

A General Field-Covariant Formulation Of Quantum Field Theory

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Abstract

In all nontrivial cases renormalization, as it is usually formulated, is not a change of integration variables in the functional integral, plus parameter redefinitions, but a set of replacements, of actions and/or field variables and parameters. Because of this, we cannot write simple identities relating bare and renormalized generating functionals, or generating functionals before and after nonlinear changes of field variables. In this paper we investigate this issue and work out a general field-covariant approach to quantum field theory, which allows us to treat all perturbative changes of field variables, including the relation between bare and renormalized fields, as true changes of variables in the functional integral, under which the functionals Z and $W = \ln Z$ behave as scalars. We investigate the relation between composite fields and changes of field variables, and we show that, if J are the sources coupled to the elementary fields, all changes of field variables can be expressed as J -dependent redefinitions of the sources L coupled to the composite fields. We also work out the relation between the renormalization of variable-changes and the renormalization of composite fields. Using our transformation rules it is possible to derive the renormalization of a theory in a new variable frame from the renormalization in the old variable frame, without having to calculate it anew. We define several approaches, useful for different purposes, in particular a linear approach where all variable changes are described as linear source redefinitions. We include a number of explicit examples.

1 Introduction

The present formulation of quantum field theory is not sufficiently general. Several properties we are interested in depend on the variables we use to formulate and quantize the theory. For example, power-counting renormalizability requires that the action should contain no parameters of negative dimensions in units of mass, but this property is spoiled by a general change of field variables. If we work in a generic field-variable setting, the only way we have to state the power-counting criterion is to demand that there should exist a field-variable frame where the theory becomes renormalizable according to the usual rules. We do not have a field-covariant formulation of quantum field theory, and we lack efficient variable-independent criteria to identify theories belonging to special classes, such as the renormalizable, conformal and finite theories. We can state, for example, that a theory is finite if all divergences can be reabsorbed by means of field redefinitions, but this is just the definition of finite theory, not a criterion to identify finite theories. So far the only general criterion we seem to have is “calculate and see”, which is clearly unsatisfactory. Similarly, we can define renormalizable theories as those whose divergences can be subtracted by means of field redefinitions and redefinitions of a finite number of physical parameters. Yet, this is not an efficient criterion to identify them. We are thus stuck with power counting and other criteria tied to special field-variable frames, and miss a broader view and a deeper insight.

The first task to overcome these difficulties is to develop a general field-covariant approach. Once this result is obtained, the second problem is to work out criteria that allow us to identify theories belonging to special subclasses in the most economic and efficient way. In this paper we investigate the first issue. The best way to search for a field-covariant formulation of quantum field theory is to study the most general perturbative changes of field variables. If we do this, we realize that whenever renormalization involves nonlinear field redefinitions the usual relation between bare and renormalized fields is not a true change of variables in the functional integral, but just a “replacement”. This means that the action is transformed according to the field redefinition, but the term $\int J\varphi$, which identifies the “elementary field” used to write Feynman rules and calculate diagrams, is not transformed, but just replaced with $\int J'\varphi'$, the analogous term in the new variables. These operations do give the transformed generating functional, but spoil its relation with the starting generating functional. Replacements are enough for a number of purposes, but they are not satisfactory for a covariant approach to quantum field theory.

Consider for example the infinitesimal redefinition $\varphi = \varphi' + b\varphi'^2$, with $b \ll 1$. When we use it as a change of variables in the functional integral, the term $\int J\varphi$ is turned into $\int J\varphi' + b \int J\varphi'^2$ and the transformed functional integral is no longer written in the usual way, by which we mean that it does not depend on J only through the term $\int J\varphi'$. To solve this problem we have the freedom to define a suitable J' , or correct the field redefinition. However, correcting the field redefinition is not helpful, since it can at most generate terms proportional to the field equations,

to the lowest order. Thus, we must search for a J' such that $\int J\varphi = \int J'\varphi'$. To the first order in b we find $J = J' - b\varphi'J'$. This J -redefinition is not acceptable, because J is an external source and φ is an integrated field. Thus, normally we jump to the new generating functional replacing $\int J\varphi$ with $\int J'\varphi'$ by brute force. In this paper we show how to overcome these problems and promote every manipulation to a true change of integration variables in the functional integral.

We show that it is always possible to reformulate the map relating bare and renormalized quantities, which we call *BR map*, as a true change of variables, plus redefinitions of parameters. Then, we study the most general perturbative changes of field variables and show how they get reflected from the classical action to the functionals Z and $W = \ln Z$. The Γ -functional requires a separate investigation, so in this paper we concentrate our attention on the Z - and W -functionals.

There is an intrinsic relation between composite fields, often called “composite operators”, and changes of field variables. Indeed, any local nonlinear change of variables maps elementary fields into composite ones. However, the field that we call elementary and with respect to which we perform the quantization is just a personal choice among infinitely many. Since the physics cannot depend on the variables we use, there are no intrinsic notions of “elementary fields” and “composite fields”. For this reason, it is convenient to treat the theory together with the set of its composite fields. Changes of field variables undergo their own renormalization, which is related to the renormalization of composite fields.

A perturbative field redefinition is a field redefinition that can be expressed as the identity map plus a perturbative series of local monomials of the fields and their derivatives. We show that, if J denote the sources coupled to the elementary fields, the most general perturbative change of field variables is a J -dependent redefinition of the sources coupled to the composite fields, and the Z - and W -functionals behave as scalars. Taking advantage of these properties, we can easily relate correlation functions before and after the variable change. In particular, our results provide a simple method to derive the renormalization of the theory in the new variables from the renormalization of the theory in the old variables, without having to calculate it anew.

We use several approaches and compare their virtues and weaknesses. In one approach, which we call *redundant*, descendants and composite fields proportional to the field equations are treated as independent composite fields. In another approach, called *essential*, they are not considered independent, therefore suppressed. In a third approach, called *linear*, we are able to linearize the source redefinitions that encode the most general changes of field variables.

For definiteness, we work using the Euclidean notation and the dimensional-regularization technique, but no results depend on these choices. To simplify the presentation, we imagine that the fields we are working with are bosonic, so we do not need to pay attention to their positions. The arguments can be immediately generalized to include fermionic fields and Grassmann variables.

The paper is organized as follows. In section 2 we address the problem. In section 3 we

define the various approaches we are going to use. In sections 4–7 we study the BR map in each approach. In sections 8–10 we study the most general changes of field variables, in each approach, at the bare and renormalized levels. In section 11 we make some important remarks about the relation between bare and renormalized changes of field variables. We give a number of explicit examples in section 12. Section 13 contains our conclusions.

2 Description of the problem

Normally we formulate quantum field theory starting from a classical action $S_c(\varphi, \lambda)$, where φ is the set of fields and λ is the set of parameters (including both couplings and masses). We define generating functionals, try to calculate them perturbatively, and find divergences. We discover that at every step of the perturbative subtraction divergences are local, and therefore can be removed with redefinitions of fields and parameters in the classical action, provided the classical action contains enough independent parameters. When it is not so, we just introduce new parameters at the tree level.

The subtraction of divergences is done as follows. Every quantity has a bare version, which is basically the classical version, and a renormalized version. The renormalized fields and parameters are those that make the generating functionals convergent. We call any map relating bare and renormalized quantities *BR map*. The subtraction of divergences makes the renormalized quantities depend on one parameter more than the bare quantities, the “dynamical scale” μ , so the BR map has the form

$$\varphi_B = \varphi_B(\varphi, \lambda, \mu), \quad \lambda_B = \lambda_B(\lambda, \mu). \quad (2.1)$$

The relations $\varphi_B = \varphi_B(\varphi, \lambda, \mu)$ are perturbatively local, but need not be polynomial. For example, they are not polynomial in a generic non-renormalizable theory, such as Einstein gravity [1].

The bare action $S_B(\varphi_B, \lambda_B)$ coincides with the classical action $S_c(\varphi, \lambda)$, once fields and parameters are replaced with the bare ones: $S_B(\varphi_B, \lambda_B) = S_c(\varphi_B, \lambda_B)$. Similarly, the renormalized action coincides with the bare action once bare fields and parameters are expressed in terms of the renormalized ones:

$$S(\varphi, \lambda, \mu) = S_B(\varphi_B, \lambda_B). \quad (2.2)$$

In the Euclidean notation the bare and renormalized Z - and W -generating functionals are

$$\begin{aligned} Z_B(J_B, \lambda_B) &= \int [d\varphi_B] \exp \left(-S_B(\varphi_B, \lambda_B) + \int \varphi_B J_B \right) = \exp(W_B(J_B, \lambda_B)), \\ Z(J, \lambda, \mu) &= \int [d\varphi] \exp \left(-S(\varphi, \lambda, \mu) + \int \varphi J \right) = \exp(W(J, \lambda, \mu)), \end{aligned} \quad (2.3)$$

and the Γ -functionals $\Gamma_B(\Phi_B, \lambda_B)$ and $\Gamma(\Phi, \lambda, \mu)$ are the Legendre transforms of W_B and W with respect to J_B and J , respectively.

Our investigation starts from the following problem: *are there relations*

$$J_B = J_B(J, \lambda, \mu), \quad \Phi_B = \Phi_B(\Phi, \lambda, \mu),$$

such that

$$Z_B(J_B, \lambda_B) = Z(J, \lambda, \mu), \quad W_B(J_B, \lambda_B) = W(J, \lambda, \mu), \quad \Gamma_B(\Phi_B, \lambda_B) = \Gamma(\Phi, \lambda, \mu) ? \quad (2.4)$$

If the renormalization of φ is multiplicative, such relations exist and are easy to find. We have

$$\varphi_B = Z_\varphi^{1/2} \varphi, \quad J_B = Z_\varphi^{-1/2} J, \quad \Phi_B = Z_\varphi^{1/2} \Phi, \quad (2.5)$$

where Z_φ is the wave-function renormalization constant of the elementary field φ . The second relation of (2.5) is obtained applying the change of variables $\varphi_B = Z_\varphi^{1/2} \varphi$ in the functional integral that defines $Z_B(J_B, \lambda_B)$. This operation gives indeed $Z(J, \lambda, \mu)$ once we define $J_B = Z_\varphi^{-1/2} J$. The third relation is obtained applying the same change of variables to go from $\Phi_B = \langle \varphi_B \rangle$ to $\Phi = \langle \varphi \rangle$.

Nevertheless, when the relation between φ_B and φ is not multiplicative, the matter is more complicated. Let us make the change of variables $\varphi_B = \varphi_B(\varphi, \lambda, \mu)$ in $Z_B(J_B, \lambda_B)$ again. Since the function $\varphi_B(\varphi, \lambda, \mu)$ is perturbatively local, using the dimensional-regularization technique the Jacobian determinant is identically 1 (because every polynomial of the momenta integrates to zero), so the functional integration measure is invariant under the BR map. The action in the exponent transforms correctly, because of (2.2), but there is no obvious way to transform the term $\int \varphi_B J_B$ into $\int \varphi J$. Thus, we *cannot* conclude $Z_B(J_B, \lambda_B) = Z(J, \lambda, \mu)$.

Formula (2.3) is a very specific way to express a functional integral, which does not survive a generic change of field variables. The entire J -dependence is encoded in the term $\int \varphi J$ appearing in the exponent of the integrand. We say that, in this case, the generating functionals Z and W are written in the *conventional form*. The role of $\int \varphi J$ is to specify which is the elementary field used to derive the Feynman rules and calculate diagrams. Clearly, the elementary field is spoiled by a nonlinear change of field variables.

A generic change of variables, including a translation $\varphi \rightarrow \varphi + a$, converts the functional integral to some unconventional form. We can also make perturbatively local non-polynomial changes of variables that depend on J . What is not obvious is how to go back to the conventional form after the variable-change. In section 9 we give a theorem that allows us to achieve this goal, in the most general perturbative setting.

To answer the question raised above we must first introduce composite fields and study their renormalization. The reason is that nonlinear field redefinitions always mix elementary fields and composite fields. The renormalization of composite fields has been extensively treated in the literature [2], but we must revisit it before proceeding. Actually, it is necessary to formulate a number of different approaches, because each of them is convenient for a different purpose.

Once this is done, we can study the BR map in a general setting. We basically have two options to describe the field redefinitions contained in BR maps. One option is to make a classical change of variables inside the action and replace the bare term $\int J_B \varphi_B$ with the renormalized one $\int J \varphi$ by brute force. This is the operation we are accustomed to, and we call it a *replacement*. The other option is to make a true change of variables inside the functional integral, which is the new operation we investigate in this paper.

Instead of working with generating functionals we could just work with sets of correlation functions $\langle O_1(\varphi(x_1)) \cdots O_n(\varphi(x_n)) \rangle$ at $J = 0$, because they are manifestly invariant under changes of field variables $\varphi' = \varphi'(\varphi)$. Indeed, if $O'_i(\varphi') \equiv O_i(\varphi)$ we obviously have

$$\langle O_1(\varphi(x_1)) \cdots O_n(\varphi(x_n)) \rangle = \langle O'_1(\varphi'(x_1)) \cdots O'_n(\varphi'(x_n)) \rangle',$$

where the primed average is calculated using the variables φ' and the unprimed average is calculated using the variables φ . Certainly the information contained in these relations allows us to do everything we need, with a suitable effort. On the other hand, working with generating functionals we gain a compact formalism that makes most of that effort for us. It would be very impractical to work without generating functionals in non-Abelian gauge theories and gravity, for example, because generating functionals give an easy control on local symmetries and their properties under renormalization.

3 Basic definitions and notation

Taking inspiration from the definitions introduced in ref. [3], we call *essential* a local composite field that is not a total derivative and is not proportional to the field equations. When we say the a composite field is “proportional to the field equations” we mean that it is equal to the product of another composite field times $\delta S/\delta\varphi$, or spacetime derivatives of $\delta S/\delta\varphi$. In all other contexts when we say that an object is “proportional to $X(\varphi)$ ” we mean that it is equal to a constant times $X(\varphi)$.

Denote the essential composite fields with $\mathcal{O}^{\bar{I}} = \mathcal{O}^{\bar{I}}(\varphi, \lambda)$. Call *descendant* a composite field that is a total derivative of an essential composite field. Define an equivalence relation stating that two essential composite fields $\mathcal{O}^{\bar{I}}$ are equivalent if they differ by a descendant. Then, for each equivalence class pick a representative \mathcal{O}^I and couple it to a source L_I . In the set of \mathcal{O}^I s we include the identity, which is the “composite field” 1, with source L_0 , and the elementary field φ itself, with source L_1 . It is convenient to keep the source J separate from the sources L , because J identifies the variables we are using to quantize the theory. Every perturbatively local function $F(\varphi)$ of φ can be expanded as a linear combination of the form

$$F(\varphi) = c_I(\lambda, \partial) \mathcal{O}^I(\varphi, \lambda) + E(\varphi). \quad (3.1)$$

where $c_I(\lambda, \partial)$ are operator-coefficients that may contain derivatives acting on the objects that appear at their rights, and $E(\varphi)$ denotes terms proportional to the field equations. The sum over repeated indices I is understood. We say that $\{\mathcal{O}^I\}$ is a *basis* of essential composite fields. We can choose a basis of essential composite fields that may depend on the couplings, which is why we use the notation $\mathcal{O}^I(\varphi, \lambda)$.

For example, we can take the \mathcal{O}^I s with $I > 1$ to be the monomials constructed with φ and its derivatives, discarding the combinations that are equal to total derivatives or terms of type $E(\varphi)$.

At the perturbative level, we write the action S as the sum $S_{\text{free}} + S_{\text{int}}$ of its free- and interaction-parts, as usual. Then, instead of identifying the terms of type $E(\varphi)$, we can equivalently identify the terms $E_{\text{free}}(\varphi)$, proportional to the free-field equations $\delta S_{\text{free}}/\delta\varphi$, and treat the differences as linear combinations of other composite fields, to be classified according to the same rules. This procedure is more convenient, because it just requires to search for factors equal to (derivatives of) $(-\square + m_s^2)\varphi$ for scalars, $(\not{\partial} + m_f)\psi$ and $\bar{\psi}(-\overleftarrow{\not{\partial}} + m_f)$ for fermions, and so on.

In some cases we may want to treat descendants and the terms of type $E(\varphi)$ or $E_{\text{free}}(\varphi)$ as independent composite fields and add them to the basis $\{\mathcal{O}^I\}$. This redundancy may be useful for several purposes. It is convenient to distinguish an *essential* approach, where the basis $\{\mathcal{O}^I\}$ contains only essential composite fields, and a *redundant* approach, where the basis $\{\mathcal{O}^I\}$ is unrestricted. We first work with the redundant approach, because it is simpler, and then discuss the essential approach in detail. Finally, we formulate a *linear* approach, where all sources L_I renormalize linearly.

We denote the classical composite fields with $\mathcal{O}_c^I(\varphi, \lambda)$. The classical basis $\{\mathcal{O}_c^I(\varphi, \lambda)\}$ can be used to define the basis of bare essential composite fields $\{\mathcal{O}_B^I(\varphi_B, \lambda_B)\}$, where the functions \mathcal{O}_B^I and \mathcal{O}_c^I are just the same, but we call them with different names to emphasize the fact that they have different arguments. Denote the bare sources with L_{IB} . We define the extended bare action S_{LB} as

$$S_{LB}(\varphi_B, \lambda_B, L_B) = S_B(\varphi_B, \lambda_B) - \int L_{IB} \mathcal{O}_B^I(\varphi_B, \lambda_B). \quad (3.2)$$

The bare Z - and W -generating functionals are then

$$Z_B(J_B, \lambda_B, L_B) = \int [d\varphi_B] \exp \left(-S_{LB}(\varphi_B, \lambda_B) + \int \varphi_B J_B \right) = \exp W_B(J_B, \lambda_B, L_B), \quad (3.3)$$

written in the conventional form.

Let us quickly review the renormalization in the presence of composite fields. Relations (2.1) hold at $L_I = 0$, so we need to concentrate on the renormalization at $L_I \neq 0$. We define the basis $\{\mathcal{O}^I\}$ of composite fields

$$\mathcal{O}^I(\varphi, \lambda, \mu) = \mathcal{O}_B^I(\varphi_B(\varphi, \lambda, \mu), \lambda_B(\lambda, \mu)) \quad (3.4)$$

at the renormalized level. Note that these objects are not the renormalized composite fields, but just the bare ones written using renormalized variables and parameters. The renormalized composite fields will be introduced later and denoted with \mathcal{O}_R^I .

We start from the classical extended action

$$S_{Lc}(\varphi, \lambda, \mu, L) = S(\varphi, \lambda, \mu) - \int L_I \mathcal{O}^I(\varphi, \lambda, \mu). \quad (3.5)$$

which is just the classical version of (3.2). Here L^I are the renormalized sources for the composite fields. Then we write the functional integral in the conventional form, using the action (3.5). Working out the Feynman rules and calculating diagrams, we realize that physical quantities are divergent. We calculate the divergent parts and subtract them away *modifying* the action as $S_{Lc} \rightarrow S_L = S_{Lc} +$ counterterms. We end up with a renormalized action $S_L(\varphi, \lambda, \mu, L)$. The renormalized generating functionals Z and W are defined as usual, once the action S is replaced with S_L :

$$Z(J, \lambda, \mu, L) = \int [d\varphi] \exp \left(-S_L(\varphi, \lambda, \mu, L) + \int \varphi J \right) = \exp W(J, \lambda, \mu, L). \quad (3.6)$$

Clearly, when we substitute S_{Lc} with $S_L = S_{Lc} +$ counterterms we are not making a change of variables, but just a replacement of actions. The theory is written in the conventional form both before and after the replacement, just because the term $\int J\varphi$ is replaced with the new one by brute force.

We later realize that the bare and renormalized actions are related by redefinitions of fields, parameters and sources, provided the classical action contains sufficiently many independent parameters. Thus S_L is nothing but S_{Lc} , or S_B , equipped with such redefinitions. This only tells us that S_B and S_L are related by such redefinitions, which contain a change of variables for the fields, but not that the functional integrals Z_B and Z are also related by those redefinitions. Indeed, we are not making any change of variables inside the functional integral. Such a change of variables would affect the term $\int J\varphi$ and convert the functional integral to some unconventional form. Instead, we are jumping from the conventional functional integral defined by the action S_B to the conventional functional integral defined by the action S_L . At this level, the BR map remains a replacement, not a true change of variables. In the next sections we show how to upgrade it to a true change of variables.

4 BR map in the redundant approach

In the redundant approach we work with a basis $\{\mathcal{O}^I\}$ containing all composite fields, including descendants (therefore also derivatives of φ) and composite fields proportional to the field equations. Then each perturbatively local function $F(\varphi)$ of φ can be expanded as a linear combination

$$F(\varphi) = c_I \mathcal{O}^I(\varphi), \quad (4.1)$$

where the c_I s are constants. The renormalized action has the form

$$S_L(\varphi, \lambda, \mu, L) = S(\varphi, \lambda, \mu) - \int L_I \mathcal{O}^I(\varphi, \lambda, \mu) + \Delta S_L(\varphi, \lambda, \mu, L), \quad (4.2)$$

where ΔS_L is a local functional that collects the counterterms belonging to the composite-field sector. Expanding ΔS_L as shown in (4.1) we can find local functions $f_I = L_I$ plus $\mathcal{O}(L)$ -radiative corrections depending on the sources L_I and their derivatives, such that

$$S_L(\varphi, \lambda, \mu, L) = S(\varphi, \lambda, \mu) - \int f_I(L, \lambda, \mu) \mathcal{O}^I(\varphi, \lambda, \mu). \quad (4.3)$$

Then we see that the replacement

$$\varphi_B = \varphi_B(\varphi, \lambda, \mu), \quad \lambda_B = \lambda_B(\lambda, \mu), \quad L_{IB} = f_I(L, \lambda, \mu), \quad \int J_B \varphi_B \leftrightarrow \int J \varphi, \quad (4.4)$$

turns the bare action (3.2) into the renormalized one (5.1) and the bare generating functionals Z_B and W_B into the renormalized ones Z and W . We call (4.4) the *BR replacement* in the redundant approach.

Observe that the first of (4.4) is just a *classical* change of variables. This means that it acts on the action as a change of variables, but it is not meant as a change of variables in the functional integral. The source L_{0B} reabsorbs field-independent counterterms.

Now we see how to relate bare and renormalized quantities by a true change of field variables. Use (2.1) to define the renormalized basis (3.4). Then, expand $\varphi_B(\varphi, \lambda, \mu)$ using (4.1):

$$\varphi_B = \varphi + c_I \mathcal{O}^I(\varphi, \lambda, \mu), \quad (4.5)$$

where the constants c_I can be treated perturbatively. Next, use this relation to make a change of field variables in the bare functional integral (3.3). Doing so, the term $\int J_B \varphi_B$ is turned into

$$\int J_B \varphi + \int c_I J_B \mathcal{O}^I(\varphi),$$

so the functional integral is not written in the conventional form anymore. We can convert it back to the conventional form if we define

$$L_{IB} = f_I(L) - c_I J, \quad J_B = J, \quad (4.6)$$

and apply these relations to (3.3), instead of using the last two formulas of (4.4). It is easy to see that we directly obtain

$$Z_B(J_B, \lambda_B, L_B) = Z(J, \lambda, \mu, L), \quad W_B(J_B, \lambda_B, L_B) = W(J, \lambda, \mu, L). \quad (4.7)$$

Thus, in the redundant approach the change of variables converting bare quantities into renormalized ones is

$$\varphi_B = \varphi_B(\varphi, \lambda, \mu) = \varphi + \sum_I c_I \mathcal{O}^I, \quad \lambda_B = \lambda_B(\lambda, \mu), \quad L_{IB} = f_I(L) - c_I J, \quad J_B = J. \quad (4.8)$$

We call it the *BR change of field variables*. Note that (4.6) are linear in J , but not in L .

The replacement (4.4) does not allow us to write the equalities (4.7). It just allows us to “jump” from the bare generating functionals to the renormalized ones. For a variety of purposes this “jump” is quite enough, yet it is not satisfactory if we want to develop a field-covariant formulation of quantum field theory, where we need operations such as (4.8), which allow us to smoothly follow every step of the transformation.

Before proceeding, let us make some observations about the functions f^I in (4.3). There exist constants Z_J^I such that

$$f_I(L, \lambda, \mu) = L_J(Z^{-1})_I^J + \mathcal{O}(L^2).$$

Indeed, at the linear level in L we do not need to consider derivatives $\partial^p L$, since integrating by parts these derivatives can be moved inside the composite fields coupled to L . The terms $\mathcal{O}(L^2)$ subtract the divergences that arise in correlation functions containing more than one insertions of composite fields.

Taking the functional derivative of Z and W with respect to L^I and then setting $L = 0$ we obtain convergent correlation functions containing one insertion of $(Z^{-1})_J^I \mathcal{O}^J$ and arbitrarily many insertions of elementary fields. Thus, $(Z^{-1})_J^I \mathcal{O}^J$ are the renormalized composite fields \mathcal{O}_R^I :

$$\mathcal{O}_R^I = (Z^{-1})_J^I \mathcal{O}^J(\varphi, \lambda, \mu). \quad (4.9)$$

Using (3.4) we get

$$\mathcal{O}_B^I = Z_J^I \mathcal{O}_R^J.$$

The quantities Z_J^I are the renormalization constants of the composite fields. It is convenient to organize the \mathcal{O}^I s in a row such that operators of equal dimensions are close to one another and operators of lower dimensions precede those of higher dimensions. In a renormalizable theory, where only parameters of non-negative dimensions are present, a composite field can mix only with composite fields of equal or smaller dimensions, so the matrix Z_J^I is block-lower triangular. If the theory is non-renormalizable we can have two situations: if only parameters of non-positive dimensions are present, then the matrix Z_J^I is block-upper triangular; in the general case, where both parameters of positive, null and negative dimensions are present, the matrix Z_J^I has no particular restriction.

5 BR replacement in the essential approach

We could be satisfied with the redundant approach just presented, however for a variety of reasons that will be appreciated later we need to upgrade it in several ways. It is often useful to restrict $\{\mathcal{O}^I\}$ to be a basis of essential composite fields. Among the other things, this allows us to better keep track of the counterterms proportional to the field equations.

We prove that there exists a field redefinition $\varphi_B = \varphi_B(\varphi, \lambda, \mu, L)$ such that the renormalized action has the form

$$S_L(\varphi, \lambda, \mu, L) = S(\varphi, \lambda, \mu, L) - \int f_I(L, \lambda, \mu) \mathcal{O}^I(\varphi, \lambda, \mu), \quad (5.1)$$

where $f_I = L_I$ plus radiative corrections are $\mathcal{O}(L)$ -local functions of the sources L_I and their derivatives and

$$S(\varphi, \lambda, \mu, L) = S_B(\varphi_B(\varphi, \lambda, \mu, L), \lambda_B(\lambda, \mu)). \quad (5.2)$$

Note that the functions f_I are not the same as in (4.3), although we use the same notation, for simplicity. At $L_I \neq 0$ the field replacement is the first of (2.1) plus $\mathcal{O}(L)$ -radiative corrections to be determined. Thus, S_L is equal to the classical extended action S_{Lc} of (3.5) plus radiative corrections.

We already know that at $L_I = 0$ renormalization is given by (2.1). At $L_I \neq 0$ we proceed iteratively in the loop expansion, as usual. At a given order, we treat the action as an expansion in powers of L . Assuming that divergences are subtracted away up to $n - 1$ loops by a renormalized action of the form (5.1), we study the n -loop $\mathcal{O}(L)$ -counterterms. It is convenient to expand them as shown in (3.1). We can distinguish counterterms proportional to the field equations and counterterms proportional to the essential fields and their derivatives. Integrating by parts we can write the n -loop $\mathcal{O}(L)$ -divergences as the sum

$$\int \Delta_I(L) \mathcal{O}^I(\varphi, \lambda, \mu) + \int U(\varphi, L) \frac{\delta S}{\delta \varphi}, \quad (5.3)$$

where $U(\varphi, L)$ is an n -loop $\mathcal{O}(L)$ -local function of φ , L and their derivatives and the $\Delta_I(L)$ s are n -loop $\mathcal{O}(L)$ -local functions of the sources L and their derivatives.

We subtract the first sum of (5.3) renormalizing the sources L_I by means of the replacements

$$L_I \rightarrow L_I + \Delta_I(L). \quad (5.4)$$

However, when we perform this replacement inside (5.1) we not only subtract the first term of (5.3), but generate also higher-loop divergent terms. They can be ignored at this stage. They will be dealt with at the subsequent steps of the subtraction algorithm. As far as the n -loop counterterms are concerned, inserting (5.4) inside (5.1) or (3.5) is the same thing.

Consider now the counterterms proportional to the field equations. We subtract them redefining the fields φ as $\varphi = \varphi' - U(\varphi', L)$ inside S_L and replacing the term $\int J\varphi$ with $\int J\varphi'$. Later we drop the prime on φ' and rename φ' as φ . Again, these operations are just replacements, not changes of variables inside the functional integral.

As above, when we perform the replacement $\varphi \rightarrow \varphi - U$ inside the action we not only subtract the last term of (5.3), but generate also divergent terms proportional to higher powers of U , to be dealt with at the subsequent steps of the subtraction algorithm. However, the replacement

$\varphi \rightarrow \varphi - U$ also affects the essential fields. Neglecting higher-order terms, we can focus on the n -loop contributions

$$- \int dx L_I(x) \frac{\delta \mathcal{O}^I(x)}{\delta \varphi(y)} U(y) dy, \quad (5.5)$$

which are now $\mathcal{O}(L^2)$. These terms can be treated like the divergences (5.3). Again, we start separating the part proportional to the field equations from the part proportional to essential composite fields and descendants, and repeat the procedure. We get an $\Delta_I(L)$ and a $U(\varphi, L)$ of order L^2 . With another L_I -replacement like (5.4) and a field replacement, we remain with n -loop divergent terms (5.5) proportional to $\mathcal{O}(L^3)$. Proceeding indefinitely in this way, we can get rid of all of them.

Then we repeat the entire procedure for the $(n + 1)$ -loop $\mathcal{O}(L)$ -counterterms, and so on. This iteration proves our statement. The replacements (5.4) build the functions f_I . The field replacements build the functions $\varphi_B(\varphi, \lambda, \mu, L)$.

Summarizing, there exist *BR replacements*

$$\varphi_B = \varphi_B(\varphi, \lambda, \mu, L), \quad \lambda_B = \lambda_B(\lambda, \mu), \quad L_{IB} = f_I(L, \lambda, \mu), \quad \int J_B \varphi_B \leftrightarrow \int J \varphi, \quad (5.6)$$

that turn the bare action (3.2) into the renormalized one (5.1) and the bare generating functionals Z_B and W_B into the renormalized ones Z and W .

We stress again that formulas (3.2) and (5.1), as well as (4.4) and (5.6) are very general, and hold even when the relation between bare and renormalized fields is non-polynomial, and the theory is non-renormalizable.

Observe that shifting some L s by constants is equivalent to modify the action $S(\varphi, \lambda, \mu)$. Using this trick we can turn on non-renormalizable vertices, for example. Renormalizable interactions can also be described this way, starting from a free-field theory. This is not surprising, because studying composite fields in a free-field theory we get enough information to reconstruct every perturbatively interacting theory, renormalizable or not. Basically, what we do is to replace the classical action S_c with the free-field action S_{free} and shift the sources L by constants, such that the extended action S_{Lc} of (3.5) at $L = 0$ gives back S_c . Similar operations can be made in the bare action (3.2) and the renormalized action (5.1). Using this trick we can also retrieve (2.1) from (5.6). For example, the massive φ^4 -theory can be described working with the free action $S_{\text{free}}(\varphi)$ and setting all sources L equal to zero except for the sources L_4 and L_2 of $\varphi^4/4!$ and $\varphi^2/2!$, which are set equal to the constants $-\lambda$ and $-m^2$, respectively. Then the relations $L_{IB} = f_I(L, \lambda, \mu)$ of (5.6) with $I = 4, 2$, become $\lambda_B = \lambda_B(\lambda, \mu)$, and $m_B^2 = m^2 Z_m(\lambda, \mu)$, while $\varphi_B = \varphi_B(\varphi, \lambda, \mu, L)$ becomes the first of (2.1), which in this case is nothing but $\varphi_B = Z_\varphi^{1/2} \varphi$.

6 BR change of variables in the essential approach

Now we study the relation between bare and renormalized quantities in the essential approach as a change of variables in the functional integral. We show that there exist local functions

$$J_B = J, \quad L_{IB} = L_{IB}(J, \lambda, \mu, L), \quad (6.1)$$

relating bare and renormalized sources, that make the bare and renormalized Z - and W -functionals coincide. Moreover, we give an algorithm to derive the functions (6.1) explicitly.

Using the first of (5.6) write the action $S(\varphi, \lambda, \mu, L)$ of (5.2). Next, expand the relation between φ_B and φ as shown in (3.1). We have

$$\varphi_B(\varphi, \lambda, \mu, L) = \varphi + c_I(\lambda, \mu, L, \partial)\mathcal{O}^I(\varphi, \lambda, \mu) + E(\varphi), \quad (6.2)$$

which is equal to φ plus perturbative corrections. Here $E(\varphi)$ is meant to be proportional to the field equations of $S(\varphi, \lambda, \mu, L)$. The coefficients c_I are local functions of the sources L and their derivatives, and can contain derivative operators acting on their right-hand sides. Similarly, inserting (6.2) in the bare composite fields, using (3.4) and expanding again, we can write

$$\mathcal{O}_B^I(\varphi_B, \lambda_B) = d_J^I(\lambda, \mu, L, \partial) (\mathcal{O}^J(\varphi, \lambda, \mu) + E^J(\varphi)),$$

where $d_J^I = \delta_J^I + \mathcal{O}(L)$. Next, use (6.2) to make a change of variables in the functional integral Z_B of (3.3). At the same time, write

$$L_{IB} = (d^{-1}(-\partial))_I^J (f_J - c_J(-\partial)J + \Delta f_J), \quad J_B = J, \quad (6.3)$$

where Δf_I are yet unknown $\mathcal{O}(\tilde{L}^2)$ -local functions of the sources $\tilde{L} = \{L, J\}$. We recall that f_I are $\mathcal{O}(L)$ -local functions of L , λ , and μ .

As usual, using the dimensional-regularization technique, which we assume here, and treating the change of variables perturbatively, the functional integration measure is invariant. Everything works as in the replacement of the previous section, but for the term $\int \varphi_B J_B$ and the corrections proportional to Δf_I . The exponent $-S_{LB} + \int \varphi_B J_B$ turns into minus (5.1) plus

$$\int J\varphi + \int \Delta f_I \mathcal{O}^I - \int U(\varphi, J, L) \frac{\delta S}{\delta \varphi}, \quad (6.4)$$

for some local function $U(\varphi, J, L) = \mathcal{O}(\tilde{L})$. The first term is the one that must be there, while the rest must be canceled out. The second integral in (6.4) is $\mathcal{O}(\tilde{L}^2)$ by assumption, but yet unknown. The last term of (6.4) collects all contributions proportional to the field equations. This object can be manipulated with the procedure described after formula (5.3). The only difference is that now the field redefinitions are true change of variables inside the functional integral. Thus, make the change of variables $\varphi = \varphi' - U(\varphi', J, L)$ and then drop the prime on φ' to rename φ' as

φ . Expanding in powers of U we cancel the last term of (6.4), but generate new terms, which, however, are all $\mathcal{O}(\tilde{L}^2)$. These terms include the object $-\int JU$ originating from $\int J\varphi$. Then we expand such terms using (3.1), and cancel essential composite fields and descendants fixing the $\mathcal{O}(\tilde{L}^2)$ -contributions to Δf_I . After this, the last term of formula (6.4) is replaced by an object of the same form, but one order of \tilde{L} higher than before. Then we can repeat the procedure. Iterating the procedure indefinitely we obtain (4.7).

Observe that in the end the redefinitions (6.1) can contain arbitrarily large powers of L and J . Moreover, combining the first of (5.6) with the further changes of variables of type $\varphi = \varphi' - U$, the final change of variables that relates bare and renormalized fields is J -dependent, so instead of the first of (5.6) we have $\varphi_B = \varphi_B(\varphi, \lambda, \mu, L, J)$.

Recapitulating, the BR change of variables in the essential approach has the form

$$\varphi_B = \varphi_B(\varphi, \lambda, \mu, L, J), \quad \lambda_B = \lambda_B(\lambda, \mu), \quad L_{IB} = L_{IB}(J, \lambda, \mu, L), \quad J_B = J, \quad (6.5)$$

where $\varphi_B = \varphi +$ radiative corrections, $L_{IB} = L_I + \mathcal{O}(\tilde{L})$ -radiative corrections, and so on. Formula (6.5) allows us to describe the relation between bare and renormalized quantities as a true change of variables in the functional integral, instead of using the replacement (5.6).

Note that the source J never renormalizes. This happens because its renormalization is moved to the renormalization of the source L_1 coupled with the elementary field φ . For example, in the massive φ^4 theory we get

$$\begin{aligned} \varphi_B &= \varphi Z_\varphi^{1/2}(\lambda, \mu), & \lambda_B &= \lambda \mu^\epsilon Z_\lambda(\lambda, \mu), & m_B^2 &= m^2 Z_m(\lambda, \mu), \\ L_{IB} &= L_1 Z_\varphi^{-1/2} + J(Z_\varphi^{-1/2} - 1), & J_B &= J, \end{aligned} \quad (6.6)$$

where Z_λ and Z_m are the renormalization constants of the coupling and the squared mass, respectively. A map similar to (6.6) holds in every multiplicatively renormalizable theory. The functional integral depends only on $J + L_1$, at this level, however in other approaches (see next section) and other applications J and L_1 play different roles, which is why we prefer to keep them distinct.

7 BR map in the linear approach

The essential approach takes advantage of the most general field- and source-redefinitions and makes us appreciate the role played by higher-powers of J in (6.1), as well as the roles played by J - and L -dependences in the relation $\varphi_B \leftrightarrow \varphi$. However, BR replacements and BR changes of variables are much simpler if we adopt the redundant approach. There the source redefinitions are linear in J , although not in L .

The reader may be worried that the redundant approach does not isolate the counterterms that can be removed by means of field redefinitions. This is true, but only in the source sector,

because the source-independent sector is taken care by (2.1). Nothing prevents us from removing the divergences proportional to the field equations by means of field redefinitions in the source-independent sector, even if we use the redundant approach. For most purposes, this is enough. Indeed, isolating terms proportional to the field equations in the source-independent sector is necessary to identify key properties of the theory, such as its finiteness, or its renormalizability with a finite number of independent parameters. Instead, isolating terms proportional to the field equations in the source sector is more a matter of aesthetics.

These arguments lead us to conclude that the redundant approach is more convenient than the essential one. We can make a step forward and define a third approach, which is even more convenient for several purposes. We call it the *linear approach*, because all source redefinitions are linear in L as well as in J .

The bare action is not written in the form (3.2), rather in the new form

$$S_{LB}(\varphi_B, \lambda_B, \tau_B, L_B) = S_B(\varphi_B, \lambda_B) - \int (L_{IB} + \tau_{vIB} \mathcal{N}_B^v(L_B, \lambda_B)) \mathcal{O}_B^I(\varphi_B, \lambda_B), \quad (7.1)$$

where $\mathcal{N}_B^v(L_B, \lambda_B) = \mathcal{O}(L_B^2)$ is a basis of independent local monomials that can be constructed with the sources L_B and their derivatives, and are at least quadratic in L_B . Each such monomial is multiplied by a new, independent coefficient τ_{vIB} . The sum over repeated indices v is understood. Note that in the linear approach the functional integral depends on J and L_1 separately. The bare Z - and W -functionals are given by (3.3), with the extended action (7.1).

The classical action is (7.1) once the subscripts B are dropped,

$$S_{cL}(\varphi, L) = S_c(\varphi) - \int L_I \mathcal{O}_c^I(\varphi) - \int \tau_{vI} \mathcal{N}^v(L) \mathcal{O}_c^I(\varphi). \quad (7.2)$$

The renormalized action $S_L(\varphi, \lambda, \tau, L)$ is derived below, check formulas (7.8) and (7.15).

Before proceeding, let us explain how the perturbative expansion must be organized. We want to be sure that radiative corrections are of higher order with respect to the classical terms. This fact is obvious when composite fields are switched off, less obvious when they are present. We describe the behavior of each quantity referring it to some parameter $\delta \ll 1$. Let us state that the coupling λ_{n_l} multiplying a vertex with n_l φ -legs is of order δ^{n_l-2} . Then when composite fields are switched off each loop carries an additional factor δ^2 . Consider an ℓ -loop diagram with E external legs, I internal legs and v_l vertices of type l . Counting legs and using the identity $\ell = I - V + 1$, we have $\sum_l n_l v_l = E + 2I = E + 2(\ell + V - 1)$, so the diagram is multiplied by an expression of order

$$\delta^{\sum_l (n_l-2)v_l} = \delta^{2\ell} \delta^{E-2}. \quad (7.3)$$

Besides the expected tree-level factor δ^{E-2} , associated with the E external φ -legs, we get a δ^2 for each loop, as claimed.

Now, assume that the composite fields $\mathcal{O}^I(\varphi, \lambda)$ are homogeneous in δ , namely

$$\mathcal{O}^I(\varphi\delta^{-1}, \lambda_I\delta^{n_I-2}) = \delta^{-n_I}\mathcal{O}^I(\varphi, \lambda)$$

for some n_I . Observe that the vertices $L_I\mathcal{O}^I$ are not multiplied by any coupling. Actually, the sources L replace the couplings in this case, so we must assume $L_I = \mathcal{O}(\delta^{n_I-2})$. Next, consider the parameters τ and observe that some contributions to their renormalization can be $\mathcal{O}(\delta^0)$, because the vertices $L_I\mathcal{O}^I$ allow us to construct diagrams with no couplings λ and at least two external L -legs. Thus, the parameters τ may carry negative orders of δ . A consistent assignment is $\tau_{vI} = \mathcal{O}(\delta^{n_I-n_v-2})$, where n_v is the δ -degree of the monomial $\mathcal{N}^v(L)$, because then the product $\tau_{vI}\mathcal{N}^v$ is $\mathcal{O}(\delta^{n_I-2})$, like L_I .

Summarizing, the δ -expansion is properly organized assuming

$$\lambda_{n_i} = \mathcal{O}(\delta^{n_i-2}), \quad L_I = \mathcal{O}(\delta^{n_I-2}), \quad \tau_{vI} = \mathcal{O}(\delta^{n_I-n_v-2}), \quad (7.4)$$

while J is $\mathcal{O}(\delta^{-1})$. A quick way to derive the correct assignments is to observe that if the fields φ are imagined to be $\mathcal{O}(\delta^{-1})$, then all terms of the classical action are $\mathcal{O}(\delta^{-2})$. In particular, if we make the substitutions

$$\varphi \rightarrow \varphi\delta^{-1}, \quad \lambda_{n_i} \rightarrow \lambda_{n_i}\delta^{n_i-2}, \quad L_I \rightarrow L_I\delta^{n_I-2}, \quad \tau_{vI} \rightarrow \tau_{vI}\delta^{n_I-n_v-2}, \quad (7.5)$$

then the action transforms as

$$S_L(\varphi, \lambda, \tau, L) \rightarrow \frac{1}{\delta^2} \left(S_{cL} + \sum_{\ell \geq 1} \delta^{2\ell} S_{\ell L} \right), \quad (7.6)$$

where the ℓ -loop contributions $S_{\ell L}$ are δ -independent.

Sometimes it may be useful to consider some couplings of orders higher than those assigned in (7.4). This is allowed, depending on the specific features of the theory, as long as the radiative corrections to those couplings are of even higher orders.

Equipped with the more involved structures (7.1) and (7.2), renormalization is now much simpler. We can define a redundant linear approach and an essential linear approach. We begin with the redundant one.

The action (7.2) contains enough independent parameters to renormalize all $\mathcal{O}(L^2)$ -divergences relating τ_B and τ . Doing so it is sufficient to renormalize the sources L_{IB} linearly, using the renormalization constants Z_{IJ} already met:

$$L_{IB} = L_J(Z^{-1})_I^J.$$

Thus, in the redundant linear approach the BR replacement that turns (3.3) into (3.6) reads

$$\begin{aligned} \varphi_B &= \varphi_B(\varphi, \lambda, \mu), & \lambda_B &= \lambda_B(\lambda, \mu), & \int J_B \varphi_B &\leftrightarrow \int J \varphi, \\ L_{IB} &= L_J(Z^{-1})_I^J, & \tau_{vIB} &= \hat{\tau}_{vJ}(\tau, \lambda, \mu)(Z^{-1})_I^J, \end{aligned} \quad (7.7)$$

instead of (4.4), where $\hat{\tau} = \tau$ plus radiative corrections. The renormalized extended action is

$$S_L(\varphi, \lambda, \mu, L) = S(\varphi, \lambda, \mu) - \int (L_I + \hat{\tau}_{vI} \mathcal{N}^v(L, \lambda, \mu)) \mathcal{O}_R^I(\varphi, \lambda, \mu), \quad (7.8)$$

where \mathcal{O}_R^I are the renormalized composite fields (4.9) and we have defined the basis of renormalized \mathcal{N}^v s as $\mathcal{N}^v(L, \lambda, \mu) = \mathcal{N}_B^v(L_B, \lambda_B)$. It is easy to check that (7.7) is consistent with the perturbative expansion governed by (7.4).

To study the BR map as a change of field variables we first make the change of variables (4.5) inside the functional integral. The constants c_I must be assumed to be of order δ^{n_I+1} . A factor δ^{n_I-1} is the tree-level assignment that makes φ and $c_I \mathcal{O}^I$ of the same order. An extra factor δ^2 comes from the fact that the c_I s are at least one loop. (In the next sections we also consider classical changes of variables, where the c_I s become of order δ^{n_I-1} .) We also make the substitutions

$$L_{IB} = L_J(Z^{-1})_I^J - c_I J + L_J h_I^J - c_J J \Delta c_I^J, \quad J_B = J, \quad \tau_{vIB} = (\hat{\tau}_{vJ} + \Delta \hat{\tau}_{vJ})(Z^{-1})_I^J, \quad (7.9)$$

where h_I^J , Δc_I^J and $\Delta \hat{\tau}_v$ are unknown constants. They have to be determined as expansions in powers of c starting with $\mathcal{O}(c)$, so they can be treated perturbatively. The exponent of the Z -integrand contains the functions \mathcal{N}^v , which generate other J -dependent terms after the replacements (7.9). Such terms have the structure

$$\int J c_I U^I(\varphi, cJ, L, h, \Delta c, \Delta \hat{\tau}), \quad (7.10)$$

where U^I are $\mathcal{O}(\tilde{L})$ -local functions of $\tilde{L} = \{L, J\}$. The exponent of the Z -integrand can be written as minus (7.8) plus $\int J \varphi$ plus

$$\int (L_J h_I^J - c_J J \Delta c_I^J) \mathcal{O}^I + \int (\hat{\tau}_{vI}(C_w^v - \delta_w^v) + \Delta \hat{\tau}_{vI} C_w^v) \mathcal{N}^w(L, \lambda, \mu) \mathcal{O}_R^I + \int J c_I U^I, \quad (7.11)$$

where we have expanded

$$\mathcal{N}_B^v(L(Z^{-1} + h), \lambda_B) = C_w^v \mathcal{N}^w(L, \lambda, \mu),$$

and $C_w^v = \delta_w^v + \mathcal{O}(h)$ are constants.

The terms (7.11) are those we must get rid of in order to obtain the renormalized generating functionals (3.6) and prove relations (4.7). Now we show that we can achieve this goal choosing h , Δc_I and $\Delta \hat{\tau}_{vI}$ appropriately.

Make the further change of variables $\varphi \rightarrow \varphi - c_I U^I$. Expanding in the basis $\{\mathcal{O}^I\}$ and $\{\mathcal{N}^v\}$, the action $S(\varphi, \lambda, \mu)$ transforms as

$$S(\varphi, \lambda, \mu) \rightarrow S(\varphi, \lambda, \mu) + \int (L_J \bar{h}_I^J - c_I J \bar{\Delta} c_I^J) \mathcal{O}^I + \int \bar{\Delta} \hat{\tau}_{vI} \mathcal{N}^v(L, \lambda, \mu) \mathcal{O}_R^I - c^2 \int J \tilde{U}(\varphi, cJ, L), \quad (7.12)$$

where the last term is written in compact form (indices being understood), $\bar{\Delta}c$, \bar{h} and $\bar{\Delta}\hat{\tau}$ are $\mathcal{O}(c)$ -constants and $\tilde{U} = \mathcal{O}(\tilde{L})$ are local functions. All these objects are perturbative expansions in powers of c , whose coefficients may also depend on h , Δc_I and $\Delta\hat{\tau}_{vI}$. An expansion similar to (7.12) can be written for the transformed $\mathcal{O}^I(\varphi, \lambda, \mu)$ s, but since these objects are always multiplied by $\mathcal{O}(\tilde{L})$, they affect the exponent only adding terms like the last two of (7.12). The corrections originating from (7.10) only give terms like the last of (7.12). Finally, the exponent of the Z -integrand is minus (7.8) plus $\int J\varphi$ plus

$$\begin{aligned} & \int (L_J(h_I^J - \bar{h}_I^J) - c_I J(\Delta c_I^J - \bar{\Delta}c_I^J)) \mathcal{O}^I + \int (\hat{\tau}_{vI}(C_w^v - \delta_w^v) + \Delta\hat{\tau}_{vI}C_w^v - \bar{\Delta}\hat{\tau}'_{vI}) \mathcal{N}^w(L, \lambda, \mu) \mathcal{O}_R^I \\ & + c^2 \int JU'(\varphi, cJ, L), \end{aligned} \quad (7.13)$$

with $U' = \mathcal{O}(\tilde{L})$. The first line of (7.13) is canceled choosing the unknowns h_I^J , Δc_I^J and $\Delta\hat{\tau}_v$, which can be done solving their equations recursively in powers of c . Actually, at this stage we can truncate the solutions at $\mathcal{O}(c)$, because higher orders must be modified anyway, to cancel the second line of (7.13). Such terms are like the term $\int JcU$ of (7.10), but one order higher in c . Then we repeat the procedure, starting from the change of variables $\varphi \rightarrow \varphi - c^2U'$, and determine higher-order corrections to the constants h_I^J , Δc_I and $\Delta\hat{\tau}_{vI}$. Proceeding indefinitely like this, we get (3.6) and relations (4.7).

Summarizing, the BR change of variables in the redundant linear approach has the form

$$\begin{aligned} \varphi_B &= \varphi_B(\varphi, \lambda, \mu, J, L), & \lambda_B &= \lambda_B(\lambda, \mu), & \tau_B &= \tilde{\tau}(\tau, \lambda, \mu), \\ L_{IB} &= (L_J - \tilde{c}_J J)(\tilde{Z}^{-1})_I^J, & J_B &= J, \end{aligned} \quad (7.14)$$

where

$$\begin{aligned} \varphi_B(\varphi, \lambda, \mu, J, L) &= \varphi + c_I \mathcal{O}^I + c_I \hat{U}^I(\varphi, cJ, L), & \hat{U}^I &= \mathcal{O}(\tilde{L}), \\ (\tilde{Z}^{-1})_I^J &= (Z^{-1})_I^J + \mathcal{O}(c), & \tilde{c}_I &= c_I + \mathcal{O}(c^2), & \tilde{\tau}_{vI} &= \hat{\tau}_{vJ}(Z^{-1})_I^J + \mathcal{O}(c). \end{aligned}$$

The Z - and W -functionals behave as scalars. The crucial property of the linear approach is that the second line of (7.14) is linear in both J and L .

Now we study the essential linear approach. Here we need to eliminate descendants and terms proportional to the field equations. Descendants are taken care of converting the constants c , Z , \tilde{c} and \tilde{Z} into derivative-operators. We do not do this explicitly, because it is straightforward. The linearity of L_{IB} in both J and L is preserved in this extended sense. This trick can also help us eliminate the terms $E(\varphi)$ proportional to the field equations, if we identify such terms as $E_{\text{free}}(\varphi)$ plus perturbative corrections. Indeed, $E_{\text{free}}(\varphi)$ are just descendants plus mass terms. If we use this trick, the redundant and essential linear approaches practically coincide.

Instead, if we want to eliminate the terms $E(\varphi)$ belonging to the source sector by means of further field redefinitions, the procedure we must apply is the same as the one we used to eliminate the last term of (6.4), expanding in powers of L or \tilde{L} . We briefly describe it here.

Let us begin with the BR replacement. The terms proportional to the field equations we want to reabsorb are $\mathcal{O}(L)$. They can be canceled by means of $\mathcal{O}(L)$ -corrections to the field replacement, which, however, generate also other $\mathcal{O}(L^2)$ -terms. These can be expanded in the basis \mathcal{O}^I , and their coefficients can be expanded in the basis \mathcal{N}^v , and canceled redefining the constants $\hat{\tau}(\tau, \lambda, \mu)$, up to $\mathcal{O}(L^2)$ -terms proportional to the field equations. Iterating in powers of L , we find that the BR replacement has the form

$$\begin{aligned} \varphi_B &= \varphi_B(\varphi, \lambda, \mu, L) = \varphi_B(\varphi, \lambda, \mu) + \mathcal{O}(L), & \lambda_B &= \lambda_B(\lambda, \mu), & \int J_B \varphi_B &\leftrightarrow \int J \varphi, \\ L_{IB} &= L_J (Z^{-1})_I^J, & \tau_B &= \tilde{\tau}(\tau, \lambda, \mu) Z^{-1}. \end{aligned}$$

The renormalized action is

$$S_L(\varphi, \lambda, \mu, L) = S(\varphi, \lambda, \mu, L) - \int (L_I + \hat{\tau}_{vI} \mathcal{N}^v(L, \lambda, \mu)) \mathcal{O}_R^I(\varphi, \lambda, \mu), \quad (7.15)$$

where $S(\varphi, \lambda, \mu, L) = S_B(\varphi_B, \lambda_B)$, and the parameter redefinitions $\tilde{\tau}_{vI}$ and $\hat{\tau}_{vI}$ need not coincide with those of (7.8) and (7.14).

Now we consider the BR change of variables. It is easy to see that L_{IB} cannot remain linear in L and J . Indeed, when we make the change of variables $\varphi_B = \varphi_B(\varphi, \lambda, \mu, L)$ in the functional integral the term $\int J_B \varphi_B$ generates objects that can be absorbed only if we introduce terms similar to the Δf_{JS} of formula (6.3), which do not depend on L and J in any simple way.

Thus, if we want to keep linearity in L and J we must use the redundant approach, or the trick mentioned above, where, besides converting c , Z , \tilde{c} and \tilde{Z} into derivative-operators, the terms proportional to the field equations are viewed as terms proportional to descendants plus mass terms and perturbative corrections.

8 Changes of field variables in the redundant approach

In this section and the next two we study the most general perturbative changes of field variables. Because of its simplicity, we prefer to concentrate on the redundant approach and drop the essential one.

We explain how a change of variables in the action S is related to a change of variables in the Z - and W -functionals, namely how it reflects from the integrand to the result of the functional integration. We do not study the change of variables inside the Γ -functional, because this investigation requires further work, which we leave to a separate paper.

Predictivity is unaffected by a change of variables. More explicitly, if the number of independent physical couplings that are necessary (together with field redefinitions) to reabsorb

divergences is finite in some variable frame, it is finite in every other variable frame. However, the change of field variables itself requires its own renormalization. We show that it is related to the renormalization of composite fields and work out this relation explicitly.

Since the composite fields \mathcal{O}_c^I form a basis for the local functions of φ and its derivatives, classically the most general perturbative change of variables can be written in the form

$$\varphi'(\varphi) = \varphi + \sum_I b_I \mathcal{O}_c^I(\varphi), \quad (8.1)$$

where $b_I = \mathcal{O}(\delta^{n_I-1})$. We treat it perturbatively in the constants b_I , so the functional integration measure is invariant. We define classical composite fields

$$\mathcal{O}_c^{I'}(\varphi') = \mathcal{O}_c^I(\varphi(\varphi')) \quad (8.2)$$

for the new variables, so the inverse of (8.1) can be simply written as

$$\varphi(\varphi') = \varphi' - \sum_I b_I \mathcal{O}_c^{I'}(\varphi'). \quad (8.3)$$

Essential variable frame

A parameter ζ is called *inessential* if the derivative of the action with respect to ζ is proportional to the field equations [3]. A convenient choice of variables is the one where the action $S(\varphi, \lambda, \mu)$ does not contain inessential parameters. Perturbatively, we can require that the action does not contain terms proportional to $\delta S_{\text{free}}/\delta\varphi$, such as $(-\square + m_s^2)\varphi$, $(\not{\partial} + m_f)\psi$, etc., and their derivatives, apart from the quadratic terms we are perturbing around. We call this reference frame the *essential variable frame*. It is useful in some applications.

The essential variable frame is preserved by renormalization. It is easy to prove this statement directly, but we can also use the derivation of the BR replacement (5.6) in the essential approach. As explained before, an equivalent way to describe the renormalization of an interacting theory with classical action S_c , at $L = 0$, is to replace S_c with the free-field action S_{free} , consider the extended action S_{Lc} of (3.5) and replace the sources L with constants, such that S_{Lc} gives back S_c . Then the renormalization of the theory is described by formula (5.6). All counterterms proportional to $\delta S_{\text{free}}/\delta\varphi$ are subtracted by the field redefinition, so the structure of the action in the essential variable frame is preserved.

Bare change of field variables

We first work at the bare level, where the change of variables is simpler, and later discuss the change of variables at the renormalized level. We start with the redundant nonlinear approach, where the bare action (3.2) is linear in the sources L_B . The bare change of variables coincides in form with the classical one (8.1), so we write

$$\varphi'_B(\varphi_B) = \varphi_B + b_{IB} \mathcal{O}_B^I(\varphi_B). \quad (8.4)$$

Making the transformation

$$L'_{IB} = L_{IB} - b_{IB}J_B, \quad J'_B = J_B, \quad (8.5)$$

and the change of variables (8.4) inside (3.3) we get

$$Z_B(J_B, L_B) = \int [d\varphi'_B] \exp \left(-S'_{LB}(\varphi'_B, L'_B) + \int \varphi'_B J'_B \right), \quad (8.6)$$

where

$$S'_{LB}(\varphi'_B, L'_B) = S'_B(\varphi'_B) - \sum_I \int L'_B{}^I \mathcal{O}'_B{}^I(\varphi'_B), \quad (8.7)$$

and

$$S'_B(\varphi'_B) = S_B(\varphi_B(\varphi'_B)).$$

Observe that (8.6) coincides with the transformed bare functional $Z'_B(J'_B, L'_B)$. Thus the functionals Z_B and W_B correctly behave as scalars:

$$Z'_B(J'_B, L'_B) = Z_B(J_B, L_B), \quad W'_B(J'_B, L'_B) = W_B(J_B, L_B). \quad (8.8)$$

As before, we can equivalently describe the map as the replacement

$$\varphi'_B = \varphi'_B(\varphi_B), \quad L'_{IB} = L_{IB}, \quad \int J'_B \varphi'_B \leftrightarrow \int J_B \varphi_B. \quad (8.9)$$

9 Renormalized changes of variables in the redundant approach

In this section we study the renormalized change of field variables in the redundant nonlinear approach and show that it provides a simple method to derive the renormalization of the theory in the new variables without having to calculate it anew.

While a replacement, by definition, simply replaces the term $\int J\varphi$ with $\int J'\varphi'$, a change of variables does transform it as any other term, therefore switches the functional integral from the conventional form to some unconventional one. We begin proving that all perturbative J -dependencies besides the term $\int J\varphi$ can be reabsorbed into a field redefinition, so it is always possible to rephrase the functional integral in the conventional form. The proof of the theorem also contains the procedure to achieve this result.

Switching from the non-conventional form to the conventional form

Theorem. *Consider a functional integral*

$$\mathcal{J} = \int [d\varphi] \exp \left(-S(\varphi) + \int J(\varphi - bU) \right),$$

where $U(\varphi, bJ)$ is a local function of φ and J , and b is a constant. Then there exists a perturbatively local change of variables

$$\varphi = \varphi(\varphi', b, bJ) = \varphi' + \mathcal{O}(b),$$

expressed as a series expansion in b , such that

$$\mathcal{J} = \int [d\varphi'] \exp \left(-S'(\varphi', b) + \int J\varphi' \right),$$

where $S'(\varphi', b) = S(\varphi(\varphi', b), 0)$.

Proof. Make the change of variables

$$\varphi_1 = \varphi - bU(\varphi, bJ) \tag{9.1}$$

in the functional integral. The functional measure is invariant, since we are treating (9.1) perturbatively in b . Call $\varphi = f_1(\varphi_1, b)$ the inverse of (9.1) at $J = 0$. We can write

$$S(\varphi) = S(f_1(\varphi_1, b)) + b^2 \int JU_1,$$

for a suitable local function $U_1(\varphi_1, bJ, b)$. Then we have

$$\mathcal{J} = \int [d\varphi_1] \exp \left(-S_1(\varphi_1, b) + \int J(\varphi_1 - b^2U_1) \right), \quad S_1(\varphi_1, b) = S(f_1(\varphi_1, b)).$$

At this point, we are in the same situation we started with, but U is replaced by bU_1 , which is one order of b higher. Repeating the step made above, we make the change of variables $\varphi_2 = \varphi_1 - b^2U_1$ and get

$$\mathcal{J} = \int [d\varphi_2] \exp \left(-S_2(\varphi_2, b) + \int J(\varphi_2 - b^3U_2) \right), \quad S_2(\varphi_2, b) = S_1(f_2(\varphi_2, b), b).$$

where $\varphi_1 = f_2(\varphi_2, b)$ is the inverse of $\varphi_2 = \varphi_1 - b^2U_1$ at $J = 0$ and $U_2(\varphi_2, bJ, b)$ is a local function. Proceeding indefinitely like this, we prove the theorem.

Renormalized change of variables

In the remainder of this section we describe the renormalized change of field variables working directly on renormalized quantities. The relation between bare and renormalized changes of variables is worked out in section 11.

Start from the generating functional (3.6), which we write in the form

$$Z(J, L) = \exp \left(\frac{1}{\hbar} W(J, L) \right) = \int [d\varphi] \exp \left(-\frac{1}{\hbar} S_L(\varphi, L) + \int J\varphi \right), \tag{9.2}$$

with the renormalized extended action (4.3), where $f_I(L) = L_I + \Delta_I(L)$, $\Delta_I = \mathcal{O}(\hbar) = \mathcal{O}(L)$ and $S(\varphi) = S_c(\varphi) + \mathcal{O}(\hbar)$ is the renormalized action, equal to the classical action S_c plus its

counterterms. We have introduced \hbar explicitly, because it is useful for our argument. For the time being we omit the dependencies on λ and μ , since they are not crucial for the arguments that follow.

Now, shift the sources L defining

$$L'_I = L_I - \hbar b_I J, \quad (9.3)$$

and perform a change of variables

$$\tilde{\varphi}(\varphi) = \varphi + b_I \mathcal{O}^I(\varphi). \quad (9.4)$$

in the functional integral (9.2). We get

$$Z(J, L) = \int [d\tilde{\varphi}] \exp \left(-\frac{1}{\hbar} \tilde{S}_L(\tilde{\varphi}, \hbar b J, L') + \int J \tilde{\varphi} \right),$$

where

$$\tilde{S}_L(\tilde{\varphi}, \hbar b J, L') = S(\varphi(\tilde{\varphi})) - \int (L'_I + \Delta_I(L' + \hbar b J)) \tilde{\mathcal{O}}^I(\tilde{\varphi}),$$

and $\tilde{\mathcal{O}}^I(\tilde{\varphi}) = \mathcal{O}^I(\varphi)$ is the basis in the tilded variables. This result is not written in the conventional form, yet, since \tilde{S}_L depends on J . However, we can use the theorem proved before to find the conventional form for the new variables.

Specifically, we write

$$\tilde{S}_L(\tilde{\varphi}, \hbar b J, L') = \tilde{S}_L(\tilde{\varphi}, 0, L') + \hbar^2 b \int J U(L', \hbar b J, \tilde{\varphi}), \quad (9.5)$$

for a suitable local function U , where b collectively denotes the parameters b_I . The second term of (9.5) is $\mathcal{O}(\hbar^2)$, because $\Delta^I(L) = \mathcal{O}(\hbar)$. The generating functional becomes

$$Z(J, L) = \int [d\tilde{\varphi}] \exp \left(-\frac{1}{\hbar} \tilde{S}_L(\tilde{\varphi}, 0, L') + \int J (\tilde{\varphi} - \hbar b U) \right).$$

The theorem proved before ensures that there exists a perturbatively local change of variables $\tilde{\varphi} = \tilde{\varphi}(\varphi', J, L') = \varphi' + \mathcal{O}(\hbar b)$ that converts the functional integral to the conventional form, such that

$$Z(J, L) = \int [d\varphi'] \exp \left(-\frac{1}{\hbar} \tilde{S}_L(\tilde{\varphi}(\varphi', 0, L'), 0, L') + \int J \varphi' \right).$$

Now it remains to expand $\tilde{\varphi}(\varphi', 0, L')$ in powers of L' . Call

$$\mathcal{O}^{I'}(\varphi') = \tilde{\mathcal{O}}^I(\tilde{\varphi}(\varphi', 0, 0)) = \mathcal{O}^I(\varphi(\tilde{\varphi}(\varphi', 0, 0))), \quad S'(\varphi') = S(\varphi(\tilde{\varphi}(\varphi', 0, 0))),$$

the new basis of composite fields and the new action, respectively. We can find local L' -dependent functions s^I_J and r^I such that

$$\mathcal{O}^I(\varphi(\tilde{\varphi}(\varphi', 0, L'))) = s^I_J(L') \mathcal{O}^{J'}(\varphi'), \quad S(\varphi(\tilde{\varphi}(\varphi', 0, L'))) = S'(\varphi') + \int r_I(L') \mathcal{O}^{I'}(\varphi').$$

Clearly, both $s_J^I - \delta_J^I$ and r_I are $\mathcal{O}(\hbar)$ and $\mathcal{O}(L')$. Finally, defining

$$S'_L(\varphi', L') = S'(\varphi') - \int f'_I(L') \mathcal{O}^{I'}(\varphi'), \quad (9.6)$$

where

$$f'_I(L') = f_J(L') s_J^I(L') - r_I(L'), \quad (9.7)$$

the generating functional reads

$$Z(J, L) = \int [d\varphi'] \exp\left(-\frac{1}{\hbar} S'_L(\varphi', L') + \int J\varphi'\right). \quad (9.8)$$

The right-hand side of this formula is precisely the generating functional $Z'(J, L')$, as it is quantized and renormalized in the new variables. We conclude that the change of field variables reads

$$\varphi' = \varphi'(\varphi, J, L'), \quad J' = J, \quad L'_I = L_I - \hbar b_I J, \quad (9.9)$$

where $\varphi' = \varphi'(\varphi, J, L')$ is the inverse of $\varphi = \varphi(\tilde{\varphi}(\varphi', J, L'))$, and the Z - and W -functionals behave as scalars:

$$Z'(J', L') = Z(J, L), \quad W'(J', L') = W(J, L). \quad (9.10)$$

Note that formula (9.7) encodes the relation between the renormalizations of composite fields before and after the change of field variables.

We also have

$$S'_L(\varphi', L') = \tilde{S}_L(\tilde{\varphi}(\varphi', 0, L'), 0, L') = S_L(\varphi(\tilde{\varphi}(\varphi', 0, L')), L'). \quad (9.11)$$

Ultimately, the change of field variables has three aspects: *i*) in the functional integral we make the change of integration variables $\varphi' = \varphi'(\varphi, J, L')$; *ii*) inside the extended action S_L we have the change of variables $\varphi' = \varphi'(\varphi, 0, L')$; *iii*) inside the action $S(\varphi)$ we just have $\varphi' = \varphi'(\varphi, 0, 0)$.

Summarizing, when we make the change of variables (9.4) we get unwanted J -dependent terms from $\int J\varphi$. We cancel them by means of the source redefinitions (9.3). However, (9.3) generate other unwanted J -dependent terms. Those are canceled upgrading the J -independent change of variables (9.4) to a J -dependent one, which is $\varphi' = \varphi'(\varphi, J, L')$.

All transformations *i*), *ii*) and *iii*) are equal to (9.4) plus appropriate counterterms. A change of variables undergoes its own renormalization, which is related to the renormalization of the composite fields it is made of. The derivation just given also teaches us how to work it out.

The most general local redefinitions of L can be considered, instead of those of (9.9). They amount to combinations of changes of variables and redefinitions of the basis \mathcal{O}^I . Relations (4.6) show that renormalization is a redefinition of this more general type, in the redundant nonlinear approach. The change of field variables is always encoded inside the J -dependence of the L -redefinitions. Redefinitions of J , instead, are never necessary, since the elementary field is also included in the basis \mathcal{O}^I .

As before, we can describe the effects of the change of variables with a replacement. We can actually give two equivalent forms of the replacement. Using (9.11) we can make

$$\varphi' = \varphi'(\varphi, 0, L'), \quad L' = L, \quad \int J' \varphi' \leftrightarrow \int J \varphi. \quad (9.12)$$

Alternatively, we can make

$$\varphi' = \varphi'(\varphi, 0, 0), \quad L' = L'(L), \quad \int J' \varphi' \leftrightarrow \int J \varphi, \quad (9.13)$$

where $L'(L)$ are the solutions of

$$f_I(L) = f'_I(L') = f_J(L') s_I^J(L') - r_I(L'),$$

which can be worked out perturbatively in \hbar .

Combining (4.4), (8.9) and (9.13) we can read the primed BR replacement

$$\varphi'_B = \varphi'_B(\varphi_B(\varphi(\varphi'))), \quad \lambda_B = \lambda_B(\lambda, \mu), \quad L'_{IB} = f'_I(L'), \quad \int J'_B \varphi'_B \leftrightarrow \int J \varphi, \quad (9.14)$$

where the function $\varphi(\varphi')$ is the inverse of $\varphi' = \varphi'(\varphi, 0, 0)$.

Note that the functions $\varphi(\varphi')$ and $L'(L)$ are in general divergent, yet the generating functionals remain convergent, because they are mapped into each other by the convergent relations (9.3) and $J' = J$. The divergences contained in $\varphi(\varphi')$ and $L'(L)$ are the extra counterterms necessary to turn the renormalization of the theory expressed in the old variables into the renormalization of the theory expressed in the new variables.

However, the replacement (9.13) is just a merely descriptive existence relation between the renormalizations in the old and the new variable frames. It is not equipped with a method to calculate the functions $\varphi' = \varphi'(\varphi, 0, 0)$ and $L' = L'(L)$. The only ways we have to derive those functions are either using the change of variables or calculating Feynman diagrams anew in the new variables.

In other words, if we do not want to recalculate Feynman diagrams from scratch, we just apply the classical variable change and then recover the conventional form of the functional integral following the steps described in this section. Thus the change of variables provides an independent way to derive the renormalization of the theory in the new variables from its renormalization in the old variables.

The explanation of this crucial difference between replacements and true changes of field variables is that only changes of variables take full advantage of composite fields. The role of composite fields in the replacement (9.13) is minor, to the extent that they can be freely switched on and off in (9.13) with practically no gain nor loss.

In the new variables the renormalization program works as in the old variables. The renormalization of parameters remains the same, but the field renormalizations can change considerably.

For example, if the fields renormalize multiplicatively before the change of variables, or do not renormalize at all, they may renormalize non-polynomially after the change of variables. Nevertheless, a theory that is predictive in some variables (which means that it can be renormalized with redefinitions of a finite number of physical parameters and local field redefinitions), remains predictive in any other variable frame. This ensures that the physics remains the same. In section 12 we give explicit examples.

The source redefinitions (9.9) are linear and encode the most general changes of field variables. However, we point out that the functional $W'(J', L') \equiv W(J, L)$ obtained applying any perturbatively local source redefinitions

$$J' = J, \quad L' = L'(J, L) = L' + \mathcal{O}(b), \quad (9.15)$$

with $L'(0, 0) = 0$, is the W -functional that we would calculate in some transformed field-variable frame. The transformed fields can be worked out applying the procedure explained in this section to recover the conventional form of the functional integral, which is spoiled by nontrivial J -dependences contained in $L'(J, L)$.

10 Changes of field variables in the linear approach

Now we examine the change of variables in the linear approach of section 7. Equipped with the experience already gained, this task is now relatively easy. We can go back to work with $\hbar = 1$.

Recall that the classical action is (7.2) and the renormalized one is (7.8). Make the substitutions (9.3) in (7.8), where $b_I = \mathcal{O}(\delta^{n_I-1})$. Such substitutions certainly leave the generating functionals Z and W convergent, but do not preserve the conventional form of the functional integral. We just have to convert the Z -integrand back to the conventional form. Then we can read the change of variables associated with (9.3) and the renormalization constants $Z' = Z + \mathcal{O}(b)$ and $\hat{\tau}' = \hat{\tau} + \mathcal{O}(b)$ that remove the divergences in the new variables.

To do this, we make the change of variables

$$\tilde{\varphi}(\varphi) = \varphi + (b_I + \Delta b_I) (Z^{-1})_J^I \mathcal{O}^J(\varphi), \quad (10.1)$$

in the functional integral, instead of (9.4), where $\Delta b_I = \mathcal{O}(b^2)$ are constants to be determined. It may be convenient to express (10.1) as

$$\varphi(\tilde{\varphi}) = \tilde{\varphi} - (b_I + \Delta b_I) (Z^{-1})_J^I \tilde{\mathcal{O}}^J(\tilde{\varphi}),$$

where $\tilde{\mathcal{O}}^I(\tilde{\varphi}) = \mathcal{O}^I(\varphi)$. The exponent of the Z -integrand can be written as

$$-S(\varphi(\tilde{\varphi})) + \int (L'_I - \Delta b_I J + \hat{\tau}_{vI} \mathcal{N}^v(L')) (Z^{-1})_J^I \tilde{\mathcal{O}}^J(\tilde{\varphi}) + \int J(\tilde{\varphi} + b_I U^I), \quad (10.2)$$

where $U^I(\tilde{\varphi}, bJ, L', b + \Delta b)$ are $\mathcal{O}(\tilde{L}')$ -local functions of $\tilde{L}' = \{L', J\}$. Now we show that Δb can be determined perturbatively in b so that (10.2) is converted to the conventional form.

Make the further change of variables $\tilde{\varphi} \rightarrow \tilde{\varphi} - b_I U^I$. The new action can be expanded as

$$S(\varphi(\tilde{\varphi})) \rightarrow S(\varphi(\tilde{\varphi})) + \int (L'_J \bar{\Delta} z_I^J - J \bar{\Delta} b_I + \bar{\Delta} \hat{\tau}_{vI} \mathcal{N}^v(L')) (Z^{-1})_K^J \tilde{\mathcal{O}}^K(\tilde{\varphi}) - b^2 \int J U^I(\tilde{\varphi}, bJ, L', b + \Delta b, b), \quad (10.3)$$

where $\bar{\Delta} z \sim b \hat{\Delta} z$, $\bar{\Delta} b \sim b^2 \hat{\Delta} b$ and $\bar{\Delta} \hat{\tau} \sim b \hat{\Delta} \hat{\tau}$, where $\hat{\Delta} z$, $\hat{\Delta} b$ and $\hat{\Delta} \hat{\tau}$ are functions of b and $b + \Delta b$, while U^I are $\mathcal{O}(\tilde{L}')$ -local functions. Now, determine Δb_I to $\mathcal{O}(b^2)$ so that $\Delta b_I = \bar{\Delta} b_I + \mathcal{O}(b^3)$ and define $Z' = Z(1 + \bar{\Delta} z) + \mathcal{O}(b^2)$. An expansion similar to (10.3) can be written for the transformed $\tilde{\mathcal{O}}^I(\tilde{\varphi})$ s, and generates additional terms like the last two of (10.3). Terms like the last-but-one of (10.3) determine the $\mathcal{O}(b)$ -corrections to $\hat{\tau}$ that define $\hat{\tau}'$. Terms like the last one of (10.3) have the same structure as the term $\int J b U$ of (10.2), but are one order higher in b . Repeating the procedure indefinitely, the complete change of variables in the functional integral gets of the form

$$\varphi' = \varphi'(\varphi, J, L) = \varphi + (b_I + \Delta b_I) (Z^{-1})_J^I \mathcal{O}^J(\varphi) + b_I \hat{U}^I(\varphi, bJ, L, b), \quad \hat{U}^I = \mathcal{O}(\tilde{L}),$$

and the exponent of the Z -integrand turns into its correct primed version, which is

$$-S'(\varphi') + \int (L'_J + \hat{\tau}'_{vJ} \mathcal{N}^v(L')) (Z'^{-1})_I^J \mathcal{O}^{I'}(\varphi') + \int J' \varphi' = -S'_L(\varphi', L') + \int J' \varphi', \quad (10.4)$$

where $S'(\varphi') = S(\varphi(\varphi', 0, 0))$ and $\mathcal{O}^{I'}(\varphi') = \mathcal{O}^I(\varphi(\varphi', 0, 0))$, while $J' = J$, $Z' = Z + \mathcal{O}(b)$ and $\hat{\tau}' = \hat{\tau} + \mathcal{O}(b)$. Thus, the renormalized change of variables in the linear redundant approach still has the form (9.9). The renormalized composite fields transform as

$$\mathcal{O}'_R{}^I(\varphi') = (Z'^{-1})_I^J \mathcal{O}^I(\varphi') = (Z'^{-1} Z)_I^J \mathcal{O}_R{}^I(\varphi).$$

We can also view the change of variables as the replacement

$$\varphi' = \varphi'(\varphi, 0, 0), \quad L'_I = L_J (Z^{-1} Z')_I^J, \quad \hat{\tau}'_{vI} = (C^{-1})_v^w \hat{\tau}_{wJ} (Z^{-1} Z')_I^J, \quad \int J' \varphi' \leftrightarrow \int J \varphi,$$

where C is the matrix such that $\mathcal{N}^v(L') = C^v_w \mathcal{N}^w(L)$. Again, the replacement is a mere description of the result, because it does not provide an independent way to calculate the quantities appearing in the transformation. Instead, the operations described with the change of integration variables do allow us to derive the renormalization of the transformed theory from the renormalization of the original one, without having to calculate diagrams from scratch in the new variables.

We learn that a finite change of variables can always be expressed with a linear source redefinition of the form (9.3), both in the linear and nonlinear approaches. On the other hand, the BR map, which includes a divergent change of field variables, can be expressed as a linear source redefinition only in the linear approach, whence the name we have given to this approach.

Let us analyze the result we have obtained in more detail. If the theory is renormalized using the minimal subtraction scheme in the variables φ , in general it will not be renormalized using the minimal subtraction scheme after the change of variables. The reason is that if, for example, $Z = 1 + \text{poles}$ in $\varepsilon = 4 - D$, where D is the continued dimension in the dimensional regularization, Z' needs not be equal to $1 + \text{poles}$. Nevertheless, we can extract the finite part writing $Z' = \bar{Z}'\bar{z}$, where $\bar{z} = 1 + \mathcal{O}(b)$ is finite and $\bar{Z}' = 1 + \text{poles}$. Similarly, although $\hat{\tau} = \tau + \text{poles}$, $\hat{\tau}'$ is equal to some finite function $\tau'(\tau, b, \lambda, \mu) = \tau + \mathcal{O}(b)$ plus poles. We can view τ' as a redefinition of τ . There also exist finite redefinitions $\tilde{b}_I(b, \lambda, \tau, \mu) = b_I + \mathcal{O}(b^2)$ such that

$$\varphi'(\varphi, 0, 0) = \varphi + \tilde{b}_I(b, \lambda, \tau, \mu)\mathcal{O}_c^I(\varphi) + \text{poles}. \quad (10.5)$$

We can preserve the minimal subtraction scheme if we include a finite change of basis $L'_I \rightarrow L'_J \bar{z}_I^J$. In other words, instead of (9.3) we define the source redefinitions as

$$L'_I = (L_J - b_J J)(\bar{z}^{-1})_I^J, \quad J' = J.$$

Then the S'_L -terms linear in L' are $\int L'_J (\bar{Z}'^{-1})_I^J \mathcal{O}'^I(\varphi')$, and the transformed renormalized composite fields are

$$\mathcal{O}'^I(\varphi') = (\bar{Z}'^{-1})_I^J \mathcal{O}'^I(\varphi') = (\bar{Z}'^{-1} Z)_I^J \mathcal{O}_R^I(\varphi). \quad (10.6)$$

More generally, we can always include a further finite change of basis $L'_I \rightarrow L'_J \tilde{z}_I^J$, $\mathcal{O}'^I \rightarrow (\tilde{z}^{-1})_I^J \mathcal{O}'^J$ and describe the change of variables as the more general map

$$\begin{aligned} \varphi' &= \varphi'(\varphi, J, L), & b' &= b'(b, \lambda, \tau, \mu), & \tau' &= \tau'(\tau, b, \lambda, \mu), \\ L'_I &= (L_J - b_J J)(z^{-1})_I^J, & J' &= J, \end{aligned} \quad (10.7)$$

where $z = \tilde{z}\bar{z}$ and the primed parameters $b'_I(b, \lambda, \tau, \mu) = b_I + \mathcal{O}(b^2)$ are finite functions obtained inverting the relation (10.5) so that it reads

$$\varphi(\varphi', 0, 0) = \varphi' - b'_I \mathcal{O}_c^I(\varphi') + \text{poles}. \quad (10.8)$$

The finite functions b' , τ' and z can be chosen to combine the change of field variables with any change of subtraction scheme in the composite-field sector. In particular they can be chosen to preserve the minimal subtraction scheme.

We have already stressed that the virtue of the linear approach is that it linearizes the map relating bare and renormalized sources. Thanks to this fact, the procedure to make a bare change of field variables is practically identical to the one just described at the renormalized level. We just present it quickly and report the result. We start from (7.1) and make the substitutions

$$L'_{IB} = (L_{JB} - b_{JB} J_B)(z_B^{-1})_I^J, \quad \tau_B = \tau'_B + \Delta\tau_B,$$

and a change of variables

$$\tilde{\varphi}_B(\varphi_B) = \varphi_B + (b_{IB} + \Delta b_{IB})\mathcal{O}_B^I(\varphi_B),$$

where $(z_B)_I^J = \delta_I^J + \mathcal{O}(b)$, $\Delta b_{IB} = \mathcal{O}(b^2)$ and $\Delta\tau_B = \mathcal{O}(b)$. We obtain an expression similar to (10.2). Then we make the further change of variables $\tilde{\varphi}_B \rightarrow \tilde{\varphi}_B - b_{IB}U_B^I$, expand the new action and the composite fields as in (10.3), and determine the first contributions to $(z_B)_I^J - \delta_I^J$, Δb_{IB} and $\Delta\tau_B$. Repeating this procedure indefinitely we arrive at the primed version of (7.1). Finally, the renormalized change of variables in the linear redundant approach has the form

$$\begin{aligned} \varphi'_B &= \varphi'_B(\varphi_B, J_B, L_B) = \tilde{\varphi}_B(\varphi_B) + b_B \hat{U}_B(\varphi_B, b_B J_B, L_B), & J'_B &= J_B, \\ L'_{IB} &= (L_{JB} - b_{JB} J_B)(z_B^{-1})_I^J, & \tau'_B &= \tau_B + \mathcal{O}(b), \end{aligned} \quad (10.9)$$

and $\hat{U}_B = \mathcal{O}_B(\tilde{L}_B)$. The basis of composite fields inherits some change of basis $\mathcal{O}_B^I(\varphi'_B) = (\tilde{z}_B^{-1})_I^J \mathcal{O}_B^J(\varphi_B)$, with $\tilde{z}_B = z_B + \mathcal{O}(b)$.

11 Relation between bare and renormalized changes of variables

Having expressed the BR map as a change of variables, now it is simple to work out the relation between bare and renormalized changes of variables. We want to close the scheme

$$\begin{array}{ccc} \text{B} & \leftrightarrow & \text{R} \\ \downarrow & & \downarrow \\ \text{B}' & \leftrightarrow & \text{R}' \end{array} \quad (11.1)$$

which gives us another way to express the map $\text{R} \leftrightarrow \text{R}'$ and clarifies some points.

We start from the nonlinear approach. Composing the renormalized change of variables (4.6) with (8.5) and the primed analogue of (4.6), we obtain $J' = J$ and

$$f'_I(L') = f_I(L) + (c'_I - c_I - b_{IB})J. \quad (11.2)$$

It is not evident how these relations can be compatible with (9.9). Using (9.7) we can solve (11.2) to find L' as functions of L and J . However, we certainly obtain a divergent transformation rule for the renormalized sources, not a relation of the form (9.9). Yet, the generating functionals are convergent and these maps must be equivalent to (9.9).

The matter can be explained as follows. The dependencies on J and the L s are related to each other, because composite fields are ultimately made of elementary fields. For example, if the bare action is written in the form (3.2) we can write

$$\frac{\delta W_B}{\delta L_{IB}} = \langle \mathcal{O}_B^I(\varphi_B) \rangle = \frac{1}{Z_B} \mathcal{O}_B^I\left(\frac{\delta}{\delta J_B}\right) Z_B. \quad (11.3)$$

Similarly, if L_{2B} and L_{4B} are the sources coupled with $\varphi_B^2/2$ and $\varphi_B^4/4!$, respectively, then

$$\frac{\delta W_B}{\delta L_{4B}} = \frac{1}{4!} \langle \varphi_B^4 \rangle = \frac{1}{6} \left\langle \left(\frac{\varphi_B^2}{2} \right)^2 \right\rangle = \frac{1}{6} \frac{\delta^2 W_B}{\delta L_{2B}^2} + \frac{1}{6} \left(\frac{\delta W_B}{\delta L_{2B}} \right)^2. \quad (11.4)$$

At the renormalized level, these identities may get corrections that compensate for the divergences originating from J - and L -derivatives at coinciding points. We call identities like (11.3), (11.4) and their renormalized counterparts *secret identities*. Due to the secret identities, there exist convergent as well as divergent redefinitions of J and L that leave the generating functionals invariant. In the next section we give an explicit example.

If some divergent J - L redefinitions leave the renormalized functionals convergent, we can find equivalent convergent redefinitions dropping the divergent parts. In the minimal subtraction scheme, where bare and renormalized quantities differ by pure poles, it is sufficient to drop the pole corrections. Secret identities ensure that the poles of such J - L redefinitions have no effect on the functionals. It is easy to see that the function U of (9.5) is divergent, therefore $s_I^J = \delta_I^J + \text{poles}$, $r_I = \text{poles}$. Moreover, $c_I = \text{poles}$ and $f_I(L) = L_I + \text{poles}$. Thus, dropping all divergent corrections from the relations (11.2) we get precisely (9.9). In another subtraction scheme the finite equivalent versions of (11.2) are, in general, a combination of (9.9) with a scheme change.

Observe that the secret identities like (11.3) and (11.4) are nonlinear in the derivatives with respect to the sources. Thus, the problem just described is absent in the linear approach. There the maps relating the sources L and J are always linear, so their composition is still linear. Composing (7.14), (10.9) and the primed version of (7.14), we get $J' = J$,

$$L'_I = (L \tilde{Z}^{-1} z_B^{-1} \tilde{Z}')_I + \left[\tilde{c}'_I - (\tilde{c} \tilde{Z}^{-1} + b_B)_J (z_B^{-1} \tilde{Z}')^J_I \right] J, \quad (11.5)$$

and some maps $\tau' = \tilde{\tau}'(\tau, \lambda, \mu)$, $\varphi' = F(\varphi, \lambda, \mu, J, L)$. Comparing (11.5) and (10.7), we find

$$z_B = \tilde{Z}' z \tilde{Z}^{-1}, \quad b_B = (b - \tilde{c} + \tilde{c}' z) \tilde{Z}^{-1}.$$

12 Examples

In this section we collect a number of examples that illustrate various properties derived in the paper.

Example 1

Consider the classical theory

$$S_c(\varphi) = \frac{1}{2} \int d^D x \left(1 + \frac{\lambda^2}{2} \varphi^2 \right) (\partial_\mu \varphi)^2.$$

This is a derivative φ^4 -theory. It is equivalent to the massless free scalar field up to the change of variables

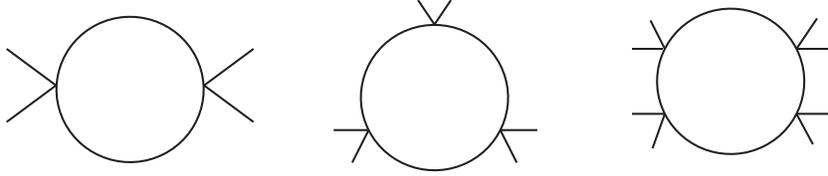
$$\varphi'_c(\varphi) = \frac{\varphi}{2} \sqrt{1 + \frac{\lambda^2}{2} \varphi^2} + \frac{1}{\lambda\sqrt{2}} \operatorname{arcsinh} \left(\frac{\lambda\varphi}{\sqrt{2}} \right). \quad (12.1)$$

Indeed,

$$S_c(\varphi) = \frac{1}{2} \int d^D x (\partial_\mu \varphi'_c(\varphi))^2 \equiv S'_c(\varphi'_c).$$

We want to study the renormalization of $S_c(\varphi)$ at one loop and verify that in the new variable frame it is still a finite theory, which means that it can be renormalized just by field redefinitions, with no redefinitions of parameters.

Calculating one-loop diagrams with four, six and eight external legs



we find the renormalized action

$$S(\varphi) = S_c(\varphi) + \frac{\hbar\lambda^4}{512\pi^2\varepsilon} \int \varphi^2 \{ 4(\square\varphi)^2 + \lambda^2\varphi(\partial_\mu\varphi)^2 [4(\square\varphi) + \lambda^2\varphi(\partial_\nu\varphi)^2] \\ + 2a\lambda^2\varphi^2(\square\varphi)^2 + b\lambda^4\varphi^3(\square\varphi)(\partial_\mu\varphi)^2 + c\lambda^4\varphi^4(\square\varphi)^2 \} + \hbar\mathcal{O}(\lambda^{10}) + \mathcal{O}(\hbar^2), \quad (12.2)$$

where a , b and c are constants that we do not need to work out. The theory is indeed finite up to the order $\mathcal{O}(\lambda^8)$ included, because the divergent terms of the first line cancel out using the field equations, while those of the second line become $\mathcal{O}(\lambda^{10})$. Observe that the cancelation is nontrivial, and occurs only because the terms appearing on the first line of (12.2) have coefficients that are related in a way to make that happen. We find

$$S(\varphi) = S_c(\varphi_\lambda(\varphi)) + \hbar\mathcal{O}(\lambda^{10}) + \mathcal{O}(\hbar^2),$$

where the field redefinition reads

$$\varphi_\lambda(\varphi) = \varphi - \frac{\hbar\lambda^4\varphi^2}{512\pi^2\varepsilon} [4(\square\varphi) + 2\lambda^2\varphi(\partial_\mu\varphi)^2 + 2(a-1)\lambda^2\varphi^2(\square\varphi) \\ + (c-a+1)\lambda^4\varphi^4(\square\varphi) + (b-a)\lambda^4\varphi^3(\partial_\mu\varphi)^2] + \hbar\mathcal{O}(\lambda^{10}) + \mathcal{O}(\hbar^2). \quad (12.3)$$

Combining the classical change of variables (12.1) and the renormalized one $\varphi_\lambda(\varphi)$, we get, up to $\hbar\mathcal{O}(\lambda^{10})$ and $\mathcal{O}(\hbar^2)$,

$$S(\varphi) = S_c(\varphi_\lambda(\varphi)) = \frac{1}{2} \int d^D x (\partial_\mu \varphi'_c(\varphi_\lambda(\varphi)))^2 = S'_c(\varphi'_c(\varphi_\lambda(\varphi))).$$

The theory remains equivalent to a free massless field after renormalization. Despite the unnecessary complications introduced by the change of variables, the physics remains the same. Observe that the renormalized field redefinition $\varphi'_c(\varphi_\lambda(\varphi))$ is not derivative-independent anymore.

Before the change of variables, the relation between bare fields φ_B and renormalized fields φ is $\varphi_B = \varphi_\lambda(\varphi)$. After the change of variables the relation between φ'_B and φ' is just $\varphi'_B = \varphi'$, because the theory is manifestly free. Thus, the bare and renormalized changes of variables are

$$\varphi'_B = \varphi'_c(\varphi_B), \quad \varphi' = \varphi'_c(\varphi_\lambda(\varphi)),$$

respectively.

Example 2

Now we want to check the first line of (12.3) using the method of section 9. Instead of calculating diagrams in the variables φ , we apply the change of variables to the renormalized theory written in the variables φ' . There the theory is free, so we just need to pay attention to the composite-field sector.

Expanding and inverting the classical change of variables (12.1), write

$$\varphi_c = \varphi' - \frac{\lambda^2}{2} \frac{\varphi'^3}{3!} + \frac{13\lambda^4}{4} \frac{\varphi'^5}{5!} + \mathcal{O}(\lambda^6).$$

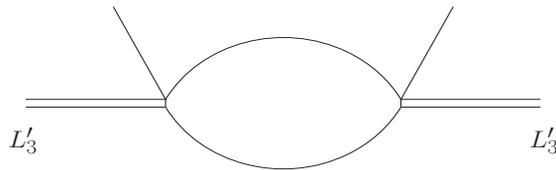
We have moved the subscript c from φ' to φ since now we are making the transformation in the opposite direction. The change of variables is thus expressed by the source redefinitions

$$L'_3 = L_3 - \frac{\lambda^2}{2} J, \quad L'_5 = L_5 + \frac{13\lambda^4}{4} J, \tag{12.4}$$

and so on, where L'_i is the source coupled to the composite field $\varphi'^i/i!$. Working at one loop in the primed variable frame, the renormalized extended action S'_L is only made of the counterterms

$$\frac{\hbar}{\varepsilon} \sum_{ij} r_{ij} \int \varphi'^{i+j-4} L'_i L'_j = \frac{\hbar}{\varepsilon} \int \varphi'^2 (r_{33} L'^2_3 + r_{35} \varphi'^2 L'_3 L'_5 + r_{55} \varphi'^4 L'^2_5) + \dots \tag{12.5}$$

where are r_{ij} numerical constants. We will see that to check the first line of (12.3) it is sufficient to calculate r_{33} , which is given by the diagram



and the one obtained exchanging the L'_3 -legs. We find $r_{33} = 1/(32\pi^2)$.

Because of (12.5), applying (12.4) we get an unprimed functional integral that is written in some unconventional form. We can set $L_i = 0$ now, since we do not need these sources anymore. The J -dependence in the exponent of the Z -integrand reads

$$\int J \left[\varphi_c - \frac{\hbar\lambda^4}{\varepsilon} \varphi_c^2 J \left(\frac{r_{33}}{4} + \frac{r_{33} - 39r_{35}}{24} \lambda^2 \varphi_c^2 \right) + \mathcal{O}(\lambda^8) \right] \equiv \int J (\varphi_c + U(\varphi_c, J)). \quad (12.6)$$

Dropping the subscript c and making the change of integration variables $\varphi \rightarrow \varphi - U$, the term (12.6) turns into $\int J\varphi + \mathcal{O}(\lambda^8)$, but we get also contributions

$$- \int \frac{\delta S_c}{\delta \varphi} U + \mathcal{O}(\lambda^8) = \frac{\hbar\lambda^4}{\varepsilon} \int \frac{\delta S_c}{\delta \varphi} \varphi^2 J \left(\frac{r_{33}}{4} + \frac{r_{33} - 39r_{35}}{24} \lambda^2 \varphi^2 \right) + \mathcal{O}(\lambda^8)$$

from the action. The integral is still written in an unconventional form, and the J -dependence in the exponent of the Z -integrand becomes

$$\int J \left[\varphi - \frac{\hbar\lambda^4}{\varepsilon} \frac{\delta S_c}{\delta \varphi} \varphi^2 \left(\frac{r_{33}}{4} + \frac{r_{33} - 39r_{35}}{24} \lambda^2 \varphi^2 \right) + \mathcal{O}(\lambda^8) \right].$$

The further change of variables

$$\varphi \rightarrow \varphi + \frac{\hbar\lambda^4}{\varepsilon} \frac{\delta S_c}{\delta \varphi} \varphi^2 \left(\frac{1}{128\pi^2} + \frac{r_{33} - 39r_{35}}{24} \lambda^2 \varphi^2 \right) + \mathcal{O}(\lambda^8)$$

finally takes us to the conventional form, up to the desired order. The field renormalization in the unprimed variables is obtained composing the changes of integration variables made so far and setting $J = 0$. We conclude that

$$\varphi_\lambda(\varphi) = \varphi + \frac{\hbar\lambda^4}{128\pi^2\varepsilon} \frac{\delta S_c}{\delta \varphi} \varphi^2 - \frac{\hbar\lambda^6}{\varepsilon} \frac{r_{33} - 39r_{35}}{24} (\square\varphi) \varphi^4 + \mathcal{O}(\lambda^8),$$

in agreement with (12.3).

Example 3

It is instructive to consider linear changes of field variables in a theory where the fields are renormalized multiplicatively. We can restrict the set of composite fields to the elementary field itself, coupled with the source L_1 , and the identity. The BR change of variables reads

$$\varphi_B = Z_\varphi^{1/2} \varphi, \quad \lambda_B = \lambda_B(\lambda, \mu), \quad L_{0B} = L_0, \quad L_{1B} = Z_\varphi^{-1/2} (L_1 + J) - J, \quad J_B = J. \quad (12.7)$$

Now, consider the bare and renormalized changes of variables

$$\varphi'_B = b_{0B} + (1 + b_{1B})\varphi_B, \quad \varphi' = b_0 + (1 + b_1)\varphi.$$

The bare change of variables is implemented by

$$L'_{0B} = L_{0B} - b_{0B}J_B, \quad L'_{1B} = L_{1B} - b_{1B}J_B,$$

while the renormalized one is given by the same formula with Bs suppressed. Closing the scheme (11.1) we find $J'_B = J'$ and

$$L'_{0B} = L'_0 + (b_0 - b_{0B})J', \quad L'_{1B} = Z_\varphi^{-1/2}L'_1 + [Z_\varphi^{-1/2}(1 + b_1) - 1 - b_{1B}]J'.$$

These relations, together with

$$\varphi'_B = b_{0B} + Z_\varphi^{1/2}(1 + b_{1B})\frac{\varphi' - b_0}{1 + b_1},$$

give the renormalization in the new variables. We are free to choose different relations between the bare and renormalized b s. Doing this we obtain equivalent ways to describe the renormalization. For example, if we choose

$$b_0 = b_{0B}, \quad b_{1B} = Z_\varphi^{-1/2}(1 + b_1) - 1,$$

we find that the field does not renormalize in the new variables: $\varphi'_B = \varphi'$. This is just because we have rescaled φ by a new parameter $1 + b_1$ and transferred the renormalization on that parameter.

Example 4

The simplest nonlinear change of variables involves a quadratic term φ^2 . Thus, we study a free massless scalar field in the presence of the composite field $\varphi^2/2$.

The renormalized generating functional is

$$Z(J, L) = e^{W(J, L)} = \int [d\varphi] \exp \left(-\frac{1}{2} \int \left\{ (\partial_\mu \varphi)^2 - L_2 \varphi^2 - \frac{\mu^{-\varepsilon}}{a} (1 + a\delta_a) L_2^2 \right\} + \int J\varphi \right), \quad (12.8)$$

where $\delta_a = -(16\pi^2\varepsilon)^{-1}$ in dimensional regularization. The functional integral is easy to work out, since it is Gaussian. The source L_2 plays the role of (minus) a spacetime dependent squared mass, so we obtain

$$W(J, L) = \frac{1}{2} \int \left\{ J \frac{1}{-\square - L_2} J + \mu^{-\varepsilon} \left(\frac{1}{a} + \delta_a \right) L_2^2 \right\} - \frac{1}{2} \text{tr} \ln(-\square - L_2). \quad (12.9)$$

Let us find the secret identity satisfied by $\delta W/\delta L_2$. From (12.8) we get

$$\frac{\delta W}{\delta L_2} = \frac{1}{2} \frac{\delta^2 W}{\delta J^2} + \frac{1}{2} \left(\frac{\delta W}{\delta J} \right)^2 + \mu^{-\varepsilon} \left(\frac{1}{a} + \delta_a \right) L_2. \quad (12.10)$$

Working out the derivatives of W , given by (12.9), it is easy to check this identity explicitly. The last term on the right-hand side of (12.10) compensates for the divergence due to the J -derivatives at coinciding points.

Example 5

Now we study the change of variables $L_2 = L'_2 + bJ$ in the previous example, to the lowest order in b . We also derive the secret identity that ensures the closure of (11.1).

We introduce sources L_0 and L_1 for the identity and the elementary field. Then the renormalized generating functional is (12.8) times $\exp \int L_0$, with J replaced by $J + L_1$. We work in the nonlinear approach. There the bare quantities are equal to the renormalized ones apart from

$$L_{0B} = L_0 + \frac{\mu^{-\varepsilon} \delta_a}{2} L_2^2.$$

Since we do not need the parameter a to reabsorb divergences, we work at $a = \infty$. The functional integral gives a $W(J, L)$ equal to (12.9) plus $\int L_0$, with $J \rightarrow J + L_1$ and $a = \infty$. The bare generating functional is formally identical with $\delta_a \rightarrow 0$.

The bare redefinitions $L_{2B} = L'_{2B} + b_B J_B$, $L_{1B} = L'_{1B}$, $L_{0B} = L'_{0B}$ are equivalent to the change of variables $\varphi'_B = \varphi_B + b_B \varphi_B^2/2$. At the renormalized level, the $L' = L'(L, J)$ -redefinition reads $L_2 = L'_2 + bJ$, $L_1 = L'_1$, $L_0 = L'_0$. It can be studied using the procedure of section 9. Doing so, we obtain new relations between primed bare and renormalized quantities, namely

$$L'_{0B} = L'_0 + \frac{\mu^{-\varepsilon} \delta_a}{2} L'_2(L'_2 - 2bL'_1), \quad L'_{1B} = L'_1 - b\mu^{-\varepsilon} \delta_a (\square L'_2 + L'^2_2), \quad L'_{2B} = L'_2. \quad (12.11)$$

Note that the field is non-renormalized also after the change of variables, to the lowest order in b , which is why the BR relations (12.11) are J -independent. Closing the scheme (11.1) we find the alternative $L' = L'(L, J)$ -redefinition

$$L'_0 = L_0 + \frac{\mu^{-\varepsilon} \delta_a}{\varepsilon} L_2(b_B J + bL_1), \quad L'_1 = L_1 + \frac{b\mu^{-\varepsilon} \delta_a}{\varepsilon} (\square L_2 + L_2^2), \quad L'_2 = L_2 - b_B J.$$

up to higher-orders in b . This redefinition differs from the one we made, which was $L_2 = L'_2 + bJ$, $L_1 = L'_1$, $L_0 = L'_0$. We can make the two coincide choosing $b_B = b$, provided we can drop the divergent corrections. Such corrections have no effect on W provided the secret identity

$$\int (\square L_2 + L_2^2) \frac{\delta W}{\delta L_1} + \int L_2 (J + L_1) \frac{\delta W}{\delta L_0} = 0$$

holds. It is easy to check that it is indeed so, since

$$\frac{\delta W}{\delta L_1} = \int \frac{1}{-\square - L_2} (J + L_1), \quad \frac{\delta W}{\delta L_0} = 1.$$

13 Conclusions

In this paper we have developed a field-covariant approach to quantum field theory, concentrating on the Z - and W -functionals. Because of the intimate relation with composite fields $\mathcal{O}^I(\varphi)$, ultimately a perturbative change of field variables can be expressed as a J -dependent redefinition

of the sources L_I coupled to the $\mathcal{O}^I(\varphi)$ s. We have defined several approaches, useful for different purposes, in particular a linear approach where all variable changes can be described as linear redefinitions $L_I \rightarrow (L_J - b_{JI})(z^{-1})_I^J$, including the map relating bare and renormalized quantities. The functionals Z and W behave as scalars. We have also seen how to convert a functional integral written in an unconventional form to the conventional form. Among the other things, this operation allows us to relate the renormalization of variable-changes to the renormalization of composite fields, and gives a simple method to derive the renormalization of the theory in the new variables from the renormalization of the theory in the old variables, without having to calculate diagrams anew.

The formalism developed here allows us to abandon the description of renormalization as a set of replacements, and view it as made of true changes of field variables, combined with parameter-redefinitions. Instead of jumping from a variable frame to another one, we can write down identities relating the generating functionals before and after a change of field variables. We regard these results as a first step to upgrade the formalism of quantum field theory to a more evolved one. Other issues, such as the effects of variable-changes on the Γ -functional, are treated in separate works.

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