}(\tilde{\lambda}(\lambda^{(I)}, \lambda'^{(I)}, \hbar), \hbar), \tag{3.9}$$

we say that the extension chain closes. Then the action $S_{\subset} \equiv S^{(I)}$ is parameter-complete. Indeed, by construction if we replace the parameter $\lambda'^{(I)}$ back with $\hbar/\varepsilon$, formula (3.9) tells us that the action

$$S^{(I+1)}(\lambda^{(I)}, \hbar/\varepsilon, \hbar) = S^{(I)}(\tilde{\lambda}(\lambda^{(I)}, \hbar/\varepsilon, \hbar), \hbar)$$

is the renormalized action associated with the classical action $S^{(I)}(\lambda^{(I)}, \hbar)$, therefore $S_{\mathrm{B}} = S^{(I)}$ is the bare action,

$$\lambda_{\mathrm{B}} = \tilde{\lambda}(\lambda^{(I)}, \hbar/\varepsilon, \hbar)$$

are the bare couplings and $\lambda^{(I)}$ are the renormalized couplings.

In power-counting renormalizable theories closure is certainly achieved after a finite number of steps, because the number of independent parameters cannot exceed the number of monomials contained in the action. When the theory is not power-counting renormalizable, instead, appropriate truncations, discussed in the next section, are necessary to achieve closure with a finite number of steps.

Reduced parameter-extension map

We can always choose an $\hbar$-independent classical action S_c, or we can replace $\hbar$ inside S_c with a new independent parameter and add it to the set of λs. Then $S_{\infty}(\lambda, \hbar/\varepsilon, \hbar)$ depends on two parameters more than S_c, so the map (3.8) extends the classical action by two parameters λ'_1 and λ'_2, which replace $\hbar/\varepsilon$ and $\hbar$, respectively. The extension chain can be constructed as before, and the parameter-complete action satisfies

$$S^{(I+1)}(\lambda^{(I+1)}) = S^{(I)}(\tilde{\lambda}(\lambda^{(I)}, \lambda'^{(I)})).$$

In this situation we can also construct a "reduced" extension map, which can be useful for some purposes. It is obtained considering $S_{\infty}(\lambda, \hbar/\varepsilon, 0)$, that is to say keeping only the maximal divergences of the renormalized action $S_{\infty}(\lambda, \hbar/\varepsilon, \hbar)$. The reduced extension map

$$S_c(\lambda) \rightarrow S_{\infty}(\lambda, \lambda'_1, 0) \tag{3.10}$$

is much easier to work out, since it is sufficient to compute the one-loop divergent parts generated by $S_c(\lambda)$ and then use standard RG techniques to resum the maximal divergences of diagrams with more loops. Using (3.10) instead of (3.8), we can construct a reduced extension chain and a reduced parameter-complete action $S_{r\subset}$. The downside is that $S_{r\subset}$ is parameter-complete only with respect to the maximal divergences. In some cases the action $S_{r\subset}$ may coincide with the final answer $S_{\subset}$, because normally new counterterms are generated already at one loop. However, we have no guarantee that it is so (and it is quite easy to construct examples where it is not so). A convenient strategy is to first construct the reduced action $S_{r\subset}$ and then check whether it is complete or not. It not, take $S_{r\subset}$ as the starting $S_c(\lambda)$ and build the complete action $S_{\subset}$ using (3.8).

4 Truncations

When we quantize a nonrenormalizable theory, or study composite fields of high dimensions in any kind of theory, we have to define a consistent perturbative expansion. In particular, we must truncate the classical action S_c so that the truncated action S_{cT} contains an arbitrarily large, but finite, number N of terms, sufficient for all practical needs. In the previous section we constructed the parameter-complete action without paying attention to this issue. Here we show how to truncate the theory and adapt the construction of the previous section so that it involves a finite number of operations for each N.

Denote the gauge coupling of minimum dimension with κ. We parametrize the starting classical action $S_c(\Phi, K, \kappa, \zeta, \xi)$ as

$$S_c(\Phi, K, \kappa, \zeta, \xi) = \frac{1}{\kappa^2} S_c'(\kappa\Phi, \kappa K, \zeta, \xi), \tag{4.1}$$

where ξ are gauge-fixing parameters, ζ are any other parameters and S_c' is polynomial in ζ and ξ. Moreover, each field Φ has a dominant kinetic term

$$\sim \frac{1}{2}\int \Phi \partial^{n_\Phi} \Phi \tag{4.2}$$

normalized to one or multiplied by a dimensionless parameter.

Before proceeding let us explain the meaning of the parametrization (4.1). Consider for example Yang-Mills theory coupled to Einstein gravity. The Yang-Mills action reads

$$\frac{1}{4}\int \sqrt{g} g^{\mu\rho} g^{\nu\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}(A, g_c), \tag{4.3}$$

where g_c is the gauge coupling. However, the gauge coupling κ of minimum dimension is not g_c, but the Newton constant κ_N. Thus (4.3) does not agree with (4.1). The right way to parametrize (4.3) is to define $g_c \equiv r_+ \kappa_N$ and rewrite (4.3) as

$$\frac{1}{4\kappa_N^2}\int \sqrt{g} g^{\mu\rho} g^{\nu\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}(\kappa_N A, r_+).$$

The action S_c' that we obtain is obviously polynomial in r_+.

When a theory contains superrenormalizable terms and massless fields Feynman diagrams usually have infrared problems. To avoid this, we assume that if superrenormalizable interactions are present, the fields are equipped with appropriate quadratic terms that cure the infrared behaviors of diagrams. For scalars and fermions we just need mass terms. For Yang-Mills theories in three dimensions we need Chern-Simons terms. For higher-derivative gravities in four dimensions, such as the theories with dominant quadratic terms

$$\frac{1}{2\kappa_N^2}\int \sqrt{g}\left(\alpha R_{\mu\nu}(D^2)^n R^{\mu\nu} + \beta R(D^2)^n R\right) \tag{4.4}$$

where α and β are dimensionless constants, $n \geqslant 0$ is an integer and D is the covariant derivative, we need either the Einstein term or the cosmological term. Writing $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa_N \phi_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is some reference metric, the fluctuation $\phi_{\mu\nu}$ has dimension $-n$. When $n > 0$ the Newton constant κ_N is a superrenormalizable parameter, so the cosmological constant must be present anyway, because radiative corrections generate it as a counterterm. Clearly, the theories (4.4) are not perturbatively unitary.

The gauge-fixing must be parametrized similarly. Let

$$S_{\text{cmin}}(\Phi, K, \kappa, \zeta) = \frac{1}{\kappa^2} S'_{\text{cmin}}(\kappa\Phi, \kappa K, \zeta)$$

denote the minimal solution of the master equation, namely $S_c(\Phi, K, \kappa, \zeta, \xi)$ where antighosts $\bar{C}$, Lagrange multipliers B and their sources $K_{\bar{C}}$, K_B are set to zero. The simplest extended solution of the master equation reads

$$S_{\text{cext}}(\Phi, K, \kappa, \zeta) = S_{\text{cmin}}(\Phi, K, \kappa, \zeta) - \int BK_{\bar{C}} = \frac{1}{\kappa^2} S'_{\text{cext}}(\kappa\Phi, \kappa K, \zeta),$$

and can be gauge-fixed using a gauge fermion Ψ of the form

$$\Psi(\Phi, K, \kappa, \xi) = \frac{1}{\kappa^2} \Psi'(\kappa\Phi, \kappa K, \xi),$$

where ξ are gauge-fixing parameters and Ψ' depends polynomially on ξ. The Ψ-contributions that do gauge-fix are actually contained in $\Psi(\Phi, 0, \kappa, \xi)$, since the K-dependent sector just describes a change of variables.

If the gauge algebra closes off shell we can choose an S_{cmin} linear in K. Taking a K-independent Ψ the gauge-fixed solution of the master equation reads

$$S_c(\Phi, K, \kappa, \zeta, \xi) = S_{\text{cext}} + (S_{\text{cext}}, \Psi) = \frac{1}{\kappa^2} S'_c(\kappa\Phi, \kappa K, \zeta, \xi). \tag{4.5}$$

For example, in Yang-Mills theory coupled with gravity a typical and simple choice is

$$\Psi(\Phi, K, \kappa, \xi) = \int \bar{C}^{\mu} \left(\eta^{\hat{a}\nu} \partial_\nu \phi_{\hat{a}\mu} + \xi_G \eta^{\hat{a}\nu} \partial_\mu \phi_{\hat{a}\nu} - \frac{\xi'_G}{2} B_\mu \right) + \int \bar{C}^{\hat{a}\hat{b}} \phi_{\hat{a}\mu} \delta^{\mu}_{\hat{b}}$$
$$+ \int \bar{C}^{a} \left(\partial^\mu A^a_\mu - \frac{\xi_g}{2} B^a \right), \tag{4.6}$$

where $\kappa\phi_{\hat{a}\mu}$ is the quantum fluctuation of the vierbein $e_{\hat{a}\mu}$ around a given background (normally flat space), $\bar{C}^{\hat{a}\hat{b}}$ and $B_{\hat{a}\hat{b}}$ are the antighosts and Lagrange multipliers of local Lorentz symmetry, $\hat{a}$, $\hat{b}$, ... are indices of the Lorentz group and the indices of ∂^μ, $\bar{C}^\mu$ and B^μ are raised and lowered with the flat metric $\eta^{\mu\nu}$.

More generally, if the gauge algebra closes only on shell we have to gauge-fix the theory defining $S_c(\Phi, K, \kappa, \zeta, \xi)$ as the action obtained from S_{cext} applying the canonical transformation generated by

$$F(\Phi, K') = \int \Phi^A K'_A + \Psi(\Phi, K', \kappa, \xi). \tag{4.7}$$

Clearly, S_c is parametrized according to the structure (4.1).

Now we can explain how the truncation works. We organize the set of parameters $\lambda = \kappa, \zeta, \xi$ into four subsets $\bar{s}$, s_0, s_+ and s_-. The first subset $\bar{s}$ contains the masses, the cosmological constant, and in general all parameters that enter the propagator and are not treated perturbatively. For example, some of them cannot be considered small because they cure infrared problems when superrenormalizable interactions are present. We express each parameter contained in $\bar{s}$ as a dimensionless constant of order one times m^d, where d is its (non-negative) dimension in units of mass.

The second set s_0 contains the parameters of vanishing dimensions. We write each of them as a constant of order one times a positive integer power of some $\sigma \ll 1$. Then we have the subset s_+ of parameters that have positive dimensions d_+ and are treated perturbatively, such as the coefficients of superrenormalizable interactions. We write them as constants of order one times $\Lambda_+^{d_+}$, where Λ_+ is some scale. Finally, the forth subset s_- contains the parameters of negative dimensions d_-, which we write as constants of order one times $\Lambda_-^{-d_-}$, where Λ_- is some other scale. The forth subset may include the coefficients of quadratic terms $\sim \phi\partial^{n'_\phi}\phi$ with $n'_\phi > n_\phi$, which have to be treated perturbatively, since we have established that the dominant quadratic terms we perturb around are (4.2).

Feynman diagrams are multiplied by various factors, but their core integrals depend only on the parameters of the subset $\bar{s}$ and external momenta. Therefore, if we assume that m and the overall energy E are of the same order, each field Φ of dimension d_Φ contributes to the amplitudes as a power $\sim E^{d_\Phi} \sim m^{d_\Phi}$.

We assume that there exists a range of energies E such that

$$\Lambda_+ \ll m \sim E \ll \Lambda_- \tag{4.8}$$

and define the ratio

$$\rho \sim \frac{E}{\Lambda_-} \sim \frac{\Lambda_+}{E} \ll 1. \tag{4.9}$$

In perturbatively unitary theories propagating fields have standard dimensions in units of mass (1 for bosons and 3/2 for fermions, i.e. $n_\Phi = 2$ and $n_\Phi = 1$, respectively). When the theory is not perturbatively unitary, like a higher-derivative theory, fields of arbitrarily negative dimensions may be present. Including these theories is useful to emphasize that our results are intrinsic properties of gauge symmetry and renormalization, and do not depend on the particular model we are working with.

The perturbative expansion is defined as the expansion in powers of ρ and σ. The truncated actions are obtained neglecting the contributions of orders $\rho^{T'}$ and $\sigma^{T'}$ with $T' > T$, and denoted with S_{cT}, $S_{\infty T}$, $S_T^{(i)}$, $S_{\subset T}$, and so on. The S_{cT}-master equation must hold within the truncation, which means $(S_{cT}, S_{cT}) = \mathcal{O}(\rho^{T+1}) + \mathcal{O}(\sigma^{T+1})$. The other identities also hold up to $\mathcal{O}(\rho^{T+1})$- and $\mathcal{O}(\sigma^{T+1})$-corrections.

We show that S_{cT} depends on a finite number of parameters and that radiative corrections are compatible with the truncation. Let us first assume that S_{cT} is $\hbar$-independent. A generic term of S_{cT} has the structure

$$(\kappa^2)^{L-1}\chi\partial^p(\kappa\Phi)^{n_\Phi}(\kappa K)^{n_K}, \tag{4.10}$$

with $L=0$, where n_Φ and n_K are non-negative integer numbers and χ is a product of parameters κ, ζ and ξ. The structure (4.10) for $L>0$ is the one of counterterms. Indeed, since the action S_c has an overall factor $1/\kappa^2$ and a κ is attached to each field and source, an n-leg vertex has at least a factor κ^{n-2}. Thus a diagram with L loops, I internal legs, E external legs and v_{jl} vertices with $n_{jl}=n_{\Phi j}+n_{Kl}$ legs, where $n_{\Phi j}$ and n_{Kl} are the numbers of Φ- and K-legs, respectively, is at least multiplied by a factor

$$\kappa^{\sum_{jl}v_{jl}(n_{jl}-2)}=(\kappa^2)^{L-1}\kappa^E,$$

in agreement with (4.10). We have used the identities $L-I+V=1$ and $\sum_{jl}v_{jl}n_{jl}=2I+E$. We also derive an inequality that we need below. Write $E=E_\Phi+E_K$, where E_Φ and E_K are the numbers of external Φ- and K-legs of the diagram. Observing that $E_K=\sum_{jl}v_{jl}n_{Kl}$, we immediately get

$$E_\Phi+E_K=\sum_{jl}v_{jl}(n_{\Phi j}+n_{Kl})-2I=E_K+\sum_{jl}v_{jl}(n_{\Phi j}-2)-2(L-1),$$

whence

$$n_{\Phi j}\leqslant E_\Phi+2L. \tag{4.11}$$

Now, depending on whether the dimension $[\kappa]$ of κ is positive or negative, we can write

$$\kappa\sim\Lambda_\pm^{[\kappa]}=m^{[\kappa]}\rho^{|[\kappa]|}.$$

If $[\kappa]=0$ we write $\kappa=\sigma$. The terms (4.10) belonging to the truncation must satisfy

$$n_\Phi,n_K,2L\leqslant 2+\frac{T}{|[\kappa]|^e},\qquad \chi_\rho\leqslant T+2|[\kappa]|^e, \tag{4.12}$$

with $e=1$ or 0 for $[\kappa]\neq 0$ and $[\kappa]=0$, respectively, while χ_ρ denotes the order of χ.

The bounds (4.12) are sufficient to show that the number of derivatives p must also be bounded from above. Indeed, in (4.10) the product χ contributes with a factor $\Lambda_+^u\Lambda_-^{-v}m^w$ for some non-negative u,v,w, and we must have $u+v\leqslant T+2|[\kappa]|^e$. Now, given n_Φ, n_K and L, the dimension of $\chi\partial^p$, which is equal to $u-v+w+p$, is also given, but then the inequalities just found imply that p must be bounded from above. This proves that the truncation can contain only a finite number of terms. We denote such number with $N(T)$.

Let us point out that the (4.12) also implies that the number of counterterms included in the truncation decreases when the order of radiative corrections increases, and eventually drops to zero. Only a finite number of Feynman diagrams contribute within the truncation, because the number of loops and the number of vertices we can use are both bounded from above.

When S_{cT} depends on $\hbar$ we must assume that the parameters, or product of parameters, that multiply the terms proportional to $\hbar^g$ have an order $\sim \kappa^{2g}$ higher than if they were tree-level. It is possible to incorporate this assignment in formula (4.10) replacing L with g. Radiative corrections are also consistent.

Now that we know how to define an appropriately truncated action, we study the extension map and make sure that it can be implemented with a finite number of steps. There is a caveat, though. When we make the replacement $\hbar/\varepsilon \rightarrow \lambda'$ we lower the order of the approximation. Indeed, by formula (4.10), a factor κ^2 appears at each loop, so defining the dimensionless constant $\tilde{\kappa} = \kappa m^{-[\kappa]}$ we should consider λ' as $\mathcal{O}(1/\tilde{\kappa}^2)$, because it replaces a $\hbar/\varepsilon$. However, it can be checked that if we really assume $\lambda' = \mathcal{O}(\tilde{\kappa}^{-2})$ then the replacement $\hbar/\varepsilon \rightarrow \lambda'$ can generate infinitely many contributions of the same order. To truncate such contributions we must assume $\lambda' = \mathcal{O}(\tilde{\kappa}^{\omega-2})$, with $\omega > 0$. At the same time, we must be sure that the radiative corrections to λ', which are $\mathcal{O}(1)$ (see (1.3)) are smaller than λ' itself, for which it is sufficient to assume $\omega < 2$. Thus we take $0 < \omega < 2$.

Because the replacement $\hbar/\varepsilon \rightarrow \lambda'$ lowers the order of the approximation, it is not sufficient to truncate the action to order T to determine the extended action to order T. Instead, we must truncate the classical action S_c to some order T_0, to determine the first extended action $S^{(1)}$ to some order T_1, so that the second extended action $S^{(2)}$ is determined to some order T_2, and so on, and guarantee that the final extended action $S_\subset$ is determined to the desired order T. We want to show that this can be done with a finite number of steps. In particular, $T_0(T)$ is finite.

Let us consider a single extension $S_{T_i}^{(i)} \rightarrow S_{T_{i+1}}^{(i+1)}$. The renormalized action $S_{\infty T_i}^{(i)}(\lambda, \hbar/\varepsilon, \hbar)$ contains terms (4.10) multiplied by

$$\left(\frac{\hbar}{\varepsilon}\right)^f \hbar^g, \qquad f + g = L.$$

When we replace $\hbar/\varepsilon$ with λ' we obtain objects

$$\hbar^g m^{(2-\omega)f[\kappa]} \kappa^{2g+\omega f-2} \chi \partial^p (\kappa\Phi)^{n_\Phi} (\kappa K)^{n_K}.$$

We want to determine all terms of this type that fall within the truncation T_{i+1}. In particular, they must satisfy

$$n_\Phi, n_K, \omega f + 2g \leqslant 2 + \frac{T_{i+1}}{|[\kappa]|^e}, \qquad \chi_\rho \leqslant T_{i+1} + 2|[\kappa]|^e,$$

therefore

$$L = f + g < f + \frac{2}{\omega} g \leqslant \frac{2}{\omega} + \frac{T_{i+1}}{\omega |[\kappa]|^e}.$$

Using (4.11) we see that all vertices v participating in the diagrams must satisfy

$$n_{\Phi j} \leqslant \left(2+\frac{T_{i+1}}{|[\kappa]|^{e}}\right)\left(1+\frac{2}{\omega}\right), \qquad n_{Kl} \leqslant 2+\frac{T_{i+1}}{|[\kappa]|^{e}}, \qquad \chi_{\rho}^{(v)} \leqslant T_{i+1}+2|[\kappa]|^{e}, \tag{4.13}$$

where $\chi_{\rho}^{(v)}$ is the order of the factor $\chi^{(v)}$ appearing in the vertex. As before, given $n_{\Phi j}$ and n_{Kl}, the number of derivatives p_v that can appear in the vertex is also bounded from above, because a large p_v would raise the order of $\chi^{(v)}$ arbitrarily. Thus, only a finite number of vertices can participate in the diagrams that contribute to $S_{T_{i+1}}^{(i+1)}$. At this point, we determine $T_i(T_{i+1})$ so that $S_{T_i}^{(i)}$ contains all such vertices.

Recall that the parameter-complete action $S_{\subset T}$ we want to determine contains a finite number of terms $N(T)$. Thus the extension chain $\{S_{T_i}^{(i)}(\lambda^{(i)},\hbar)\}$ contains at most $N(T)$ elements, because each step adds at least one independent parameter, and there cannot be more independent parameters than Lagrangian terms. Consequently, $T_0(T)$ is finite, as we wished to prove. Thus, after a finite number of operations we achieve closure within the truncation and determine the parameter-complete action $S_{\subset T}$. That action is equipped with all parameters that are necessary to renormalize divergences by means of parameter-redefinitions, without using cohomological properties, within the truncation.

Observe that choosing ω small the bounds (4.13) become larger, which means that to determine the exended action more and more precisely as a function of the new parameters λ' we must work harder and harder. These facts emphasize that our extension procedure is mainly a theoretical tool. On the one side it is conceptually simple, on the other side it appears to be prohibitive from the practical point of view, unless ad hoc parameter-redefinitions and other tricks are found case-by-case to reduce the effort.

5 Parameter-completion and canonical transformations

So far we have used the approach i), where the extension algorithm is applied after introducing redundant parameters to renormalize all kinds of divergences, including those proportional to the field equations, by means of parameter-redefinitions, instead of using both parameter-redefinitions and canonical transformations of fields and sources. Now it is relatively easy to explain how to proceed in the standard approach ii). We understand that we are working with truncated actions where necessary, although we do not make it explicit all the time.

The parameter-extension map is unchanged. Making the source- and field-dependences explicit, we write (3.8) as

$$S_c(\Phi,K,\lambda,\hbar)\rightarrow S_{\infty}(\Phi,K,\lambda,\lambda',\hbar).$$

The extension chain $\{S^{(0)},S^{(1)},S^{(2)}\cdots\}$ is obtained taking $S^{(0)}$ as the starting classical action $S_c(\lambda,\hbar)$, and $S^{(i)}$, $i>0$, as the parameter extension of $S^{(i-1)}$. It is often convenient to express each action $S^{(i)}$ in some specific field- and source-variables, which we denote with $\Phi^{(i)}, K^{(i)}$. For example, when the classical action is written in some standard form, we may want to preserve that form throughout the extension process. This can be obtained updating the field- and source-variables from $\{\Phi^{(i-1)}, K^{(i-1)}\}$ to $\{\Phi^{(i)}, K^{(i)}\}$ by means of canonical transformations. A common option is to choose the "essential" form [5, 6], where the dominant kinetic terms of the field equations (e.g. $\Box\phi$ and $\partial\!\!\!/\psi$ for bosons ϕ and fermions ψ in perturbatively unitary theories) are contained only in the dominant kinetic terms of the action (up to total derivatives), and removed from every other place by means of field redefinitions.

As before, the parameters contained in $S^{(i)}$ are denoted with $\lambda^{(i)}$, and we have $\{\lambda^{(i)}\} \subset \{\lambda^{(i+1)}\} = \{\lambda^{(i)}, \lambda'^{(i)}\}$. Now we state that the chain closes if there exists an $i = I$ such that the parameter extension of $S^{(I)}$ is stable in the sense that there exist parameter-redefinitions $\tilde{\lambda}(\lambda^{(I)}, \lambda'^{(I)}, \hbar)$ and canonical transformations

$$\tilde{\Phi}(\Phi, K, \lambda^{(I)}, \lambda'^{(I)}, \hbar), \qquad \tilde{K}(\Phi, K, \lambda^{(I)}, \lambda'^{(I)}, \hbar),$$

such that

$$S^{(I+1)}(\Phi, K, \lambda^{(I+1)}, \hbar) = S^{(I)}(\tilde{\Phi}, \tilde{K}, \tilde{\lambda}(\lambda^{(I)}, \lambda'^{(I)}, \hbar), \hbar). \tag{5.1}$$

When these operations are combined with the truncations explained above, the extension chain closes after a finite number of manipulations. The action $S_{\subset} = S^{(I)}$ is then parameter-complete within the truncations.

Recapitulating, $S^{(I+1)}$ is the renormalized action and $S_{\rm B} = S^{(I)}$ is the bare action. Indeed, setting $\lambda'^{(I)} = \hbar/\varepsilon$ we obtain

$$S^{(I+1)}(\Phi, K, \lambda^{(I)}, \hbar/\varepsilon, \hbar) = S_{\rm B}(\Phi_{\rm B}, K_{\rm B}, \lambda_{\rm B}, \hbar),$$

where

$$\Phi_{\rm B} = \tilde{\Phi}(\Phi, K, \lambda^{(I)}, \hbar/\varepsilon, \hbar), \qquad K_{\rm B} = \tilde{K}(\Phi, K, \lambda^{(I)}, \hbar/\varepsilon, \hbar), \qquad \lambda_{\rm B} = \tilde{\lambda}(\lambda^{(I)}, \hbar/\varepsilon, \hbar)$$

are the relations between bare and renormalized fields, sources and parameters.

6 Proper formalism and parameter-completion without cohomology

In the usual formalism of quantum field theory, based on the generating functional Γ of one-particle irreducible diagrams, any time the canonical transformations are nonlinear, or contain field-dependent source-transformations, they cannot be interpreted as true changes of field variables in the functional integral, but only as replacements of integrands [5]. To overcome this issue, in ref. [5] we developed a field-covariant formalism for quantum field theory and in [7] we introduced a new generating functional Ω of one-particle irreducible diagrams, called "master functional", that behaves as a scalar under arbitrary perturbative changes of field variables (namely field redefinitions that can be expressed as local perturbative series around the identity). The master functional supersedes the functional Γ, which does not transform in a simple way. In ref. [2] the formalism was generalized to gauge theories.

The set of integrated fields is enlarged from Φ to a set of "proper fields" Φ, N. Similarly, the set of sources K is enlarged to the "proper sources" K, H. The extra fields N^I are associated with local composite fields $\mathcal{O}^I(\Phi)$, while the extra sources H^I are associated with the $\mathcal{O}^I$-gauge transformations. The master functional Ω and its classical action S_{cN}, called "proper action", are functionals of the proper fields and the proper sources.

In ref. [2] it was shown that when cohomological properties such as (2.3) hold in the usual formalism, they can be generalized to the proper formalism and the master functional for gauge theories. Doing so, a "proper cohomology" emerges, based on the *squared antiparentheses*

$$\lfloor X, Y \rfloor \equiv \int \left(\frac{\delta_r X}{\delta \Phi^A}\frac{\delta_l Y}{\delta K_A} + \frac{\delta_r X}{\delta N^I}\frac{\delta_l Y}{\delta H_I} - \frac{\delta_r X}{\delta K_A}\frac{\delta_l Y}{\delta \Phi^A} - \frac{\delta_r X}{\delta H_I}\frac{\delta_l Y}{\delta N^I} \right) \tag{6.1}$$

between two functionals X and Y of Φ, K, N and H. The squared antiparentheses satisfy identities analogous to the ones satisfied by the usual antiparentheses, and can be used to extend the Batalin-Vilkovisky formalism to the sector of composite fields.

Given a classical action $S_c(\Phi, K)$ that satisfies (3.1), it is possible to construct a proper classical action $S_{cN}(\Phi, K, N, H)$ that satisfies the proper master equation

$$\lfloor S_{cN}, S_{cN} \rfloor = 0 \tag{6.2}$$

and is such that the extra fields N have "propagator" equal to one, and $S_{cN} = S_c$ at $H_I = 0$, $\delta_l S_{cN}/\delta N^I = 0$. The master functional Ω collects the one-particle irreducible diagrams generated by S_{cN}.

In [2] we used these tools to show that if (2.3) holds then S_{cN} can be extended till it becomes parameter-complete. In the proper approach a parameter-complete action contains enough independent parameters so that all divergences can be removed by means of parameter-redefinitions and "proper" canonical transformations, which are special source-independent linear transformations of the proper fields Φ, N, combined with Φ-N-independent source-transformations (see formula (6.9)).

In the derivation of [2] cohomological properties played a crucial role. On the other hand, in the previous sections we proved that cohomological properties are not really essential for renormalization. In this section we show that this fact remains true in the field-covariant proper approach and construct a parameter-complete proper action $S_{\subset N}$ without relying on cohomological assumptions.

As before, we assume that the starting classical action $S_c(\lambda, \hbar)$ satisfies the master equation (3.1) exactly at the regularized level, which ensures that gauge anomalies are manifestly absent. Then the proper classical action S_{cN} also satisfies the proper master equation (6.2) at the regularized level (see below for the proof). To make the presentation more understandable, we first work with gauge algebras that close off shell, and later generalize the results to gauge algebras that close only on shell.

If the gauge algebra closes off shell we choose $S_c(\Phi, K)$ of the form (2.1). Then the starting proper action is

$$S_{cN}(\Phi, K, N, H) = S_c(\Phi, K) - \int R^I_{\mathcal{O}}(\Phi) H_I + \frac{1}{2}\int \tilde{N}^I A_{IJ} \tilde{N}^J + \int \rho_{vI} \mathcal{N}^v(\tilde{N}) \mathcal{O}^I_{\text{inv}}(\Phi), \tag{6.3}$$

where $\tilde{N}^I = N^I - \mathcal{O}^I(\Phi)$, A_{IJ} and ρ_{vI} are constants,

$$R^I_{\mathcal{O}}(\Phi) \equiv \int R^A(\Phi) \frac{\delta_l \mathcal{O}^I}{\delta \Phi^A}$$

are the gauge transformations of the composite fields $\mathcal{O}^I$, $\mathcal{O}^I_{\text{inv}}$ are the gauge-invariant composite fields and $\mathcal{N}^v(\tilde{N})$ denotes a basis of local monomials, at least quadratic in $\tilde{N}$, constructed with $\tilde{N}$ and its derivatives. The functional integral is over both Φ and N, and the "improvement term"

$$\frac{1}{2}\int N^I A_{IJ} N^J, \tag{6.4}$$

provides propagators for the fields N. All other quadratic terms coming from the last contribution to (6.3) must be treated perturbatively with respect to (6.4). It is easy to check that (3.1) indeed implies (6.2).

The generating functional of Green functions Z and the generating functional of connected Green functions W are defined from

$$Z(J, K, L, H) = \int [\mathrm{d}\Phi \mathrm{d}N] \exp\left(-S_{cN}(\Phi, K, N, H) + \int \Phi^A J_A + \int N^I L_I\right) = \exp W(J, K, L, H). \tag{6.5}$$

Since the N-propagator is equal to 1, the N-integral can be done exactly in dimensional regularization, and amounts to make the Legendre transform of S_{cN} with respect to N^I. Defining

$$-\tilde{S}(\Phi, K, L, H) = -S_{cN}(\Phi, K, N, H) + \int N^I L_I, \qquad L_I = \frac{\delta_l S_{cN}}{\delta N^I},$$

we get

$$\tilde{S} = \mathcal{S}(\Phi) - \int \left(R^A(\Phi) K_A + R^I_{\mathcal{O}}(\Phi) H_I\right) - \frac{1}{2}\int L_I (\tilde{A}^{-1})^{IJ} L_J - \int \tau_{vI} \mathcal{N}^v(L) \mathcal{O}^I_{\text{inv}}(\Phi) - \int \mathcal{O}^I L_I,$$

where the τ_{vI}s are parameters equal to ρ_{vI} plus perturbative corrections and $\tilde{A}$ is the A-transpose. Thus we have

$$Z = \int [\mathrm{d}\Phi] \exp\left(-\tilde{S}(\Phi, K, L, H) + \int \Phi^A J_A\right). \tag{6.6}$$

We see that L_I play the role of sources coupled to the composite fields $\mathcal{O}^I$. Observe that the equations

$$L_I = \frac{\delta_l S_{cN}}{\delta N^I} = 0,$$

which switch composite fields off, are solved by $\tilde{N}^I = 0$. Moreover, the conditions H_I =constants amount to a change of gauge-fixing.

The master functional Ω is defined as the Legendre transform of W with respect to both Φ and L, while K and H remain inert. We have

$$\Omega(\Phi, K, N, H) = -W(J, K, L, H) + \int \left(\Phi^A J_A + N^I L_I\right), \tag{6.7}$$

with

$$\begin{aligned} \Phi^A &= \frac{\delta_r W}{\delta J_A}, \qquad N^I = \frac{\delta_r W}{\delta L_I}, \qquad \frac{\delta_r W}{\delta K_A} = -\frac{\delta_r \Omega}{\delta K_A}, \\ J_A &= \frac{\delta_l \Omega}{\delta \Phi^A}, \qquad L_I = \frac{\delta_l \Omega}{\delta N^I}, \qquad \frac{\delta_r W}{\delta H_I} = -\frac{\delta_r \Omega}{\delta H_I}. \end{aligned} \tag{6.8}$$

Since no confusion is expected to arise, we use the same names for the proper fields Φ, N inside both S_N and Ω, while strictly speaking the latter are averages of the former: $\langle \Phi_S \rangle = \Phi_\Omega$, $\langle N_S \rangle = N_\Omega$. In the Legendre transform (6.7) the improvement term (6.4) must again be treated as dominant with respect to all other quadratic terms involving the fields N^I.

If the proper classical action S_{cN} satisfies (6.2), then the master functional Ω satisfies the proper master equation

$$\lfloor \Omega, \Omega \rfloor = 0.$$

The proof [2], which we do not repeat here, follows from the generalization of identity (3.3) to the proper formalism.

Changes of field variables and changes of gauge-fixings can be easily implemented as "proper" canonical transformations, namely canonical transformations for the proper fields and sources with generating functional

$$F(\Phi, K', N, H') = \int (\Phi^A + N^I b_I^A) K'_A + \int N^I z_I^J (H'_J - \xi_J), \tag{6.9}$$

where b_I^A, z_I^J and ξ_I are constants, which can be both c-numbers and Grassmann variables. More explicitly, a proper transformation reads

$$\begin{aligned} \Phi^{A\prime} &= \frac{\delta F}{\delta K'_A} = \Phi^A + N^I b_I^A, \qquad K_A = \frac{\delta F}{\delta \Phi^A} = K'_A, \\ N^{I\prime} &= \frac{\delta F}{\delta H'_I} = N^J z_J^I, \qquad H_I = \frac{\delta F}{\delta N^I} = z_I^J (H'_J - \xi_J) + b_I^A K'_A. \end{aligned} \tag{6.10}$$

Let us briefly describe how the field redefinitions contained in (6.10) work. If we write

$$\Phi^{A\prime} = \Phi^A + \mathcal{O}^I(\Phi)b_I^A + \tilde{N}^I b_I^A, \tag{6.11}$$

when we set $\tilde{N}^I = K = 0$, H =constant, to switch off the sectors of composite fields and gauge transformations, (6.11) becomes $\Phi^{A\prime} = \Phi^A + \mathcal{O}^I(\Phi)b_I^A$, which is the expansion of the most general local perturbative change of field variables. However, the conditions $\tilde{N}^I = 0$ switch off the composite-field sector only before the transformation. Indeed, due to the term $\tilde{N}^I b_I^A$ in (6.11) after the transformation the solutions of $L_I' = \delta_l S_N'/\delta N^{I\prime} = 0$ at $K = 0$, H =constant, are no longer $\tilde{N}^I = 0$, but some new $\tilde{N}^{I\prime} = 0$. Working out $\tilde{N}^{I\prime}$ it is found that at $\tilde{N}^{I\prime} = K = 0$, H =constant the effective change of variables is corrected by $\mathcal{O}(b^2)$-terms and finally reads

$$\Phi^{A\prime} = \Phi^A + \mathcal{O}^I(\Phi)\tilde{b}_I^A,$$

where $\tilde{b}_I^A = b_I^A + \mathcal{O}(b^2)$ is some calculable power series in b. More details can be found in refs. [2, 7].

Now we generalize the arguments of the previous sections to the proper formalism for gauge theories. The raw renormalization algorithm is immediately generalized replacing the classical action S_c with the classical proper action S_{cN}, the antiparentheses with the squared antiparentheses, the master equation with the proper master equation and the Γ-functional with the master functional Ω. We do not repeat the derivation here, because it was already given in section 5 of ref. [2]. Calling S_{Nn} and Ω_n the proper action and the master functional renormalized up to n loops (with $S_{N0} = S_{cN}$), we subtract the order-$(n+1)$ divergent part $\Omega_{n\,\mathrm{div}}^{(n+1)}$ of Ω_n (in the minimal subtraction scheme) and define

$$S_{N\,n+1} = S_{Nn} - \Omega_{n\,\mathrm{div}}^{(n+1)}. \tag{6.12}$$

Iterating this operation we construct the renormalized proper action $S_{N\infty}$ and prove that it satisfies the master equation

$$\lfloor S_{N\infty}, S_{N\infty}\rfloor = 0.$$

Now, if λ denotes the parameters contained in S_{cN}, the map

$$S_{cN}(\Phi, K, N, H, \lambda, \hbar) \rightarrow S_{N\infty}(\Phi, K, N, H, \lambda, \hbar/\varepsilon, \hbar) \tag{6.13}$$

sends a solution of the proper master equation into an extended solution. Thus we can define the parameter-extension map

$$S_{cN}(\Phi, K, N, H, \lambda, \hbar) \rightarrow S_{N\infty}(\Phi, K, N, H, \lambda, \lambda', \hbar), \tag{6.14}$$

and build an extension chain $\{S_N^{(0)}, S_N^{(1)}, S_N^{(2)} \cdots\}$ of proper classical actions, where $S_N^{(0)} = S_{cN}$, and $S_N^{(i)}$, $i > 0$, is the parameter extension of $S_N^{(i-1)}$. We state that the extension chain closes if there exists an $i = I$ such that

$$S_N^{(I+1)}(\Phi, K, N, H, \lambda^{(I+1)}, \hbar) = S_N^{(I)}(\tilde{\Phi}, \tilde{K}, \tilde{N}, \tilde{H}, \tilde{\lambda}(\lambda^{(I)}, \lambda'^{(I)}, \hbar), \hbar), \tag{6.15}$$

where the tilded proper fields and sources are related to the untilded ones by a proper canonical transformation that depends on $\lambda^{(I)}$, $\lambda'^{(I)}$ and $\hbar$. Within the truncations, the extension chain closes after a finite number of operations, and $S^{(I)}$ identifies the parameter-complete proper action $S_{\subset N}$.

Although we have assumed that the gauge algebra closes off shell, so far, this assumption did not enter the key-steps of our arguments. Its main purpose was to let us use the simple and explicit form (6.3) of the starting proper action S_{cN}. Relaxing the assumption of off-shell closure, we can start from any classical action $S_c(\Phi, K)$ that satisfies (3.1). Then we pick a basis $\{\mathcal{O}^I(\Phi)\}$ of local composite fields and make a canonical transformation with generating function

$$F_c(\Phi, K') = \int \Phi^A K'_A - \int \mathcal{O}^I(\Phi) H_I.$$

The transformed action reads

$$\tilde{S}_c(\Phi, K, H) = S_c(\Phi, K + \int \frac{\delta_l \mathcal{O}^I}{\delta \Phi} H_I) \tag{6.16}$$

and obviously satisfies $(\tilde{S}_c, \tilde{S}_c) = 0$. Define the proper action

$$S_{cN}(\Phi, K, N, H) = \tilde{S}_c(\Phi, K, H) + \frac{1}{2} \int \tilde{N}^I A_{IJ} \tilde{N}^J, \tag{6.17}$$

where $\tilde{N}^I = N^I - \mathcal{O}^I$. It is easy to prove that S_{cN} satisfies the proper master equation (6.2). First observe that $\lfloor \tilde{S}_c, \tilde{S}_c \rfloor = (\tilde{S}_c, \tilde{S}_c) = 0$, therefore

$$\lfloor S_{cN}, S_{cN} \rfloor = \lfloor \tilde{S}_c, \int \tilde{N} A \tilde{N} \rfloor.$$

Next, working out the squared antiparentheses explicitly, formulas (6.16) and (6.17) give

$$\lfloor \tilde{S}_c, \tilde{N}^I \rfloor = \int \frac{\delta_r \tilde{S}_c}{\delta K_A} \frac{\delta_l \mathcal{O}^I}{\delta \Phi^A} - \frac{\delta_r \tilde{S}_c}{\delta H_I} = 0,$$

whence $\lfloor S_{cN}, S_{cN} \rfloor = 0$ immediately follows.

The generating functionals are defined as in (6.5) and (6.7). Integrating over the extra fields N we obtain (6.6) with

$$\tilde{S}(\Phi, K, H, L) = \tilde{S}_c(\Phi, K, H) - \frac{1}{2} \int L_I (\tilde{A}^{-1})^{IJ} L_J - \int \mathcal{O}^I L_I.$$

All other derivations are identical to the ones given before, since the explicit forms of S_c and S_{cN} play no role in those. Again, we have the parameter-extension map (3.7), which allows us to construct the extension chain. We define closure by formula (6.15), where the canonical transformation must be proper. The truncations ensure that the chain closes after a finite number I of steps. The parameter-complete proper action $S_{\subset N} = S^{(I)}$ can thus be worked out with a finite number of calculations, and is such that its divergences can be subtracted redefining parameters and making proper canonical transformations.

7 Remarks

In this section we collect a few observations that can make us better appreciate some properties of the constructions presented so far.

Every time we extend the solution with the operations described above, we introduce a new parameter λ', obtained replacing $\hbar/\varepsilon$. Since the factors $\hbar/\varepsilon$ multiply powers of other parameters λ coming from Feynman diagrams, the new parameter λ' also multiplies various powers of λ. Thus the extended actions are parametrized in non-standard ways.

Call the coefficient of a Lagrangian monomial constructed with the fields, the sources and their derivatives, *parameter-singlet* if it is made of a single parameter. Call it *parameter-product* if it is made of a product of parameters, with various (possibly negative) exponents. It may be convenient to organize the action so that, proceeding order-by-order along with the truncations defined previously, the first time a new parameter appears it appears as a parameter-singlet. To achieve this goal, the first time we find a Lagrangian term multiplied by a new independent coefficient equal to a parameter-product, say $\lambda\lambda'$, we redefine that coefficient as a new parameter-singlet α, and re-express α everywhere else in terms of λ and λ'. If the coefficient is a sum of more parameter-products, to avoid complicated functions of parameters we call α one parameter-product of the sum, randomly chosen. When we do these operations, we very likely generate negative powers of parameters, which makes the new parametrization also non-standard. Ultimately, the original parametrization $S_{\subset}(\Phi, K, \lambda, \hbar)$ may be the most convenient one, because at least it guarantees that all parameters appear polynomially.

Let us emphasize that if we search for the most general solution $S_{\sqsubset}$ of the master equation (see next section) we get coupled quadratic equations (8.2) that can lead to even more involved non-standard parametrizations, when cohomological properties do not hold.

Another remark concerns possible modifications of the gauge symmetry. When the gauge algebra closes off shell and the cohomological property (2.3) holds, the symmetry transformations are not affected by radiative corrections in any observable way, and the renormalized action is really equivalent to the starting classical action S_c, extended as shown in (2.5), up to canonical transformations and parameter-redefinitions. Instead, when (2.3) does not hold there is no obvious reason why the gauge symmetry should remain the same after renormalization. Radiative corrections can modify it in a physically observable way. Moreover, even when the starting gauge symmetry, encoded in S_c, closes off shell, the final one, encoded in $S_{\subset}$ may close only on shell.

We do not have examples of non-trivial parameter extensions induced by renormalization, since the theories we normally deal with do satisfy (2.3). Nevertheless, to clarify the remark just made it may be helpful to think of an interacting gauge theory as a parameter extension of its free-field limit. For example, switching off antighosts $\bar{C}$ and Lagrange multipliers B, the solution of the master equation of an Abelian gauge theory is just

$$S_{\mathrm{Ab}} = \frac{1}{4}\int (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \int K_\mu \partial_\mu C.$$

If we assume that the number of photons is appropriate, we can write the action S_{nAb} of non-Abelian Yang-Mills theory as

$$S_{\mathrm{nAb}} = S_{\mathrm{Ab}} + \omega(A) - \int (\Delta R^A) K_A,$$

where

$$\omega(A) = g f^{abc} \int (\partial_\mu A^a_\nu) A^b_\mu A^c_\nu + \frac{g^2}{4}\int (f^{abc} A^b_\mu A^c_\nu)^2,$$
$$-\int (\Delta R^A) K_A = g \int \left[K^a_\mu f^{abc} A^b_\mu C^c + \frac{1}{2} K^a_C f^{abc} C^b C^c \right].$$

The first line is a one-parameter extension of the action, while the second line encodes the corresponding extensions of the symmetry transformations.

This observation illustrates what may happen in more involved theories, where new interactions may be created by the extension. In that case the symmetry transformations may be modified accordingly, to preserve the master equation.

8 Search for the most general solution of the master equation

In the previous sections we proved that a manifestly anomaly-free gauge theory always admits a parameter-complete classical action $S_{\subset}$. A related issue that remains to be addressed is the search for the most general solution $S_{\sqsubset}$ of the master equation. In this section we make some remarks about this topic and compare the properties of the actions $S_{\sqsubset}$ and $S_{\subset}$.

Certainly $S_{\sqsubset}$ is parameter-complete, because it includes $S_{\subset}$. Another way to prove the parameter-completeness of $S_{\sqsubset}$ is to take $S_{\sqsubset}$ as the classical action and use the raw renormalization algorithm to work out the renormalized action $S_{\infty\sqsubset}$. Since $S_{\sqsubset}$ is the most general solution of the master equation, $S_{\infty\sqsubset}$ must be related to $S_{\sqsubset}$ by means of parameter-redefinitions and canonical transformations. Observe that both these arguments do not make use of cohomological properties.

Let us now see how $S_{\sqsubset}$ can be built. Start from any solution $S_c(\Phi, K, \lambda)$ of the master equation and let $\{\mathcal{L}^i(\Phi, K, \lambda)\}$ denote a basis of Lagrangian terms $\mathcal{L}^i$ constructed with the fields, the sources and their derivatives. For simplicity, we can take a basis made of monomials. Composite fields $\mathcal{O}^I$ can be included also, coupled to external sources L_I. The most general extension of the starting classical action S_c can be parametrized as

$$S_{\sqsubset}(\Phi, K, \lambda, \tau) = S_c(\Phi, K, \lambda) + \tau_i \Delta^i(\Phi, K, \lambda), \qquad \Delta^i = \int \mathcal{L}^i, \tag{8.1}$$

where the sum over i is understood and the τ_is are constants. Requiring that $S_{\sqsubset}$ solve the master equation $(S_{\sqsubset}, S_{\sqsubset}) = 0$ we obtain the condition

$$2\tau_i(S_c, \Delta^i) + \tau_i \tau_j (\Delta^i, \Delta^j) = 0$$

for the constants τ. The objects (Δ^i, Δ^j) are local functionals, equal to the integrals of local composite fields of dimension 5 and ghost number one. Let Δ_5^a denote a basis of such functionals. Then there exist constants C_a^{ij} such that

$$(\Delta^i, \Delta^j) = C_a^{ij} \Delta_5^a,$$

where the sum over a is understood. Expanding S_c as $\sigma_i \Delta^i$, where σ_i are known constants, the equations we must solve can be written as

$$(\sigma_i + \tau_i) C_a^{ij} (\sigma_j + \tau_j) = 0, \qquad \sigma_i C_a^{ij} \sigma_j = 0. \tag{8.2}$$

The latter is a constraint on the σ_is, following from $(S_c, S_c) = 0$. Clearly, $\tau_i = h\sigma_i$, where h is an overall constant, are solutions, but they just give $S_{\sqsubset} = (1 + h) S_c$.

If we do not assume cohomological properties such as (2.3), the problem remains quadratic. The coupled quadratic algebraic equations (8.2) can be very difficult to solve, and it is not even evident how to solve them perturbatively. Thus, the search for $S_{\sqsubset}$ might not be a practically viable strategy.

If we ignore the difficulties to build $S_{\sqsubset}$ and just assume that $S_{\sqsubset}$ is known, we can investigate its cohomological properties. Denote the $S_{\sqsubset}$-independent parameters with ρ_i. Differentiating $(S_{\sqsubset}, S_{\sqsubset}) = 0$ with respect to ρ, we find

$$\left(S_{\sqsubset}, \frac{\partial S_{\sqsubset}}{\partial \rho_i} \right) = 0. \tag{8.3}$$

If $S_{\sqsubset}$ depends on $\hbar$ what is actually important is the $\hbar \to 0$-limit of this equation, which reads

$$\left(\bar{S}_0, \frac{\partial \bar{S}_0}{\partial \rho_i} \right) = 0, \tag{8.4}$$

$\bar{S}_0$ being the $\hbar \rightarrow 0$-limit of $S_{\sqsubset}$. We see that the ρ-derivatives of $\bar{S}_0$ are solutions of the cohomological problem (2.2) with S_c replaced by $\bar{S}_0$. However, there is no guarantee that all non-trivial solutions of that problem are contained in the set $\{\partial \bar{S}_0/\partial \rho_i\}$.

Take $S_{\sqsubset}$ as the starting classical action. As usual, call S_n the action renormalized up to n loops, with $S_0 = S_{\sqsubset}$, and assume that it satisfies the master equation $(S_n, S_n) = 0$ exactly. Then the $(n+1)$-loop divergences $\Gamma^{(n+1)}_{n\,\text{div}}$ satisfy the cohomological problem $(\bar{S}_0, \Gamma^{(n+1)}_{n\,\text{div}}) = 0$. If we want to remove $\Gamma^{(n+1)}_{n\,\text{div}}$ redefining the ρs and making canonical transformations, we need to know either that *a*) the set $\{\partial \bar{S}_0/\partial \rho_i\}$ contains all the non-trivial solutions of the cohomological problem, or anyway *b*) it contains the solutions generated by renormalization, namely there exist constants $\Delta_{n+1}\rho_i$ and local functionals χ_{n+1} such that

$$\Gamma^{(n+1)}_{n\,\text{div}} = \sum_i \Delta_{n+1}\rho_i \frac{\partial \bar{S}_0}{\partial \rho_i} + (\bar{S}_0, \chi_{n+1}).$$

However, it is not enough to know that $S_{\sqsubset}$ is the most general solution of the master equation to prove *a*). In principle, there might be solutions of the cohomological problem that cannot be embedded into any extension of the classical action. As far as statement *b*) is concerned, the parameter-completeness of $S_{\sqsubset}$ proves that it does hold, but, again, this argument does not use cohomological properties.

To conclude, instead of trying to solve the problem in a purely algebraic way, it is more convenient to let renormalization build the extended action, as we have done in this paper. Then we discover that a parameter-complete solution $S_{\subset}$ always exists and obeys property *b*) (with $\bar{S}_0$ replaced by the $\hbar \rightarrow 0$-limit of $S_{\subset}$), but not necessarily property *a*). The most general solution $S_{\sqsubset}$ of the master equation is also parameter-complete, and obeys *b*). The action $S_{\sqsubset}$ may coincide with $S_{\subset}$ or be more general than $S_{\subset}$. In either case, statement *a*) is not guaranteed to hold. Ultimately, only cohomological theorems can ensure a property as strong as *a*), but renormalization does not need that much.

9 Conclusions

In this paper we have shown that in a manifestly anomaly-free gauge theory it is always possible to extend the classical action S_c into a parameter-complete action $S_{\subset}$ that satisfies the master equation and is able to remove all divergences redefining its own parameters and making canonical transformations. The construction was also extended to the master functional and the field-covariant proper formalism for gauge theories, where renormalization works by means of parameter-redefinitions and proper canonical transformations. Such canonical transformations are true changes of variables in functional integrals and generating functionals, rather than mere replacements of integrands.

The compatibility between gauge symmetry and renormalization is encoded in an intrinsically quadratic problem, because the master equation is quadratic in the action. The main virtue of our algorithm is that it solves the quadratic problem even when it is not possible to reduce it to a much simpler, linear (cohomological) problem. Cohomological properties can linearize the quadratic problem, but their proofs must be worked out case-by-case and normally demand a remarkable effort. Sufficiently powerful cohomological theorems might not hold in the theory we are insterested in. Even when they hold, we might just not want to use them. It is interesting to know that, without assuming cohomological properties, whenever gauge anomalies are manifestly absent the classical action can be iteratively extended till it becomes parameter-complete. In other words, quantum field theory, renormalization and renormalization-group invariance are intrinsically compatible with gauge symmetry.

At the practical level, we start from any classical action $S_c(\Phi, K)$. In case we want to work with the proper formalism and the master functional, we construct the proper action (6.17). Then we calculate the renormalization of the theory, which allows us to define the maps (3.7) or (6.13). At this point we discover that renormalization is able to build an extended classical action, which still satisfies the master equation, but contains a new independent parameter. Taking advantage of this fact and iterating the extension till it closes, we end up with the parameter-complete action, $S_\subset$ or $S_{N\subset}$, which satisfies the master equation and is able to renormalize all divergences by means of parameter-redefinitions and (proper) canonical transformations. The renormalization algorithm defined by this procedure is conceptually simpler than any previously known one.

The results of this paper lead us to conjecture that if the theory is potentially plagued with gauge anomalies, but admits anomaly cancellation at one loop, it is always possible to find a parameter-complete extension of the classical action such that the functionals Γ and Ω satisfy their master equations exactly in the physical limit. However, we have to leave the investigation of this issue to a separate work.

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