IFUP-TH 2012/09

# A Master Functional For Quantum Field Theory

*Damiano Anselmi*

*Dipartimento di Fisica "Enrico Fermi", Università di Pisa,*
*Largo B. Pontecorvo 3, I-56127 Pisa, Italy,*

damiano.anselmi@df.unipi.it



**Abstract**

We study a new generating functional of one-particle irreducible diagrams in quantum field theory, called master functional, which is invariant under the most general perturbative changes of field variables. The usual functional $\Gamma$ does not behave as a scalar under the transformation law inherited from its very definition as the Lagendre transform of $W = \ln Z$, although it does behave as a scalar under an unusual transformation law. The master functional, on the other hand, is the Legendre transform of an improved functional $W$ with respect to the sources coupled to both elementary and composite fields. The inclusion of certain improvement terms in $W$ and $Z$ is necessary to make the new Legendre transform well defined. The master functional behaves as a scalar under the transformation law inherited from its very definition. Moreover, it admits a proper formulation, obtained extending the set of integrated fields to so-called proper fields, which allows us to work without passing through $Z$, $W$ or $\Gamma$. In the proper formulation the classical action coincides with the classical limit of the master functional, and correlation functions and renormalization are calculated applying the usual diagrammatic rules to the proper fields. Finally, the most general change of field variables, including the map relating bare and renormalized fields, is a linear redefinition of the proper fields.

# 1 Introduction

Renormalization, as it is usually formulated, is not a change of variables in the functional integral, combined with parameter redefinitions, but a simple replacement of variables and parameters inside the action. More precisely, the action is correctly transformed according to the field redefinition, but the term $\int J\varphi$, which identifies the "elementary field" used to write Feynman rules and calculate diagrams, is not transformed, rather just replaced with $\int J'\varphi'$, the analogous term for the new variables. In simple power-counting renormalizable theories, such as ordinary Yang-Mills theory, where the renormalization of fields and sources is multiplicative, it is straightforward to turn replacements into true changes of field variables. Instead, in theories such as Yang-Mills theory with an unusual action, or with composite fields turned on, as well as effective field theories and gravity, the relation between bare and renormalized fields can be non-linear [1]. In those cases replacements are convenient shortcuts that allow us to avoid certain lengthy manipulations. However, they are not completely satisfactory, since the do not really allow us to write precise identities relating generating functionals before and after the changes of variables.

A perturbative field redefinition is a field redefinition that can be expressed as the identity map plus a perturbative series of local monomials of the fields and their derivatives. In ref. [2] we studied how a general perturbative change of integration variables in the functional integral reflects on the generating functionals $Z$ and $W = \ln Z$. Due to the intimate relation between composite fields $\mathcal{O}^I(\varphi)$ and changes of field variables, it is convenient to include sources $L_I$ coupled to the $\mathcal{O}^I(\varphi)$s, besides the sources $J$ coupled to the elementary fields $\varphi$. In a particularly convenient approach, called *linear approach*, all perturbative changes of field variables, including the *BR map*, which is the map relating bare and renormalized fields, are expressed as linear source redefinitions of the form

$$L_I = L'_J z_I^J + b_I J', \qquad J = J', \tag{1.1}$$

where $z_I^J$ and $b_I$ are constants. The $Z$- and $W$- functionals behave as scalars,

$$Z'(J', L') = Z(J, L), \qquad W'(J', L') = W(J, L). \tag{1.2}$$

The $L$-$J$ mixing of formula (1.1) reflects the fact that a general change of field variables mixes the elementary field with composite fields. The transformations (1.1) are associated with certain changes of integration variables $\varphi' = \varphi'(\varphi, J, L)$ in the functional integral, combined with parameter redefinitions.

We say that the functional integral is written in the *conventional form* when the entire $J$-dependence is encoded in the term $\int J\varphi$ that appears in the exponent of the $Z$-integrand. Clearly, a non-linear change of integration variables turns the functional integral into some unconventional form. However, it was shown in ref. [2] that the conventional form can be recovered applying a nontrivial set of manipulations. The renormalization of the theory in the new variables does not need to be calculated anew. It can be derived from the renormalization in the old variables applying the operations that switch the functional integral back to the conventional form. The change of integration variables undergoes its own renormalization, which is related to the renormalization of composite fields.

In this paper we extend the investigation of ref. [2] to the generating functionals of one-particle irreducible diagrams. The usual generating functional $\Gamma(\Phi, L)$ is the Legendre transform of $W$ with respect to $J$. This kind of operation, however, must be treated with caution, because it is not covariant. The first consequence of this fact is that $\Gamma$ does not behave as a scalar under the field-transformation law derived from its very definition as a $W$-Legendre transform. Yet, we prove that there exists a corrected field-transformation law under which $\Gamma$ does behave as a scalar.

The second consequence is that there must exist a better generating functional of one-particle irreducible diagrams, which does transform as expected. We call it *master functional* and denote it with $\Omega(\Phi, N)$. Roughly, it is the Legendre transform of $W$ with respect to both $J$ and $L$. However, the naive Legendre transform with respect to $L$ does not exist, so we must first "improve" the $W$-functional in a suitable way, and then make the Legendre transform of the improved $W$.

We said that the Legendre transform is not covariant under general source redefinitions. Nevertheless, it is covariant under linear source redefinitions. If we use the linear approach, where linear source redefinitions encode the most general perturbative changes of field variables, we do not lose generality. Approaches alternative to the linear one have also been defined in ref. [2], but they are less efficient for the purposes of this paper. For this reason here we mostly use the so-called redundant linear approach, although we also include comments on the other approaches. "Redundant" means that the basis $\{\mathcal{O}^I\}$ of composite fields is unrestricted. In particular, it contains also descendants, composite fields proportional to the field equations, the identity and the elementary field itself. In the redundant approach divergent terms proportional to the field equations can still be subtracted by means of field redefinitions, in the source-independent sector. Doing so is useful, for example, to identify finite theories, whose divergences can be subtracted by means of sole field redefinitions, and renormalizable theories, whose divergences can be subtracted by means of field redefinitions and redefinitions of a finite number of physical parameters. In the source-dependent sector, instead, we take advantage of the redundancy to simplify the formal structure as much as possible.

The master functional admits a convenient *proper formulation*, where the set of integrated fields $\varphi$ is extended to the *proper fields* $\varphi$-$N$, where $N$ are partners of the sources $L$. In the proper formulation the classical action coincides with the classical limit $S_N$ of the master functional $\Omega$ and radiative corrections are determined from the classical limit with the usual diagrammatic rules. Moreover, the conventional form of the functional integral is manifestly preserved during any change of field variables, including the BR map. In this way, it is possible to work directly on $\Omega$ without referring to its definition from $W$.

For definiteness, we work using the Euclidean notation and the dimensional regularization, but no results depend on these choices. To simplify the presentation, we imagine that the fields we are working with are bosonic, but the arguments can be immediately generalized to include fermionic fields.

The paper is organized as follows. In section 2 we investigate how the changes of field variables (1.1) reflect inside the Γ-functional, and show that Γ does not behave as a scalar under this operation. We work out the correct field-transformation law under which Γ does behave as a scalar. In section 3 we motivate the search for a better generating functional of one-particle irreducible diagrams and describe how to overcome the most basic difficulties. In section 4 we define the master functional and investigate its main properties. We also calculate it in an explicit example. In section 5 we study the perturbative changes of field variables in the master functional, and apply them to the example of section 4. In section 6 we study restrictions on the master functional, one of which gives the Γ-functional itself. In section 7 we work out the proper formulation, while in section 8 we study the renormalization of the master functional. In section 9 we describe some generalizations obtained "covariantizing" the notion of Legendre transform. Section 10 contains the conclusions, while in the appendix we recall a theorem used in the paper about field redefinitions.

## 2 Changes of field variables in the Γ-functional

In this section we study perturbative changes of field variables in the Γ-functional $\Gamma(\Phi, L) = -W(J, L) + \int J\Phi$, where $\Phi = \delta W/\delta J$ and $L$ are the sources coupled to the composite fields. We first show that the transformation does not work as expected. Precisely, under the transformation law derived from its very definition, Γ does not behave as a scalar. This fact has an intuitive explanation. The notion of one-particle irreducibility is not compatible with general field redefinitions, because a non-linear change of field variables mixes elementary fields with composite fields, therefore one-particle irreducibility with many-particle irreducibility. We show that Γ does transform as a scalar once we compose the expected change of field variables with a further change of field variables.

Inside the functional $\Gamma(\Phi, L)$ the change of variables that follows from (1.1), (1.2) and the definition of Legendre transform reads

$$\Phi'(\Phi, L) = \frac{\delta W'(J', L')}{\delta J'} = \frac{\delta W(J, L)}{\delta J} + b_I \frac{\delta W(J, L)}{\delta L_I} = \Phi - b_I \frac{\delta \Gamma(\Phi, L)}{\delta L_I}. \tag{2.1}$$

We also have

$$\frac{\delta \Gamma(\Phi, L)}{\delta L_I} = -\frac{\delta W(J, L)}{\delta L_I} = -\frac{\delta W'(J, (L - bJ)z^{-1})}{\delta L_I} = -z^{-1}\frac{\delta W'(J', L')}{\delta L'_I} = z^{-1}\frac{\delta \Gamma'(\Phi', L')}{\delta L'_I}. \tag{2.2}$$

To visualize the change of variables more explicitly it is helpful to switch composite fields off for a moment, setting $L = 0$. Then the derivatives with respect to the renormalized sources $L_I$ generate insertions of renormalized composite fields $\mathcal{O}^I_{\mathrm{R}}$, so we get

$$\Phi'(\Phi, 0) = \Phi + b_I \left.\frac{\delta W}{\delta L_I}\right|_{L=0} = \langle \varphi + b_I \mathcal{O}^I_{\mathrm{R}}(\varphi)\rangle_{L=0}.$$

Dropping also radiative corrections we see that the classical change of variables is practically $\varphi' = \varphi + b_I \mathcal{O}^I_c(\varphi)$, where $\{\mathcal{O}^I_c\}$ is a basis of classical composite fields, which coincide with the classical limits of the $\mathcal{O}^I_{\mathrm{R}}$s. Nevertheless, this result is only partially correct, because the conditions $L = 0$ switch composite fields off *before* the change of variables. After the change of variables we should impose $L' = 0$. It can be shown [2] that when we take this fact into account the correct classical change of variables becomes

$$\varphi' = \varphi + \tilde{b}_I \mathcal{O}^I_c(\varphi),$$

where $\tilde{b}_I = b_I + \mathcal{O}(b^2)$ is a calculable power series in $b$.

Using (1.1) and $J = \delta\Gamma/\delta\Phi$ we can express $L'$ as a function of $\Phi$ and $L$. To express $\Phi$ and $L$ as functions of $\Phi'$ and $L'$, we can write

$$\Phi = \Phi' + b_I \frac{\delta\Gamma(\Phi, L)}{\delta L_I} = \Phi' + bz^{-1}\frac{\delta\Gamma'(\Phi', L')}{\delta L'_I} = \Phi(\Phi', L'), \tag{2.3}$$

$$L = L'z + bJ' = L'z + b\frac{\delta\Gamma'(\Phi', L')}{\delta\Phi'} = L(\Phi', L'). \tag{2.4}$$

Obviously, these relations, as well as (2.1) and (2.2), are not linear and not even local.

Using (1.2), (2.1) and the definitions of $\Gamma$ and $\Gamma'$ we can work out the relation between the $\Gamma$-functionals. Keeping $\Phi$ and $L'$ fixed and expanding in powers of $b$ we find

$$\begin{aligned}\Gamma'(\Phi', L') &= -W(J, L) + \int J\frac{\delta W}{\delta J}(J, L) + \int b_I J\frac{\delta W}{\delta L_I}(J, L) = \Gamma(\Phi, L) - \int b_I J\frac{\delta\Gamma(\Phi, L)}{\delta L_I} \\ &= \Gamma(\Phi, L'z) - \sum_{n=2}^{\infty}\frac{n-1}{n!}\int J(bz^{-1})_{I_1}\cdots J(bz^{-1})_{I_n}\frac{\delta^n\Gamma(\Phi, L'z)}{\delta L'_{I_1}\cdots\delta L'_{I_n}}.\end{aligned} \tag{2.5}$$

We could also expand in powers of $b$ keeping $\Phi$ and $L$ fixed, instead, but it would give an equivalent result. The important thing is that the same sources, in our case $L'$, appear on the left- and right-hand sides of (2.5). Then, setting $L' = 0$ we can switch composite fields off both in the left- and right-hand sides of the equation and compare $\Gamma'(\Phi', 0)$ with $\Gamma(\Phi, 0)$.

Formula (2.5) shows that the transformation rule we expect, $\Gamma'(\Phi', 0) = \Gamma(\Phi, 0)$, does not hold, because other terms appear on the right-hand side. Thus the change of variables in the $\Gamma$-functional does not work as expected.

Nevertheless, we can show that the extra terms that appear in the last line of (2.5) can be reabsorbed inside a further change of variables. Recalling that

$$J = J' = \frac{\delta\Gamma'(\Phi', L')}{\delta\Phi'}, \tag{2.6}$$

we can manipulate the last line of (2.5) and get

$$\Gamma(\Phi, L'z) = \Gamma'(\Phi', L') + \sum_{n=2}^{\infty} \frac{n-1}{n!} \int \frac{\delta\Gamma'}{\delta\Phi'}(bz^{-1})_{I_1} \cdots \frac{\delta\Gamma'}{\delta\Phi'}(bz^{-1})_{I_n} \left. \frac{\delta^n \Gamma(\Phi, L'z)}{\delta L'_{I_1} \cdots \delta L'_{I_n}} \right|_{\Phi=\Phi(\Phi', L')}. \tag{2.7}$$

Observe that the corrections to $\Gamma'$ on the right-hand side are at least quadratic in the field equations of $\Gamma'$. Thanks to this fact, we can apply a theorem of ref. [3], recalled in the appendix. That theorem ensures that the corrections of (2.7) can be reabsorbed inside $\Gamma'$ by means of a further (still non-local) change of variables $\tilde{\Phi}(\Phi', L')$. The change of variables is encoded in formulas (A.2) and (A.4) of the appendix, while the structure of the action and the transformation law are given by formulas (A.1) and (A.3). We obtain

$$\Gamma(\Phi, L'z) = \Gamma'(\tilde{\Phi}(\Phi', L'), L'). \tag{2.8}$$

Defining $\Gamma''(X, \tilde{L}) = \Gamma'(X, L')$ and the corrected change of variables

$$\Phi''(\Phi, \tilde{L}) \equiv \tilde{\Phi}(\Phi'(\Phi, L'), L'), \tag{2.9}$$

where $\tilde{L} = L'z$ and $\Phi'(\Phi, L')$ is obtained inverting (2.3), we get

$$\Gamma(\Phi, \tilde{L}) = \Gamma''(\Phi'', \tilde{L}), \tag{2.10}$$

namely the $\Gamma$-functional transforms as a scalar under the corrected transformation law. Now we can forget about the origin of the sources $\tilde{L}$ and just pay attention to the fact that they are the same on both sides of the equation.

In particular, when composite fields are switched off ($\tilde{L} = 0$) the transformation law for the $\Gamma$-functional reads

$$\Gamma(\Phi) = \Gamma(\Phi, 0) = \Gamma''(\Phi''(\Phi, 0), 0) = \Gamma''(\Phi''),$$

where in the last expression it is understood that $\Phi''$ is $\Phi''(\Phi, 0)$.

Now we prove that (2.9) is the correct change of field variables for the functional $\Gamma$. To do so we must carefully analyze the structure of the $\Gamma$-functional and the properties of its changes of field variables. Non-local field redefinitions are tricky, because they give us an enormous freedom and can even relate theories that are not physically equivalent to each other, if we do not apply them correctly. Acceptable changes of field variables are only those that do relate physically equivalent theories.

The $\Gamma(\Phi, L)$-functional can be decomposed into the sum of a local tree-level action $S_{cL}(\Phi, L)$, which coincides with the classical action, plus (non-local) radiative corrections $\hbar\Gamma_{\text{non-loc}}$. The radiative corrections are determined by the tree-level action itself. Precisely, $\hbar\Gamma_{\text{non-loc}}$ collects the one-particle irreducible diagrams that are constructed with the vertices and propagators determined by $S_{cL}$, multiplied by appropriate coefficients. We write

$$\Gamma(\Phi, L) = S_{cL}(\Phi, L) + \hbar\Gamma_{\text{non-loc}}(\Phi, L). \tag{2.11}$$

A non-local field redefinition $\Phi'' = \Phi''(\Phi, \tilde{L})$ maps physically equivalent theories when it is a perturbative field redefinition at the tree level and $\Gamma''(\Phi'', \tilde{L}) = \Gamma(\Phi, \tilde{L})$ has a structure analogous to (2.11). We can decompose it as

$$\Phi'' = \Phi''(\Phi, \tilde{L}) = \Phi''_c(\Phi, \tilde{L}) + \hbar\Phi''_{\text{non-loc}}(\Phi, \tilde{L}),$$

where $\Phi''_c(\Phi, \tilde{L})$ is local. Moreover, the field redefinition must be such that the new non-local radiative corrections $\hbar\Gamma''_{\text{non-loc}}$ collect the one-particle irreducible diagrams determined by the new classical action $S''_{cL}$, multiplied by the correct coefficients.

Now we prove that the non-local change of variables (2.9) satisfies these requirements. First, let us recall the form of the classical action $S_{cL}$ in the redundant linear approach, because we are going to use it in the proof. It is given by

$$S_{cL}(\varphi, L) = S_c(\varphi) - \int L_I \mathcal{O}^I_c(\varphi) - \int \tau_{vI}\mathcal{N}^v(L)\mathcal{O}^I_c(\varphi), \tag{2.12}$$

where $\mathcal{N}^v(L)$ is a basis of independent local monomials that can be constructed with the sources $L$ and their derivatives, and are at least quadratic in $L$, while the $\tau_{vI}$s are constants. The reason why composite fields are multiplied by the most general $\mathcal{O}(L)$-structure is that doing so it is possible to linearize also the BR map, which can be expressed as a source redefinition of the form (1.1) combined with parameter redefinitions.

Now, let us consider the map $\Phi'(\Phi, L')$. We can work it out inverting formula (2.3) perturbatively in $b$. If we are just interested in the tree-level contributions to $\Phi'(\Phi, L')$ we can use (2.2) and (2.6) to replace $\delta\Gamma'/\delta L'_I$ and $\delta\Gamma'/\delta\Phi'$ with $\delta\Gamma/\delta L_I$ and $\delta\Gamma/\delta\Phi$ in (2.3) and (2.4). Then we can replace $\Gamma$ with $S_{cL}$, given by (2.12), then iterate (2.4) to express $L$ as a function of $\Phi$ and $L'$, insert the result in (2.3), and finally invert (2.3). Clearly, the result is a perturbative field redefinition. We conclude that $\Phi' = \Phi'(\Phi, L')$ is a perturbative field redefinition plus radiative corrections. Next, consider the map $\tilde{\Phi}(\Phi', L')$. Observe that, expressed in the variables $\Phi$-$L'z$ the coefficients of the $\delta\Gamma'/\delta\Phi'$-powers in (2.7) are local at the tree level, and at higher orders they involve only one-particle irreducible diagrams with multiple composite-field insertions. These properties hold even after expressing $\Phi$ as a function of $\Phi'$ and $L'$, which is done using (2.3). Thus, using formulas (A.2) and (A.4), we see that $\tilde{\Phi}'(\Phi', L')$ shares the same properties. Composing this transformation with $\Phi' = \Phi'(\Phi, L')$, we find that $\Phi''(\Phi, \tilde{L})$ is the sum of a tree-level perturbative field redefinition plus radiative corrections that involve only one-particle irreducible diagrams, as we wished to prove.

The second requirement, that the radiative corrections are determined by the tree-level action with the usual diagrammatic rules, is also satisfied. Indeed, definition $\Gamma''(X, \tilde{L}) = \Gamma'(X, L')$ tells us that the transformed functional $\Gamma''(\Phi'', \tilde{L})$ is just the functional $\Gamma'(\Phi', L')$ with $\Phi'$ replaced by $\Phi''$. We know that $\Gamma'$ has the correct structure, which is the primed version of (2.11), therefore $\Gamma''$ also has the correct structure.

We conclude that (2.9) is an acceptable change of variables for the $\Gamma$-functional, which behaves as a scalar.

Observe that all non-localities involved in the change of field variables are those typical of one-particle irreducible diagrams. Nowhere the non-localities typical of the $W$-functional (such as propagators with external momenta) enter the game. The change of variables itself is one-particle irreducible.

So far we have used the linear redundant approach, taking (2.12) as the classical action and assuming that the most general change of field variables is encoded in the source redefinitions (1.1). Nevertheless, the argument can be easily generalized to the essential approach and the other non-linear approaches studied in ref. [2]. In the essential approach, which is inspired by the classification of couplings made in ref. [4], we work with a basis of composite fields that does not contain descendants (i.e. derivatives of other composite fields) and objects proportional to the field equations. Then some changes of field variables, for example those appearing in the BR map, require to make non-linear source transformations in $W$. Something similar occurs with the other approaches of ref. [2]. Consider the most general perturbatively local finite redefinitions

$$L_I' = L_I'(J, L) = L_I' + \mathcal{O}(b), \qquad J' = J, \tag{2.13}$$

that can be expanded in powers of some parameters $b$ and satisfy the initial conditions $L_I'(0,0) = 0$. We recall that every transformed functional $W'(J', L') = W(J, L)$, obtained applying (2.13), is the $W$-functional that we would calculate in some transformed field-variable frame. The transformed fields can be worked out applying the procedure explained in ref. [2] to recover the conventional form of the functional integral, which is spoiled by any nontrivial $J$-dependence contained in $L'(J, L)$.

At the level of the $\Gamma$-functional, the expected field transformation reads

$$\Phi'(\Phi, L) = \Phi - \int \frac{\delta L_I(J', L')}{\delta J'} \frac{\delta \Gamma(\Phi, L)}{\delta L_I}. \tag{2.14}$$

Expanding in powers of $J$ we can write

$$\Gamma'(\Phi', L') = \Gamma(\Phi, L) - \int J' \frac{\delta L_I(J', L')}{\delta J'} \frac{\delta \Gamma(\Phi, L)}{\delta L_I} = \Gamma(\Phi, L(L')) - \int J' \mathcal{M}(\Phi, L') J', \tag{2.15}$$

where $L(L') = L(0, L')$ and $\mathcal{M}(\Phi, L')$ is an order $b^2$-sum of a tree-level local functional plus one-particle irreducible radiative corrections. Then we use (2.13) and (2.14) to express $\Phi$ as a function of $\Phi'$ and $L'$ inside $\mathcal{M}$, move the last term (2.15) to the left-hand side, realize that the correction to $\Gamma'$ is quadratically proportional to the $\Gamma'$-field equations, and reabsorb such a correction into a further change of variables $\tilde{\Phi}(\Phi')$, applying the theorem recalled in the appendix. All arguments proceed as above, with straightforward modifications, and lead us to conclude that the final change of variables (2.9) is correct, because it preserves the structure (2.11) of the $\Gamma$-functional, which expresses $\Gamma$ as the sum of a local function plus one-particle irreducible radiative corrections, determined by the tree-level part with the usual diagrammatic rules.

Summarizing, the final change of variables for $\Gamma$ is not the one inherited by the very definition of $\Gamma$ as the Legendre transform of $W$, namely (2.1) or (2.14), but instead it is (2.9). Nevertheless, the result we have found proves that a correct change of field variables for $\Gamma$ does exist. We just need to bear in mind that it is not the expected one.

In the next sections we show that there exists a better functional that still collects one-particle irreducible diagrams, but also transforms as expected, and very simply (that is to say linearly, in the linear approach), under arbitrary changes of field variables.



# 3 Master functional: motivation and introductory observations

A change of field variables in the functionals $Z$ and $W$ is a redefinition of the sources $J$ and $L$. Although the functionals $Z$ and $W$ are non-local, the $J$- and $L$-redefinitions must be local, since the exponent of the $Z$-integrand

$$Z(J, L) = \int [\mathrm{d}\varphi] \exp\left(-S_L(\varphi, L) + \int J\varphi\right) \tag{3.1}$$

must remain local. In the linear approach, the $J$- and $L$-redefinitions are local *and* linear. Moreover, since we include the elementary field in the set of composite fields, we can work in a framework where $J$ is unmodified and the entire transformation is encoded in the $L$-redefinition, as shown in (1.1).

On the other hand, we have observed that under changes of field variables the generating functional $\Gamma(\Phi, L)$ of one-particle irreducible diagrams does not transform as expected from its very definition as the Legendre transform of $W$. We have been able to find a more involved change of variables that compensates for this fact and that is satisfactory for most purposes. The complete field redefinition is itself non-local. This is not surprising, because $\Gamma$ is a non-local functional.

Nevertheless, since the source redefinitions (1.1) for $Z$ and $W$ are local, and linear in the linear approach, we are tempted to think that there should exist a better generating functional

$\Omega$ of one-particle irreducible diagrams that works similarly, namely such that the most general field transformations can be expressed locally, and linearly in the linear approach. Moreover, the transformations should be the ones following from the very definition of $\Omega$. In this section we collect a number of remarks that help us identify the desired generating functional.

Intuitively, the problem of $\Gamma$ is that a non-linear change of variables mixes the elementary field with composite fields, therefore one-particle irreducibility with many-particle irreducibility. This argument suggests that maybe we should work with many-particle irreducible generating functionals [5]. Recall, however, that those generating functionals are defined coupling non-local sources $K_n(x_1, \ldots, x_n)$ with strings $\varphi(x_1) \cdots \varphi(x_n)$ of elementary-field insertions located at distinct points, which are non-local composite fields. When we want to study local changes of variables we need to shift the sources $K$ by local terms proportional to $J$. For example, the shift

$$K_2(x, y) \to K_2(x, y) + b\delta(x - y)J(x), \tag{3.2}$$

allows us to study the change of variables $\varphi \to \varphi + b\varphi^2$. However, non-local sources do not capture the renormalization of local composite fields. Thus, the local shift of (3.2) causes the appearance of new divergences, those associated with the composite field $\varphi^2$, which need to be calculated anew in this approach. For this reason, we do not pursue the use of generating functionals of many-particle irreducible diagrams and look for a different solution.

Since a change of variables mixes the elementary field with (local) composite fields, it sounds natural to treat all of them on the same footing. This suggests to define a functional $\Omega(\Phi, N)$ as the Legendre transform of the functional $W(J, L)$ with respect to all sources $J$ and $L$, not just with respect to $J$. However, the Legendre transform of $W$ with respect to the sources $L$ does not exist, in general. The two-point functions of composite fields in momentum space form a matrix $G^{IJ}$ that is not invertible, due to some sort of "gauge" symmetries obeyed by the sources $L$.

Before moving forward, let us illustrate this important point in more detail. Composite fields proportional to the field equations give zero or a contact term, when they are inserted in a two-point function. On the other hand, descendants give two-point functions proportional to the ones of their primaries, so if the matrix $G^{IJ}$ contains both primaries and descendants it is degenerate. We might think that in the essential approach, where descendants and composite fields proportional to the field equations are not contained in the basis of composite fields, $G^{IJ}$ is invertible. This is not true, however.

Consider a free massless scalar field $\varphi$ in Euclidean space and the composite fields $\mathcal{O}^J = \varphi^J/J!$. The two-point functions $G^{IJ} = \langle \mathcal{O}^I \mathcal{O}^J \rangle_{L=0}$ can be easily calculated in momentum space integrating one loop after another. The result is, using the dimensional-regularization technique,

$$\langle \mathcal{O}^I(k)\ \mathcal{O}^J(-k) \rangle = \frac{\delta^{IJ}}{J!} \frac{\Gamma\left(J - D\frac{J-1}{2}\right)\left[\Gamma\left(\frac{D}{2} - 1\right)\right]^J}{(4\pi)^{(J-1)D/2}\Gamma\left(\frac{JD}{2} - J\right)} (k^2)^{(J-1)D/2-J}, \tag{3.3}$$

where $D = 4 - \varepsilon$ is the continued spacetime dimension. Subtracting the divergent part at coincident points, we find

$$\langle \mathcal{O}^I(k)\ \mathcal{O}^J(-k)\rangle\big|_{\text{finite}} = \delta^{IJ}\frac{(-1)^{J-1}(k^2)^{J-2}(\ln k^2 + \text{constant})}{(4\pi)^{2J-2}J!(J-1)!(J-2)!}. \tag{3.4}$$

The matrix (3.4) is a diagonal block of $G^{IJ}$ and is invertible. Nevertheless, observe that it would be problematic to use its inverse. The reason is that the first nontrivial contributions to (3.4) are one-loop, so its inverse introduces negative powers of $\hbar$. At the bare level, the matrix is even divergent, so its inverse introduces objects of order $\varepsilon$, very difficult to handle.

Now we calculate the $G^{IJ}$-block made of the composite fields $\{\mathcal{O}^1, \mathcal{O}^2\} = \{(1/2)\varphi^2, (1/2)\varphi\partial_\mu\partial_\nu\varphi\}$. We find, in momentum space,

$$\begin{aligned}\langle\varphi^2\ \varphi^2\rangle &= -\frac{\ln(k^2/\mu^2)}{32\pi^2}, \qquad \langle\varphi^2\ \mathcal{O}_M\rangle = -\frac{\ln(k^2/\mu^2)}{96\pi^2}M_{\mu\nu}k_\mu k_\nu, \\ \langle\mathcal{O}_M\ \mathcal{O}_M\rangle &= -\frac{\ln(k^2/\mu^2)}{3840\pi^2}\left((k^2)^2M_{\mu\nu}^2 - 2k^2(M_{\mu\nu}k_\nu)^2 + 14(M_{\mu\nu}k_\mu k_\nu)^2\right),\end{aligned} \tag{3.5}$$

where $M_{\mu\nu}$ is a constant traceless matrix and $\mathcal{O}_M = (M_{\mu\nu}/2)\varphi\partial_\mu\partial_\nu\varphi$. This $G^{IJ}$-block is not invertible. A quick way to prove this statement is to check that the vector $\{(k^2)^2, \delta_{\mu\nu}k^2 - 4k_\mu k_\nu\}$ is a null vector.

We can interpret this singularity as the consequence of a gauge symmetry. Although $\varphi\partial_\mu\partial_\nu\varphi$ is not a descendant of $\varphi^2$, the two composite fields $\varphi\partial_\mu\partial_\nu\varphi$ and $\varphi^2$ have a descendant in common, up to terms proportional to the field equations. Indeed,

$$\partial_\nu(\varphi\partial_\mu\partial_\nu\varphi) = \frac{1}{4}\partial_\mu\Box(\varphi^2) + \varphi(\partial_\mu\Box\varphi) - \frac{1}{2}\partial_\mu(\varphi\Box\varphi).$$

The action

$$S_L = \frac{1}{2}\int(\partial\varphi)^2 - \frac{1}{2}\int L\varphi^2 - \frac{1}{2}\int L_{\mu\nu}(\varphi\partial_\mu\partial_\nu\varphi) \tag{3.6}$$

is invariant with respect to the infinitesimal "gauge" transformation

$$\delta\varphi = -\partial\cdot(\varphi\ell) + \frac{1}{2}\varphi(\partial\cdot\ell), \qquad \delta L_{\mu\nu} = \partial_\mu\ell_\nu + \partial_\nu\ell_\mu, \qquad \delta L = -\frac{1}{2}\Box(\partial\cdot\ell), \tag{3.7}$$

to the lowest order in $L$, where $\ell_\mu$ are arbitrary functions. This is why the block cannot be invertible.

The symmetry (3.7) can be extended to the complete action

$$S_L = \frac{1}{2}\int(\partial\varphi)^2 - \int L_I\mathcal{O}^I,$$

assuming that $\{\mathcal{O}^I\}$ is the basis of composite fields. We must cancel the terms

$$-\int L_I\frac{\delta\mathcal{O}^I}{\delta\varphi}\delta\varphi, \tag{3.8}$$

which can be done as follows. Expanding (3.8) in the basis $\{\mathcal{O}^I\}$, we can write (3.8) as

$$-\int P_I(L,\ell)\mathcal{O}^I(\varphi),$$

where $P_I(L,\ell)$ are bilinear local functions of $L$ and $\ell$, or their derivatives. Then to reabsorb (3.8) it is sufficient to correct the $\delta L_I$-transformations of (3.7) as

$$\delta L_I \rightarrow \delta L_I - P_I(L,\ell).$$

Note that the $L$-transformations remain $\varphi$-independent, as it must be, otherwise it would be impossible to apply them inside the functional integral.

Clearly, similar arguments can be used to relate most of the composite fields containing derivatives. We learn that the renormalized two-point functions $G^{IJ}$, which are equal to $\langle \mathcal{O}^I_{\mathrm{R}}\mathcal{O}^J_{\mathrm{R}}\rangle_{L=0}$ plus counterterms taking care of coinciding points, do *not* form an invertible matrix in momentum space, not even if the set $\{\mathcal{O}^I\}$ is restricted to the essential fields. Thus, the Legendre transform of the $\Gamma$-functional with respect to the sources $L$ does not exist, in general.

At the same time, we learn that this problem is due to the presence of a special class of gauge symmetries. One way to solve it is to gauge-fix those gauge symmetries. However, since the sources $L$ are just formal tools, we do not need to worry about the propagation of unphysical "$L$-degrees of freedom". Therefore, more simply, we can just break the symmetries (3.7) explicitly.

We need to choose the most convenient symmetry-breaking term. We can show that the unique term that is compatible with all properties we need (some of which we have not mentioned, yet) is

$$T(L) = \frac{1}{2}\int L_I(A^{-1})^{IJ}L_J, \tag{3.9}$$

where $A$ is a constant invertible matrix. We call (3.9) *improvement term*. It must be included in $-S_L$, and therefore also $W(J,L)$, if it is not already present. It provides otherwise missing tree-level quadratic contributions for the $L$-sector. If we proceed in the way explained below, this trick is enough to make the $W$-Legendre transform with respect to the sources $L$ well-defined.

Now we must face another key problem: the Legendre transform is not a covariant operation. Given a function $f(x^\mu)$, define $y_\mu = \mathrm{d}f/\mathrm{d}x^\mu$ and the Legendre transform $g(y) = -f(x(y)) + x^\mu(y)y_\mu$. Consider a general change of coordinates $x' = x'(x)$ and study how it reflects from $f$ to $g$. To do this, it is useful to write $g$ as a function of $x$:

$$g = -f(x) + x^\mu \frac{\mathrm{d}f}{\mathrm{d}x^\mu}. \tag{3.10}$$

If $f$ transforms as a scalar, then $\mathrm{d}f/\mathrm{d}x^\mu$ transforms as a vector. However, $x^\mu$ does not transform as a vector, so $g$ is not a scalar.

There is one exception: the Legendre transform $g$ does behave as a scalar when the change of coordinates $x' = x'(x)$ is linear. If we use the linear approach, where all changes of field variables can be expressed as linear transformations of $L$ and $J$, we can define a completely invariant $\Omega$.

This is encouraging, yet still not enough for our purposes. The main virtue of the functional $\Gamma$ is that its diagrams obey the theorem of locality of counterterms. Because of this, it is relatively easy to have control on renormalization working on $\Gamma$. It is more difficult working, for example, directly on $W$, where local divergences can be multiplied by propagators and generate non-local divergent expressions.

Thus, the functional $\Omega$ must be a collection of one-particle irreducible diagrams. Better, it must be a collection of one-particle irreducible diagrams in all variable frames. To achieve this result it is sufficient to require that the "propagators" of the sources $L$ be equal to the identity. In this way, $L$-insertions are glued together at the same point and no $W$-type of non-localities are generated. More details on this issue are given in the next section.

The desired type of $L$-propagators are given by the improvement term (3.9), therefore it is sufficient to state that all other $\mathcal{O}(L^2)$-terms belonging to the $L$-sector must be treated perturbatively with respect to (3.9). We show below that it is consistent to do so.

In the end, we are able to build a functional $\Omega$ that meets our requirements. It is invariant with respect to the most general changes of field variables, it is one-particle irreducible in all field-reference frames and it obeys the theorem of locality of counterterms. Moreover, it contains all pieces of information we need, since we can always reconstruct $W$ and $Z$ (and also $\Gamma$) from $\Omega$. Finally, we can renormalize the theory working directly on $\Omega$ instead of $\Gamma$.

We think that the functional $\Omega$ can play a key role in the general field-covariant approach to quantum field theory. This is the reason why we call it the master functional.

We have already noted that the linear approach is very convenient for our purposes, because there all changes of field variables, including the BR map, are described by linear source redefinitions, which are transparent to the Legendre transform. Moreover, the linear approach provides the improvement term (3.9) naturally, because it is contained inside the terms $\tau_{v0} \int \mathcal{N}^v(L)$ that multiply the identity operator in the classical extended action $S_{cL}$ (2.12). In case that term is not already present, we just add it. Actually, for future use it is better to shift $\tau_{v0} \int \mathcal{N}^v(L)$ by $T(L)$, even if this operation may introduce some redundancy.

The classical action (2.12) is now turned into

$$S_{cL}(\varphi, \lambda, L) = S_c(\varphi, \lambda) - \int L_I \mathcal{O}_c^I(\varphi, \lambda) - T(L) - \int \tau_{vI} \mathcal{N}^v(L, \lambda) \mathcal{O}_c^I(\varphi, \lambda), \tag{3.11}$$

where $\lambda$ are the masses, the coupling constants and all other parameters of the theory. The bare action is formally identical, with bare quantities replacing classical quantities: $S_{L\mathrm{B}}(\varphi_{\mathrm{B}}, \lambda_{\mathrm{B}}, L_{\mathrm{B}}) =$

$S_{cL}(\varphi_{\rm B}, \lambda_{\rm B}, L_{\rm B})$, $S_{\rm B}(\varphi_{\rm B}, \lambda_{\rm B}) = S_c(\varphi_{\rm B}, \lambda_{\rm B})$, $\mathcal{O}^I_{\rm B}(\varphi_{\rm B}, \lambda_{\rm B}) = \mathcal{O}^I_c(\varphi_{\rm B}, \lambda_{\rm B})$. Finally, let

$$\varphi_{\rm B} = \varphi_{\rm B}(\varphi, \lambda, \mu), \qquad \lambda_{\rm B} = \lambda_{\rm B}(\lambda, \mu),$$

be the relation between bare and renormalized fields and couplings when composite fields are switched off. The renormalized action reads [2]

$$S_L(\varphi, \lambda, \mu, L) = S(\varphi, \lambda, \mu) - T(L) - \int \left(L_I + \hat{\tau}_{vI}\mathcal{N}^v(L, \lambda, \mu)\right) \mathcal{O}^I_{\rm R}(\varphi, \lambda, \mu), \tag{3.12}$$

where $\hat{\tau} = \tau$ plus counterterms, $S$ is the renormalized action at $L = 0$ and $\mathcal{O}^I_{\rm R}$ are the renormalized composite fields. We have

$$S(\varphi, \lambda, \mu) = S_{\rm B}(\varphi_{\rm B}, \lambda_{\rm B}), \qquad \mathcal{O}^I_{\rm R} = (Z^{-1})^I_J \mathcal{O}^J(\varphi, \lambda, \mu) = (Z^{-1})^I_J \mathcal{O}^I_{\rm B}(\varphi_{\rm B}, \lambda_{\rm B}),$$

where $Z^I_J$ is the matrix of renormalization constants for the composite fields. If counterterms of type $T(L)$ are necessary, we include them in $\tau_{v0}\int \mathcal{N}^v$ and keep $T(L)$ unrenormalized.

Next, we state that when we make the Legendre transform with respect to the sources $L$, $\tau_{v0}\int \mathcal{N}^v$ must be treated perturbatively with respect to (3.9). This is achieved as follows. In ref. [2] it was shown that the perturbative expansion is well organized if we assume

$$\lambda_{n_l} = \mathcal{O}(\delta^{n_l-2}), \qquad L_I = \mathcal{O}(\delta^{n_I-2}), \qquad A_{IJ} = \mathcal{O}(\delta^{n_I+n_J-2}), \qquad \tau_{vI} = \mathcal{O}(\delta^{n_I-n_v-2}), \tag{3.13}$$

where $\delta$ is some reference parameter $\ll 1$. Here $\lambda_{n_l}$ is the coupling, or product of couplings, that multiplies a monomial with $n_l$ $\varphi$-legs, $n_I$ is such that $\mathcal{O}^I(\varphi\delta^{-1}, \lambda_l\delta^{n_l-2}) = \delta^{-n_I}\mathcal{O}^I(\varphi, \lambda)$ and $n_v$ is the $\delta$-degree of $\mathcal{N}^v(L)$. With these assignments all radiative corrections carry an extra factor $\delta^{2\ell}$, where $\ell$ is the number of loops. However, for the present purposes we need to slightly modify the assignment (3.13), in a way that makes $A^{-1}$ more important than the $\tau$s and does not affect the statements derived so far. For example we can assume that $\tau_{vI}$ is $\mathcal{O}(\delta^{n_I-n_v-1})$, while $A_{IJ}$ remains $\mathcal{O}(\delta^{n_I+n_J-2})$. In this way all $\tau_{vI}$s remain leading with respect to their radiative corrections, so the assignment modification is consistent with our previous arguments. Summarizing, the perturbative expansion is properly organized assuming

$$\lambda_{n_l} = \mathcal{O}(\delta^{n_l-2}), \qquad L_I = \mathcal{O}(\delta^{n_I-2}), \qquad A_{IJ} = \mathcal{O}(\delta^{n_I+n_J-2}), \qquad \tau_{vI} = \mathcal{O}(\delta^{n_I-n_v-1}), \tag{3.14}$$

instead of (3.13). Consistently with (3.14), we also have $J = \mathcal{O}(\delta^{-1})$, since both $J$ and $L_1$ are sources for the elementary field. These assignments are easy to remember, because if we rescale every object by a factor $\delta^n$, where $n$ it its $\delta$-degree, and in addition rescale $\varphi$ by $1/\delta$, then the action $S_L$ rescales as

$$S_L \to \frac{1}{\delta^2}\bar{S}_L,$$

where $\bar{S}_L$ has a factor $\delta$ for each $\tau$ and a factor $\delta^2$ for each loop, but is $\delta$-independent everywhere else.

Before concluding this section, let us explain why (3.9) is unique for our purposes. Under a Legendre transform the coefficients of quadratic terms are turned into their reciprocals. If, for example, (3.9) were replaced with $L$-quadratic terms containing polynomials in derivatives, the $L$-propagators would be non-local. Then the master functional would contain unphysical poles, one-particle irreducibility would be destroyed and the theorem of locality of counterterms would be difficult to apply. To avoid all this, the $L$-propagators must be local. Now, assume that (3.9) is replaced with a non-local improvement term, such that the $L$-propagators are still local. A non-local improvement term of this type is acceptable inside $W$, which is non-local, but not acceptable in the exponent of the $Z$-integrand, which must be local. However, in these two places the improvement term is just the same. We conclude that both the improvement term and the $L$-propagators derived from it should be local, which leaves just (3.9).

## 4 Master functional: definition and basic properties



Now we are ready to define the master functional and study its structure. As said, we use the redundant linear approach. Moreover, we work at the renormalized level, because the arguments extend to bare quantities with little modifications. Let us first recall that the $\Gamma$-functional is the Legendre transform of $W(J,L)$ with respect to $J$,

$$\Gamma(\Phi, L) = -W(J,L) + \int J\Phi, \qquad \Phi = \frac{\delta W}{\delta J}.$$

In this operation, the sources $L$ are just spectators, so we have $\delta\Gamma/\delta L_I = -\delta W/\delta L_I$.

Now, assuming that the functional $W$ is the improved one, we define the master functional $\Omega(\Phi, N)$ as the Legendre transform of $W(J,L)$ with respect to both $J$ and $L$, namely

$$\Omega(\Phi, N) = -W(J,L) + \int J\Phi + \int L_I N^I, \tag{4.1}$$

where

$$\Phi = \frac{\delta W}{\delta J}, \qquad N^I = \frac{\delta W}{\delta L_I}. \tag{4.2}$$

Clearly, $\Omega$ is also the Legendre transform of minus $\Gamma(\Phi, L)$ with respect to $L$:

$$\Omega(\Phi, N) = \Gamma(\Phi, L) + \int L_I N^I, \tag{4.3}$$

where

$$N^I = -\frac{\delta\Gamma}{\delta L_I}. \tag{4.4}$$

We have the inverse formulas

$$J = \frac{\delta\Omega}{\delta\Phi}, \qquad L_I = \frac{\delta\Omega}{\delta N^I}. \tag{4.5}$$

Let us show that $\Omega$ is indeed well-defined and collects one-particle irreducible diagrams. To achieve this goal, it is convenient to view $\Omega$ as the Legendre transform (4.3) of minus $\Gamma$ with respect to $L$. We can use (4.4) to expand $N(\Phi, L)$ in powers of $L$. The coefficients of this expansion are the (renormalized) connected, one-particle irreducible correlation functions $\langle \mathcal{O}_{\mathrm{R}}^{I_1} \cdots \mathcal{O}_{\mathrm{R}}^{I_n} \rangle_{1\mathrm{PI}, L=0}$ (plus counterterms taking care of coinciding points), containing single or multiple insertions of renormalized composite operators $\mathcal{O}_{\mathrm{R}}^{I}$. Using (3.12) we get

$$N^I = (A^{-1})^{IJ} L_J + \langle \mathcal{O}_{\mathrm{R}}^I \rangle + \int \hat{\tau}_{vJ} \frac{\delta \mathcal{N}^v(L)}{\delta L_I} \langle \mathcal{O}_{\mathrm{R}}^J \rangle, \tag{4.6}$$

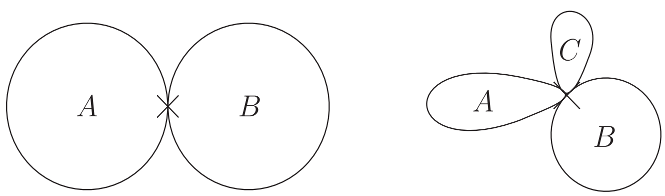
whence

$$\tilde{N}^I \equiv N^I - \langle \mathcal{O}_{\mathrm{R}}^I \rangle_{1\mathrm{PI}, L=0} = (A^{-1})^{IJ} L_J + \int \langle \mathcal{O}_{\mathrm{R}}^I \mathcal{O}_{\mathrm{R}}^J \rangle_{1\mathrm{PI}, L=0} L_J + \int \hat{\tau}_{vJ} \frac{\delta \mathcal{N}^v(L)}{\delta L_I} \langle \mathcal{O}_{\mathrm{R}}^J \rangle_{1\mathrm{PI}, L=0} + \mathcal{O}(L^2). \tag{4.7}$$

Formula (3.14) tells us that the quantities $\tilde{N}^I$ are $\mathcal{O}(\delta^{-n_I})$. The improvement term (3.9) is responsible for the contribution $A^{-1}L$ appearing on the right-hand side of (4.7), which is crucial for the invertibility of (4.7). Expanding in orders of $\delta$ we can invert (4.7) and find

$$L_I(\Phi, N) = A_{IJ} \tilde{N}^J - A_{IJ} A_{KH} \int \langle \mathcal{O}_{\mathrm{R}}^J \mathcal{O}_{\mathrm{R}}^K \rangle_{1\mathrm{PI}, L=0} \tilde{N}^H - \int \hat{\tau}_{vK} \frac{\delta \mathcal{N}^v(A\tilde{N})}{\delta \tilde{N}^I} \langle \mathcal{O}_{\mathrm{R}}^K \rangle_{1\mathrm{PI}, L=0} + \mathcal{O}(A^3) \mathcal{O}(\tilde{N}). \tag{4.8}$$

Now we are ready to prove that the master functional $\Omega$ just contains one-particle irreducible diagrams glued together as shown in the pictures




(4.9)

Here $A$, $B$ and $C$ can be any correlation functions $\langle \mathcal{O}_{\mathrm{R}}^{I_1} \cdots \mathcal{O}_{\mathrm{R}}^{I_n} \rangle_{1\mathrm{PI}, L=0}$, while the symbol $\times$ denotes that two or more composite-field insertions are "locally connected" using vertices provided by $\mathcal{N}^v(L)$ and the "identity propagators" provided by (3.9).

Consider first $\Omega = \Omega(\Phi, N(\Phi, L))$ as a functional of $\Phi$ and $L$, as given by the right-hand side of (4.3). This expression is a generating functional of one-particle irreducible diagrams in the same way as $\Gamma$ is. Indeed, because of (4.4), the right-hand side of (4.3) collects the same correlation functions that are contained inside $\Gamma$, however multiplied by different coefficients.

Now we express the sources $L$ as functions of $\Phi$ and $N$. Using (4.8) we see that we get precisely the objects depicted in the pictures (4.9). In momentum space we have just products of correlation functions $\langle \mathcal{O}_{\mathrm{R}}^{I_1} \cdots \mathcal{O}_{\mathrm{R}}^{I_n} \rangle_{1\mathrm{PI}, L=0}$ and polynomials. This argument proves that the master functional obeys the theorem of locality of counterterms. For the moment we are satisfied with this result.

Later, in section 7, we develop a "proper formalism" that allows us to study $\Omega$ using diagrammatic rules analogous to the ones we normally use for $\Gamma$, in particular calculate the renormalization of $\Omega$ working directly on $\Omega$ without using the definitions (4.1) and (4.3) based on $W$ and $\Gamma$.

Let us discuss how $\Omega$ depends on $A$ and $\tilde{N}$. Because of (4.8), $L$ contains only powers $A^m\tilde{N}^n$ with $m \geqslant n$. More precisely, $L_I = A_{IJ}\tilde{N}^J$ plus a sum of powers $A^m\tilde{N}^n$ with $m \geqslant n+1$. Instead, due to the improvement term (3.9) the $\tilde{N}$-dependence inside $\Omega$ has the form of monomials $A^m\tilde{N}^n$ with $m \geqslant n-1$. More precisely, we can write

$$\Omega(\Phi, N) = \Gamma(\Phi) + T_\Omega(\tilde{N}) + \Delta_2\Omega(\Phi, \tilde{N}). \tag{4.10}$$

where

$$T_\Omega(\tilde{N}) = \frac{1}{2}\int \tilde{N}^I A_{IJ}\tilde{N}^J \tag{4.11}$$

is the $\Omega$-improvement term and $\Delta_2\Omega$ is a sum of monomials of the form

$$(A_{I_1J_1}\tilde{N}^{J_1})\cdots(A_{I_nJ_n}\tilde{N}^{J_n})X^{I_1\cdots I_n}(\Phi)$$

with $n \geqslant 2$, where the $X$s are power series in $A$, and can contain derivatives acting on $\Phi$ and on the $\tilde{N}$s. Of course, $\Delta_2\Omega$ is of higher order in $\delta$ than $T_\Omega$. Note that the term linear in $\tilde{N}$ is missing in (4.10). Actually, we introduced $\tilde{N}$ precisely to make this happen.

The functional $\Gamma(\Phi)$ is the minimum of $\Omega$ with respect to the $N^I$s. Indeed, the conditions

$$\frac{\delta\Omega}{\delta N^I} = 0 \tag{4.12}$$

are nothing but $L_I = 0$. The solutions of (4.12) determine $N^I$ as functions of $\Phi$. Formula (4.10) immediately gives $\tilde{N}^I = 0$, or $N^I = \langle \mathcal{O}^I_{\mathrm{R}}\rangle_{L=0}$, so finally

$$\Gamma(\Phi) = \Omega(\Phi, \langle \mathcal{O}^I_{\mathrm{R}}\rangle_{L=0}).$$

Another way to derive $\Gamma(\Phi)$ from $\Omega(\Phi, N)$ is to take the limit $A \to 0$, which is regular in $\Omega$ and is equivalent to set $L_I = 0$:

$$\Gamma(\Phi) = \lim_{A\to 0}\Omega(\Phi, N).$$

So far we have been working with renormalized quantities, but every argument can be applied to bare quantities with obvious modifications.

**Example**

To give an explicit example, we consider a free massless scalar field and the composite field $\varphi^2/2$ coupled to the source $L_2$. We want to work out the master functional to the order $\tilde{N}^3$. Let $L_0$ and $L_1$ denote the sources coupled with the identity operator and the elementary field, as usual. We choose $A = \mathrm{diag}(a_0\mu^{-\varepsilon}, a_1, a_2\mu^{\varepsilon})$, where the factors $\mu^{\varepsilon}$ are introduced to make the dimensions of $a_0$, $a_1$ and $a_2$ integer. The functional $W$ is easy to calculate (check for example section 12 of [2]). We find

$$W(J,L) = \frac{1}{2}\int \left\{ (J+L_1)\frac{1}{-\Box - L_2}(J+L_1) + \mu^{-\varepsilon}\left(\frac{1}{a_2}+\delta_a\right)L_2^2 \right\}$$
$$-\frac{1}{2}\mathrm{tr}\ln(-\Box - L_2) + \int L_0 + \mu^{\varepsilon}\int \frac{L_0^2}{2a_0} + \int \frac{L_1^2}{2a_1},$$

where $\delta_a = -(16\pi^2\varepsilon)^{-1}$. Then

$$\Phi = \int \frac{1}{-\Box - L_2}(J+L_1), \qquad N_0 = 1 + \mu^{\varepsilon}\frac{L_0}{a_0}, \qquad N_1 = \Phi + \frac{L_1}{a_1},$$

and, in momentum space,

$$\tilde{N}_2(k) = N_2(k) - \frac{\Phi^2(k)}{2} = \frac{\mu^{-\varepsilon}}{a_2(k)}L_2(k) + \frac{1}{2}\int \mathrm{d}k' G_3(k,k')L_2(k')L_2(k-k') + \mathcal{O}(L_2^3), \qquad (4.13)$$

where $G_3 = \langle \varphi^2\varphi^2\varphi^2\rangle/8$, $\mathrm{d}k'$ stands for $\mathrm{d}^D k'/(2\pi)^D$ and we have defined the running coupling

$$\frac{1}{a_2(k)} = \frac{1}{a_2} - \frac{1}{32\pi^2}\ln\frac{k^2}{\mu^2}.$$

Inverting the $N$-$L$ relations we find $L_0 = \mu^{-\varepsilon}a_0(N_0 - 1)$, $L_1 = a_1(N_1 - \Phi)$ and

$$L_2(k) = \mu^{\varepsilon}a_2(k)\tilde{N}_2(k) - \frac{1}{2}\mu^{3\varepsilon}a_2(k)\int \mathrm{d}k' G_3(k,k')a_2(k')\tilde{N}_2(k')a_2(k-k')\tilde{N}_2(k-k') + \mathcal{O}(a_2^4)\mathcal{O}(\tilde{N}_2^3).$$

The functional $\Omega$ is

$$\Omega(\Phi,N) = \int \frac{1}{2}(\partial_\mu\Phi)^2 + \frac{a_0\mu^{-\varepsilon}}{2}\int (N_0-1)^2 + \frac{a_1}{2}\int (N_1-\Phi)^2 + \frac{\mu^{\varepsilon}}{2}\int \mathrm{d}k\,\tilde{N}_{2(-k)}a_2(k)\tilde{N}_{2k}$$
$$-\frac{\mu^{3\varepsilon}}{6}\int \mathrm{d}k\mathrm{d}k' G_3(k,k')a_2(k)\tilde{N}_{2k}a_2(k')\tilde{N}_{2k'}a_2(k-k')\tilde{N}_{2(k-k')} + \mathcal{O}(a_2^4)\mathcal{O}(\tilde{N}_2^4). \qquad (4.14)$$

Clearly, the limit $a \to 0$ gives back the $\Gamma$-functional of the free-field theory.



## 5 Changes of field variables in the master functional

In this section we study the changes of field variables in the master functional, using the redundant linear approach. Again, we work with renormalized quantities, since the analysis of bare changes of variables is practically identical.

In ref. [2] it was explained that a change of field variables is made of the source-redefinitions (1.1), or

$$L_I' = (L_J - b_J J)(z^{-1})_I^J, \qquad J' = J, \qquad (5.1)$$

in the $Z$- and $W$-functionals, and that such functionals behave as scalars. To make $L_J$ and $b_J J$ of the same $\delta$-order in (5.1), we must assume $b_I = \mathcal{O}(\delta^{n_I-1})$. It is very simple to work out how (5.1) reflects in the $\Omega$-functional. From (1.2) we have $W'(J', L') = W(J, L'_J z^J_I + b_I J)$, so definitions (4.2) give

$$\Phi' = \left.\frac{\delta W'}{\delta J'}\right|_{L'} = \Phi + b_I N^I, \qquad N^{I\prime} = \left.\frac{\delta W'}{\delta L'_I}\right|_{J'} = z^I_J N^J. \tag{5.2}$$

Then (4.1) gives

$$\Omega'(\Phi', N') = -W'(J', L') + \int J'\Phi' + \int L'_I N^{I\prime} = -W(J, L) + \int J\Phi + \int L_I N^I = \Omega(\Phi, N),$$

which shows that the master functional, differently from $\Gamma$, does transform as expected. Note that the transformations (5.2) are linear in $\Phi$ and $N$.

In [2] it was also shown that redefinitions (5.1) are associated with a change of variables $\varphi' = \varphi'(\varphi, \lambda, J, L)$ in the functional integral and a number of parameter-redefinitions, e.g. $b' = b'(b, \tau, \lambda, \mu)$, $\tau' = \tau'(b, \tau, \lambda, \mu)$. Of course such reparametrizations must be finite, because they act on a convergent functional.

In this paper we have split the set of parameters $\tau$ into $A^{-1}$ plus the rest, and the rest was still called $\tau$. The two subsets play a different role, because the improvement term is dominant with respect to the other terms belonging to the source sector. Because of this, we have also modified the $\delta$-assignments into (3.14). Thus, the parameter-redefinitions associated with (5.1) now read $A'^{-1} = A'^{-1}(b, A^{-1}, \tau, \lambda, \mu)$, $\tau' = \tau'(b, A^{-1}, \tau, \lambda, \mu)$, etc., and must be determined carefully, because the change of variables makes $A$-denominators spread out everywhere. We must determine $A'$ and $\tau'$ such that all $A'$-denominators cancel out inside $\Omega'$. Then the limit $A' \to 0$ of $\Omega'$ gives $\Gamma'$.

Separating the improvement term $T$ from the rest let us write

$$W(J, L) = \tilde{W}(J, L) + T(L), \tag{5.3}$$

where $\tilde{W}$ does not depend on $A$. When we make the substitutions (5.1) we obtain

$$W'(J', L') = W(J, L) = \tilde{W}(J', L'z + bJ) + \frac{1}{2}\int (L'z + bJ')_I (A^{-1})^{IJ} (L'z + bJ')_J. \tag{5.4}$$

The last term of this formula contains powers $b^m A^{-n}$, with $m \geqslant n$. Working out $\Omega'$ from its definition (4.1) these powers spread out everywhere inside the transformed master functional. From the point of view of the expansion in powers of $\delta$, negative $A$-powers are not a problem, since in any case the orders of $\delta$ organize correctly. However, we want to be able to treat the change of variables perturbatively, while $A$ is also treated perturbatively. For example, it is sufficient to imagine that each $b_I$ carries an extra small parameter $\zeta$ besides the order of $\mathcal{O}(\delta^{n_I-1})$ assigned to it, and expand in $\zeta$ before expanding in $\delta$.

We can also view the problem of negative $A$-powers in the field transformations (5.2). Those transformations do leave $\Omega'$ regular for $A \rightarrow 0$, but they do not preserve the structure (4.10). In particular, they generate terms linear in $\tilde{N}$, which are absent in (4.10). To recover the primed version of (4.10) we must redefine $\tilde{N}$. However, it is easy to see that when we do this, powers $b^m A^{-n}$, with $m \geqslant n$, propagate from the improvement term to $\Gamma'(\Phi')$, $T'_\Omega(\tilde{N}')$ and $\Delta_2\Omega'(\Phi', \tilde{N}')$. To completely determine $\Omega'(\Phi', N')$ we must determine the parameters $b'$, $A'$, $\tau'$ and the constants $z$ as functions of $A$, $b$, and $\tau$, so that they absorb away all negative $A$-powers and turn the structure of $\Omega'(\Phi', N')$ into the primed version of (4.10), where $\tilde{N}'$ is worked out solving $\delta\Omega'/\delta N' = 0$. Note that the matrix $z$ is not uniquely determined, because after eliminating the negative $A$-powers we can always make a further change of composite-field basis.

Finally, we can also view this problem inside the functional integral, going through section 10 of ref. [2]. If the starting functional integral is written in the conventional form, as we assume, the redefinition (5.1) turns it into some unconventional form. We can recover the conventional form applying the theorem proved in section 9 of ref. [2], but then it is easy to see that powers $b^m A^{-n}$, with $m \geqslant n$, propagate inside the change of field variables $\varphi' = \varphi'(\varphi, \lambda, J, L)$, as well as in $z$, $A'$ and $\tau'$.

Now we give a step-by-step procedure to work out the reparametrization that must accompany the change of field variables (5.2) to reabsorb the negative $A$-powers. We work directly on the master functional, bypassing $Z$ and $W$. At the end of this section we illustrate the procedure with an explicit example.

1) First we make the substitutions $\Phi = \Phi' - bz^{-1}N'$, $N = z^{-1}N'$ inside $\Omega(\Phi, N)$. They do give the transformed functional $\Omega'$, but this $\Omega'$ is still written in the old parametrization. Next, we solve the conditions $\delta\Omega'/\delta N'^I = 0$ and insert the solutions $N'(\Phi')$ back into $\Omega'$. This operation gives $\Gamma'(\Phi')$, still written in the old parametrization. We know, from the analysis of section 2, that there exists a non-local change of field variables $\Phi'(\Phi)$ such that $\Gamma'(\Phi') = \Gamma(\Phi)$. The classical limits of $\Gamma(\Phi)$ and $\Gamma'(\Phi')$ are the classical actions $S_c(\varphi)$ and $S'_c(\varphi')$, before and after the change of variables. They are related by the classical limit $\varphi'(\varphi)$ of $\Phi'(\Phi)$. Inverting this relation and writing it as

$$\varphi(\varphi') = \varphi' - b'_I \mathcal{O}'^I_c(\varphi'),$$

we determine the constants $b'_I$. They make $S'_c(\varphi')$ free of $A$-denominators, because $S_c(\varphi)$ is independent of $A$.

2) At this point, we consider again the solutions $N'(\Phi')$ of $\delta\Omega'/\delta N'^I = 0$. These are the average values $\langle \mathcal{O}^{I\prime}_{\mathrm{R}} \rangle'$ in the new variable frame, at $L' = 0$, and must also be regular. We determine the constants $z$ canceling the negative $A$-powers of the $\langle \mathcal{O}^{I\prime}_{\mathrm{R}} \rangle'$-classical limits.

3) Finally, we are ready to consider $\Omega'(\Phi', N')$. The new parameters $A'$ and $\tau'$ are determined matching its structure with the primed version of (4.10), again in the classical limit. Once we

express $z$ and $A$, $b$, and $\tau$ as functions of $A'$, $b'$, and $\tau'$, everywhere, we obtain the correctly parametrized $\Omega'$.

Observe that at each step we determine the desired reparametrizations working with classical limits. Indeed, the reparametrization is fully determined by those limits, in the same way as the entire functional $\Omega$ is fully determined by the classical action, by means of Feynman rules and Feynman diagrams (see section 7). When the classical limits are matched, radiative corrections automatically turn out to be right. Moreover, they are consistent with the perturbative expansion in $\delta$.

We could also find the desired reparametrizations working with the renormalized actions $S_L$ and $S'_L$, instead of working with $\Omega$. However, it would not make much difference: the divergent parts cannot enter the reparametrizations, which are finite, and once we drop them we end up again matching the classical limits.

Summarizing, a change of variables in the master functional is the linear redefinition

$$\Phi' = \Phi + b_I N^I, \qquad N^{I\prime} = z^I_J N^J, \tag{5.5}$$

under which $\Omega$ behaves as a scalar, $\Omega'(\Phi', N') = \Omega(\Phi, N)$. To find the correct structure of $\Omega'$ we must accompany (5.5) with a set of reparametrizations that can be worked out with the procedure outlined above.

Now we illustrate the main issues with the help of an example.



**Example**

We consider again the free theory of a massless scalar field, with the composite field $\varphi^2/2$ coupled to the source $L_2$. We want to study the change of variables $L = L'z + bJ$ to the order $b^2$ in the functionals $\Omega$ and $\Gamma$ and check the results computing the associated Feynman diagrams. We treat $\tilde{N}$ as an $\mathcal{O}(b)$-object and truncate the $\Omega$-functional to the first line of (4.14). In this approximation the field transformation and the functional $\Gamma'$ can be calculated up to $\mathcal{O}(b^2)$, while $\tilde{N}'$ can be worked out up to $\mathcal{O}(b)$.

The change of variables reads

$$\Phi' = \Phi + b_2 N_2 + b_1 N_1, \qquad N'_2 = z_{22} N_2 + z_{21} N_1, \qquad N'_1 = z_{11} N_1 + z_{12} N_2, \qquad N'_0 = N_0, \tag{5.6}$$

and the transformed $\Omega$-functional $\Omega'$ is $\Omega(\Phi, N)$ once (5.6) are implemented. To find the correct reparametrizations, we first solve the conditions $\delta\Omega'/\delta N'_2 = \delta\Omega'/\delta N'_1 = \delta\Omega'/\delta N'_0 = 0$. Inserting the solutions $\tilde{N}'(\Phi')$ back inside $\Omega'$ we get $\Gamma'(\Phi')$. Then it is relatively easy to check that

$$\Gamma'(\Phi') = \frac{1}{2} \int \mathrm{d}^D x \, (\partial_\mu \Phi(\Phi'))^2, \tag{5.7}$$

where

$$\Phi(\Phi') = \Phi'(1 - b'_1) - \frac{b'_2}{2}\Phi'^2 + \frac{b'^2_2}{2}\Phi'^3 - \frac{b'^2_2 \mu^{-\varepsilon}}{64\pi^2}\left(\ln \frac{-\Box}{\mu^2}\right)(\Box\Phi') + \mathcal{O}(b^3). \tag{5.8}$$

The relations between $b$ and $b'$ are

$$b_1 = b_1' + b_1'^2 + \frac{1}{2}\left(\frac{b_1'^2}{a_1} + \frac{b_2'^2\mu^{-\varepsilon}}{a_2}\right)\Box + \mathcal{O}(b^3), \qquad b_2 = b_2' + 3b_1'b_2' + \mathcal{O}(b^3).$$

Boxes appear inside our "constants" because we work in an approach where descendants, such as $\Box\varphi$, $\Box\varphi^2$, $\Box^2\varphi$, etc., are not viewed as independent composite fields, but treated altogether with their primaries. This amounts to promote the constants to polynomials in derivatives. Note that formula (5.8) contains also the cubic power of the field. Since we have not introduced an independent source for the composite field $\varphi^3$, the coefficient of $\Phi'^3$ in (5.8) is not independent, but a function of $b_1'$ and $b_2'$.

Clearly, the classical limits of (5.7) and (5.8) are local. It is easy to check by explicit computation that the radiative corrections of (5.7) are determined by the classical limit of (5.7) in the usual way. There is just one one-loop diagram to compute, the scalar self-energy made with two vertices $(b_2'/2)\varphi'^2\Box\varphi'$.

Observe that (5.8) is also the appropriate non-local variable change of the $\Gamma$-functional, that is to say (2.9) at $L'=0$ (upon converting the notation of that formula to the one used here).

We have worked out the reparametrizations $b'(b)$ that make all $a$-denominators disappear from $\Gamma'(\Phi')$. The next task is to find the values of $z_{ij}$ that reabsorb the $a$-denominators contained in the averages $\langle \mathcal{O}_{\mathrm{R}}^{I\prime}\rangle'_{L'=0}$. This is straightforward, since we already have such averages from the solutions of $\delta\Omega'/\delta N^{I\prime}=0$. Proceeding order-by-order in $b$ we find

$$\mathcal{O}^{1\prime} = \varphi', \qquad \mathcal{O}^{2\prime} = \frac{\varphi'^2}{2} - \frac{b_2'}{2}\mu^{\varepsilon/2}\varphi'^3 + \mathcal{O}(b^2), \tag{5.9}$$

$$\langle \mathcal{O}_{\mathrm{R}}^{2\prime}\rangle'_{L'=0} = \frac{\Phi'^2}{2} - \frac{b_2'}{2}\Phi'^3 + \frac{b_2'\mu^{-\varepsilon}}{32\pi^2}\left(\ln\frac{-\Box}{\mu^2}\right)(\Box\Phi') + \mathcal{O}(b^2), \tag{5.10}$$

together with

$$z_{11} = 1 + b_1' + \frac{b_1'}{a_1}\Box + \mathcal{O}(b^2), \quad z_{12} = b_2' + \mathcal{O}(b^2), \quad z_{21} = \frac{b_2'}{a_2}\mu^{-\varepsilon}\Box + \mathcal{O}(b^2), \quad z_{22} = 1 + 2b_1' + \mathcal{O}(b^2).$$

Again, it is easy to check by explicit computation that the radiative corrections contained in (5.10) are those predicted by the new classical action and the new composite fields (5.9).

The final task is to find the reparametrizations $A'$ and $\tau'$ that make $\Omega'$ have the correct dependence on $A'$ and $\tilde{N}'$, which is encoded in the primed version of formula (4.10). In our approximation we have to stop at the terms that are quadratic in $\tilde{N}'$ and $\mathcal{O}(b^0)$. We find

$$\Omega'(\Phi', N') = \Gamma'(\Phi') + \frac{1}{2}\int \tilde{N}'^I A'_{IJ}\tilde{N}'^J + \mathcal{O}(b)\mathcal{O}(\tilde{N}'^2) + \mathcal{O}(\tilde{N}'^3),$$

where $A'_{IJ} = \mathrm{diag}(a_0\mu^{-\varepsilon}, a_1, a_2(k)\mu^{\varepsilon}) + \mathcal{O}(b)$.

As expected, the new parametrization, obtained matching only tree-level contributions, makes all terms regular inside $\Omega'$, including radiative corrections.

## 6 Restrictions

The sources $L_I$ and their "Legendre-partners" $N^I$ are useful tools to study composite fields and field redefinitions, but at some point we may want to get rid of them choosing suitable restrictions and define some sort of "quantum action" $\Omega(\Phi)$ depending only on the fields $\Phi$. In this section we consider some options of this kind. The $\Gamma$-functional can be viewed as one of them.

Choose a restriction $N^I = N^I(\Phi)$, where the functions $N^I(\Phi)$ are unspecified for the moment, and define $\Omega(\Phi) = \Omega(\Phi, N(\Phi))$. Because of (5.5) the transformed restriction is $N^I(\Phi) = (z^{-1})^I_J N^{J\prime}(\Phi')$. The change of variables reads

$$\Phi'(\Phi) = \Phi + b_I N^I(\Phi)$$

and the restricted master functional transforms correctly,

$$\Omega'(\Phi') = \Omega'(\Phi', N'(\Phi')) = \Omega(\Phi, N(\Phi)) = \Omega(\Phi).$$

A simple restriction is $N^{I\prime} = N^I = 0$, however in this case the field redefinition (5.5) is just the identity $\Phi' = \Phi$. The restriction $L_I = 0$ or, equivalently, $N^I = \langle \mathcal{O}^I_{\mathrm{R}} \rangle_{L=0} = N^I(\Phi)$, gives the functional $\Gamma(\Phi)$. In that case the change of variables becomes

$$\Phi'(\Phi) = \Phi + b_I \langle \mathcal{O}^I_{\mathrm{R}} \rangle_{L=0} \tag{6.1}$$

and we have

$$\Gamma(\Phi) = \Omega(\Phi) = \Omega(\Phi, N(\Phi)) = \Omega'(\Phi', N'(\Phi')) = \Omega'(\Phi').$$

However, the last expression does not coincide with $\Gamma'(\Phi')$. Indeed, we know that, although the restricted master functional does transform correctly, $\Gamma$ does not transform as expected. We get the correct transformed $\Gamma$-functional $\Gamma'(\Phi')$ when the restriction reads $N^{I\prime} = \langle \mathcal{O}^{I\prime}_{\mathrm{R}} \rangle_{L'=0}$ in the new variables, or $L'_I = 0$, but (1.1) shows that $L_I = 0$ cannot imply $L'_I = 0$. Applying the change of variables we find instead that the transformed restriction reads $N^{I\prime} = z^I_J \langle \mathcal{O}^J_{\mathrm{R}} \rangle_{L=0}$.

To recover the correct transformed $\Gamma$-functional we must make an additional step, similar to the one explained in section 2. Consider the difference

$$\begin{aligned}\tilde{N}^{I\prime} &= z^I_J \langle \mathcal{O}^J_{\mathrm{R}} \rangle_{L=0} - \langle \mathcal{O}^{I\prime}_{\mathrm{R}} \rangle_{L'=0} = z^I_J \left.\frac{\delta W}{\delta L_I}\right|_{L=0} - \left.\frac{\delta W'}{\delta L'_I}\right|_{L'=0} \\ &= \left.\frac{\delta W'}{\delta L'_I}\right|_{L'=-bz^{-1}J} - \left.\frac{\delta W'}{\delta L'_I}\right|_{L'=0} = \left.\frac{\delta \Gamma'(\Phi', L')}{\delta L'_I}\right|_{L'=0} - \left.\frac{\delta \Gamma'(\Phi', L')}{\delta L'_I}\right|_{L'=-bz^{-1}J}.\end{aligned} \tag{6.2}$$

Now, observe that at $L_I = 0$, $J$ coincides with the field equations $\delta\Gamma(\Phi)/\delta\Phi$. Using (6.1) we can view the right-hand side of (6.2) as a function of $\Phi$. Clearly, this function is proportional to

$J = \delta\Gamma(\Phi)/\delta\Phi$ and the "coefficient" of $J$ is a collection of one-particle irreducible diagrams. Then, by the primed version of (4.10) the difference

$$\Gamma(\Phi) - \Gamma'(\Phi') = \Omega'(\Phi') - \Gamma'(\Phi') = T'_{\Omega}(\tilde{N}'(\Phi')) + \Delta_2\Omega'(\Phi', \tilde{N}'(\Phi')).$$

is quadratically proportional to $\tilde{N}'$. By (6.2), when expressed as a function of $\Phi$ it has the form

$$-\int \frac{\delta\Gamma(\Phi)}{\delta\Phi}\mathcal{M}(\Phi)\frac{\delta\Gamma(\Phi)}{\delta\Phi},$$

namely it is quadratically proportional to the field equations $\delta\Gamma(\Phi)/\delta\Phi$. Moreover, the "coefficient of proportionality" $\mathcal{M}(\Phi)$ collects one-particle irreducible diagrams and is local at the tree level. Then we can use the theorem recalled in the appendix and absorb the difference $\Gamma - \Gamma'$ inside a further change of variables $\tilde{\Phi}(\Phi)$, which is the sum of a tree-level perturbative field redefinition plus one-particle irreducible radiative corrections. Finally, we get $\Gamma'(\Phi') = \Gamma(\tilde{\Phi}(\Phi(\Phi'))) = \tilde{\Gamma}(\tilde{\Phi})$, if we define $\Gamma \equiv \tilde{\Gamma}$. We find, as in section 2, that the functionals $\Gamma$ and $\Gamma'$ are mapped into each other, but the correct field transformation is not just (6.1), rather $\tilde{\Phi}(\Phi(\Phi'))$. Clearly, this map preserves the structure (2.11).

Other restrictions $N^I(\Phi)$ may be useful for different purposes. For example, if we choose $L_I = \ell_I$ =constants, we turn the classical action $S_c(\varphi)$ into $S_c(\varphi) - \sum_I \ell_I \mathcal{O}^I(\varphi)$. In this way we can study all actions, therefore all theories with the same field content, at the same time.

# 7 Proper formulation

In this section we show that with the help of a simple trick we can work with the master functional in a more economic way. The action $S_L(\varphi, L)$ appearing in the $Z$-integrand is not sufficiently similar to the master functional $\Omega(\Phi, N)$ and the classical action $S_{cL}(\varphi, L)$ does not coincide with the classical limit of $\Omega$. In particular, $S_L$ depends on "mixed" variables, since the sources $L$ are, strictly speaking, arguments of the functionals $Z$ and $W$, together with $J$, not arguments of an action. We want an action $S_N(\varphi, N_S)$ that coincides with the master functional in the classical limit, therefore it must depend on $\varphi$ and some new "fields" $N_S$, such that $\Phi = \langle\varphi\rangle$ and $N = \langle N_S\rangle$. We call this formulation the *proper formulation* of the master functional. Among the other things, it allows us to work directly on the master functional from the very beginning, without passing from $Z$, $W$ or $\Gamma$. To study the renormalization of $\Omega$ it is sufficient to write the Feynman rules of the *proper action* $S_N(\varphi, N_S)$ and work out their one-particle irreducible Feynman diagrams. Finally, in the proper formulation the conventional form of the functional integral is manifestly preserved during a general change of field variables.

To begin with, it is easy to see that the $Z$-functional (3.1) can be expressed in the form

$$Z(J, L) = \int [\mathrm{d}\varphi \mathrm{d}N_S \mathrm{d}\tilde{L}] \exp\left(-S_L(\varphi, \tilde{L}) + \int J\varphi + \int (L_I - \tilde{L}_I)N_S^I\right). \tag{7.1}$$

Indeed, the $N_S$-integral gives a functional $\delta$-function $\delta(L_I - \tilde{L}_I)$ and the further $\tilde{L}$-integral gives back (3.1). Now, define the proper action $S_N(\varphi, N_S)$ from the formula

$$\exp\left(-S_N(\varphi, N_S)\right) \equiv \int [\mathrm{d}L] \exp\left(-S_L(\varphi, L) - \int L_I N_S^I\right). \tag{7.2}$$

Inserting (7.2) with $L \to \tilde{L}$ in (7.1) we can express the $Z$- and $W$-functionals as

$$Z(J, L) = \exp W(J, L) = \int [\mathrm{d}\varphi \mathrm{d}N_S] \exp\left(-S_N(\varphi, N_S) + \int J\varphi + \int L_I N_S^I\right). \tag{7.3}$$

Here each composite field is associated with an integrated variable $N_S$ and an external source $L$. Both $\varphi$ and $N_S$ are regarded as elementary fields, called *proper fields*.

The exponent $-S_N$ on the left-hand side of (7.2) can be viewed as the $W$-functional associated with the functional integral appearing on the right-hand side of the same formula, where the fields $\varphi$ are treated as external variables and the $L$-propagators are those provided by the improvement term contained in $S_L$. The $L$-functional integral of (7.2) is a purely algebraic operation, because the $L$-propagators are equal to the identity in momentum space. The loop diagrams are integrals of the form

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} P(p),$$

where $P(p)$ is a polynomial, so they vanish using the dimensional regularization. Thus the action $S_N$ receives only tree-level contributions, therefore it is local.

We can work out $S_N$ explicitly using the saddle-point approximation, which is actually exact in the case of the functional integral (7.2). Let $L_I = L_I^*(\varphi, N_S)$ denote the perturbative solutions of

$$N_S^I = -\frac{\delta S_L(\varphi, L)}{\delta L_I}.$$

Then, writing $\tilde{L} = L - L^*$ and expanding the integrand of (7.2) around $L^*(\varphi, N_S)$, the right-hand side of (7.2) becomes

$$\int [\mathrm{d}\tilde{L}] \exp\left(-S_L(\varphi, L^*) - \int L_I^* N_S^I + \mathcal{O}(\tilde{L}^2)\right) = \exp\left(-S_L(\varphi, L^*) - \int L_I^* N_S^I\right).$$

The last expression is proved observing that the $\tilde{L}$-propagators are equal to the identity, and the $\tilde{L}$-functional integral involves only vertices that have at least two $\tilde{L}$-legs. So, it can receive contributions only from loop diagrams, which however vanish. Finally, we get

$$S_N(\varphi, N_S) = S_L(\varphi, L^*(\varphi, N_S)) + \int L_I^*(\varphi, N_S) N_S^I. \tag{7.4}$$

In practice, $S_N$ coincides with the Legendre transform of $-S_L$ with respect to $L$. In particular, we have the relation

$$\frac{\delta S_N}{\delta N^I} = L_I^*(\varphi, N_S).$$

The inverse of formula (7.2) reads

$$\exp\left(-S_L(\varphi, L)\right) = \int [\mathrm{d}N_S] \exp\left(-S_N(\varphi, N_S) + \int L_I N_S^I\right).$$

The integral over $N_S$ can be calculated like the $L$-integral of (7.2), and receives only tree-level contributions because the $N_S$-propagators are also proportional to the identity. Alternatively, to go from $S_N$ to $S_L$ we can use the inverse Legendre transform.

The proper formulation is convenient for several reasons, which we now illustrate. The generating functionals $Z$ and $W$ associated with the extended action $S_L$ (where the fields $\varphi$ are integrated and $L$ are external sources) can also be viewed as the generating functionals $Z$ and $W$ associated with the proper action $S_N$ (where both $\varphi$ and $N_S$ are integrated fields).

On the other hand, the master functional $\Omega$ can be viewed as the $\Gamma$-functional of the proper approach. Indeed, the master functional $\Omega$ is the Legendre transform of $W$ with respect to both $J$ and $L$. In the proper approach this is precisely the $\Gamma$-functional, because now the integrated fields are both $\varphi$ and $N_S$, while $J$ and $L$ are the sources coupled with them. Clearly, the classical limit of the master functional $\Omega(\Phi, N)$ coincides with the classical action $S_{cN}(\Phi, N)$ of the proper approach, and $\Phi = \langle\varphi\rangle$, $N = \langle N_S\rangle$, as promised. Moreover, the master functional has the structure (2.11), which means that its radiative corrections follows from its classical limit $S_{cN}$ according to the usual rules.

When no confusion can arise, we drop the subscript $S$ in $N_S$ and use the symbol $N$ for the variables of $S_N$. Some other times we may denote the $N$-variables of $\Omega$ with $N_\Omega$.

As a first example, we work out $S_N$ for the basic $S_L$-action

$$S_{0L}(\varphi, L) = S(\varphi) - \int L_I \mathcal{O}_{\mathrm{R}}^I(\varphi) - \frac{1}{2}\int L_I (A^{-1})^{IJ} L_J.$$

The functional integral of (7.2) is Gaussian and gives

$$S_{0N}(\varphi, N) = S(\varphi) + \frac{1}{2}\int \tilde{N}^I A_{IJ} \tilde{N}^J,$$

where $\tilde{N}^I = N^I - \mathcal{O}_{\mathrm{R}}^I(\varphi)$. More generally, we can work out $S_N$ either using (7.4) or expanding around $S_{0N}$. Decompose the complete action $S_L$ (3.12) as

$$S_L(\varphi, L) = S_{0L}(\varphi, L) - \int \tau_{vI} \mathcal{N}^v(L) \mathcal{O}_{\mathrm{R}}^I(\varphi), \tag{7.5}$$

where $S_{0L}$ is the part we expand around, while the terms $\tau_{vJ}\mathcal{N}^v\mathcal{O}_{\mathrm{R}}^J$ are treated perturbatively. The action $S_N$ is equal to $S_{0N}$ plus corrections that we now describe. Inserting (7.5) in (7.2) and observing that each $L$-insertion can be traded for minus the functional derivative $\delta/\delta N$ and moved outside of the functional integral, we can write a formula that implicitly gives $S_N$. Precisely,

$$\exp\left(-S_N(\varphi, N)\right) = \exp\left(\int \tau_{vJ} \mathcal{N}^v(-\delta/\delta N) \mathcal{O}_{\mathrm{R}}^J(\varphi)\right) \exp\left(-S_{0N}(\varphi, N)\right).$$

Next, observe that $\delta S_{0N}/\delta N^I = A_{IJ}\tilde{N}^J$, so the structure of $S_N$ is

$$S_N(\varphi, N) = S_{0N}(\varphi, N) + \sum_{n\geqslant 0}(A_{I_1J_1}\tilde{N}^{J_1})\cdots(A_{I_nJ_n}\tilde{N}^{J_n})\tilde{X}_I^{I_1\cdots I_n}\mathcal{O}_{\mathrm{R}}^I(\varphi), \tag{7.6}$$

where the $\tilde{X}$s are power series in $A$ and can contain derivatives acting on the $\tilde{N}$s. The terms with $n = 0, 1$ do not contribute to the sum and can be dropped. Indeed, write

$$\exp\left(-S_N(\varphi, N) + S(\varphi)\right) = \int [\mathrm{d}L] \exp\left(\int T(L) + \int \tau_{vJ}\mathcal{N}^v(L)\mathcal{O}_{\mathrm{R}}^J(\varphi) - \int L_I\tilde{N}^I\right).$$

It is easy to check that the exponent of the right-hand side vanishes for $\tilde{N}^I = 0$. To see this we must focus on connected diagrams that do not have external $L$-legs. Since all vertices have at least two $L$-legs, all such diagrams are at least one-loop, so they vanish. This proves that $S_N(\varphi, N) = S(\varphi)$ when $N^I = \mathcal{O}_{\mathrm{R}}^I(\varphi)$, therefore the term with $n = 0$ can be dropped from the sum of (7.6). Similarly, the derivative with respect to $N$, calculated at $\tilde{N}^I = 0$, collects the set of connected diagrams with one external $L$-leg, which must also contain at least one loop. Thus, the terms with $n = 1$ of (7.6) also vanish.

We conclude that $S_N$ has a structure similar to the structure (4.10) of $\Omega$:

$$S_N(\varphi, N) = S(\varphi) + \frac{1}{2}\int \tilde{N}^I A_{IJ}\tilde{N}^J + \sum_{n\geqslant 2}(A_{I_1J_1}\tilde{N}^{J_1})\cdots(A_{I_nJ_n}\tilde{N}^{J_n})\tilde{X}_I^{I_1\cdots I_n}\mathcal{O}_{\mathrm{R}}^I(\varphi). \tag{7.7}$$

This is the general structure of the classical, bare and renormalized actions in the proper approach.

Let us compare this action with the action (3.12), which is written using the "improper variables" $\varphi, L$. The terms of $S_L$ linear in $L_I$ and the terms of $S_N$ linear in $A_{IJ}N^J$ are multiplied by (minus) the renormalized composite fields $\mathcal{O}_{\mathrm{R}}^I(\varphi)$, therefore allow us to identify them. The improvement terms

$$T(L) = \frac{1}{2}\int L_I(A^{-1})^{IJ}L_J, \qquad T_N(\tilde{N}) \equiv \frac{1}{2}\int \tilde{N}^I A_{IJ}\tilde{N}^J,$$

correspond to each other. Similarly, the terms $\int \hat{\tau}_{vI}\mathcal{N}^v\mathcal{O}_{\mathrm{R}}^I$ correspond to the last sum in (7.7). The constants $\tilde{X}$ are equal to the $\tau$s plus perturbative corrections. Clearly, there are as many $\tilde{X}$s as $\tau$s, so we can invert the $\tilde{X}$-$\tau$ relations and consider the $\tilde{X}$s as independent parameters. Expanding the monomials quadratically proportional to $\tilde{N}$ using the same basis $\mathcal{N}^v$ we used for the monomials quadratically proportional to $L$, we conclude that the most general proper classical action $S_{cN}$ has the form

$$S_{cN}(\varphi, N) = S_c(\varphi) + \frac{1}{2}\int \tilde{N}_c^I A_{IJ}\tilde{N}_c^J + \int \rho_{vI}\mathcal{N}^v(\tilde{N}_c)\mathcal{O}_c^I(\varphi), \tag{7.8}$$

where $\rho_{vI}$ are constants and $\tilde{N}_c^I = N^I - \mathcal{O}_c^I(\varphi)$. The proper renormalized action is then

$$S_N(\varphi, N) = S(\varphi) + \frac{1}{2}\int \tilde{N}^I A_{IJ}\tilde{N}^J + \int \hat{\rho}_{vI}\mathcal{N}^v(\tilde{N})\mathcal{O}_{\mathrm{R}}^I(\varphi), \tag{7.9}$$

where $\hat{\rho}_{vI} = \rho_{vI}$ plus perturbative corrections. Recall that all counterterms of type $T_N(\tilde{N})$ are moved to $\int \hat{\rho}_{v0}\mathcal{N}^v(\tilde{N})$, so the matrix $A$ is unrenormalized.

From (3.14), we find that the perturbative expansion is correctly organized if we assume that the constants $\rho_{vI}$ are $\mathcal{O}(\delta^{n_I - n_v - 1})$, where $n_v$ is the $\delta$-degree of $\mathcal{N}^v(\tilde{N})$.

## 7.1 Changes of variables in the proper action

Now we study how the proper action $S_N$ transforms under a change of variables. Inserting (5.1) into (7.3) the identity $W(J, L) = W'(J', L')$ follows defining

$$\varphi' = \varphi + b_I N^I, \qquad N^{I\prime} = z^I_J N^J, \tag{7.10}$$

which gives

$$S'_N(\varphi', N') = S_N(\varphi, N), \qquad \int J\varphi + \int L_I N^I = \int J'\varphi' + \int L'_I N'^I.$$

As before, we have dropped the subscript $S$ in the integrated fields $N^I_S$.

We see that using the proper approach a change of variables (7.10) in the functional integral looks exactly as it looks in the master functional, where we have formula (5.5). Enlarging the set of integrated fields from $\varphi$ to the proper variables $\varphi, N$ we have linearized the change of variables also at the level of integrated fields, and gained a lot of simplicity and clarity. We call (7.10) a *proper field redefinition*.

Moreover, in the proper approach both the action $S_N(\varphi, N)$ and the term $\int J\varphi + \int L_I N^I$ behave as scalars, without talking to each other. This means that a proper functional integral written in the conventional form remains written that way at all stages of the variable change. Because of this, replacements and true changes of variables are practically the same thing. We recall that, instead, when we work with improper variables, where we have only $\int J\varphi$ instead of $\int J\varphi + \int L_I N^I$, lengthy procedures are necessary to retrieve the conventional form after the change of variables [2].

Nevertheless, from (7.10) it is not evident what the $\varphi$-change of field variables truly is, once we eliminate the $N$s. To make it more explicit it is sufficient to apply (7.10) and then reconvert the transformed action into its proper form (7.9). The operations necessary to achieve this goal are very similar to the manipulations met in ref. [2], now viewed from the viewpoint of the master functional.

Let $f(\varphi') = \varphi' + \mathcal{O}(b)$ denote the recursive solution to the equation

$$f(\varphi') = \varphi' - b_I \mathcal{O}^I_{\mathrm{R}}(f(\varphi')). \tag{7.11}$$

Using (7.10) and (7.11), we can write

$$\varphi = f(\varphi') - b_I \bar{N}^I, \tag{7.12}$$

where

$$\bar{N}^I \equiv N^I - \mathcal{O}_{\mathrm{R}}^I(f(\varphi')). \tag{7.13}$$

We have

$$\tilde{N}^I = N^I - \mathcal{O}_{\mathrm{R}}^I(\varphi) = N^I - \mathcal{O}_{\mathrm{R}}^I(f(\varphi') - b_J\bar{N}^J) = \bar{N}^I + F^I(\bar{N},\varphi'), \tag{7.14}$$

where $F^I$ are local functions of order $b$ and order $\bar{N}$.

Inserting (7.14) and (7.12) in $S_N(\varphi,N)$ and expanding in the basis of composite fields, we get

$$S_N(\varphi,N) = S(f(\varphi')) + \int \bar{N}^I E_I(f(\varphi')) + \frac{1}{2}\int \bar{N}^I\bar{A}_{IJ}\bar{N}^J + \int \bar{\rho}_{vI}\mathcal{N}^v(\bar{N})\mathcal{O}_{\mathrm{R}}^I(f(\varphi')), \tag{7.15}$$

where $\bar{A}_{IJ} = A_{IJ} + \mathcal{O}(b)$ and $\bar{\rho}_{vI} = \hat{\rho}_{vI} + \mathcal{O}(b)$ are new constants and $E_I$ are $\mathcal{O}(b)$-local composite fields proportional to (derivatives of) the field equations $\delta S(f(\varphi'))/\delta\varphi'$. For later convenience, we focus our attention on $\delta S(f(\varphi'))/\delta\varphi'$ rather than $\delta S(\varphi)/\delta\varphi|_{\varphi=f(\varphi')}$.

Formula (7.15) is not written in the form we want, since it contains terms linear in $\bar{N}^I$. We must work out $\tilde{N}'^I = \bar{N}^I + \mathcal{O}(b)$, so that (7.15) turns into the primed version of (7.9). A crucial fact is that the terms linear in $\bar{N}^I$ are also proportional to the field equations of $S(f(\varphi'))$.

Calculate the derivative of (7.15) with respect to $\bar{N}$ and set it to zero. This condition can be written as

$$\bar{N}^I = -(\bar{A}^{-1})^{IJ}E_J(f(\varphi')) - (\bar{A}^{-1})^{IJ}\bar{\rho}_{vK}\int \frac{\delta\mathcal{N}^v(\bar{N})}{\delta\bar{N}^J}\mathcal{O}_{\mathrm{R}}^K(f(\varphi'))$$

and solved recursively. The solution $\bar{N}^I = Y^I(\varphi') = \mathcal{O}(b)$ is local and proportional to the field equations $\delta S(f(\varphi'))/\delta\varphi'$. Now, define

$$\bar{N}'^I = \bar{N}^I - Y^I(\varphi') \tag{7.16}$$

and use this definition to replace $\bar{N}^I$ inside (7.15). We get

$$S_N(\varphi,N) = \bar{S}(\varphi') + \frac{1}{2}\int \bar{N}'^I\bar{A}'_{IJ}\bar{N}'^J + \int \bar{\rho}'_{vI}\mathcal{N}^v(\bar{N}')\mathcal{O}_{\mathrm{R}}^I(f(\varphi')),$$

where $\bar{A}'_{IJ} = A_{IJ} + \mathcal{O}(b)$ and $\bar{\rho}'_{vI} = \rho_{vI} + \mathcal{O}(b)$ are new constants. The term linear in $\bar{N}'$ is absent by construction and

$$\bar{S}(\varphi') = S_N(\varphi,N)|_{\bar{N}=Y^I(\varphi')} = S(f(\varphi')) + \int \frac{\delta S(f(\varphi'))}{\delta\varphi'}\mathcal{M}(f(\varphi'))\frac{\delta S(f(\varphi'))}{\delta\varphi'},$$

where $\mathcal{M}(\varphi') = \mathcal{O}(b^2)$ is local and can contain derivatives acting to its left and to its right. Now we can apply the theorem recalled in the appendix, which tells us that there exists a perturbatively local function $g(\varphi') = \varphi' + \mathcal{O}(b^2)$, such that

$$\bar{S}(\varphi') = S(f(g(\varphi'))).$$

Write

$$\varphi(\varphi') \equiv f(g(\varphi')) = \varphi' - b_I \mathcal{O}^I_{\mathrm{R}}(\varphi') + \mathcal{O}(b^2).$$

This formula is the renormalized variable change associated with (7.10). Inserting the inverse $\varphi' = \varphi'(\varphi)$ of this relation in (7.13) and (7.16), expanding in the basis of composite fields, and then using the second of (7.10), we can write

$$\begin{aligned}\tilde{N}'^I &= N^I - \mathcal{O}^I_{\mathrm{R}}(f(\varphi')) - Y^I(\varphi') = N^I - w^I_J \mathcal{O}^J_{\mathrm{R}}(\varphi(\varphi')) \\ &= (z^{-1})^I_J \left(N'^J - (zw)^J_K \mathcal{O}^K_{\mathrm{R}}(\varphi(\varphi'))\right) = (z^{-1})^I_J \left(N'^J - \mathcal{O}'^J_{\mathrm{R}}(\varphi')\right) = (z^{-1})^I_J \tilde{N}'^J,\end{aligned}$$

where $w^I_J = \delta^I_J + \mathcal{O}(b)$ are constants and the formula

$$\mathcal{O}'^I_{\mathrm{R}}(\varphi') = (zw)^I_J \mathcal{O}^J_{\mathrm{R}}(\varphi(\varphi')) \tag{7.17}$$

tells us how the basis of composite fields is transformed by the change of variables. Formula (7.17) can also be used to work out how the renormalization constants of composite fields are affected. Finally,

$$S_N(\varphi, N) = S'(\varphi') + \frac{1}{2}\int \tilde{N}'^I A'_{IJ} \tilde{N}'^J + \int \rho'_{vI} \mathcal{N}^v(\tilde{N}') \mathcal{O}'^I_{\mathrm{R}}(\varphi') = S'_N(\varphi', N'),$$

where $S'(\varphi') = \bar{S}(\varphi') = S(\varphi(\varphi'))$ is the transformed action and $A'_{IJ} = A_{IJ} + \mathcal{O}(b)$ and $\rho'_{vI} = \rho_{vI} + \mathcal{O}(b)$ are new constants.

Observe that the procedure just described allows us to work out the renormalization of the theory in the new variables without having to calculate it anew. It is sufficient to know the renormalization (of the action *and* composite fields) in some variable frame to derive it in any other variable frame using the change of variables.

We have learned that an operation as simple as (7.10) corresponds to a complex list of operations on the action. Nevertheless, those operations are not completely new to us, since they resemble the operations we had to do in ref. [2] when we studied the changes of field variables working with the $Z$- and $W$-functionals. These observations show once again that the master functional is the correct one-particle-irreducible partner of the $Z$- and $W$-functionals, while $\Gamma$ behaves in its own peculiar way.

## 8 Renormalization of the master functional

In this section we study the renormalization of the master functional. We first derive it from the renormalization of $W$. However, this method does not make us appreciate the virtues of the master functional. Moreover, the theorem of locality of counterterms can be applied in a much simpler way on generating functionals of one-particle irreducible diagrams rather than on

$W$. Therefore, we also derive the renormalization of $\Omega$ working directly on $\Omega$, using the proper approach, without referring to the definition of $\Omega$ from $W$.

The renormalization of $W$ in the linear redundant approach is encoded in formula (7.14) of ref. [2] and amounts to the source transformation

$$L_{I\mathrm{B}} = (L_J - \tilde{c}_J J)(\tilde{Z}^{-1})_I^J, \qquad J_{\mathrm{B}} = J, \tag{8.1}$$

plus parameter-redefinitions that we do not need to report here. Deriving the renormalization of $\Omega$ from the one of $W$ is straightforward. The transformation (8.1) is a particular case of (5.1), so we know that it corresponds to a linear $\Phi$-$N$ redefinition of the form (5.5) in $\Omega$ and an identical redefinition of the form (7.10) in the proper action $S_N(\varphi, N)$.

This could be the end of the story, but we want to rederive these results working directly on $\Omega$, to emphasize that the formulation of quantum field theory using the master functional is completely autonomous. The proper approach is very useful for our present purpose. If we forget about the derivation just given, imported from the $W$-functional, it is not obvious that the renormalization of $\Omega$ is just a linear redefinition of the form (7.10) of the proper variables, plus a redefinition of parameters. It is instructing to see how these properties emerge from $\Omega$.

As usual, we proceed inductively. We assume that renormalization works by means of proper field redefinitions

$$\varphi \rightarrow \varphi + b_I N^I, \qquad N^I \rightarrow z_J^I N^J,$$

and parameter redefinitions up to $n$-loops and prove that then it works the same way at $n+1$ loops. Call $\Omega_n$ the $\Omega$-functional renormalized up to $n$ loops. Denote its proper fields with $\varphi_n$ and $N_n$, the parameters with $\lambda_n$ and $\rho_n$, the composite fields with $\mathcal{O}_n^I(\varphi_n)$ and the $n$-loop renormalized proper action with $S_{Nn}$. Using (7.9), we can write

$$S_{Nn}(\varphi_n, N_n, \lambda_n, \rho_n) = S_n(\varphi_n, \lambda_n, \rho_n) + \frac{1}{2}\int \tilde{N}_n^I A_{IJ} \tilde{N}_n^J + \int \hat{\rho}_{vIn} \mathcal{N}^v(\tilde{N}_n) \mathcal{O}_{\mathrm{R}n}^I(\varphi_n), \tag{8.2}$$

where $\tilde{N}_n^I = N_n^I - \mathcal{O}_{\mathrm{R}n}^I(\varphi_n)$. As usual, we do not need to renormalize the constants $A_{IJ}$, as counterterms for the improvement term are provided by $\int \hat{\rho}_{v0} \mathcal{N}^v(\tilde{N}_n)$.

Recalling that the master functional is just the $\Gamma$-functional of the proper variables, we can apply the theorem of locality of counterterms, which tells us that the $(n+1)$-loop divergent part $\Omega_{n\,\mathrm{div}}^{(n+1)}$ of $\Omega_n$ is a local functional. Organize $\Omega_{n\,\mathrm{div}}^{(n+1)}$ as an expansion in powers of $\tilde{N}_n^I$:

$$\Omega_{n\,\mathrm{div}}^{(n+1)}(\varphi_n, N_n, \lambda_n, \rho_n) = \omega_n(\varphi_n) + \int \frac{\delta S_n(\varphi_n)}{\delta \varphi_n} q_{In} \mathcal{O}_{\mathrm{R}n}^I(\varphi_n) + \int \tilde{N}_n^I \zeta_{IJn} \mathcal{O}_{\mathrm{R}n}^J(\varphi_n)$$
$$+ \int \sigma_{vIn} \mathcal{N}^v(\tilde{N}_n) \mathcal{O}_{\mathrm{R}n}^I(\varphi_n),$$

where $q_{In}$, $\zeta_{IJn}$ and $\sigma_{vIn}$ are constants of order $(n+1)$-loop. We have separated the contributions at $\tilde{N}_n^I = 0$ into two sets: the terms proportional to the field equations, whose coefficients are also expanded in the basis $\mathcal{O}^I_{\mathrm{R}n}$ of composite fields, and the terms $\omega_n(\varphi)$ that must be reabsorbed redefining the parameters $\lambda_n$ inside $S_c(\varphi)$. Now, the action $S_{Nn+1}$ that renormalizes the theory up to $n+1$ loops must be equal to $S_{Nn} - \Omega^{(n+1)}_{n\,\mathrm{div}}$ up to higher orders (which means $(n+2)$-loop or higher), and its fields and parameters must then carry the subscript $n+1$. We write

$$S_{Nn+1}(\varphi_{n+1}, N_{n+1}, \lambda_{n+1}, \rho_{n+1}) = S_{Nn}(\varphi_{n+1}, N_{n+1}, \lambda_{n+1}, \rho_{n+1}) - \Omega^{(n+1)}_{n\,\mathrm{div}}(\varphi_{n+1}, N_{n+1}, \lambda_{n+1}, \rho_{n+1}), \tag{8.3}$$

up to higher orders, which for the moment remain unspedified. It is clear that the master functional $\Omega_{n+1}$ defined by the action (8.3) is convergent up to $n+1$ loops, since $\Omega_{n+1} = \Omega_n - \Omega^{(n+1)}_{n\,\mathrm{div}}$ up to that order. We want to show that once field and parameters are converted to $\varphi_n$, $N_n$, $\lambda_n$ and $\rho_n$, by means of the proper field redefinitions

$$\varphi_{n+1} = \varphi_n + q_{In} N^I_n, \qquad N^I_{n+1} = z^I_{nJ} N^J_n, \tag{8.4}$$

and certain parameter redefinitions,

$$\lambda_{n+1} = \lambda_n + \Delta_n \lambda_n, \qquad \rho_{n+1} = \rho_n + \Delta_n \rho_n, \tag{8.5}$$

where the unknown constants $z^I_{nJ} - \delta^I_J$, $\Delta_n\lambda_n$ and $\Delta_n\rho_n$ are $(n+1)$-loop, then the right-hand side of formula (8.3) coincides with $S_{Nn}(\varphi_n, N_n, \lambda_n, \rho_n)$ up to higher orders. Note that we can also write

$$\varphi_{n+1} = \varphi_n + q_{nI}\mathcal{O}^I_{\mathrm{R}n}(\varphi_n) + q_{nI}\tilde{N}^I_n. \tag{8.6}$$

The redefinitions of fields and parameters may be implemented writing

$$\begin{aligned} &S_{Nn}(\varphi_n + \Delta_n\varphi_n, N_n + \Delta_n N_n, \lambda_n + \Delta_n\lambda_n, \rho_n + \Delta_n\rho_n) = S_{Nn}(\varphi_n, N_n, \lambda_n, \rho_n) \\ &\qquad + \left( \int \Delta_n\varphi_n \frac{\delta}{\delta\varphi_n} + \int \Delta_n N_n \frac{\delta}{\delta N_n} + \Delta_n\lambda_n \frac{\partial}{\partial\lambda_n} + \Delta_n\rho_n \frac{\partial}{\partial\rho_n} \right) S_{cN}(\varphi_n, N_n, \lambda_n, \rho_n) \end{aligned} \tag{8.7}$$

plus higher orders. In the corrections that appear on the right-hand side we have replaced $S_{Nn}$ with the classical proper action $S_{cN}$ (7.8). This is allowed since the difference is again made of higher order terms.

As said, there must exist redefinitions $\lambda_{n+1}$ of the parameters $\lambda_n$ inside $S_c(\varphi)$ that reabsorb $\omega_n(\varphi_n)$. Then, using (8.7) and neglecting higher-orders, we can write the right-hand side of (8.3) in the form

$$\begin{aligned} &S_{Nn}(\varphi_n, N_n, \lambda_n, \zeta_n) + \left( \int \Delta_n\varphi_n \frac{\delta}{\delta\varphi_n} + \int \Delta_n N_n \frac{\delta}{\delta N_n} + \Delta_n\rho_n \frac{\partial}{\partial\rho_n} \right) S_{cN}(\varphi_n, N_n, \lambda_n, \rho_n) \\ &\qquad - \tilde{\Omega}^{(n+1)}_{n\,\mathrm{div}}(\varphi_n, N_n, \lambda_n, \rho_n). \end{aligned} \tag{8.8}$$

where

$$\tilde{\Omega}^{(n+1)}_{n\,\mathrm{div}}(\varphi_n, N_n, \lambda_n, \rho_n) = \int \frac{\delta S_c(\varphi_n)}{\delta\varphi_n} q_{In}\mathcal{O}^I_{\mathrm{R}}(\varphi_n) + \int \tilde{N}^I_n \tilde{\zeta}_{IJn}\mathcal{O}^J_{\mathrm{R}}(\varphi_n) + \int \tilde{\sigma}_{vIn}\mathcal{N}^v(\tilde{N}_n)\mathcal{O}^I_{\mathrm{R}}(\varphi_n).$$

The constants in front of the last two divergent terms have been modified, since the $\lambda_n$-redefinitions applied to (7.8) may also affect those terms if the composite fields depend on $\lambda$. Thus, (8.8) becomes

$$S_{Nn}(\varphi_n, N_n, \lambda_n, \zeta_n) - \int \tilde{N}_n^I \bar{\Delta}_{nIJ} \mathcal{O}_{\mathrm{R}}^J(\varphi_n) + \int (\Delta_n \rho_{vIn} - \bar{\sigma}_{vIn}) \mathcal{N}^v(\tilde{N}_n) \mathcal{O}_{\mathrm{R}}^I(\varphi_n), \tag{8.9}$$

plus higher orders, where

$$\bar{\Delta}_{nIJ} = A_{IK}(1 - z_n)_J^K + \rho_{vKn} C_{LIJ}^{vKM}(1 - z_n)_M^L + d_{IJ}$$

and $C_{LIJ}^{vKM}$, $d_{IJ}$ and $\bar{\sigma}_{vIn}$ are $(n+1)$-loop $\Delta_n \rho_{vIn}$-independent constants, $C_{LIJ}^{vKM}$ and $d_{IJ}$ being also $z_n$-independent. Finally, we can choose $z_n$ so that $\bar{\Delta}_{nIJ} = 0$ and set $\Delta_n \rho_{vIn} = \bar{\sigma}_{vIn}$. Then (8.3) coincides with $S_{Nn}(\varphi_n, N_n, \lambda_n, \zeta_n)$ up to higher orders, which is the desired result.

Now we can upgrade formula (8.3), where higher-order contributions remained unspecified, and define $S_{Nn+1}$ by the exact identity

$$S_{Nn+1}(\varphi_{n+1}, N_{n+1}, \lambda_{n+1}, \zeta_{n+1}) = S_{Nn}(\varphi_n, N_n, \lambda_n, \zeta_n). \tag{8.10}$$

This formula encodes the correct order-by-order renormalization, made of proper field redefinitions (7.10) and parameter redefinitions.

We conclude that renormalization can be worked out directly on the master functional following rules entirely similar to the ones we are accustomed to. The advantage is that now we have a general field-covariant approach. Moreover, all field redefinitions, including those that are part of the BR map, are linear and there is no practical difference between replacements and true changes of field variables.



# 9 Generalizations

The master functional, as defined so far, is well suited for the linear approach. There all changes of field variables, including the BR map, are simple linear redefinitions of $\Phi$ and $N$. We have pointed out that the Legendre transform is indeed invariant only under linear transformations. Nevertheless, in ref. [2] we have also been able to work with the essential approach in the $W$-functional, and in section 2 we have been able to do that in the $\Gamma$-functional. Thus, it must be possible to generalize the definition of master functional to make it work with the most general approach and the most general redefinitions of $\Phi$ and $N$. In this section we elaborate a little bit on this issue.

Let us go back to formula (3.10). We have pointed out that its lack of covariance is due to the fact that $x^\mu$ does not transform as a vector under general coordinate transformations. Let us define a more general transform, where $x^\mu$ is replaced by a vector $v^\mu(x)$. We have

$$g(y) = -f(x) + v^\mu(x) \frac{\mathrm{d}f}{\mathrm{d}x^\mu}, \qquad y^\mu(x) = \frac{\mathrm{d}f}{\mathrm{d}x^\mu}.$$

Now $g(y)$ does transform correctly as a scalar, if $f$ does.

Let $V_I(J,L)$ denote perturbatively local functions of the sources. In general, we assume that $V_I$ is equal to $L_I$ plus a perturbative series in some expansion parameters. We call such parameters $\kappa$. Moreover, we assume that $V_I$ is a vector in source space, which means that it transforms as

$$V_I' = \int V_J \frac{\delta L_I'}{\delta L_J} + \int J \frac{\delta L_I'}{\delta J}, \tag{9.1}$$

under a perturbatively local change of variables (2.13).

Define $\Phi$ and $N$ as in (4.2), but replace the definition (4.1) of the master functional with

$$\Omega(\Phi,N) = -W(J,L) + \int J\Phi + \int V_I N^I. \tag{9.2}$$

On $\Phi$ and $N$ the change of variables reads

$$\Phi' = \Phi + \int N^I \frac{\delta L_I}{\delta J'}, \qquad N^{I\prime} = \int \frac{\delta L_J}{\delta L_I'} N^J,$$

where however $L'$ and $J'$ must still be replaced by the appropriate functions of $\Phi$ and $N$. Since the relations $J(\Phi,N)$ and $L(\Phi,N)$ are in general non-local, the change of variables is non-local in the space $\Phi$, $N$. Of course, it must be the sum of local tree-level functions plus radiative corrections. We have

$$\Omega'(\Phi',N') = \Omega(\Phi,N),$$

as desired. We can also write

$$\Omega(\Phi,N) = \Gamma(\Phi,L) + \int V_I N^I.$$

Since $\Gamma(\Phi,L)$ collects one-particle irreducible diagrams, and $V_I = L_I$ plus local perturbative corrections, $\Omega(\Phi,N)$ also collects one-particle irreducible diagrams. Nevertheless, in general $\Omega$ does not have the typical structure (2.11), in the sense that its radiative corrections do not follow from its classical limit with the usual rules, and the classical limit of $\Omega$ is not necessarily the classical action.

For example, we can take $V_I = L_I$ in the essential frame, which is the variable frame where the action does not contain terms proportional to the field equations, apart from those containing the free kinetic terms [2]. Then $\Omega$ is the Legendre transform of $W$ with respect to $J$ and $L$ in the essential frame, and has the structure (2.11). In every other frame we define $V_I$ as given by (9.1). With this convention the vectors $V_I$ are inherited by a change of variables from the essential frame.

The inverse formulas read

$$J = \frac{\delta\Omega}{\delta\Phi} - \frac{\delta}{\delta\Phi}\int (V_I - L_I)N^I, \qquad L_I = \frac{\delta\Omega}{\delta N^I} - \frac{\delta}{\delta N^I}\int (V_J - L_J)N^J. \tag{9.3}$$

If $\Omega$ were a Legendre transform its inverse would be a Legendre transform. Instead, the procedure to obtain $W$ from $\Omega$ is more complicated, and we cannot implement it unless we know the vector $V_I(J,L)$. Assuming that we have this knowledge, and recalling that $V_I - L_I = \mathcal{O}(\kappa)$, we can solve formulas (9.3) recursively in powers of $\kappa$. This procedure gives us the functions $J(\Phi,N)$ and $L_I(\Phi,N)$. Once we have them we are ready to invert (9.2) and find

$$W(J,L) = -\Omega(\Phi,N) + \int J\Phi + \int V_I N^I.$$

A similar procedure can be used to extract the expectation values of elementary and composite fields from the master functional. These are the constant solutions of the conditions $J(\Phi,N) = L_I(\Phi,N) = 0$. Formulas (9.3) give

$$\frac{\delta\Omega}{\delta\Phi} = \frac{\delta}{\delta\Phi}\int (V_I - L_I)N^I, \qquad \frac{\delta\Omega}{\delta N^I} = \frac{\delta}{\delta N^I}\int (V_J - L_J)N^J.$$

Since the right-hand sides are $\mathcal{O}(\kappa)$, these equations can be solved recursively in powers of $\kappa$. The zeroth-order expectation values are the constant solutions of $\delta\Omega/\delta\Phi = \delta\Omega/\delta N^I = 0$.

## 10 Conclusions

In this paper we have defined and studied a new generating functional of one-particle irreducible diagrams, called master functional, which is invariant with respect to the most general perturbative changes of field variables.

A perturbative change of field variables starts with a redefinition of the fields $\varphi$ in the action $S$. Inside the functionals $Z(J,L)$ and $W(J,L)$ it becomes a local perturbative redefinition of the sources $J$ and $L$ coupled to elementary and composite fields, under which $Z$ and $W$ behave as scalars. In a particularly convenient approach, the linear one, such a source redefinition is linear. The functional $\Gamma(\Phi,L)$, on the other hand, does not behave as a scalar under the transformation law inherited from its very definition. Nevertheless, there exists an unusual field transformation under which $\Gamma$ does behave as a scalar. Instead, the master functional $\Omega(\Phi,N)$ behaves as a scalar under the transformation law derived from its very definition, which is linear in $\Phi$ and $N$. We have worked out the relations among these three ways to describe changes of field variables in quantum field theory and studied the BR map as a particular case.

One obstruction to construct the master functional was that the Legendre transform of $W$ with respect to the sources $L$ does not exist, in general. We have solved this problem adding a certain "improvement term" to the functional $W$, which equips the sources $L$ with suitable quadratic terms. Then the master functional $\Omega(\Phi,N)$ is defined as the Legendre transform of the improved $W(J,L)$ with respect to both $J$ and $L$. We must organize the perturbative expansion so that the "$L$-propagators" are equal to unity. Then the master functional collects one-particle irreducible diagrams. The lack of covariance of the Legendre transform is naturally overcome in the linear approach, where all field redefinitions, including those of the BR map, can be expressed linearly.

The master functional admits a very economic "proper formulation", where the set of integrated fields is extended from $\varphi$ to the proper variables $\varphi$-$N^I$, the $N^I$s being partners of the sources $L_I$ for composite fields. In this formulation the master functional is the ordinary $\Gamma$-functional for the proper variables. The proper classical action coincides with the classical limit of the master functional and radiative corrections are the one-particle irreducible Feynman diagrams of the proper formulation. Thus, they can be calculated working directly on the master functional, without passing through $Z$, $W$ or $\Gamma$. Finally, the conventional form of the functional integral is manifestly preserved during a general change of field variables, so replacements and true changes of field variables are practically the same thing.

An interesting subject for a future investigation is the generalization to non-perturbative changes of field variables, which we have not considered here.

## Appendix Field redefinitions and field equations



We know that if we perturb the action adding a local term proportional to the field equations, we can reabsorb such a term inside the action by means a local field redefinition to the first order of the Taylor expansion. It is interesting to know that if we perturb the action adding a local term quadratically proportional to the field equations, we can perturbatively reabsorb it inside the action to *all* orders by means of a local field redefinition. In this appendix we briefly rederive this result and its generalization to non-local functionals and non-local field redefinitions. The theorem was proved in ref. [3], where a number of applications and explicit examples can be found.

**Theorem 1** *Consider an action $S$ depending on fields $\phi_i$, where the index $i$ labels both the field type, the component and the spacetime point. Add a term quadratically proportional to the field equations $S_i \equiv \delta S/\delta\phi_i$ and define the modified action*

$$S'(\phi_i) = S(\phi_i) + S_i F_{ij} S_j, \tag{A.1}$$

*where $F_{ij}$ is symmetric and can contain derivatives acting to its left and to its right. Summation over repeated indices (including the integration over spacetime points) is understood. Then there exists a field redefinition*

$$\phi_i' = \phi_i + \Delta_{ij} S_j, \tag{A.2}$$

*with $\Delta_{ij}$ symmetric, such that, perturbatively in $F$ and to all orders in powers of $F$,*

$$S'(\phi_i) = S(\phi_i'). \tag{A.3}$$

*Proof.* The condition (A.3) can be written as

$$S(\phi_i) + S_i F_{ij} S_j = S(\phi_i + \Delta_{ij} S_j) = S(\phi_i) + \sum_{n=1}^{\infty} \frac{1}{n!} S_{k_1 \cdots k_n} \prod_{l=1}^{n} (\Delta_{k_l m_l} S_{m_l}),$$

after a Taylor expansion, where $S_{k_1 \cdots k_n} \equiv \delta^n S/(\delta\phi_{k_1} \cdots \delta\phi_{k_n})$. This equality is verified if

$$\Delta_{ij} = F_{ij} - \Delta_{ik_1} \left[ \sum_{n=2}^{\infty} \frac{1}{n!} S_{k_1 k_2 k_3 \cdots k_n} \prod_{l=3}^{n} (\Delta_{k_l m_l} S_{m_l}) \right] \Delta_{k_2 j}, \tag{A.4}$$

where the product is meant to be equal to unity when $n = 2$. Equation (A.4) can be solved recursively for $\Delta$ in powers of $F$. The first terms of the solution are

$$\Delta_{ij} = F_{ij} - \frac{1}{2} F_{ik_1} S_{k_1 k_2} F_{k_2 j} + \cdots \tag{A.5}$$

This result is very general. It works both for local and non-local theories. If $S(\phi_i)$ and $F_{ij}$ are perturbatively local, namely they can be perturbatively expanded so that every order of the expansion is local, the field redefinition (A.2) and the action $S'(\phi_i)$ are perturbatively local. If both $S(\phi_i)$ and $F_{ij}$ are local, in general (A.2) and $S'(\phi_i)$ are only perturbatively local. Actually, the resummation of the expansion can produce a non-local field redefinition. Finally, if $S(\phi_i)$ and $F_{ij}$ are local or perturbatively local at the classical level, then (A.2) and $S'(\phi_i)$ are perturbatively local at the classical level.

## References


[1] In the case of gravity, see for example G. 't Hooft and M. Veltman, One-loop divergences in the theory of gravitation, Ann. Inst. Poincarè, 20 (1974) 69;

M.H. Goroff and A. Sagnotti, The ultraviolet behavior of Einstein gravity, Nucl. Phys. B 266 (1986) 709;

A.E.M. van de Ven, Two loop quantum gravity, Nucl. Phys. B 378 (1992) 309.

[2] D. Anselmi, A general field-covariant formulation of quantum field theory, Eur.Phys.J. C73 (2013) 2338 and arXiv:1205.3279 [hep-th].

[3] D. Anselmi, Renormalization and causality violations in classical gravity coupled with quantum matter, JHEP 0701 (2007) 062 and arXiv:hep-th/0605205.

[4] S. Weinberg, Ultraviolet divergences in quantum theories of gravitation, in *An Einstein centenary survey*, Edited by S. Hawking and W. Israel, Cambridge University Press, Cambridge 1979.

[5] J.M. Cornwall, R. Jackiw and E. Tomboulis, Effective action for composite operators, Phys. Rev. D 10 (1974) 2428.