

Renormalization Of High-Energy Lorentz Violating Four Fermion Models

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Abstract

We study the one-loop renormalization of high-energy Lorentz violating four fermion models. We derive general formulas and then consider a number of specific models. We study the conditions for asymptotic freedom and give a practical method to determine the asymptotic-freedom domain. We also point out that in some models the RG flow contains “rational” Zimmermann trajectories that might hide new symmetries.

1 Introduction

Although Lorentz symmetry is one of the most precise symmetries in nature [1], the possibility that it might be violated at high energies or very large distances is still open and has been extensively investigated. A Lorentz symmetry violation at high energies allows us, among the other things, to renormalize vertices that are otherwise non-renormalizable. This result is achieved using a modified power counting criterion, which weights space and time differently [2]. Modified dispersion relations improve the large-momentum behavior of propagators in such a way that, in the common perturbative framework, the theory remains unitary, local, polynomial and causal.

Using this knowledge, it is possible to formulate a Standard Model extension [3, 4] that is CPT invariant, but violates Lorentz symmetry at high energies, and contains two scalar-two fermion vertices, as well as four fermion vertices, at the fundamental level. The inclusion of CPT violating terms is also possible. Four fermion vertices are important for a variety of reasons. On the one hand, they can be used to explain proton decay. On the other hand, they can trigger a Nambu–Jona-Lasinio mechanism and give masses to fermions and gauge fields even if the elementary Higgs boson is suppressed [4]. In its simplest version, the scalarless Lorentz violating Standard Model schematically reads

$$\mathcal{L}_{\text{noH}} = \mathcal{L}_Q + \mathcal{L}_{\text{kinf}} - \sum_{I=1}^5 \frac{1}{\Lambda_L^2} g \bar{D} \bar{F} (\bar{\chi}_I \bar{\gamma} \chi_I) + \frac{Y_f}{\Lambda_L^2} \bar{\chi} \chi \bar{\chi} \chi - \frac{g}{\Lambda_L^2} \bar{F}^3, \quad (1.1)$$

where the quadratic terms are

$$\begin{aligned} \mathcal{L}_Q &= \frac{1}{4} \sum_G (2F_{\mu\nu}^G F_{\mu\nu}^G - F_{\mu\nu}^G \tau^G(\tilde{\Upsilon}) F_{\mu\nu}^G), \\ \mathcal{L}_{\text{kinf}} &= \sum_{a,b=1}^3 \sum_{I=1}^5 \bar{\chi}_I^a i \left(\delta^{ab} \hat{D} - \frac{b_0^{Iab}}{\Lambda_L^2} \hat{D}^3 + b_1^{Iab} \hat{D} \right) \chi_I^b \end{aligned} \quad (1.2)$$

and the vertices are denoted symbolically, namely without listing all possible field differentiations and index contractions. In our notation hats are used to denote time, bars to denote space. The weight of time is equal to -1 , while the weight of each space coordinate is $-1/3$. The gauge couplings g have weight $1/3$. The weighted dimension of space-time is 2, so the Lagrangian contains only terms of weights ≤ 2 . Moreover, $\chi_1^a = L^a = (\nu_L^a, \ell_L^a)$, $\chi_2^a = Q_L^a = (u_L^a, d_L^a)$, $\chi_3^a = \ell_R^a$, $\chi_4^a = u_R^a$ and $\chi_5^a = d_R^a$, $\nu^a = (\nu_e, \nu_\mu, \nu_\tau)$, $\ell^a = (e, \mu, \tau)$, $u^a = (u, c, t)$ and $d^a = (d, s, b)$. The sum \sum_G is over the gauge groups $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$. Finally, $\tilde{\Upsilon} \equiv -\bar{D}^2/\Lambda_L^2$, where Λ_L is the scale of Lorentz violation, and τ^G are polynomials of degree 2.

The models of [3, 4] are anomaly-free, because gauge anomalies cancel out exactly as in the Standard Model [3]. The “boundary conditions” such that Lorentz invariance is recovered at low energies are that b_1^{Iab} tend to δ^{ab} and τ^G tend to 1 (one such condition can be trivially fulfilled normalizing the space coordinates \bar{x}).

An important consequence of the high-energy Lorentz violation is that all gauge interactions are super-renormalizable, therefore asymptotically free. Moreover, since fermions have weight 1/2, the four fermion interactions are strictly renormalizable. At energies much larger than Λ_L , vectors decouple and the model (1.1) reduces to a four fermion model in two weighted dimensions,

$$\mathcal{L}_{\text{HE}} = \sum_{a,b=1}^3 \sum_{I=1}^5 \bar{\chi}_I^a i \left(\delta^{ab} \hat{\partial} - \frac{b_0^{Iab}}{\Lambda_L^2} \bar{\partial}^3 \right) \chi_I^b + \frac{Y_f}{\Lambda_L^2} \bar{\chi} \chi \bar{\chi} \chi, \quad (1.3)$$

plus free fields. The purpose of this paper is to study this type of model, its one-loop beta functions and the conditions for asymptotic freedom. We stress that if the high-energy model (1.3) is asymptotically free, then the full Standard Model extension (1.1) is, as well as its other versions of ref.s [3, 4].

We work out a method to determine the domain of asymptotic freedom in quantum field theories with more couplings and apply it to some of our models. Our approach is to study the asymptotic expansion of the running couplings around the free fixed point. The domain of asymptotic freedom \mathcal{D}_{AF} is determined by the arbitrary constants contained in the expansion. The dimension of \mathcal{D}_{AF} is equal the number of positive eigenvalues (including multiplicities) of a certain matrix \mathcal{N} , which depends only on the one-loop coefficients of the beta functions.

Finally, we point out the presence of special RG trajectories that might hide new symmetries. Indeed, if we apply Zimmermann's "reduction of couplings" [5] to our beta functions we find that some solutions of the RG equations exhibit features that normally appear only in the presence of hidden symmetries.

The paper is organized as follows. In section 2 we classify the four fermion vertices and present the most general CPT- and rotation invariant four fermion model. In section 3 we work out general formulas for its one-loop beta functions. In section 4 we study some explicit examples. In section 5 we formulate our method to determine the domain of asymptotic freedom. In section 6 we recall Zimmermann's reduction of couplings, and explain why some of our models might possess hidden symmetries. Section 7 contains our conclusions.

2 Four fermion model

Using charge conjugation we can use only left-handed fermions, which we collect into a vector ℓ_j . We orthonormalize the kinetic terms $\bar{\ell}_j i \gamma^\mu \hat{\partial}_\mu \ell_k$ and simultaneously diagonalize the quadratic terms $\bar{\ell}_j i \gamma^\mu \bar{\partial}_\mu \bar{\partial}^2 \ell_k$ by means of a unitary transformation.

The most general CPT invariant, rotation invariant four fermion vertices are of the form $L^* L^* L L$ and $LLLL$, plus its conjugate $L^* L^* L^* L^*$. We do not have to include vertices $L^* L L L$. Indeed, in the four-component notation they can be constructed only with an odd number of

space-time indices, so they violate CPT. Precisely, a term L^*LLL plus its Hermitian conjugate reads

$$a(\bar{L}_1UL_2)(\bar{L}_3^cVL_4) + a^*(\bar{L}_2\gamma^0U\gamma^0L_1)(\bar{L}_4\gamma^0V\gamma^0L_3^c), \quad (2.1)$$

where U can be γ^0 or γ^i , V can be 1, σ^{0i} or σ^{ij} , and a is a constant. However, the combination (2.1) is CPT odd.

On the other hand, all combinations of terms $LLLL$, $L^*L^*L^*L^*$ and L^*L^*LL are CPT even. Explicitly, we have the structures

$$(\bar{L}_1VL_2^c)(\bar{L}_3^cVL_4), \quad (\bar{L}_1UL_2)(\bar{L}_3UL_4),$$

for L^*L^*LL , and

$$(\bar{L}_1^cVL_2)(\bar{L}_3^cVL_4),$$

for $LLLL$, plus their Hermitian conjugates.

Using Fierz identities, we can show that every $LLLL$ vertex is Lorentz invariant and of the form

$$(\bar{L}_1^cL_2)(\bar{L}_3^cL_4). \quad (2.2)$$

Similarly, all four fermion vertices of type L^*L^*LL have the form

$$(L_1^\dagger L_2)(L_3^\dagger L_4). \quad (2.3)$$

In general, this structure is Lorentz violating, but the combination

$$(L_1^\dagger L_2)(L_3^\dagger L_4) + (L_1^\dagger L_4)(L_3^\dagger L_2) = (\bar{L}_1L_3^c)(\bar{L}_2^cL_4) = \frac{1}{2}(\bar{L}_1\gamma_\mu L_2)(\bar{L}_3\gamma^\mu L_4), \quad (2.4)$$

is Lorentz invariant, which can be easily proved using the Fierz identity $\sigma^i_{\alpha\beta}\sigma^i_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\gamma\beta} - \delta_{\alpha\beta}\delta_{\gamma\delta}$.

At this point it is convenient to switch to the two-component spinor notation. Because of (2.2) and (2.3), the most general four fermion vertices are constructed with the contractions

$$\ell_i^\dagger \ell_j \equiv \ell_i^\dagger \alpha \ell_j^\alpha, \quad \ell_k^T \varepsilon \ell_m \equiv \ell_k^\alpha \varepsilon_{\alpha\beta} \ell_m^\beta, \quad \ell_k^\dagger \varepsilon \ell_m^* \equiv \ell_k^{*\alpha} \varepsilon_{\alpha\beta} \ell_m^{*\beta}, \quad (2.5)$$

α, β, \dots being spinor indices and T denoting transposition. The most general CPT- and rotation invariant interaction Lagrangian reads

$$\mathcal{L}_{4f} = \frac{1}{2\Lambda_L^2} \sum_{ijkl} \left[(\ell_i^\dagger \ell_j)(\ell_k^\dagger \ell_m) g_{ijkl} + (\ell_i^T \varepsilon \ell_j)(\ell_k^T \varepsilon \ell_m) f_{ijkl} + (\ell_i^\dagger \varepsilon \ell_j^*)(\ell_k^\dagger \varepsilon \ell_m^*) f_{ijkl}^* \right], \quad (2.6)$$

where the couplings f and g satisfy the symmetry properties

$$g_{ijkl} = g_{kmij} = g_{jimk}^*, \quad f_{ijkl} = f_{kmij} = f_{jikm}.$$

Because of (2.4), the most general Lorentz invariant interaction Lagrangian is (2.6) if the couplings g_{ijklm} satisfy the additional symmetry property

$$g_{ijklm} = g_{imkj}.$$

Finally, the most general high-energy Lorentz violating four fermion model that we are going to study reads

$$\mathcal{L} = \sum_j \ell_j^\dagger i \left(\hat{\partial} + b_j \sigma \cdot \bar{\partial} \frac{\partial^2}{\Lambda_L^2} \right) \ell_j + \mathcal{L}_{4f}, \quad (2.7)$$

where σ are the Pauli matrices. This model is renormalizable. The quadratic terms $\bar{\chi}_I^a i b_1^{Iab} \bar{\partial} \chi_I^b$ of (1.2) have been switched off, because they are not important at high energies. Nevertheless, the correlation functions at generic external momenta have no infrared divergences in two weighted dimensions. Therefore, we can study the model (2.7) in itself.

Assigning the axial charges $+1$ to ℓ_j and -1 to ℓ_j^\dagger , the g -terms are axially symmetric, while the f -terms explicitly violate the axial symmetry. Suppressing the f -terms we obtain a restricted model

$$\mathcal{L}_r = \sum_j \ell_j^\dagger i \left(\hat{\partial} + b_j \sigma \cdot \bar{\partial} \frac{\partial^2}{\Lambda_L^2} \right) \ell_j + \frac{1}{2\Lambda_L^2} \sum_{ijklm} (\ell_i^\dagger \ell_j) (\ell_k^\dagger \ell_m) g_{ijklm}, \quad (2.8)$$

that is still renormalizable. This restriction is interesting for merely theoretical purposes. However, for phenomenological applications we include the full set (2.4) of interactions. Integrating the Higgs field out in the Standard Model produces f -terms, which generate g -terms by renormalization. Moreover, our Lorentz-violating theories can include four fermion vertices that describe proton decay, and such terms are of both f - and g -types [6].

It should be emphasized that a high-energy Lorentz violation does not imply that the proton must decay. In this respect, we have two classes of renormalizable models. The models of the first class are described by B -invariant Lagrangians. Then, B -violating vertices are not generated back as counterterms by renormalization¹, so this choice is consistent. The models of the second class contain B -violating four fermion vertices at the classical level. Consistency with existing experimental bounds on proton decay imply that the energy scale of Lorentz violation Λ_L must then be greater than or equal to 10^{14-15} GeV (see for example [6]). At present, there is no reason to expect that Λ_L is much smaller than this value. However, if neutrino masses have the Lorentz-violating origin suggested by one of us in [3], namely they are explained by the renormalizable dimension-5 vertex

$$\frac{1}{\Lambda_L} (LH)^2,$$

¹The B -violations due to the $B + L$ anomaly are non-perturbative, and do not affect the renormalization structure of the theory.

where H is the Higgs field, then Λ_L could be around 10^{14}GeV , which is still compatible with the bound coming from proton decay².

3 One-loop renormalization

In this section we study the one-loop renormalization in the most general model (2.7). It is convenient to use the background field method. Replace ℓ with $\ell + \psi$, where ψ denotes the background field. Then expand \mathcal{L} in ψ and keep only the quadratic part in ℓ . The result can be written as

$$\mathcal{L}_\psi = \frac{1}{2} \sum_{np} \left(\ell_n^\dagger, \ell_n^T \right) (Q_n \delta_{np} + H_{np}) \begin{pmatrix} \ell_p \\ \ell_p^* \end{pmatrix},$$

where Q_n collects the kinetic terms and H_{np} is a matrix quadratic in ψ . In momentum space,

$$Q_n = p^0 + b_n \frac{\mathbf{p}^2}{\Lambda_L^2} \begin{pmatrix} \mathbf{p} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \mathbf{p} \cdot \boldsymbol{\sigma}^* \end{pmatrix}.$$

Thus, the one-loop contribution to the effective action reads

$$\Gamma_1 = -\frac{i}{2} \text{tr} \ln [Q_n \delta_{np} + H_{np}] = \frac{i}{4} \text{tr} [Q^{-1} H Q^{-1} H] + \text{constant} + \text{finite}. \quad (3.1)$$

Observe that the tadpole $-i \text{tr} [Q^{-1} H]/2$ vanishes, since the propagator Q^{-1} is odd in p . For this reason there is no wave-function renormalization at one loop.

We use a dimensional regularization where only the dimensions of space are continued to complex values $3 - \varepsilon_2$. There is no reason to continue also the time dimension, since the integrals over $\hat{p} = p^0$ converge. The divergent parts of just two integrals are necessary to evaluate expression (3.1), namely

$$\int \frac{dp_4 d^{3-\varepsilon_2} \bar{p}}{(2\pi)^4} \frac{p_4^2}{D_n D_p} = \frac{\Lambda_L^2}{4\pi^2 \varepsilon_2 (|b_n| + |b_p|)} + \text{finite},$$

and

$$\int \frac{dp_4 d^{3-\varepsilon_2} \bar{p}}{(2\pi)^4} \frac{1}{\Lambda_L^4} \frac{(\bar{p}^2)^3}{D_n D_p} = \frac{\Lambda_L^2}{4\pi^2 \varepsilon_2 |b_n b_p| (|b_n| + |b_p|)} + \text{finite},$$

where

$$D_i = p_4^2 + b_i^2 \frac{(\bar{p}^2)^3}{\Lambda_L^4} + \delta^2$$

(in Euclidean space). The fictitious mass δ is introduced to avoid IR problems at vanishing external momenta and can be set to zero after the evaluation. The calculations are performed

²A difference of one or two orders of magnitude can always be due to the dimensionless couplings that multiply the vertices.

using Feynman parameters to integrate over p_4 . This isolates the pole of the \bar{p} -integral, therefore the divergent part.

Using these formulas we easily find

$$\Gamma_1 = \frac{\Lambda_L^2}{(4\pi)^2 \varepsilon_2} \sum_{np} t_{np} \left\{ \text{tr}[H_{np}H_{pn}] - \frac{s_{np}}{3} \text{tr} \left[\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^{i*} \end{pmatrix} H_{np} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^{i*} \end{pmatrix} H_{pn} \right] \right\}.$$

We can convert this formula from the dimensional regularization to a conventional cut-off $\bar{\Lambda}$ on the \bar{p} -integral replacing $1/\varepsilon_2$ with $\ln \bar{\Lambda} + \text{constant}$. Now, since $\bar{\Lambda}$ has weight $1/3$ and, by definition, the high-energy dynamical scale μ has weight 1 , matching weights and dimensions we find the identification

$$\frac{1}{\varepsilon_2} = \frac{1}{3} \ln \frac{\bar{\Lambda}^3}{\mu \Lambda_L^2}.$$

The relation between the one-loop bare and renormalized Lagrangians is

$$\mathcal{L}_B = \mathcal{L}_R - \Gamma_1.$$

The beta functions are found equating the μ -derivative of \mathcal{L}_B to zero and performing a number of straightforward manipulations. We find

$$\begin{aligned} \beta_g^{ijkm} = & -\frac{1}{36\pi^2} \sum_{np} t_{np} [6(1 - s_{np})g_{ijnp}(g_{pnkm} - g_{pmkn}) - 2s_{np}g_{inpj}g_{kpnm} \\ & + 2s_{np}(g_{inkp}g_{pjnm} + 16f_{jpmn}f_{inkp}^*) - (3 + s_{np})(g_{inpm}g_{kpnj} - g_{inkp}g_{pmnj} - 16f_{mpjn}f_{inkp}^*) \\ & + 48(1 + s_{np})(f_{jmnt}f_{iknp}^* - f_{mpjn}f_{iknp}^* - f_{jmnt}f_{kpin}^*)]_{(i,j) \leftrightarrow (k,m)}, \end{aligned} \quad (3.2)$$

$$\beta_f^{ijkm} = -\frac{1}{18\pi^2} \sum_{np} t_{np} [2(3 + s_{np})g_{nipk}f_{mpjn} + 3(1 + s_{np})g_{nipj}f_{kmnp} + 4s_{np}g_{nipk}f_{jpmn}]_{(i,j) \leftrightarrow (k,m)}^{i \leftrightarrow j, k \leftrightarrow m},$$

and $\beta_{b_i} = 0$, where $s_{np} = b_n b_p / |b_n b_p|$ and $t_{np} = 1/(|b_n| + |b_p|)$. The expressions on the right-hand sides of (3.2) have to be symmetrized as follows:

$$\begin{aligned} [X_{ijkm}]_{(i,j) \leftrightarrow (k,m)} & \equiv \frac{1}{2} (X_{ijkm} + X_{kmij}), \\ [X_{ijkm}]^{i \leftrightarrow j, k \leftrightarrow m} & \equiv \frac{1}{4} (X_{ijkm} + X_{jikm} + X_{ijmk} + X_{jimk}). \end{aligned}$$

4 Explicit examples

In this section we consider some particular cases in detail. A separate section is devoted to the conditions for asymptotic freedom.

$U(N_L)$ **model** A simple model is the $U(N_L)$ -symmetric model of left-handed fermions L ,

$$\mathcal{L}_L = L^\dagger i \left(\hat{\partial} + b\sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) L^i + \frac{g_1}{2\Lambda_L^2} (L^\dagger i L^i)^2 + \frac{g_2}{2\Lambda_L^2} (L^\dagger i L^j)(L^\dagger j L^i).$$

Here we can set $b = 1$ rescaling the space coordinates, the fields and the couplings. We have a model (2.7) with

$$g_{ijkm} = g_1 \delta_{ij} \delta_{km} + g_2 \delta_{im} \delta_{kj}, \quad f_{ijkm} = 0,$$

so the one-loop beta functions read

$$\beta_1 = \frac{g_1^2}{36\pi^2} (N_L - 2), \quad \beta_2 = \frac{g_2^2}{36\pi^2} (2N_L - 1). \quad (4.1)$$

The solutions of the RG equations read

$$g_1(t) = g_1(0) + \frac{N_L - 2}{2N_L - 1} (g_2(t) - g_2(0)), \quad g_2(t) = \frac{g_2(0)}{1 - (2N_L - 1)g_2(0)t/(36\pi^2)},$$

where $t = -\ln(|x|\mu)$ and x is some scale.

In the ultraviolet limit ($t \rightarrow \infty$) we have

$$g_1(t) \sim g_1(0) - \frac{N_L - 2}{2N_L - 1} g_2(0) - \frac{36\pi^2(N_L - 2)}{(2N_L - 1)^2 t}, \quad g_2(t) \sim -\frac{36\pi^2}{(2N_L - 1)t}.$$

In particular,

$$g_1(\infty) \sim g_1(0) - \frac{N_L - 2}{2N_L - 1} g_2(0).$$

If $g_1(\infty) \neq 0$ the UV fixed point is interacting, if $g_1(\infty) = 0$ it is free. However, an interacting fixed point is not guaranteed to survive beyond the one-loop approximation. Thus, we must restrict to the subspace with $g_1(\infty) = 0$.

It is easy to prove that the model with $g_2 \equiv 0$ is renormalizable, for example introducing an auxiliary field χ of weight 1 and writing the Lagrangian in the form

$$\mathcal{L}'_L = L^\dagger i \left(\hat{\partial} + b\sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) L^i + \chi (L^\dagger i L^i) - \frac{\Lambda_L^2}{2g_1} \chi^2. \quad (4.2)$$

Such a model has vanishing one-loop beta function. However, it is unlikely to be finite, since no symmetry appears to forbid higher-order corrections.

We have just pointed out a general feature of our four fermion models: there exist combinations of couplings that have zero one-loop beta functions. Nevertheless, we are unable to use this observation to prove the existence of interacting fixed points. That would require more knowledge about higher-order corrections. Using only one-loop results the best we can do is to project onto a suitable subspace of the space of couplings, using a method inspired by Zimmermann's "reduction

of couplings" [5], and study the conditions for asymptotic freedom in that subspace. We first prove that there exists an analytic solution of the RG equations of the form

$$g_1 = c_1 g_2 + \sum_{i=2}^{\infty} c_i g_2^i \quad (4.3)$$

to all orders in the perturbative expansion. Consistence with the RG equations gives

$$\frac{dg_1}{dg_2} = \frac{\beta_1}{\beta_2},$$

which in turn uniquely determines all coefficients c_i 's. We find

$$c_1 = \frac{N_L - 2}{2N_L - 1},$$

plus recurrence relations of the form

$$c_i = P_i(c_{j < i}),$$

where $P_i(c_{j < i})$ are well-defined polynomials depending only on the coefficients c_j 's with $j < i$. The Zimmermann solution (4.3) restricts the two-parameter space to a curve. There our theory has a unique coupling, g_2 , and its one-loop beta function is still given by the second formula of (4.1). The condition of asymptotic freedom is thus

$$g_2 < 0.$$

“Electroweak” model Now we consider a four fermion model containing one family of the electroweak model. We have the left-handed doublet $L^a = (\nu_L, e_L)$ and the right-handed electron e_R . The high-energy four fermion Lagrangian reads

$$\begin{aligned} \mathcal{L}_{\text{EW}} = & L^{\dagger a} i \left(\hat{\partial} + b_L \sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) L^a + e_R^\dagger i \left(\hat{\partial} - b_R \sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) e_R + \frac{\lambda}{\Lambda_L^2} (L^{\dagger a} e_R) (e_R^\dagger L^a) \\ & + \frac{g_{1L}}{2\Lambda_L^2} (L^{\dagger a} L^a)^2 + \frac{g_{2L}}{2\Lambda_L^2} (L^{\dagger a} L^b) (L^{\dagger b} L^a) + \frac{g_R}{2\Lambda_L^2} (e_R^\dagger e_R)^2 + \frac{g_{LR}}{\Lambda_L^2} (L^{\dagger a} L^a) (e_R^\dagger e_R). \end{aligned}$$

Because of hypercharge conservation f -terms are not allowed, so this model is of restricted type (2.8). Define $\ell^i = (\nu_L, e_L, e_R^c)$, where c denotes the charge conjugate. Then we have

$$\begin{aligned} g_{ijkm} = & g_{1L} \hat{\delta}_{ij} \hat{\delta}_{km} + g_{2L} \hat{\delta}_{im} \hat{\delta}_{jk} + g_R \delta_{i3} \delta_{j3} \delta_{k3} \delta_{m3} \\ & + (\lambda - g_{LR}) (\hat{\delta}_{ij} \delta_{k3} \delta_{m3} + \delta_{i3} \delta_{j3} \hat{\delta}_{km}) + \lambda (\hat{\delta}_{im} \delta_{k3} \delta_{j3} + \delta_{i3} \delta_{m3} \hat{\delta}_{jk}), \end{aligned}$$

where $\hat{\delta}_{ij} = \delta_{ij} - \delta_{i3} \delta_{j3}$. Applying (3.2) we obtain the beta functions

$$\begin{aligned} \beta_{1L} = & \frac{\lambda^2}{36\pi^2 |b_R|}, & \beta_{2L} = & \frac{1}{36\pi^2} \left(3 \frac{g_{2L}^2}{|b_L|} + 2 \frac{\lambda^2}{|b_R|} \right), & \beta_R = & \frac{\lambda^2}{6\pi^2 |b_L|}, \\ \beta_{LR} = & \frac{\lambda}{36\pi^2} \left(u + \frac{\lambda(s-3)}{|b_L| + |b_R|} \right), & \beta_\lambda = & \frac{\lambda}{18\pi^2} \left(u - \frac{\lambda(2s+3)}{|b_L| + |b_R|} \right), \end{aligned} \quad (4.4)$$

where $s \equiv b_L b_R / (|b_L| |b_R|)$ and

$$u \equiv \frac{g_{1L}}{|b_L|} + 2 \frac{g_{2L}}{|b_L|} + \frac{g_R}{|b_R|} + \frac{4s g_{LR}}{|b_L| + |b_R|}.$$

Recall that b_L and b_R do not run at one loop.

We see that the beta functions depend only on three couplings, precisely λ , g_{2L} and the combination u . The conditions for asymptotic freedom are studied in section 5. There exist RG-flow trajectories with special properties, possibly hidden symmetries. This topic is discussed in section 6.

$U(N_L) \times U(N_R)$ model This is a generalization of the electroweak model, where the left- and right-handed fermions are in the fundamental representations of $U(N_L)$ and $U(N_R)$, respectively. The Lagrangian is

$$\begin{aligned} \mathcal{L}_{N_L N_R} = & L^{\dagger a} i \left(\hat{\partial} + b_L \sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) L^a + E_R^{\dagger I} i \left(\hat{\partial} - b_R \sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) E_R^I \\ & + \frac{g_{1L}}{2\Lambda_L^2} (L^{\dagger a} L^a)^2 + \frac{g_{2L}}{2\Lambda_L^2} (L^{\dagger a} L^b)(L^{\dagger b} L^a) + \frac{g_{1R}}{2\Lambda_L^2} (E_R^{\dagger I} E_R^I)^2 + \frac{g_{2R}}{2\Lambda_L^2} (E_R^{\dagger I} E_R^J)(E_R^{\dagger J} E_R^I) \\ & + \frac{g_{1LR}}{\Lambda_L^2} (L^{\dagger a} L^a)(E_R^{\dagger I} E_R^I) + \frac{g_{2LR}}{\Lambda_L^2} (L^{\dagger a} E_R^I)(E_R^{\dagger I} L^a), \end{aligned}$$

where $a, b = 1, \dots, N_L$ and $I, J = 1, \dots, N_R$. Define $\ell^i = (L^a, E^{cI})$, with $i = (a, I)$. We have a restricted model (2.8) with couplings

$$\begin{aligned} g_{ijkm} = & g_{1L} \hat{\delta}_{ij} \hat{\delta}_{km} + g_{2L} \hat{\delta}_{im} \hat{\delta}_{jk} + g_{1R} \bar{\delta}_{ij} \bar{\delta}_{km} + g_{2R} \bar{\delta}_{im} \bar{\delta}_{jk} \\ & + (g_{2LR} - g_{1LR})(\hat{\delta}_{ij} \bar{\delta}_{km} + \hat{\delta}_{km} \bar{\delta}_{ij}) + g_{2LR}(\hat{\delta}_{im} \bar{\delta}_{jk} + \hat{\delta}_{jk} \bar{\delta}_{im}), \end{aligned}$$

where $\hat{\delta}$ and $\bar{\delta}$ are the Kronecker tensors of $U(N_L)$ and $U(N_R)$, respectively. The beta functions are

$$\begin{aligned} \beta_{1L} &= \frac{1}{36\pi^2} \left((N_L - 2) \frac{g_{2L}^2}{|b_L|} + N_R \frac{g_{2LR}^2}{|b_R|} \right), & \beta_{2L} &= \frac{1}{36\pi^2} \left((2N_L - 1) \frac{g_{2L}^2}{|b_L|} + 2N_R \frac{g_{2LR}^2}{|b_R|} \right), \\ \beta_{1R} &= \frac{1}{36\pi^2} \left((N_R - 2) \frac{g_{2R}^2}{|b_R|} + N_L \frac{g_{2LR}^2}{|b_L|} \right), & \beta_{2R} &= \frac{1}{36\pi^2} \left((2N_R - 1) \frac{g_{2R}^2}{|b_R|} + 2N_L \frac{g_{2LR}^2}{|b_L|} \right), \\ \beta_{1LR} &= \frac{g_{2LR}}{36\pi^2} \left(u + \frac{g_{2LR}(s-3)}{|b_L| + |b_R|} \right), & \beta_{2LR} &= \frac{g_{2LR}}{18\pi^2} \left(u - \frac{g_{2LR}(2s+3)}{|b_L| + |b_R|} \right), \end{aligned} \quad (4.5)$$

where now

$$u = \frac{g_{1L}}{|b_L|} + \frac{g_{1R}}{|b_R|} + N_L \frac{g_{2L}}{|b_L|} + N_R \frac{g_{2R}}{|b_R|} + \frac{4s g_{1LR}}{|b_L| + |b_R|}.$$

Dirac fermions So far, we have considered only explicit examples of reduced type (2.8). Now we consider a model of N Dirac fermions, which involves also f terms and new types of g terms. We impose the flavor symmetry $U(N)$ and parity invariance, which allows us to set $b_L = b_R = 1$.

As usual, write the Dirac fermions ψ^i as (ℓ_1^i, ℓ_2^{ci}) , where ℓ_1^i denote the left-handed components and ℓ_2^i are the charge-conjugates of the right-handed components. The action of parity reads

$$P\ell_1^i = \varepsilon\ell_2^{i*}, \quad P\ell_2^i = -\varepsilon\ell_1^{i*}.$$

Moreover, ℓ_1^i and ℓ_2^i belong to the fundamental and anti-fundamental $U(N)$ representations, respectively.

The Lagrangian (2.7) becomes

$$\begin{aligned} \mathcal{L}_N = & \ell_1^{\dagger i} i \left(\hat{\partial} + \sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) \ell_1^i + \ell_2^{\dagger i} i \left(\hat{\partial} + \sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) \ell_2^i \\ & + \frac{g_1}{2\Lambda_L^2} \left[(\ell_1^{\dagger i} \ell_1^i)^2 + (\ell_2^{\dagger i} \ell_2^i)^2 \right] + \frac{g_2}{2\Lambda_L^2} \left[(\ell_1^{\dagger i} \ell_1^j)(\ell_1^{\dagger j} \ell_1^i) + (\ell_2^{\dagger i} \ell_2^j)(\ell_2^{\dagger j} \ell_2^i) \right] \\ & + \frac{g_3}{\Lambda_L^2} (\ell_1^{\dagger i} \ell_1^i)(\ell_2^{\dagger j} \ell_2^j) + \frac{g_4}{\Lambda_L^2} (\ell_1^{\dagger i} \ell_2^j)(\ell_2^{\dagger j} \ell_1^i) + \frac{g_5}{\Lambda_L^2} (\ell_1^{\dagger i} \ell_1^j)(\ell_2^{\dagger i} \ell_2^j) + \frac{g_6}{\Lambda_L^2} (\ell_1^{\dagger i} \ell_2^j)(\ell_2^{\dagger i} \ell_1^j) \\ & + \frac{f_1}{\Lambda_L^2} \left[(\ell_1^{T i} \varepsilon \ell_2^i)(\ell_1^{T j} \varepsilon \ell_2^j) + (\ell_1^{\dagger i} \varepsilon \ell_2^{*i})(\ell_1^{\dagger j} \varepsilon \ell_2^{*j}) \right] + \frac{f_2}{\Lambda_L^2} \left[(\ell_1^{T i} \varepsilon \ell_2^j)(\ell_1^{T j} \varepsilon \ell_2^i) + (\ell_1^{\dagger i} \varepsilon \ell_2^{*j})(\ell_1^{\dagger j} \varepsilon \ell_2^{*i}) \right], \end{aligned} \quad (4.6)$$

and all couplings are real. A possible third vertex of f -type, namely

$$(\ell_1^{T i} \varepsilon \ell_1^j)(\ell_2^{T i} \varepsilon \ell_2^j)$$

is not included, because it is not independent of the other two. Indeed, a Fierz rearrangement gives the identity

$$(\ell_1^{T i} \varepsilon \ell_2^i)(\ell_1^{T j} \varepsilon \ell_2^j) + (\ell_1^{T i} \varepsilon \ell_2^j)(\ell_1^{T j} \varepsilon \ell_2^i) + (\ell_1^{T i} \varepsilon \ell_1^j)(\ell_2^{T i} \varepsilon \ell_2^j) = 0.$$

We report only the beta functions in the large N limit, which simplify considerably:

$$\begin{aligned} \beta_{g_1} &= \frac{N(g_2^2 + g_4^2)}{36\pi^2}, & \beta_{g_2} &= \frac{N(g_2^2 + g_4^2)}{18\pi^2}, & \beta_{g_3} &= \frac{Ng_2g_4}{18\pi^2}, & \beta_{g_4} &= \frac{Ng_2g_4}{9\pi^2}, \\ \beta_{g_5} &= -\frac{N}{18\pi^2} (g_5^2 + g_5g_6 + g_6^2 + 12f_1^2 - 12f_1f_2 + 4f_2^2), \\ \beta_{g_6} &= -\frac{N}{36\pi^2} (g_5^2 + 4g_5g_6 + g_6^2 + 24f_1^2 - 24f_1f_2 + 4f_2^2), \\ \beta_{f_1} &= \frac{N}{18\pi^2} [f_2(g_5 + 2g_6) - 3f_1(g_5 + g_6)], & \beta_{f_2} &= \frac{N}{18\pi^2} f_2(g_6 - g_5). \end{aligned}$$

Observe that the beta functions do not depend on g_1 and g_3 . Moreover, the couplings separate in two groups, g_{1-4} and $g_{5,6}-f_{1,2}$, which do not talk to each other. These, however, are only features of the large N limit.

5 Asymptotic freedom

In this section we study the conditions for asymptotic freedom in the presence of more than one coupling. The idea is to search for solutions of the RG equations as expansions around the ultraviolet free fixed point. The domain of asymptotic freedom \mathcal{D}_{AF} is then determined by the free parameters contained in the solution. A different approach to asymptotic freedom with more couplings is due to Zimmermann [7]. An investigation that is in part related to this problem can be found in [8].

We first illustrate our method in the case of a single coupling α with beta function

$$\beta_\alpha = \dot{\alpha} = \alpha \sum_{n=1}^{\infty} \beta_n \alpha^n.$$

where the dot denotes the derivative with respect to $t = -\ln(|x|\mu)$, $|x|$ being some scale. If $\beta_1 \neq 0$ the asymptotic expansion around the ultraviolet limit $t \rightarrow \infty$ reads

$$\alpha(t) = \frac{1}{t} \sum_{n=0}^{\infty} \frac{b_n(\ln t)}{t^n}, \quad (5.1)$$

where b_n are polynomials of degree n in $\ln t$. Inserting this expansion into the RG equation we get $b_0 = -1/\beta_1$ and the recursion relations

$$(n-1)b_n - b'_n = \delta_{n<} \quad (5.2)$$

for $n > 0$, where $\delta_{n<}$ is a linear combination of monomials $\prod_i b_{n_i}^{k_i}$ with $\sum_i n_i k_i \leq n$ and depends only on the coefficients b_m with $0 < m < n$. Consider first $n = 1$, and observe that $\delta_{1<}$ contains no logarithms, so b_1 is a polynomial of degree 1. The coefficient of $\ln t$ in b_1 is uniquely determined, while $b_1(0) \equiv b$ remains arbitrary. For $n > 1$ the relations (5.2) can be solved recursively:

$$b_n = \frac{1}{n-1}(\delta_{n<} + b'_n) = \frac{1}{n-1}\delta_{n<} + \frac{1}{(n-1)^2}\delta'_{n<} + \frac{1}{(n-1)^3}\delta''_{n<} + \dots \quad (5.3)$$

Clearly, the sum ends after a finite number of terms, since $\delta_{n<}$ is a polynomial.

Thus, the asymptotic solution (5.1) is well-defined and uniquely determined as a function of the arbitrary constant b . To the lowest orders, we find

$$\alpha(t) = -\frac{1}{\beta_1 t} - \frac{\beta_2 \ln t}{\beta_1^3 t^2} + \frac{b}{t^2} - \frac{\beta_2^2 \ln^2 t}{\beta_1^5 t^3} + \frac{\beta_2}{\beta_1^3} (2b\beta_1^3 + \beta_2) \frac{\ln t}{t^3} + \mathcal{O}(t^{-3}). \quad (5.4)$$

If $\beta_1 = 0$ but $\beta_2 < 0$ we have the expansion

$$\alpha(t) = \frac{1}{t^{1/2}} \sum_{n=0}^{\infty} \frac{b_n(\ln t)}{t^{n/2}} = \frac{1}{\sqrt{-2\beta_2 t}} + \frac{\beta_3}{2\beta_2^2 t} + \frac{\beta_3^2 - \beta_2\beta_4}{4\sqrt{2}(-\beta_2)^{7/2} t^{3/2}} + \frac{b}{t^{3/2}} + \mathcal{O}(t^{-2} \ln t). \quad (5.5)$$

If the first non-vanishing coefficient is β_n then the expansion begins with $t^{-1/n}$.

Now we generalize this result to the case of more couplings. Consider a theory with s couplings $g = \{g_i\}$ and beta functions $\beta_i(g)$. To be specific, we assume

$$\beta_i(g) = \sum_{n=2}^{\infty} \frac{1}{n!} c_{ij_1 \dots j_n} g_{j_1} \cdots g_{j_n} = \frac{1}{2} c_{ijk} g_j g_k + \mathcal{O}(g^3), \quad (5.6)$$

where the constants c_{ijk} are the one-loop coefficients. We look for asymptotic solutions of the RG equations starting from the ‘‘Zimmermann trajectories’’

$$g_i(t) \sim -\frac{a_i}{t} \quad (5.7)$$

in the limit $t \rightarrow \infty$. Inserting (5.7) into the RG equations

$$\dot{g}_i(t) = \beta_i(g(t)), \quad (5.8)$$

where β_i are given by (5.6), and keeping only the leading terms, we see that the constants a_i are determined by the quadratic equations

$$a_i = \frac{1}{2} c_{ijk} a_j a_k, \quad (5.9)$$

which have, in general, a discrete set of solutions. Normally the solutions just have to be real (if the couplings are parametrized to be real, as we assume), but in some cases further physical restrictions might apply. For example, stability (positive-definiteness of the action in Euclidean space) might require that some couplings be positive. We do not consider such restrictions here and assume that all real solutions are physical acceptable. It is straightforward to adapt our conclusions to specific situations.

Around a Zimmermann trajectory, we continue the expansion as

$$g_i(t) \sim -\frac{1}{t} \left(a_i + \frac{b_i}{t^\gamma} \right), \quad (5.10)$$

assuming that γ is a positive number. Then γ and b_i are an eigenvalue and an eigenvector of the real matrix

$$\mathcal{N}_{ij} = c_{ikj} a_k - \delta_{ij},$$

respectively.

The matrix \mathcal{N} is crucial for our discussion. The dimension d_{AF} of the domain \mathcal{D}_{AF} of asymptotic freedom is equal to the number of \mathcal{N} -positive eigenvalues γ , including multiplicities. Observe that because of (5.9) one eigenvalue is always equal to 1, with eigenvector a_i . If the Zimmermann trajectory exists, the dimension d_{AF} is at least 1. If $\gamma = 1$ is the unique positive eigenvalue, the form of the expansion is (5.1).

The most general solution reads

$$g_i(t) = -\frac{1}{t} \left(a_i + \sum_{\gamma \cdot \mathbf{n}, n > 0} \frac{b_{i, \gamma \cdot \mathbf{n}}(\ln t)}{t^{\gamma \cdot \mathbf{n}}} \right). \quad (5.11)$$

Here γ is a vector collecting the positive eigenvalues of the matrix \mathcal{N} , while \mathbf{n} is a vector of non-negative integer entries. The condition $n > 0$ means that \mathbf{n} must not vanish identically. Two vectors \mathbf{n} and \mathbf{n}' such that $\gamma \cdot \mathbf{n} = \gamma \cdot \mathbf{n}'$ are equivalent, and associated with a unique numerator $b_{i, \gamma \cdot \mathbf{n}}$. The sum is ordered for increasing values of $\gamma \cdot \mathbf{n}$. Finally, the $b_{i, \gamma \cdot \mathbf{n}}(\ln t)$'s are polynomials of certain finite degrees in $\ln t$.

Inserting (5.11) into the RG equations (5.8) and isolating the coefficients of the powers $t^{-2+\gamma \cdot \mathbf{n}}$, we obtain equations for the polynomials $b_{i, \gamma \cdot \mathbf{n}}$. It is immediate to find that such equations have the form

$$[(\gamma \cdot \mathbf{n})\delta_{ij} - \mathcal{N}_{ij}] b_{j, \gamma \cdot \mathbf{n}} - b'_{i, \gamma \cdot \mathbf{n}} = \delta_{i, \gamma \cdot \mathbf{n} <}, \quad (5.12)$$

where $\delta_{i, \gamma \cdot \mathbf{n} <}$ is a sum of monomials

$$\prod_k b_{j, \gamma \cdot \mathbf{n}_k}$$

with $\gamma \cdot \mathbf{n}_k < \gamma \cdot \mathbf{n}$ and $\sum_k \gamma \cdot \mathbf{n}_k \leq \gamma \cdot \mathbf{n}$.

Clearly, $\delta_{i, \gamma \cdot \mathbf{n} <}$ contains a finite number of terms. Now we want to show that equations (5.12) allow us to recursively determine the $b_{i, \gamma \cdot \mathbf{n}}$'s.

For intermediate purposes, it is convenient to turn to the basis where the matrix \mathcal{N}_{ij} has a real canonical Jordan form. Quantities in this basis are denoted with a tilde. Specifically, $\tilde{\mathcal{N}}$ is block-diagonal. Its first blocks are associated with the real eigenvalues λ and have the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \ddots & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \quad (5.13)$$

while the last blocks are associated with the complex eigenvalues μ and have the same form as (5.13), where however the λ 's are replaced by 2×2 blocks

$$\begin{pmatrix} \operatorname{Re} \mu & \operatorname{Im} \mu \\ -\operatorname{Im} \mu & \operatorname{Re} \mu \end{pmatrix},$$

the 1's are replaced by 2×2 identity matrices and the 0's are replaced by 2×2 matrices with vanishing entries. All matrices $\tilde{\mathcal{N}}_{ij} - (\gamma \cdot \mathbf{n})\delta_{ij}$ are then in canonical Jordan form.

Let M_{ij} be such that $\mathcal{N} = M^{-1} \tilde{\mathcal{N}} M$ and $\tilde{b}_{i, \gamma \cdot \mathbf{n}} \equiv M_{ij} b_{j, \gamma \cdot \mathbf{n}}$. Multiplying equation (5.12) by M to the left, we can rewrite it in the form

$$[(\gamma \cdot \mathbf{n})\delta_{ij} - \tilde{\mathcal{N}}_{ij}] \tilde{b}_{j, \gamma \cdot \mathbf{n}} - \tilde{b}'_{i, \gamma \cdot \mathbf{n}} = \tilde{\delta}_{i, \gamma \cdot \mathbf{n} <}, \quad (5.14)$$

By induction, if we assume that the polynomials $\tilde{b}_{i,\gamma\cdot\mathbf{n}'}$ with $\gamma\cdot\mathbf{n}' < \gamma\cdot\mathbf{n}$ are known, we conclude that the $\tilde{\delta}_{i,\gamma\cdot\mathbf{n}'}$'s are polynomials of certain finite degrees in $\ln t$. When the matrix $\tilde{\mathcal{N}}_{ij} - (\gamma\cdot\mathbf{n})\delta_{ij}$ is invertible, the polynomials $\tilde{b}_{i,\gamma\cdot\mathbf{n}}$ are uniquely determined. Calling \mathcal{U}_{ij} the inverse matrix of $\tilde{\mathcal{N}}_{ij} - (\gamma\cdot\mathbf{n})\delta_{ij}$, we have, similarly to (5.3),

$$\tilde{b}_{i,\gamma\cdot\mathbf{n}} = -\mathcal{U}_{ij} \left(\tilde{\delta}_{j,\gamma\cdot\mathbf{n}' <} + \tilde{b}'_{j,\gamma\cdot\mathbf{n}} \right) = -\mathcal{U}_{ij} \tilde{\delta}_{j,\gamma\cdot\mathbf{n}' <} + \mathcal{U}_{ij} \mathcal{U}_{jk} \tilde{\delta}'_{k,\gamma\cdot\mathbf{n}' <} - \mathcal{U}_{ij} \mathcal{U}_{jk} \mathcal{U}_{kl} \tilde{\delta}''_{l,\gamma\cdot\mathbf{n}' <} + \dots \quad (5.15)$$

Again, the sum ends after a finite number of terms, since the $\tilde{\delta}_{i,\gamma\cdot\mathbf{n}' <}$'s are polynomials.

When $\tilde{\mathcal{N}}_{ij} - (\gamma\cdot\mathbf{n})\delta_{ij}$ is not invertible, one of its blocks has $m_0 > 0$ zeros on the diagonal. Assume that this block is the one with $\bar{i} \leq i, j < \bar{i} + m_0$ and proceed as follows. The block-structure of $\tilde{\mathcal{N}}$ allows us to split equation (5.14) into: *a*) the equation for the $\tilde{b}_{i,\gamma\cdot\mathbf{n}}$'s with $i < \bar{i}$; *b*) the equation for the $\tilde{b}_{i,\gamma\cdot\mathbf{n}}$ ' with $\bar{i} \leq i, j < \bar{i} + m_0$; *c*) the equation for the $\tilde{b}_{i,\gamma\cdot\mathbf{n}}$'s with $i \geq \bar{i} + m_0$. Equations *a*) and *c*) are solved by formulas similar to (5.15). Equation *b*) has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ \zeta_1 & \ddots & 0 \\ 0 & \zeta_{m_0-1} & 0 \end{pmatrix}_{ij} \tilde{b}_{j,\gamma\cdot\mathbf{n}} - \tilde{b}'_{i,\gamma\cdot\mathbf{n}} = \tilde{\delta}_{i,\gamma\cdot\mathbf{n}' <},$$

where the ζ_i 's can be equal to 0 or 1. The right-hand side is made of recursively known polynomials of some degrees d_i in $\ln t$. Then the $\tilde{b}_{i,\gamma\cdot\mathbf{n}}$'s are polynomials of finite degrees greater than d_i , and each of them is uniquely determined up to an arbitrary additional constant. Therefore, in total we have m_0 arbitrary constants.

Thus, equations (5.12)-(5.14) can be solved recursively to determine the polynomials $\tilde{b}_{i,\gamma\cdot\mathbf{n}}$, and therefore the $b_{i,\gamma\cdot\mathbf{n}}$'s. The solutions (5.11) contain a number of arbitrary constants equal to the number of times the matrices $\tilde{\mathcal{N}}_{ij} - (\gamma\cdot\mathbf{n})\delta_{ij}$ become degenerate, including multiplicities. This number is equal to the number of positive \mathcal{N} -eigenvalues, including multiplicities. Indeed, recalling that different \mathbf{n} 's with the same $\gamma\cdot\mathbf{n}$ are equivalent, each equation $\gamma_i = \gamma\cdot\mathbf{n}$ admits precisely one solution, and no other degeneracies are possible.

The set of arbitrary constants contained in the asymptotic expansion (5.11) determines the domain of asymptotic freedom \mathcal{D}_{AF} . We conclude that the dimension of \mathcal{D}_{AF} is equal to the number of positive eigenvalues γ of the matrix \mathcal{N} , including multiplicities.

In practice, we have to look for the Zimmermann trajectory around which the asymptotic expansion (5.11) has the maximal number of positive eigenvalues. In most cases the other Zimmermann trajectories also play a role. They can determine the boundary of \mathcal{D}_{AF} , if it is two-dimensional, or its edges, if it more than two dimensional.

Now we consider two examples in detail.

Dirac model First we consider the Dirac model (4.6) in the large N limit, focusing on the g_2 - g_4 subsystem. The RG flow is given by the equations

$$\beta_2 = \dot{g}_2 = \kappa(g_2^2 + g_4^2) + \mathcal{O}(g^3), \quad \beta_4 = \dot{g}_4 = 2\kappa g_2 g_4 + \mathcal{O}(g^3),$$

where $\kappa = N/(18\pi^2) > 0$. We find the constants

$$c_{222} = c_{244} = c_{424} = c_{442} = 2\kappa,$$

while all other entries c_{ijk} vanish. The Zimmermann trajectories are given by $(a_2, a_4) = (1, 0)/\kappa$ and $(a_2, a_4) = (1, \pm 1)/(2\kappa)$. Expanding around the trajectories with $(a_2, a_4) = (1, 0)/\kappa$ we find that \mathcal{N} is equal to the identity matrix, so $\gamma = 1$ with multiplicity 2. This means that \mathcal{D}_{AF} has dimension 2. Two arbitrary constants appear at order t^{-2} .

Precisely, we find

$$g_2(t) = -\frac{1}{\kappa t} + \frac{1}{t^2}(\xi_2 \ln t + b_2) + \mathcal{O}(t^{-3} \ln^2 t), \quad g_4(t) = \frac{1}{t^2}(\xi_4 \ln t + b_4) + \mathcal{O}(t^{-3} \ln^2 t), \quad (5.16)$$

where $b_{2,4}$ are arbitrary constants, while $\xi_{2,4}$ are uniquely determined by the cubic terms of the beta functions. Since all other matrices $\mathcal{N}_{ij} - (\gamma \cdot \mathbf{n})\delta_{ij}$ are invertible the higher-order corrections are uniquely determined. Thus, the two-dimensional domain of asymptotic freedom is parametrized by the arbitrary constants $b_{2,4}$. The other two Zimmermann trajectories, given by $(a_2, a_4) = (1, \pm 1)/(2\kappa)$, are the boundary of \mathcal{D}_{AF} .

At one loop we can check our results solving the system explicitly. Call $\lambda_{\pm} = g_2 \pm g_4$. We have

$$\beta_{\pm} = \dot{\lambda}_{\pm} = \kappa \lambda_{\pm}^2,$$

wherefrom

$$\lambda_{\pm}(t) = \frac{\lambda_{\pm 0}}{1 - \kappa \lambda_{\pm 0} t}. \quad (5.17)$$

In the ultraviolet limit $t \rightarrow \infty$ the asymptotic behaviors are

$$g_2(t) = -\frac{1}{\kappa t} - \frac{1}{2\kappa^2 t^2} \frac{\lambda_{+0} + \lambda_{-0}}{\lambda_{+0} \lambda_{-0}} + \mathcal{O}(t^{-3}), \quad g_4(t) = \frac{1}{2\kappa^2 t^2} \frac{\lambda_{+0} - \lambda_{-0}}{\lambda_{+0} \lambda_{-0}} + \mathcal{O}(t^{-3}).$$

Here the logarithmic terms of (5.16) are absent, because we are neglecting higher-order corrections to the beta functions (check also (5.4)).

Electroweak model Now we apply our method to the electroweak four fermion model (4.4) with $b_L = b_R = 1$. We first restrict to the subspace (g_{2L}, λ, u) . At one loop we have

$$\beta_{2L} = \frac{1}{(6\pi)^2}(3g_{2L}^2 + 2\lambda^2), \quad \beta_{\lambda} = \frac{\lambda}{(6\pi)^2}(2u - 5\lambda), \quad \beta_u = \frac{1}{(6\pi)^2}(9\lambda^2 + 6g_{2L}^2 + 2\lambda u). \quad (5.18)$$

The conditions $\beta_{2L} = \beta_\lambda = \beta_u = 0$ are solved by $g_{2L} = \lambda = 0$, while u remains arbitrary. Since a non-trivial fixed point cannot be trusted within our approximation, we need to project onto a suitable subspace of parameter space. Such a projection is automatic in our approach.

The Zimmermann trajectories are

$$g_{2L} \sim -\frac{a_1}{t}, \quad \lambda \sim -\frac{a_2}{t}, \quad u \sim -\frac{a_3}{t},$$

with

$$(a_1, a_2, a_3) = \left(\frac{1}{3}, 0, \frac{2}{3}\right), \left(\frac{1}{5}, \frac{1}{5}, 1\right), (0.046, -0.141, 0.149). \quad (5.19)$$

The third trajectory is given by complicated irrational coefficients of which we just give the approximate numerical values. Two complex solutions to (5.9) also exist, but must be discarded.

Expanding around the first Zimmermann trajectory, we find

$$\mathcal{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 4 & \frac{4}{3} & -1 \end{pmatrix}, \quad (5.20)$$

so the positive eigenvalues are $\gamma = 1/3$ and $\gamma = 1$. The Jordan canonical form of this matrix is diagonal: $\tilde{\mathcal{N}} = \text{diag}(1, 1/3, -1)$. The domain of asymptotic freedom is two-dimensional and the arbitrary constants appear at orders $t^{-4/3}$ and t^{-2} . Using the one-loop truncated beta functions, the asymptotic expansion of the solution reads

$$\begin{aligned} (6\pi)^{-2} g_{2L}(t) &= -\frac{1}{3t} + \frac{6a^2}{t^{5/3}} + \frac{1}{2t^2}(72a^3 \ln t + 27a^3 + b) - \frac{1566a^4}{t^{7/3}} - \frac{3a^2}{t^{8/3}}(360a^3 \ln t + 2907a^3 + 5b) \\ &\quad - 3888a^6 \frac{\ln^2 t}{t^3}, \\ (6\pi)^{-2} \lambda(t) &= \frac{a}{t^{4/3}} + \frac{9a^2}{t^{5/3}} + \frac{63a^3}{t^2} - \frac{a}{t^{7/3}}(144a^3 \ln t + 2b - 495a^3) \\ &\quad - \frac{a^2}{t^{8/3}} \left(1620a^3 \ln t - \frac{138267}{14}a^3 + \frac{45}{2}b\right), \\ (6\pi)^{-2} u(t) &= -\frac{2}{3t} + \frac{a}{t^{4/3}} + \frac{15a^2}{t^{5/3}} + \frac{1}{t^2}(72a^3 \ln t + b) - \frac{a}{t^{7/3}} \left(144a^3 \ln t + \frac{24669}{7}a^3 + 2b\right) \\ &\quad - \frac{3a^2}{t^{8/3}} \left(900a^3 \ln t + \frac{70587}{14}a^3 + \frac{25}{2}b\right) - 7776a^6 \frac{\ln^2 t}{t^3}, \end{aligned} \quad (5.21)$$

up to $\mathcal{O}(t^{-3} \ln t)$, where a and b are the arbitrary constants.

The cubic corrections to the beta functions start with terms $\sim 1/t^3$, so they give extra contributions of the form $\sim c_i/t^2$ to the solutions, where c_i are uniquely determined functions of a and b . The corrections do not affect the terms proportional to $(\ln t)/t^2$. Thus, the complete solution

has the form

$$\begin{aligned}
(6\pi)^{-2}g_{2L}(t) &= -\frac{1}{3t} + \frac{6a^2}{t^{5/3}} + \frac{1}{2t^2}(72a^3 \ln t + 27a^3 + b + \xi_1) + \mathcal{O}(t^{-7/3} \ln t), \\
(6\pi)^{-2}\lambda(t) &= \frac{a}{t^{4/3}} + \frac{9a^2}{t^{5/3}} + \frac{63a^3 + \xi_2}{t^2} + \mathcal{O}(t^{-7/3} \ln t), \\
(6\pi)^{-2}u(t) &= -\frac{2}{3t} + \frac{a}{t^{4/3}} + \frac{15a^2}{t^{5/3}} + \frac{1}{t^2}(72a^3 \ln t + b + \xi_3) + \mathcal{O}(t^{-7/3} \ln t),
\end{aligned} \tag{5.22}$$

where ξ_i , $i = 1, 2, 3$, are calculable numbers, depending on the cubic corrections to the beta functions. The other beta functions of (4.4) give

$$(6\pi)^{-2}g_{1L} = -\frac{3a^2}{5t^{5/3}} + \frac{\xi_4}{t^2} + \mathcal{O}(t^{-7/3}), \quad (6\pi)^{-2}g_R = -\frac{18a^2}{5t^{5/3}} + \frac{\xi_5}{t^2} + \mathcal{O}(t^{-7/3}),$$

where $\xi_{4,5}$ are calculable numbers.

We have thus found a two dimensional domain of asymptotic freedom.

One-loop degeneracies In special cases, not frequent in physical problems, the one-loop coefficients c_{ijk} can have degeneracies that make the expansions of some couplings start from powers $t^{-1/n}$ instead of $1/t$, similarly to what happens in (5.5) when $\beta_1 = 0$ for a single coupling. Some higher loop contributions can be as important as the one-loop ones, or even more important than the one-loop ones. Then the expansions of the beta functions in powers of the couplings have to be accordingly reordered. For example, consider the system

$$\dot{g}_1 = g_1^2 + \kappa^2 g_1 g_2^2, \quad \dot{g}_2 = \frac{1}{4} g_1 g_2,$$

where κ is a constant. Observe that the terms g_1^2 and $\kappa^2 g_1 g_2^2$ are one- and two-loop, respectively. Nevertheless, they are equally important in the asymptotic expansion. The Zimmermann trajectories are

$$(g_1, g_2) = \left(-\frac{1}{t}, 0\right), \left(-\frac{2}{t}, \pm \frac{1}{\kappa\sqrt{t}}\right), \tag{5.23}$$

to which we must add the line of fixed points $g_1 \equiv 0$. The procedure described above has to be applied using the trajectories (5.23). For example, expanding around the second pair of trajectories we find that the matrix \mathcal{N} has eigenvalue 1 with degeneracy 2 and the expansions read

$$\begin{aligned}
g_1(t) &= -\frac{1}{t} \left(2 + \frac{a-b}{t} \ln t - \frac{a}{t}\right) + \mathcal{O}(t^{-3} \ln^2 t), \\
g_2(t) &= \pm \frac{1}{4\kappa\sqrt{t}} \left(4 + \frac{a-b}{t} \ln t - \frac{b}{t}\right) + \mathcal{O}(t^{-5/2} \ln^2 t),
\end{aligned}$$

where a and b are the arbitrary constants. All higher-order terms are uniquely determined. The domain of asymptotic freedom is two-dimensional.

Another way to find \mathcal{D}_{AF} Here we give an alternative method that can be useful to determine the domain \mathcal{D}_{AF} of asymptotic freedom when the origin is an isolated fixed point (possibly after a suitable projection in parameter space). We define the radius ρ in parameter space as

$$\rho = \sqrt{\sum_{i=1}^N g_i^2},$$

and the radial velocity v as

$$v = \frac{d\rho}{dt} = \frac{1}{\rho} \sum_{i=1}^N g_i \beta_i.$$

Let \mathcal{D} denote a domain in parameter space and $\overline{\mathcal{D}}$ its closure. A theory is asymptotically free in \mathcal{D}_{AF} if the origin belongs to $\overline{\mathcal{D}_{\text{AF}}}$ (but not \mathcal{D}_{AF}) and every trajectory passing through \mathcal{D}_{AF} remains in \mathcal{D}_{AF} and flows to the origin in the ultraviolet limit. The trajectories that satisfy $v < 0$ *asymptotically* for $t \rightarrow \infty$ in a neighborhood of the origin belong to \mathcal{D}_{AF} . The Zimmermann trajectories, in particular, belong to \mathcal{D}_{AF} .

When the origin $g = 0$ is an isolated fixed point \mathcal{D}_{AF} can be also characterized as follows:

- i) find the domain \mathcal{D} around the origin where $v < 0$;
- ii) call $\partial_0 \mathcal{D}$ the boundary of \mathcal{D} minus the origin, and consider the trajectories crossing it;
- iv) if all such trajectories enter \mathcal{D} , then $\mathcal{D}_{\text{AF}} = \mathcal{D}$; if not, \mathcal{D}_{AF} is \mathcal{D} minus the trajectories leaving \mathcal{D} through the boundary $\partial_0 \mathcal{D}$.

Indeed, the remaining trajectories cannot leave \mathcal{D} , so they must flow to the origin, because it is the unique fixed point.

The condition $v < 0$ is necessary, but not sufficient, because some trajectories intersecting \mathcal{D} can cross its boundary and run away, instead of flowing to the origin. The good feature of \mathcal{D} is that it can be easily determined, but \mathcal{D}_{AF} is only a subset of \mathcal{D} .

Now, observe that \mathcal{D} depends on the parametrization of the couplings, while \mathcal{D}_{AF} of course does not. Call $h_i(g)$ a reparametrization of the couplings and \mathcal{D}_h the domain where the velocity

$$v_h \equiv \frac{\sum_{i=1}^N h_i \beta_{h_i}}{\sqrt{\sum_{j=1}^N h_j^2}}$$

is negative in a neighborhood of the origin. An efficient way to estimate \mathcal{D}_{AF} (and in most cases determine it) is to take the intersection of the \mathcal{D}_h 's, for all reparametrizations h .

We illustrate this method in the g_2 - g_4 subsystem of the Dirac model in the large N limit, using one-loop truncated beta functions. Observe that the origin is the unique fixed point. We have

$$\rho v = g_2 \beta_2 + g_4 \beta_4 = a g_2 (g_2^2 + 3g_4^2).$$

The domain \mathcal{D} is just $g_2 < 0$. Consider now the one-parameter family of reparametrizations

$$h_2 = g_2 + \alpha g_4, \quad h_4 = \alpha g_2 + g_4.$$

We find

$$\rho_h v_h = \frac{g_2^3}{r^3} [r(1 + \alpha^2)(3 + r^2) + 2\alpha(1 + 3r^2)],$$

where $r = g_2/g_4$. Now we study the condition $v_h < 0$, knowing that g_2 must be negative. Varying α to obtain the best result, we find

$$|r| < 1, \quad \text{i.e.} \quad |g_2| < |g_4|,$$

which together with $g_2 < 0$ gives our best estimate of the domain \mathcal{D}_{AF} .

We can check this estimate using the explicit one-loop solution (5.17). We see that inside \mathcal{D} only the trajectories with $|g_2| \leq |g_4|$ flow to the origin. All others cross the boundary $\partial_0 \mathcal{D}$ (that is the line $g_2 = 0$), enter the region with $v > 0$ and run away. Thus, \mathcal{D}_{AF} is given by $g_2 < 0$, $|g_2| \leq |g_4|$. We conclude that our method gives an accurate estimate of \mathcal{D}_{AF} , since it misses only its boundary, namely the trajectories with $|g_2| = |g_4|$.

6 Zimmermann trajectories and hidden symmetries

In the previous section we have seen that the Zimmermann trajectories play an important role in the study of the domain of asymptotic freedom. Some such trajectories (see for example (5.19)) involve only rational coefficients, others very complicated irrational numbers. Normally, rational trajectories appear when the theory has more symmetries. In this section we point out that the existence of rational Zimmermann trajectories appears to be a general feature of high-energy Lorentz violating four fermion models.

First, we briefly recall Zimmermann's "reductions of couplings" [5, 9]. Assume that a theory has couplings λ_I , $I = 1, \dots, N$. Zimmermann's idea is to parametrize the couplings in terms of a smaller set of independent parameters α_j , $j = 1, \dots, M < N$. Write $\lambda_I = \lambda_I(\alpha_j)$. Consistence with the renormalization group demands

$$\beta_I = \frac{\partial \lambda_I}{\partial \alpha_j} \beta_j.$$

The most interesting case is $N = 2$, $M = 1$. Normally, if the one-loop beta functions are quadratic in the couplings, as in our case, Zimmermann's equations admit two power-series solutions of the form

$$\bar{\lambda} = c\alpha + \alpha \sum_{n=1}^{\infty} d_n \alpha^n,$$

where c and d_k are calculable, generically irrational, numbers. If the solution exists at one loop (namely, if c is real), then it exists to all orders. The most general solution to Zimmermann's equations is not analytic, but has the form

$$\bar{\lambda}' = \bar{\lambda} + \sum_{m,n=1}^{\infty} d_{mn} \alpha^{m\xi+n},$$

where ξ is typically irrational, d_{11} is arbitrary and the other coefficients d_{mn} are uniquely determined once d_{11} is given.

In special situations, such as when the “reduced” model has additional symmetries, the power-series solution $\bar{\lambda}$ shrinks to the monomial $c\alpha$, with a rational coefficient c , and $\bar{\lambda}'$ becomes analytic.

For example, the (Lorentz invariant) model of a spinor ψ and a pseudoscalar field A interacting with the Lagrangian

$$\mathcal{L}_I = igA\bar{\psi}\gamma_5\psi - \frac{\lambda}{4!}A^4$$

admits reductions

$$\lambda' = \frac{1}{3}(1 \pm \sqrt{145})g^2 + d_1g^4 + \dots + d_{11}g^{\frac{2}{5}\sqrt{145}+2} + \dots$$

On the other hand, the massless model with interaction

$$\mathcal{L}_I = g\bar{\psi}(A + i\gamma_5 B)\psi - \frac{\lambda}{2}(A^2 + B^2)^2$$

admits the rational reduction

$$\lambda = g^2, \tag{6.1}$$

which reveals the existence of a symmetry. Indeed, when the couplings are related by formula (6.1) we have the supersymmetric Wess-Zumino model.

We now analyze Zimmermann's trajectories at one loop in some of our Lorentz violating four fermion models. Consider again the electroweak model. We look for RG trajectories where all couplings are proportional to one another: $g_{1L} = ax$, $g_{2L} = bx$, $g_R = cx$, $g_{LR} = dx$ and $\lambda = ex$. The constants a , b , c , d and e can be worked out matching the beta functions (4.4). The absolute value of b_L (or b_R) can be set to 1 rescaling the space coordinates. On the other hand, the ratio b_L/b_R is free and does not run at one loop.

We choose $|b_L| = |b_R| = 1$ and consider the cases $s = \pm 1$. We find only three real solutions. One of them is just $g_{1L} = g_R = g_{LR} = \lambda = 0$, with only g_{2L} non-vanishing. Of the other two solutions, only one has rational coefficients, and reads

$$g_{1L} = \frac{\lambda}{5}, \quad g_{2L} = \lambda, \quad g_R = \frac{6}{5}\lambda, \quad g_{LR} = \frac{5+3s}{10}\lambda. \tag{6.2}$$

If we choose $|b_L| \neq |b_R|$ we generically do not find rational trajectories.

The existence of a trajectory with rational coefficients is unexpected and offers evidence that the reduced model might have hidden symmetries. Its Lagrangian reads

$$\begin{aligned} \mathcal{L}_{\text{EWred}} = & L^{\dagger i} \left(\hat{\partial} + \sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) L^i + e_R^{\dagger i} \left(\hat{\partial} - s\sigma \cdot \bar{\partial} \frac{\bar{\partial}^2}{\Lambda_L^2} \right) e_R - \frac{g}{\Lambda_L^2} (L^{\dagger i} e_R) (e_R^{\dagger} L^i) \\ & - \frac{g}{10\Lambda_L^2} (L^{\dagger i} L^i)^2 - \frac{g}{2\Lambda_L^2} (L^{\dagger i} L^j) (L^{\dagger j} L^i) - \frac{3g}{5\Lambda_L^2} (e_R^{\dagger} e_R)^2 - \frac{(5+3s)g}{10\Lambda_L^2} (L^{\dagger i} L^i) (e_R^{\dagger} e_R), \end{aligned}$$

where $g = -\lambda > 0$, and its one-loop beta function is

$$\beta_g = -\frac{5g^2}{(6\pi)^2}.$$

If the hidden symmetry is simple, we expect that the relations (6.2) are preserved by higher-loop corrections. However, this is not a necessary requirement for a hidden symmetry.

The rational RG trajectory exists also in the $U(N_L) \times U(N_R)$ model for $|b_L| = |b_R| = 1$. We find

$$g_{1L} = g_{1R} = -\frac{N_L + N_R - 2}{2N_L + 2N_R - 1}g, \quad g_{2L} = g_{2R} = g_{2LR} = -g, \quad g_{1LR} = -\frac{2N_L + 2N_R + 3s - 1}{2(2N_L + 2N_R - 1)}g,$$

the beta function being

$$\beta_g = -\frac{g^2}{36\pi^2} (2N_L + 2N_R - 1).$$

Curiously, the model of N Dirac fermions in the large N limit admits *only* rational trajectories. We have already studied the g_2 - g_4 subset in the previous section. In the $g_{5,6}$ - $f_{1,2}$ subset we find 15 rational trajectories:

$$\begin{aligned} g_5 &= (-2, \pm 1, 1, -1, -5, -2, -2, -7/5)g_6, \\ f_1 &= (0, 0, \pm 1/2, \pm 1/2, \pm 3/2, \pm 1, \pm 1/2, \pm 1/10)g_6, \\ f_2 &= (0, 0, 0, \pm 1, \pm 3, \pm 3/2, \pm 3/2, 0)g_6. \end{aligned}$$

One such trajectory ($g_5 = g_6 = 2f_1$, $f_2 = 0$) gives the Lorentz violating ‘‘Gross-Neveu’’ model [10], whose interaction reads

$$\frac{\lambda}{2\Lambda_L^2} (\bar{\psi}^i \psi^i)^2 = \frac{\lambda}{\Lambda_L^2} (\ell_1^{\dagger i} \ell_1^j) (\ell_2^{\dagger i} \ell_2^j) + \frac{\lambda}{\Lambda_L^2} (\ell_1^{\dagger i} \ell_2^j) (\ell_2^{\dagger i} \ell_1^j) + \frac{\lambda}{2\Lambda_L^2} \left[(\ell_1^{Ti} \varepsilon \ell_2^i) (\ell_1^{Tj} \varepsilon \ell_2^j) + (\ell_1^{\dagger i} \varepsilon \ell_2^{*i}) (\ell_1^{\dagger j} \varepsilon \ell_2^{*j}) \right],$$

which is renormalizable by the same argument used in (4.2). In some sense, this is an example of ‘‘hidden symmetry’’ associated with the rational trajectory.

7 Conclusions

If Lorentz symmetry is violated at high energies, then the Standard Model admits a ultraviolet completion that is renormalizable despite it contains four fermion vertices at the fundamental level. In this scenario, four fermion models play a key role, because all other interactions, being super-renormalizable, disappear at energies much higher than the scale of Lorentz violation. In this paper we have studied the one-loop renormalization of CPT-invariant Lorentz violating four fermion models and their RG flows.

We have first considered the most general case, working out formulas for the beta functions, and then analyzed particular models in detail. We have formulated a method to determine the domain of asymptotic freedom expanding the running couplings around the free fixed point. We emphasize that if the four fermion model (1.3) is asymptotically free then the entire Lorentz violating Standard Model is.

Moreover, we found that the RG flow admits a number of special “rational” trajectories that, in the spirit of Zimmermann’s reduction of couplings, might hide some new symmetries.

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