

Weighted Power Counting And Lorentz Violating Gauge Theories.

II: Classification

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Abstract

We classify the local, polynomial, unitary gauge theories that violate Lorentz symmetry explicitly at high energies and are renormalizable by weighted power counting. We study the structure of such theories and prove that renormalization does not generate higher time derivatives. We work out the conditions to renormalize vertices that are usually non-renormalizable, such as the two scalar-two fermion interactions and the four fermion interactions. A number of four dimensional examples are presented.

1 Introduction

Lorentz symmetry is a fundamental assumption behind the Standard Model of particle physics. Experimental bounds on the parameters of the Lorentz violating Standard-Model extension [1] are often very precise [2]. Nevertheless, several authors, inspired by different considerations, have suggested that Lorentz symmetry and CPT could be broken at very high energies [3]. The problem of Lorentz violation has attracted a lot of interest, in cosmology, astrophysics, high-energy physics. If Lorentz symmetry were not exact our understanding of Nature would change considerably.

We can imagine that the Standard Model is corrected by Lorentz violating terms of higher dimensions, multiplied by inverse powers of a scale Λ_L , which can be understood as the scale of Lorentz violation. If Λ_L is sufficiently large, the corrected model can be organized so that it agrees with all present experimental data, yet it predicts violations of Lorentz symmetry starting from energies $\sim \Lambda_L$.

If we do not assume exact Lorentz invariance at arbitrarily high energies, yet demand locality and unitarity, several theories that are not renormalizable by the usual power counting become renormalizable in the framework of a “weighted power counting” [4], which assigns different weights to space and time. The large momentum behavior of propagators is improved by quadratic terms containing higher space derivatives. The set of vertices is arranged so that no higher time derivatives are generated by renormalization, in agreement with (perturbative) unitarity. Scalar and fermion theories of this type have been studied in [4, 5]. In ref. [6], to which we refer as “paper I” from now on, the basic properties of Lorentz violating gauge theories have been derived. Here we give an exhaustive classification of gauge theories, investigate their structure and study a number of four dimensional examples.

We search for theories that are local and polynomial, free of infrared divergences in the Feynman diagrams, and renormalizable by weighted power counting. To avoid the presence of certain spurious subdivergences, originated by the peculiar form of the gauge-field propagator, spacetime is split into space and time and other restrictions are imposed.

The paper is organized as follows. In section 2 we review the weighted power counting. In section 3 we study the structure of renormalizable theories containing gauge fields and matter. We work out the conditions for polynomiality and renormalizability, and prove that higher time derivatives are absent. In section 4 we derive sufficient conditions for the absence of spurious subdivergences. In section 5 we study the conditions to renormalize vertices that are usually non-renormalizable, in particular the two scalar-two fermion interactions and the four fermion interactions, and illustrate a number of four dimensional examples. In section 6 we consider the most general type of Lorentz violations. Section 7 contains our conclusions. In appendix A we recall the form of the gauge-field propagator and the dispersion relations. In appendix B we study the renormalizability of our theories to all orders, using the Batalin-Vilkovisky formalism.

2 Weighted power counting

In this section we review the weighted power counting criterion of refs. [4, 5] and a number of results from paper I. The simplest framework to study the Lorentz violations is to assume that the d -dimensional Lorentz group $O(1, d-1)$ is broken to a residual symmetry $O(1, \hat{d}-1) \times O(\bar{d})$. The d -dimensional spacetime manifold $M = \mathbb{R}^d$ is split into the product $\hat{M} \times \bar{M}$ of two submanifolds, a \hat{d} -dimensional submanifold $\hat{M} = \mathbb{R}^{\hat{d}}$, containing time and possibly some space coordinates, and a \bar{d} -dimensional space submanifold $\bar{M} = \mathbb{R}^{\bar{d}}$. The partial derivative ∂ is decomposed as $(\hat{\partial}, \bar{\partial})$, where $\hat{\partial}$ and $\bar{\partial}$ act on the subspaces \hat{M} and \bar{M} , respectively. Coordinates, momenta and spacetime indices are decomposed similarly. We first study renormalization in this simplified framework and later generalize our results to more general breakings (see section 6). For simplicity, we assume separate invariances under C, P and T throughout this paper.

Consider a free scalar theory with (Euclidean) lagrangian

$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\hat{\partial}\varphi)^2 + \frac{1}{2\Lambda_L^{2n-2}}(\bar{\partial}^n\varphi)^2, \quad (2.1)$$

where Λ_L is an energy scale and n is an integer > 1 . It is invariant under the weighted rescaling

$$\hat{x} \rightarrow \hat{x} e^{-\Omega}, \quad \bar{x} \rightarrow \bar{x} e^{-\Omega/n}, \quad \varphi \rightarrow \varphi e^{\Omega(\bar{d}/2-1)}, \quad (2.2)$$

where $\bar{d} = \hat{d} + \bar{d}/n$ is the ‘‘weighted dimension’’. Note that Λ_L is not rescaled.

The interacting theory is defined as a perturbative expansion around (2.1). For the purposes of renormalization, the masses and the other quadratic terms can be treated perturbatively, since the counterterms depend polynomially on them. Denote the ‘‘weight’’ of an object \mathcal{O} by $[\mathcal{O}]$ and assign weights to coordinates, momenta and fields as follows:

$$[\hat{x}] = -1, \quad [\bar{x}] = -\frac{1}{n}, \quad [\hat{\partial}] = 1, \quad [\bar{\partial}] = \frac{1}{n}, \quad [\varphi] = \frac{\bar{d}}{2} - 1, \quad (2.3)$$

while Λ_L is weightless. The lagrangian terms of weight \bar{d} are strictly renormalizable, those of weights smaller than \bar{d} super-renormalizable and those of weights greater than \bar{d} non-renormalizable. The weighted power counting criterion amounts to demand that the theory contains no parameter of negative weight. The considerations just recalled are easily generalized to fermions, whose weight is $(\bar{d}-1)/2$.

The gauge field $A_\mu = A_\mu^a T^a$, with T^a anti-Hermitian, is decomposed as $A = (\hat{A}, \bar{A})$. The covariant derivative

$$D = (\hat{D}, \bar{D}) = (\hat{\partial} + g\hat{A}, \bar{\partial} + g\bar{A}) \quad (2.4)$$

induces the weight assignments

$$[g\hat{A}] = [\hat{D}] = 1, \quad [g\bar{A}] = [\bar{D}] = \frac{1}{n},$$

where g is the gauge coupling. On the other hand, the weight- \bar{d} kinetic term $\sim (\hat{\partial}\hat{A})^2$ gives $[\hat{A}] = \bar{d}/2 - 1$, so $[g] = 2 - \bar{d}/2$. The field strength is split as

$$\hat{F}_{\mu\nu} \equiv F_{\hat{\mu}\hat{\nu}}, \quad \tilde{F}_{\mu\nu} \equiv F_{\tilde{\mu}\tilde{\nu}}, \quad \bar{F}_{\mu\nu} \equiv F_{\bar{\mu}\bar{\nu}}. \quad (2.5)$$

We find

$$[\hat{A}] = \frac{\bar{d}}{2} - 1, \quad [\bar{A}] = \frac{\bar{d}}{2} - 2 + \frac{1}{n}, \quad [\hat{F}] = \frac{\bar{d}}{2}, \quad [\tilde{F}] = \frac{\bar{d}}{2} - 1 + \frac{1}{n}, \quad [\bar{F}] = \frac{\bar{d}}{2} - 2 + \frac{2}{n}. \quad (2.6)$$

In the presence of gauge interactions the renormalizable theories are still those that do not contain parameters of negative weights. To single out the super-renormalizable theories we can refine this requirement, demanding that no parameters have weights smaller than some non-negative constant χ . Indeed, if that happens Feynman diagrams are certainly multiplied by coefficients of weights greater than or equal to χ , so no new counterterms are turned on by renormalization. Applying the refined requirement to the gauge coupling g we find

$$0 \leq \chi \leq 2 - \bar{d}/2. \quad (2.7)$$

In particular, we must have $\bar{d} \leq 4$.

It is convenient to write the gauge-field action

$$\mathcal{S}_0 = \int d^d x (\mathcal{L}_Q + \mathcal{L}_I) \equiv \mathcal{S}_Q + \mathcal{S}_I, \quad (2.8)$$

as the sum of two contributions \mathcal{S}_Q and \mathcal{S}_I : \mathcal{S}_Q collects the gauge-invariant quadratic terms of weight $\leq \bar{d}$, constructed with two field strengths and possibly covariant derivatives, while \mathcal{S}_I collects the vertex terms of weights $\leq \bar{d} - \chi$, constructed with at least three field strengths and possibly covariant derivatives.

Up to total derivatives the quadratic part \mathcal{L}_Q of the lagrangian reads (in the Euclidean framework)

$$\mathcal{L}_Q = \frac{1}{4} \left\{ \hat{F}_{\mu\nu}^2 + 2F_{\hat{\mu}\hat{\nu}}\eta(\bar{\Upsilon})F_{\tilde{\mu}\tilde{\nu}} + F_{\tilde{\mu}\tilde{\nu}}\tau(\bar{\Upsilon})F_{\bar{\mu}\bar{\nu}} + \frac{1}{\Lambda_L^2} (D_{\hat{\rho}}F_{\tilde{\mu}\tilde{\nu}})\xi(\bar{\Upsilon})(D_{\bar{\rho}}F_{\bar{\mu}\bar{\nu}}) \right\}. \quad (2.9)$$

The proof can be found in paper I. Here $\bar{\Upsilon} \equiv -\bar{D}^2/\Lambda_L^2$ and η , τ and ξ are polynomials of degrees $n - 1$, $2n - 2$ and $n - 2$, respectively. We have expansions

$$\eta(\bar{\Upsilon}) = \sum_{i=0}^{n-1} \eta_{n-1-i} \bar{\Upsilon}^i, \quad [\eta_j] = \frac{2j}{n}, \quad (2.10)$$

and similar, where η_i are dimensionless constants of non-negative weights.

The free action is positive definite if and only if

$$\eta > 0, \quad \tilde{\eta} \equiv \eta + \frac{\bar{k}^2}{\Lambda_L^2} \xi > 0, \quad \tau > 0, \quad (2.11)$$

where now η , τ and ξ are functions of \bar{k}^2/Λ_L^2 . Furthermore, we assume

$$\eta_0 > 0, \quad \tau_0 > 0, \quad \tilde{\eta}_0 = \eta_0 + \xi_0 > 0, \quad \eta_{n-1} > 0, \quad \tau_{2n-2} > 0. \quad (2.12)$$

The first three conditions ensure that the propagators have the best UV behaviors. The other two conditions, together with

$$d \geq 4, \quad (2.13)$$

ensure that the Feynman diagrams are free of IR divergences at non-exceptional external momenta, despite the fact that the gauge fields are massless. The reason is that, under the mentioned assumptions, the IR behavior of Feynman diagrams is governed by the low-energy theory

$$\mathcal{L}_{\text{IR}} = \frac{1}{4} [(F_{\mu\nu}^a)^2 + 2\eta_{n-1}(F_{\mu\nu}^a)^2 + \tau_{2n-2}(F_{\mu\nu}^a)^2], \quad (2.14)$$

which has an ordinary power counting.

The BRST symmetry [7] coincides with the usual one,

$$\begin{aligned} sA_\mu^a &= D_\mu^{ab}C^b = \partial_\mu C^a + gf^{abc}A_\mu^b C^c, & sC^a &= -\frac{g}{2}f^{abc}C^b C^c, \\ s\bar{C}^a &= B^a, & sB^a &= 0, & s\psi^i &= -gT_{ij}^a C^a \psi^j, \end{aligned}$$

etc., with the weight assignments

$$[C] = [\bar{C}] = \frac{\bar{d}}{2} - 1, \quad [s] = 1, \quad [B] = \frac{\bar{d}}{2}. \quad (2.15)$$

We choose the gauge-fixing

$$\mathcal{L}_{\text{gf}} = s\Psi, \quad \Psi = \bar{C}^a \left(-\frac{\lambda}{2} B^a + \mathcal{G}^a \right), \quad \mathcal{G}^a \equiv \hat{\partial} \cdot \hat{A}^a + \zeta(\bar{v}) \bar{\partial} \cdot \bar{A}^a, \quad (2.16)$$

where λ is a dimensionless, weightless constant, $\bar{v} \equiv -\bar{\partial}^2/\Lambda_L^2$ and ζ is a polynomial of degree $n-1$. Compatibly with (2.12) we assume

$$\zeta > 0, \quad \zeta_0 > 0, \quad \zeta_{n-1} > 0. \quad (2.17)$$

The total gauge-fixed action is

$$\mathcal{S} = \int d^d x (\mathcal{L}_Q + \mathcal{L}_I + \mathcal{L}_{\text{gf}}) \equiv \mathcal{S}_0 + \mathcal{S}_{\text{gf}}. \quad (2.18)$$

The propagator is reported in appendix A, together with the dispersion relations.

For the purposes of renormalization, we can treat the weightful parameters η_i , τ_i , ξ_i and ζ_i , $i > 0$, perturbatively, because the divergent parts of Feynman diagrams depend polynomially on

them. In this framework, the propagators we use in the high-energy analysis of the diagrams are (A.2)-(A.4) with the replacements

$$\eta \rightarrow \eta_0 \left(\frac{\bar{k}^2}{\Lambda_L^2} \right)^{n-1}, \quad \tau \rightarrow \tau_0 \left(\frac{\bar{k}^2}{\Lambda_L^2} \right)^{2(n-1)}, \quad \xi \rightarrow \xi_0 \left(\frac{\bar{k}^2}{\Lambda_L^2} \right)^{n-2}, \quad \zeta \rightarrow \zeta_0 \left(\frac{\bar{k}^2}{\Lambda_L^2} \right)^{n-1},$$

every other term being treated as a vertex. Intermediate masses can be added to the denominators, to avoid IR problems, and removed immediately after calculating the divergent parts.

We recall that $P_{k,n}(\hat{p}, \bar{p})$ is a weighted polynomial in \hat{p} and \bar{p} , of degree k , where k is a multiple of $1/n$, if $P_{k,n}(\xi^n \hat{p}, \xi \bar{p})$ is a polynomial of degree kn in ξ . A propagator is regular if it is the ratio

$$\frac{P_r(\hat{k}, \bar{k})}{P'_{2s}(\hat{k}, \bar{k})} \tag{2.19}$$

of two weighted polynomials of degrees r and $2s$, where r and s are integers, such that the denominator $P'_{2s}(\hat{k}, \bar{k})$ is non-negative (in the Euclidean framework), non-vanishing when either $\hat{k} \neq 0$ or $\bar{k} \neq 0$ and has the form

$$P'_s(\hat{k}, \bar{k}) = \hat{\omega}(\hat{k}^2)^s + \bar{\omega}(\bar{k}^2)^{ns} + \dots, \tag{2.20}$$

with $\hat{\omega} > 0$, $\bar{\omega} > 0$, where the dots collect the terms $(\hat{k}^2)^{j-m}(\bar{k}^2)^{mn}$ with $j < s$, $0 \leq m \leq j$, and $j = s$, $0 < m < s$. The regularity conditions ensure that the derivatives with respect to \hat{k} improve the large- \bar{k} behavior (because $\bar{\omega} \neq 0$), besides the large- \hat{k} and overall ones, and the derivatives with respect to \bar{k} improve the large- \hat{k} behavior (because $\hat{\omega} \neq 0$), besides the large- \bar{k} and overall ones. For this reason, the \hat{k} -subdivergences are local in \bar{k} and the \bar{k} -subdivergences are local in \hat{k} . The \hat{k} -subintegrals and the \bar{k} -subintegrals, which cannot behave worse than the \hat{k} - \bar{k} -integrals, are automatically cured by the counterterms that subtract the overall divergences of the \hat{k} - \bar{k} -integrals. Such counterterms are, for example, the first terms of the “weighted Taylor expansion” around vanishing external momenta [4].

A propagator that does not satisfy (2.19) can generate spurious ultraviolet subdivergences in Feynman diagrams when \hat{k} tends to infinity at \bar{k} fixed, or viceversa. The gauge and ghost propagators (A.2), (A.3) are regular at non-exceptional momenta, because the positivity conditions (2.11) and (2.17) ensure that the denominators are positive-definite in the Euclidean framework. Moreover, the conditions (2.12) ensure that all such propagators but $\langle \bar{A}\bar{A} \rangle$ satisfy (2.19)-(2.20) in the Feynman gauge (A.4). Instead, $\langle \bar{A}\bar{A} \rangle$ is regular when \bar{k} tends to infinity at \hat{k} fixed, but not when \hat{k} tends to infinity at \bar{k} fixed, where it behaves like $\sim 1/\hat{k}^2$. To ensure that no spurious subdivergence is generated by the \hat{k} -subintegrals, a more careful analysis must be performed, to which we devote section 4. The result is that the sufficient conditions to ensure the absence of spurious subdivergences include

$$\hat{d} = 1, \quad d = \text{even} \quad n = \text{odd}, \tag{2.21}$$

plus other restrictions stated at the end of section 4. In particular, spacetime is split into space and time. In section 6 we prove that, because of the spurious subdivergences, more general type of Lorentz violations ($\hat{d} > 1$) are disfavored.

The absence of spurious subdivergences ensures the locality of counterterms. Consider a diagram G_r equipped with the subtractions that take care of its diverging proper subdiagrams. Differentiating G_r a sufficient number of times with respect to any components \hat{p}_i, \bar{p}_i of the external momenta p_i , we can arbitrarily reduce the overall degree of divergence and eventually produce a convergent integral. Therefore, overall divergences are polynomial in all components of the external momenta.

3 Structure of renormalizable theories

In this section we investigate renormalizable and super-renormalizable theories in detail. We study the conditions for renormalizability and polynomiality, and investigate the time-derivative structure. In section 4 we study the spurious subdivergences, while section 5 is devoted to explicit examples, mainly four dimensional.

We know that the theories contain only parameters of weights $\geq \chi$, where χ satisfies (2.7). Call λ_i the coupling multiplying the i -th vertex belonging to the physical sector and denote the number of its external legs by n_i . Clearly, $n_i \geq 3$ and $[\lambda_i] \geq \chi$. By polynomiality, the number of physical vertices is finite, so we can take $\chi \equiv \min_i [\lambda_i]$. Define

$$\kappa \equiv \min_i \frac{[\lambda_i]}{n_i - 2}.$$

Since the gauge coupling multiplies three-leg vertices, we have

$$[\lambda_i] \geq (n_i - 2)\kappa \quad \forall i, \quad \text{and } 0 \leq \kappa \leq 2 - \frac{\hat{d}}{2}, \quad (3.1)$$

and $\chi > 0$ if and only if $\kappa > 0$. Introduce a coupling \bar{g} of weight κ and write $\lambda_i = \bar{\lambda}_i \bar{g}^{n_i - 2}$. Then (3.1) ensures $[\bar{\lambda}_i] \geq 0$. The theory can be reformulated in the “ $1/\bar{\alpha}$ form” ($\bar{\alpha} = \bar{g}^2$), namely as

$$\mathcal{L}_{1/\bar{\alpha}} = \frac{1}{\bar{\alpha}} \bar{\mathcal{L}}_r(\bar{g}A, \bar{g}\varphi, \bar{g}\psi, \bar{g}\bar{C}, \bar{g}C, \bar{\lambda}), \quad (3.2)$$

where φ and ψ are matter fields (scalars and fermions, respectively) and the reduced lagrangian $\bar{\mathcal{L}}_r$ depends polynomially on \bar{g} and the $\bar{\lambda}$'s. The gauge coupling can be parametrized as $g = \bar{g}\rho$, where ρ has a non-negative weight and is included in the set of the $\bar{\lambda}$'s. A generic vertex of (3.2) has the structure

$$\bar{\lambda}_i \bar{g}^{n_i - 2} \hat{\partial}^k \bar{\partial}^m \hat{A}^p \bar{A}^q \bar{C}^r C^r \varphi^s \bar{\psi}^t \psi^t, \quad (3.3)$$

where $n_i = p + q + 2r + s + 2t$ and p, q, r, k, m, s and t are integers. Formula (3.3) and analogous expressions in this paper are meant “symbolically”, which means that we pay attention to the

field- and derivative-contents of the vertices, but not where the derivatives act and how Lorentz, gauge and other indices are contracted.

Every counterterm generated by (3.2) fits into the structure (3.2). Indeed, consider a L -loop diagram with E external legs, I internal legs and v_i vertices of type i . The leg-counting gives $\sum_i n_i v_i = E + 2I = E + 2(L + V - 1)$, so the diagram is multiplied by a product of couplings

$$\bar{g}^{\sum_i (n_i - 2)v_i} \prod_i \bar{\lambda}_i^{v_i} = \bar{\alpha}^L \bar{g}^{E-2} \prod_i \bar{\lambda}_i^{v_i}. \quad (3.4)$$

We see that a \bar{g}^{E-2} factorizes, as expected. Moreover, each loop order carries an additional weight of at least 2κ .

When $\kappa = 2 - \bar{d}/2$ we can take $\bar{g} = g$, which gives the $1/\alpha$ theories considered in paper I. They have a lagrangian of the form

$$\mathcal{L}_{1/\alpha} = \frac{1}{\alpha} \mathcal{L}_r(gA, g\varphi, g\psi, g\bar{C}, gC, \lambda). \quad (3.5)$$

The class (3.2) is much richer than the class (3.5), yet it does not cover the most general case.

To move a step forward towards the most general class of theories, it is useful to show how to gauge scalar-fermion theories. Express the matter theory in $1/\bar{\alpha}$ form, namely

$$\mathcal{L}_{\text{matter}} = \frac{1}{\bar{\alpha}} \bar{\mathcal{L}}_{sf}(\bar{g}\varphi, \bar{g}\psi, \bar{\lambda}_{sf}). \quad (3.6)$$

We assume that $[g] \geq [\bar{g}]$ and write $g = \bar{g}\rho$, with $[\rho] \geq 0$. In this way, the gauge interactions can be switched off letting ρ tend to zero. Covariantize the derivatives contained in (3.6) and add the $1/\alpha$ pure gauge theory, plus extra terms allowed by the weighted power counting. We obtain a mixed theory of the form

$$\mathcal{L} = \frac{1}{\alpha} \mathcal{L}_g(gA, g\bar{C}, gC, \lambda_g) + \frac{1}{\bar{\alpha}} \bar{\mathcal{L}}_{sf}(\bar{g}\varphi, \bar{g}\psi, \bar{\lambda}_{sf}) + \frac{1}{\bar{\alpha}} \Delta\mathcal{L}(gA, g\bar{C}, gC, \bar{g}\varphi, \bar{g}\psi, \lambda). \quad (3.7)$$

Here $\Delta\mathcal{L}$ contains both the terms necessary to covariantize $\bar{\mathcal{L}}_{sf}$ and the mentioned extra terms. Consider a diagram G with E external legs and L loops. Using the $\bar{g}\text{-}\rho$ parametrization and repeating the argument that leads to (3.4) we find that G is multiplied by $\bar{\alpha}^L \bar{g}^{E-2}$, so it agrees with the structure (3.7). On the other hand, every vertex of \mathcal{L}_g used to construct G provides at least two internal legs. Therefore, every external A -, \bar{C} - and C -leg of G is multiplied by at least one power of ρ . This proves that the structure (3.7) is renormalizable. The theory is polynomial if $[\bar{g}\varphi], [\bar{g}\psi] > 0$, namely $\bar{d} > 2 - 2\kappa$ if scalar fields are present, $\bar{d} > 1 - 2\kappa$ if the matter sector contains only fermions.

Now we are ready to introduce the most general class of theories, where different fields can carry different \bar{g} 's. Call \bar{g}_i , $i = 1, 2, 3$, the ones of vectors, fermions and scalars, respectively¹.

¹A more general situation where different subsets of fields with the same spin have different \bar{g} 's is also possible. This generalization is straightforward and left to the reader.

As in (3.2), \bar{g}_1 needs not coincide with g . Call $\bar{\gamma}_k$, $k = 1, 2, 3$, the coupling of minimum weight between \bar{g}_i and \bar{g}_j , where $k \neq i, j$. Call \bar{g} the coupling of minimum weight among the \bar{g}_i 's. Define $\bar{\alpha}_i = \bar{g}_i^2$, $\bar{a}_i = \bar{\gamma}_i^2$.

The lagrangian has the weight structure

$$\begin{aligned} \mathcal{L} = & \frac{1}{\bar{\alpha}_1} \mathcal{L}_1(\bar{g}_1 A) + \frac{1}{\bar{\alpha}_2} \mathcal{L}_2(\bar{g}_2 \psi) + \frac{1}{\bar{\alpha}_3} \mathcal{L}_3(\bar{g}_3 \varphi) + \frac{1}{\bar{a}_3} \mathcal{L}_{12}(\bar{g}_1 A, \bar{g}_2 \psi) \\ & + \frac{1}{\bar{a}_2} \mathcal{L}_{13}(\bar{g}_1 A, \bar{g}_3 \varphi) + \frac{1}{\bar{a}_1} \mathcal{L}_{23}(\bar{g}_2 \psi, \bar{g}_3 \varphi) + \frac{1}{\bar{\alpha}} \mathcal{L}_{123}(\bar{g}_1 A, \bar{g}_2 \psi, \bar{g}_3 \varphi). \end{aligned} \quad (3.8)$$

In A we collectively include also ghosts and antighosts. Any other parameters λ contained in (3.8) must have non-negative weights. The \bar{g}_i -factors appearing in formula (3.8) are mere tools to keep track of the weight structure. For example, instead of $\bar{g}_2 \psi$ we can have any $\bar{g}_i \psi$, as long as $[\bar{g}_i] \geq [\bar{g}_2]$. Similarly, the denominators $1/\bar{\alpha}_i$, $1/\bar{a}_i$ and $1/\bar{\alpha}$ are devices that lower the weights of appropriate amounts.

Every \mathcal{L} on the right-hand side of (3.8) must be polynomial in the fields and parameters. Moreover, we assume

$$[g] \geq [\bar{g}_1], \quad [g\bar{g}_1] \geq [\bar{g}_2^2], \quad [g\bar{g}_1] \geq [\bar{g}_3^2]. \quad (3.9)$$

These inequalities ensure that (3.8) is compatible with the covariant structure. Indeed, because of (3.9), the vertices generated by covariant derivatives are multiplied by factors of weights not smaller than the ones appearing in (3.8), so they can fit into one of the structures (3.8). Observe that (3.9) implies $[g] \geq [\bar{g}_i]$ for every i .

Again, it is easy to prove that the structure (3.8) is preserved by renormalization. Assume, for example, that $[\bar{g}_1] \geq [\bar{g}_2] \geq [\bar{g}_3]$ (the other cases can be treated symmetrically, because (3.9) plays no role here) and write $\bar{g}_1 = \rho\sigma\bar{g}$, $\bar{g}_2 = \sigma\bar{g}$, $\bar{g}_3 = \bar{g}$, with $[\rho] \geq 0$, $[\sigma] \geq 0$. In the parametrization \bar{g} - ρ - σ the \bar{g} -powers in front of counterterms can be counted as in (3.4). Moreover, vertices contain a factor σ for every A - and ψ -leg, save two legs in φ -independent vertices. Since at least two legs of every vertex enter the diagrams, counterterms contain at least a factor σ for every external A - and ψ -leg. A similar argument applies to ρ -factors and external A -legs. Thus, every diagram with $L \geq 1$ loops, E external legs, E_A external A -legs and E_ψ external ψ -legs is multiplied at least by a factor $\bar{\alpha}^L \bar{g}^{E-2} \rho^{E_A} \sigma^{E_A+E_\psi}$ and therefore fits into the structure (3.8).

This argument proves also that the one-loop counterterms generated by (3.8) have the weight structure

$$\Delta_1 \mathcal{L}(\bar{g}_1 A, \bar{g}_2 \psi, \bar{g}_3 \varphi), \quad (3.10)$$

while at L loops there is an additional factor of $\bar{\alpha}^{L-1}$. Simplified versions of our theories can be obtained dropping vertices and quadratic terms of (3.8) that are not contained in (3.10), because renormalization is unable to generate them back. The quadratic terms that cannot be dropped are those that control the behavior of propagators. Of course, the simplified model must also contain the vertices related to such quadratic terms by covariantization.

Polynomiality Now we derive the conditions to have polynomiality. Consider first the physical (i.e. non gauge-fixing) sectors of the lagrangian (3.8). Apart from the factors $1/\bar{\alpha}_i$, $1/\bar{a}_i$ and $1/\bar{\alpha}$, they depend only on the products $\bar{g}_1\bar{F}$, $\bar{g}_2\psi$, $\bar{g}_3\varphi$, and their covariant derivatives, so polynomiality is ensured when these objects have positive weights. Let us focus for the moment on the gauge sector. From (2.6) we see that if $\bar{d} > 1$ the most meaningful condition is $[\bar{g}_1\bar{F}] > 0$. If instead $\bar{d} = 1$ the most meaningful condition is $[\bar{g}_1\tilde{F}] > 0$, because $\bar{F} \equiv 0$. However, because of (2.21) and (2.13) we have to concentrate on the former case. We conclude that pure gauge theories are polynomial in the physical sector if and only if

$$4 - \frac{4}{n} - 2\kappa_1 < \bar{d},$$

having written $[\bar{g}_i] = \kappa_i$. In the presence of scalars and fermions we must have

$$4 - \frac{4}{n} - 2\kappa_1 < \bar{d}, \quad 1 - 2\kappa_2 < \bar{d}, \quad 2 - 2\kappa_3 < \bar{d}. \quad (3.11)$$

Observe that (3.11) and $n \geq 2$ ensure that the weight of $\bar{g}_1\hat{A}$ is strictly positive. Thus the theory is certainly polynomial in \hat{A} . For the same reason, it is polynomial also in \bar{C} and C . On the other hand, the weight of $\bar{g}_1\bar{A}$ can be negative, because (3.11) ensures only $[\bar{g}_1\bar{A}] > -1/n$. This means that, in principle, the gauge-fixing sector can be non-polynomial. Now we show that if the tree-level gauge fixing is (2.16), then the theory is polynomial also in the gauge-fixing sector. Note that in some cases (see appendix B) the gauge-fixing sector does not preserve the simple form (2.16), but can acquire new types of vertices by renormalization.

We need to prove that beyond the tree level, in both the physical and gauge-fixing sectors, the field \bar{A} appears only in the combinations

$$g\bar{A}, \quad \bar{g}_1\hat{\partial}\bar{A}, \quad \bar{g}_1g\hat{A}\bar{A}, \quad \bar{g}_1\bar{\partial}\bar{A}, \quad \bar{g}_1g\bar{A}\bar{A}. \quad (3.12)$$

First observe that at the tree level this statement is true up to the factors $1/\bar{\alpha}_i$, $1/\bar{a}_i$ and $1/\bar{\alpha}$ appearing in (3.8). Indeed, \bar{A} appears only in the following locations: *i*) in $(\mathcal{G}^a)^2$, which contributes only to the propagator; *ii*) inside the covariant derivative \bar{D} (also in the ghost action); *iii*) inside the field strength. In case *ii*) \bar{A} is multiplied by g and gives the first term of (3.12). In case *iii*) the field strength carries an extra factor \bar{g}_1 : $\bar{g}_1\tilde{F}$ gives the second and third terms of (3.12), while $\bar{g}_1\bar{F}$ gives the fourth and fifth terms.

Next, consider an L -loop Feynman diagram G and assume that the \bar{A} -structure of the renormalized action is (3.12) up to the order $L - 1$ included, with the tree-level caveat just mentioned. The factors $1/\bar{\alpha}_i$, $1/\bar{a}_i$ and $1/\bar{\alpha}$ of (3.8) are simplified by the internal legs of G , which are at least two for every vertex. Consider the \bar{A} -external legs of G . In the first case of (3.12) the \bar{A} -leg is accompanied by a factor g and in the second case by a \bar{g}_1 and a derivative $\hat{\partial}$ acting on it. In the third case it carries a factor g (the \bar{g}_1 being left for the \hat{A} -leg, in case it is external), in the fourth

case a \bar{g}_1 and a derivative $\bar{\partial}$. In the fifth case both \bar{A} 's or just one \bar{A} can be external, with factors $\bar{g}_1 g$ and g , respectively. Therefore, diagrams and counterterms contain \bar{A} only in the combinations (3.12), so the property (3.12) is inductively promoted to all orders.

Under the conditions (3.11) $\bar{g}_1 \hat{A}$ and each combination (3.12) have positive weights. The \bar{A} external legs are always equipped with enough \bar{g}_1 - g -factors and/or derivatives to raise the weight by a finite amount. Thus, the total renormalized lagrangian is polynomial, gauge-fixing sector included.

In conclusion, recalling (2.13), (2.21) and (3.11), consistent renormalizable gauge theories with a non-trivial super-renormalizable subsector require

$$n = \text{odd}, \quad d = \text{even} \geq 4, \quad \hat{d} = 1, \quad 4 - \frac{4}{n} < \bar{d} + 2\kappa_1, \quad 1 < \bar{d} + 2\kappa_2, \quad 2 < \bar{d} + 2\kappa_3, \quad (3.13)$$

plus other restrictions summarized at the end of section 4 to ensure the absence of spurious subdivergences. Moreover, (3.9) gives

$$\kappa_1 \leq 2 - \frac{\bar{d}}{2}, \quad \kappa_{2,3} \leq 1 + \frac{\kappa_1}{2} - \frac{\bar{d}}{4}, \quad (3.14)$$

and of course we must have $\kappa_i \geq 0$.

The same argument that leads to (3.12) proves that the counterterms contain the field \hat{A} only in the combinations

$$g\hat{A}, \quad \bar{g}_1 \hat{\partial}\hat{A}, \quad \bar{g}_1 g \hat{A}\hat{A}, \quad \bar{g}_1 \bar{\partial}\hat{A}, \quad \bar{g}_1 g \hat{A}\bar{A}. \quad (3.15)$$

Again, at the tree level this statement is true up to the factors $1/\bar{\alpha}_i$, $1/\bar{\alpha}_i$ and $1/\bar{\alpha}$ appearing in (3.8).

Time-derivative structure To ensure (perturbative) unitarity it is crucial to prove that the lagrangian contains no terms with higher time derivatives. We now prove that it is so and give a complete classification of the $\hat{\partial}$ -structure.

Using the information encoded in (3.12) and (3.15) a generic lagrangian term can be schematically written as

$$\begin{aligned} & \frac{\bar{\lambda}_i}{\bar{g}'^2} (\hat{\partial} + g\hat{A})^k (\bar{\partial} + g\bar{A})^m (\bar{g}_1 \hat{\partial}\hat{A} + \bar{g}_1 g \hat{A}\hat{A})^p (\bar{g}_1 \hat{\partial}\bar{A} + \bar{g}_1 \bar{\partial}\hat{A} + \bar{g}_1 g \hat{A}\bar{A})^q \\ & (\bar{g}_1 \bar{\partial}\bar{A} + \bar{g}_1 g \bar{A}\bar{A})^h (\bar{g}_1^2 \bar{C}C)^r (\bar{g}_3 \varphi)^s (\bar{g}_2^2 \bar{\psi}\psi)^t, \end{aligned} \quad (3.16)$$

where $[\bar{\lambda}_i] \geq 0$ and \bar{g}' is the \bar{g} of minimum weight among those appearing in the vertex. We find the inequality

$$\begin{aligned} & k + \frac{m}{n} + (2p + q) \left(1 - \frac{1}{n}\right) + (p + q + h) \left(\frac{\bar{d}}{2} - 2 + \frac{2}{n} + \kappa_1\right) \\ & + r(\bar{d} - 2 + 2\kappa_1) + s\left(\frac{\bar{d}}{2} - 1 + \kappa_3\right) + t(\bar{d} - 1 + 2\kappa_2) - \bar{d} - 2\kappa' \leq 0. \end{aligned} \quad (3.17)$$

Moreover, we know that $\kappa' = [\bar{g}']$ is not larger than any of the other κ 's appearing in the inequality. Observe that every quantity between parenthesis is non-negative.

First we study the vertices, then the quadratic terms. Consider the vertices containing fermions ($t \geq 1$). We have two possibilities: *i*) $p = q = h = r = s = 0$, $t = 1$ and $k + m/n \leq 1$; or *ii*)

$$k + \frac{m}{n} + (2p + q) \left(1 - \frac{1}{n}\right) < 1.$$

Case *i*) gives no vertex with time derivatives. In case *ii*) we have immediately $k = p = 0$, $q \leq 1$. Time derivatives are contained only in terms of the form

$$X'_1 \equiv f'_1(\bar{A}, \varphi, \psi, \bar{C}, C, \bar{\partial})(\hat{\partial}\bar{A}), \quad (3.18)$$

where $\bar{\partial}$ can act anywhere.

From now on we can neglect the fermions. Consider the vertices with two or more scalars. Again, we have two cases: *iii*) $p = q = h = r = 0$, $s = 2$ and $k + m/n \leq 2$; or *iv*)

$$k + \frac{m}{n} + (2p + q) \left(1 - \frac{1}{n}\right) < 2. \quad (3.19)$$

In case *iii*) vertices can have at most one time derivative and fall in the class

$$X_1 \equiv \hat{\partial}f_1(\hat{A}, \bar{A}, \varphi, \bar{C}, C, \bar{\partial}), \quad (3.20)$$

where the $\hat{\partial}$ -and $\bar{\partial}$ -derivatives are allowed to act anywhere. In case *iv*) we must have $k \leq 1$. For $k = 1$, we have either $p = 0$, $q = 1$, $m = 0$, which is not $O(\bar{d})$ -invariant, or $p = q = 0$, which is not $O(1, \hat{d} - 1)$ -invariant. For $k = 0$ we have $p = 1$, $q = 0$, which is of the form (3.20), or $p = 0$, $q = 2$, which is of the form (3.20) or

$$X_2 \equiv f_2(\bar{A}, \varphi, \bar{C}, C, \bar{\partial})(\hat{\partial}\bar{A})(\hat{\partial}\bar{A}), \quad (3.21)$$

where only the $\bar{\partial}$ -derivatives can act anywhere.

Next, consider the vertices with one scalar. If $r \geq 1$ we have again (3.19), therefore vertices of the form (3.20) or (3.21). If $r = 0$ consider first the case $p + q + h \geq 2$. Then we have

$$k + \frac{m}{n} + (2p + q - 4) \left(1 - \frac{1}{n}\right) < 0, \quad (3.22)$$

so either $p = 1$ or $p = 0$. If $p = 1$ we can have only $k = q = 0$, which has the form (3.20). If $p = 0$ we can have $k = 1, 0$. If $k = 1$ then $q = 1$, so the vertex is of the form (3.20), (3.21) or

$$X'_2 \equiv f'_2(\bar{A}, \varphi, \bar{\partial})(\hat{\partial}^2\bar{A}). \quad (3.23)$$

If $k = 0$ then $q \leq 2$ so the vertex is of the form (3.20) or (3.21). It is easy to see that also the vertices with $s = 1$, $r = 0$ and $p + q + h < 2$ fall in the classes (3.20), (3.21) or (3.23).

Now consider the vertices with neither scalars nor fermions. Here $\kappa' = \kappa_1$. If $r \geq 1$ we have either $v)$ $p = q = h = 0, r = 1$; or $vi)$ (3.19). These cases are the same as $iii)$ and $iv)$ above, with two scalar fields replaced by ghosts. The vertices they give fall in the classes listed so far. We remain with the vertices with $s = r = t = 0$. We have the cases: $vii)$ $p + q + h > 2$; $viii)$ $p + q + h \leq 2$. In case $vii)$ we have (3.22) again, therefore $k \leq 3, 2p + q < 4$. If $k = 3$ then $q = 1$, by $O(1, \hat{d} - 1)$ invariance, but it violates (3.22). If $k = 2$ we have $p = q = 0$, while if $k = 1$ we have $q = 1, p = 0$. In either case the vertices fall in the classes (3.20), (3.23) and (3.23). If $k = 0$ then $p = 1, q = 0$, which gives (3.20), or $p = 0, q = 2$, which gives (3.20) or (3.21). It is easy to show that case $viii)$ does not produce new types of vertices with time derivatives.

Finally, the quadratic terms that do not fall in the classes (3.20), (3.21) and (3.23) have the forms

$$(\hat{\partial} \hat{A})^2, \quad \bar{C} \hat{\partial}^2 C, \quad \varphi \hat{\partial}^2 \varphi, \quad \bar{\psi} \hat{\partial} \psi, \quad (3.24)$$

as expected. Every other term is $\hat{\partial}$ -independent. We conclude, in particular, that the theory is free of higher time derivatives.

4 Absence of spurious subdivergences

In this section we derive sufficient conditions to ensure the absence of spurious subdivergences. We generalize the proof given in paper I, which was specific for $1/\alpha$ theories. We use the Feynman gauge (A.4) and the dimensional-regularization technique. We proceed by induction and assume that counterterms corresponding to diverging proper subdiagrams are included. Moreover, we assume $\hat{d} = 1, n = \text{odd}$ and that the spacetime dimension is even. We also assume that the theory does not contain vertices of type X'_1 , see 3.18), which is true in most physical applications. Other restrictions will emerge along with the analysis. The complete sets of sufficient conditions are recapitulated at the end of the section.

Consider a generic N -loop integral

$$\int \frac{d\hat{k}_1}{(2\pi)^{\hat{d}}} \int \frac{d^{\bar{d}}\bar{k}_1}{(2\pi)^{\bar{d}}} \cdots \int \frac{d\hat{k}_N}{(2\pi)^{\hat{d}}} \int \frac{d^{\bar{d}}\bar{k}_N}{(2\pi)^{\bar{d}}}, \quad (4.1)$$

with loop momenta (k_1, \dots, k_N) . We have to prove that all subintegrals, in all parametrizations (k'_1, \dots, k'_N) of the momenta, are free of subdivergences. By the inductive assumption, all divergent subintegrals

$$\prod_{j=1}^M \int \frac{d\hat{k}'_j}{(2\pi)^{\hat{d}}} \int \frac{d^{\bar{d}}\bar{k}'_j}{(2\pi)^{\bar{d}}}, \quad (4.2)$$

where $M < N$, are subtracted by appropriate counterterms. We need to consider subintegrals where some hatted integrations are missing and the corresponding barred integrations are present, and/or viceversa.

The proof given in paper I is divided in three steps: structure of integrals, \hat{k} -subintegrals and mixed subintegrals. The first step does not have to be repeated here, since it applies unchanged. It proves that we can focus on the subintegrals containing some \hat{k}'_a -integrations without the corresponding \bar{k}'_a -integrations. We generalize the second step of the proof and the third one.

\hat{k} -subintegrals Now we prove that the subintegrals over hatted components of momenta have no spurious subdivergences. More precisely, we prove, under very general assumptions, that one-dimensional integrals have no logarithmic divergences, namely their renormalization-group flow is trivial. This property ensures that using the dimensional-regularization technique, which kills the power-like divergences automatically, the \hat{k} -subintegrals are convergent.

Consider ‘‘Feynman integrals’’ in one dimension, and assume that: *i*) the propagators are regular everywhere; *ii*) when p is large they behave as $1/(p^2)^N$ times some polynomial in p , for some $N < \infty$; *iii*) they tend to a constant for $p \rightarrow 0$. In Lorentz violating gauge theories such assumptions hold with N equal to 1, but our proof is more general. Consider a diagram G with L loops, V vertices and I internal legs. Denote the loop momenta with p_i . We have an integral that for large p_i 's looks like

$$\mathcal{I}(L, V, \omega) = \int \prod_{i=1}^L dp_i \frac{P'_\omega(p)}{\left(\prod_{j=1}^L (p_j^2)^N\right) P_{V-1}((\Delta p^2)^N)}.$$

We have used $I = L + V - 1$. Here P_{V-1} is a polynomial of degree $V - 1$ in $(\Delta p^2)^N$, where Δp are linear combinations of the p 's with coefficients ± 1 . The numerator P'_ω is a polynomial of degree ω in the p 's. To have a potential overall divergence we need

$$\omega \geq L(2N - 1) + 2N(V - 1). \quad (4.3)$$

If $V = 1$ the integral factorizes into L one-loop integrals, which cannot contain logarithmic divergences. Assume $V > 1$. Then (4.3) implies that each monomial of $P'_\omega(p)$ contains at least $2N$ powers of some p_i , say p_1 , which ‘‘simplify’’ a propagator. Actually they produce a regular function of the form

$$\frac{p_1^{2N}}{p_1^{2N} + \sum_{j=1}^{2N} c_j p_1^{2N-j}} = 1 - \frac{\sum_{j=1}^{2N} c_j p_1^{2N-j}}{p_1^{2N} + \sum_{j=1}^{2N} c_j p_1^{2N-j}}. \quad (4.4)$$

Consider first the ‘‘1’’ on the right-hand side of this equation. It gives

$$\int \prod_{i=1}^L dp_i \frac{P'_{\omega-2N}(p)}{\left(\prod_{j=2}^L (p_j^2)^N\right) P_{V-1}((\Delta p^2)^N)}.$$

We can distinguish two cases: *i*) P_{V-1} does not depend on p_1 ; *ii*) P_{V-1} depends on p_1 . In case *i*) the p_1 -integral factorizes and cannot produce logarithmic divergences. We remain with a

$\mathcal{I}(L-1, V, \omega-2N)$. In case *ii*), after a p_1 -translation we obtain an integral $\mathcal{I}(L, V-1, \omega-2N)$. The translation can cost at worst another $\mathcal{I}(L-1, V, \omega-2N)$. Now consider the second term on the right-hand side of (4.4): it gives *iii*) a $\mathcal{I}(L, V, \omega')$ with $\omega' < \omega$.

In all cases we can repeat the arguments made so far, with fewer loops or vertices, or with a smaller ω . At each step either an integral factorizes, or a propagator simplifies, or ω decreases. We end up with zero loops, namely no integral, or one vertex, namely L factorized integrals, or an ω violating (4.3), i.e. an overall convergent integral. Proceeding this way we find that there cannot be logarithmic divergences. If there are no logarithmic divergences at $\hat{d} = 1$ there are no divergences at all reaching $\hat{d} = 1$ from complex dimensions $1 - \varepsilon_1$.

Mixed subintegrals Consider subintegrals of the form

$$\prod_{i=1}^L \int \frac{d\hat{k}'_i}{(2\pi)^{\hat{d}}} \left[\prod_{j=L+1}^{L+M} \int \frac{d\hat{k}'_j}{(2\pi)^{\hat{d}}} \int \frac{d\bar{d}\bar{k}'_j}{(2\pi)^{\bar{d}}} \right], \quad (4.5)$$

which are “incomplete” in L barred directions. The complete subintegrals in square brackets can be regarded as products of (nonlocal, but one-particle irreducible) “subvertices”. Let r label the subvertices with $\tilde{n}_{\hat{A}r}, \tilde{n}_{\bar{A}r}, \tilde{n}_{Cr}, \tilde{n}_{fr}, \tilde{n}_{sr}$ external legs of types \hat{A}, \bar{A} , ghost, fermion and scalar, respectively. Since subvertices are at least one-loop, each leg has a factor $\bar{g}_{1,2,3}$ attached to it (see (3.10)). Thus, the weight $\tilde{\delta}_r$ of the subvertices of type r satisfies the bound

$$\begin{aligned} \tilde{\delta}_r \leq & \bar{d} - (\tilde{n}_{\hat{A}r} + \tilde{n}_{Cr}) \left(\frac{\bar{d}}{2} - 1 + \kappa_1 \right) - \tilde{n}_{\bar{A}r} \left(\frac{\bar{d}}{2} - 2 + \frac{1}{n} + \kappa_1 \right) \\ & - \tilde{n}_{fr} \left(\frac{\bar{d} - 1}{2} + \kappa_2 \right) - \tilde{n}_{sr} \left(\frac{\bar{d}}{2} - 1 + \kappa_3 \right). \end{aligned} \quad (4.6)$$

Consider a subintegral (4.5) corresponding to a subdiagram with L loops, v_1 vertices of type X_1 , v_2 vertices of type X_2 and X'_2 , Δv_i vertices of other types with $2i$ fermionic legs, I_B internal bosonic legs (including ghosts) and I_F internal fermionic legs, E_F external fermionic legs and \tilde{v}_r subvertices of type r . We know that every bosonic propagator behaves at least like $1/\hat{k}^2$, for \hat{k} large, while the fermionic propagator behaves like $1/\hat{k}$. The subintegral has a superficial degree of divergence equal to

$$\hat{\omega}(G) = L + v_1 + 2v_2 - 2I_B - I_F + \sum_r \tilde{v}_r \tilde{\delta}_r. \quad (4.7)$$

Moreover, the topological identity $L - I + V = 1$ gives

$$L = 1 + I_B + I_F - v_1 - v_2 - \sum_{i=0}^{i_{\max}} \Delta v_i - \sum_r \tilde{v}_r. \quad (4.8)$$

Counting the fermionic legs of the subdiagram we have

$$2I_F + E_F = \sum_{i=0}^{i_{\max}} 2i\Delta v_i + \sum_r \tilde{v}_r \tilde{n}_{fr}. \quad (4.9)$$

Combining (4.7), (4.8) and (4.9) we get

$$\hat{\omega}(G) = 2 - L - v_1 - \frac{E_F}{2} + \sum_{i=0}^{i_{\max}} (i-2)\Delta v_i + \sum_r \tilde{v}_r \left(\tilde{\delta}_r - 2 + \frac{\tilde{n}_{fr}}{2} \right). \quad (4.10)$$

We know that in the realm of the usual power counting, odd-dimensional integrals do not have logarithmic divergences. In the realm of the weighted power counting, such a property generalizes as follows: if $\hat{d} = 1$, $d = \text{even}$ and $n = \text{odd}$, then odd-dimensional (weighted) integrals do not have logarithmic divergences. The proof is simple and left to the reader. Thus, the case $L = 1$ is excluded. Sufficient conditions to have $\hat{\omega}(G) \leq 0$ are then $i_{\max} \leq 2$ (i.e. vertices with more than four fermionic legs should be absent) and

$$\tilde{\delta}_r - 2 + \frac{1}{2}\tilde{n}_{fr} < 0 \quad \text{for every } r. \quad (4.11)$$

Indeed, if such conditions hold (4.10) gives $\hat{\omega}(G) < 0$ unless all subvertices are absent, which is the case considered previously. Using (4.6), sufficient conditions to have (4.11) are

$$\bar{d} - (\tilde{n}_{\hat{A}r} + \tilde{n}_{Cr})c_1 - \tilde{n}_{\bar{A}r}\bar{c}_1 - \tilde{n}_{fr}c_2 - \tilde{n}_{sr}c_3 < 2, \quad (4.12)$$

where

$$c_i = \frac{\bar{d}}{2} - 1 + \kappa_i, \quad \bar{c}_1 = \frac{\bar{d}}{2} - 2 + \frac{1}{n} + \kappa_1. \quad (4.13)$$

Finally, the most general mixed subintegrals have the form

$$\prod_{i=1}^L \int \frac{d\hat{k}'_i}{(2\pi)^{\hat{d}}} \left[\prod_{j=L+1}^{L+M} \int \frac{d\hat{k}'_j}{(2\pi)^{\hat{d}}} \int \frac{d\bar{k}'_j}{(2\pi)^{\bar{d}}} \prod_{m=L+M+1}^{L+M+P} \int \frac{d\bar{k}'_m}{(2\pi)^{\bar{d}}} \right].$$

They can be treated as above, considering the integrals between square brackets as subvertices. Now formula (4.10) has an extra $-P$ on the right-hand side, since P hatted intergations are missing. The situation, therefore, can only improve. The only caveat is that now L can also be one (if P is odd). Even in that case, however, $2 - L - P \leq 0$, since $P \geq 1$.

Sufficient conditions for the absence of spurious subdivergences Now we work out sufficient conditions to fulfill (4.11). First, we require that the coefficients c_i , \bar{c}_1 in (4.12) be strictly positive, so that the bound (4.12) improves when the number of legs increases. It is easy to check that under such conditions vertices X'_1 and vertices with more than four fermionic legs are

automatically forbidden. We can have various cases, according to which of the c_i , \bar{c}_1 is minimum.
i) If

$$2 - \frac{1}{n} - \frac{\bar{d}}{2} < \kappa_1 \leq \kappa_{2,3} + 1 - \frac{1}{n} \quad (4.14)$$

the minimum coefficient is \bar{c}_1 . Then the worst case for the bound (4.6) is $\tilde{n}_{\bar{A}r} = 2$, $\tilde{n}_{\hat{A}r} = \tilde{n}_{Cr} = \tilde{n}_{fr} = \tilde{n}_{sr} = 0$, so sufficient conditions to ensure (4.12) are

$$1 - \frac{1}{n} < \kappa_1. \quad (4.15)$$

The combination of (4.14) and (4.15) is

$$1 - \frac{1}{n} < \kappa_1 \leq \kappa_{2,3} + 1 - \frac{1}{n}, \quad \kappa_1 > 2 - \frac{1}{n} - \frac{\bar{d}}{2}. \quad (4.16)$$

Repeating the argument for the other cases, we find *ii)*

$$1 - \frac{\bar{d}}{2} < \kappa_2 \leq \kappa_3, \quad 1 - \frac{1}{n} + \kappa_2 \leq \kappa_1, \quad (4.17)$$

with minimum coefficients c_2 , and *iii)* (4.17) with κ_2 and κ_3 interchanged, with minimum coefficient c_3 .

The case $\bar{d} \leq 2$ is important for physical applications, so we treat it apart. We prove that sufficient conditions for the absence of spurious subdivergences are

$$\bar{d} \leq 2, \quad \kappa_1 > 2 - \frac{1}{n} - \frac{\bar{d}}{2}, \quad \kappa_2 \geq 1 - \frac{\bar{d}}{2}. \quad (4.18)$$

The second and third inequalities of this list ensure $\bar{c}_1 > 0$ and $c_2 \geq 0$, respectively, while $c_3 > 0$ is already ensured by the last inequality of (3.13). Again, it is easy to prove that under such conditions neither vertices X'_i , nor vertices with more than four fermionic legs are allowed. Moreover, four fermion vertices cannot have other types of legs.

Now, (4.18) imply $\hat{\omega}(G) \leq 0$. If $\bar{d} < 2$ or $\kappa_2 > 1 - \bar{d}/2$, then $\hat{\omega}(G) = 0$ only for $\tilde{v}_r = 0$. Instead, if $\bar{d} = 2$ and $\kappa_2 = 0$, then $\hat{\omega}(G) = 0$ for $\tilde{v}_r = 0$ or

$$L = 2 - P, \quad v_1 = E_F = \Delta v_0 = \Delta v_1 = \tilde{n}_{\hat{A}r} = \tilde{n}_{\bar{A}r} = \tilde{n}_{Cr} = \tilde{n}_{sr} = 0, \quad \tilde{n}_{fr}, v_2, \Delta v_2 = \text{arbitrary.}$$

The subdiagrams with such features do not contain vertices with both fermionic and bosonic legs, have no external fermionic leg and their subvertices have only fermionic legs. Thus, either $\tilde{v}_r = 0$ and the diagram falls in the \hat{k} -subintegral class discussed above, or $v_2 = 0$ and the diagram has no external leg, therefore it is trivial.

Concluding, if $\hat{d} = 1$, $n = \text{odd}$ and d is even spurious subdivergences are absent if either (4.16), or (4.17), or (4.17) with $\kappa_2 \leftrightarrow \kappa_3$, or (4.18) hold.

5 Renormalizable theories

In this section we study examples of renormalizable theories and look for four dimensional models that contain two scalar-two fermion interactions and four fermion interactions.

The simplest models are those that have the smallest values of n (≥ 2) and the largest values of $\kappa_{1,2,3}$. Compatibly with (3.14) the largest value of all κ_i 's is $2-\bar{d}/2$, which gives the $1/\alpha$ theories considered in paper I. Those models exhibit, in a simplified framework, several properties of Lorentz violating gauge theories, but are not particularly rich from a phenomenological point of view, because they cannot renormalize vertices that are otherwise non-renormalizable, apart from those that contain some unusual dependences on \bar{A} and $\bar{\partial}$.

The simplest four dimensional $1/\alpha$ theory [6] has $n = 2$, $\bar{d} = 5/2$ and the (Euclidean) lagrangian

$$\mathcal{L}_{1/\alpha} = \mathcal{L}_Q + \frac{g}{\Lambda_L^2} f_{abc} \left(\lambda \tilde{F}_{\mu\nu}^a \tilde{F}_{\mu\rho}^b + \lambda' \bar{F}_{\mu\nu}^a \bar{F}_{\mu\rho}^b \right) \bar{F}_{\nu\rho}^c + \frac{g}{\Lambda_L^4} \sum_j \lambda_j \bar{D}^2 \bar{F}^3_j + \frac{\alpha}{\Lambda_L^4} \sum_k \lambda'_k \bar{F}^4_k, \quad (5.1)$$

where j labels the independent gauge invariant terms constructed with two covariant derivatives \bar{D} acting on three field strengths \bar{F} , and k labels the terms constructed with four \bar{F} 's. The last two terms are symbolic.

Let us investigate the $1/\bar{\alpha}$ extensions of (5.1). The maximal extension is the one with $\kappa = 0$. The theory contains the additional vertices

$$\sum_{p=2}^4 \frac{\lambda_p}{\Lambda_L^{9p/2}} \tilde{F}^2 \bar{F}^p + \sum_{q=4}^6 \frac{\lambda'_q}{\Lambda_L^{9q/2-7}} \bar{D}^2 \bar{F}^q + \sum_{r=5}^{10} \frac{\lambda''_r}{\Lambda_L^{9r/2-9}} \bar{F}^r. \quad (5.2)$$

Larger values of κ can reduce the set of vertices in various ways. For example, for $5/12 < \kappa \leq 3/4$ the theory is still (5.1). For $1/4 < \kappa \leq 5/12$ we have a unique additional vertex, \bar{F}^5 . For $3/20 < \kappa \leq 1/4$ we have also $\tilde{F}^2 \bar{F}^2$, $\bar{D}^2 \bar{F}^4$ and \bar{F}^6 . For $1/12 < \kappa \leq 3/20$ we have also \bar{F}^7 , and so on. However, because n is even the model (5.1) and its extensions (5.2) may have spurious subdivergences. Going through the previous section it is possible to show that such subdivergences appear only at three loops. The first completely consistent model is thus the theory with $\hat{d} = 1$, $n = 3$, $\bar{d} = 2$. Its simplest renormalizable lagrangian is the sum of \mathcal{L}_Q plus \bar{F}^3 .

Theories with two scalar-two fermion vertices Two scalar-two fermion interactions

$$\frac{\bar{g}_2^2 \bar{g}_3^2}{\bar{a}_1} \varphi^2 \bar{\psi} \psi$$

are renormalizable if and only if

$$\kappa_{2,3} \leq \frac{3}{2} - \frac{\bar{d}}{2}$$

and of course $\kappa_{2,3} \geq 0$. Let us choose the largest values of $\kappa_{1,2,3}$ compatible with this bound and (3.13)-(3.14), namely

$$\bar{g}_1 = g, \quad \bar{g}_2 = \bar{g}_3 = \bar{g}, \quad \kappa_2 = \kappa_3 = \frac{3}{2} - \frac{\bar{d}}{2}.$$

We can take, for example, $n = 2$, $\bar{d} = 5/2$, with gauge group $SU(2) \times U(1)$ and matter fields in the fundamental representation of $SU(2)$. Then we have the theory

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{1/\alpha} + \bar{\psi} \left(\hat{D} + \frac{\eta_f}{\Lambda_L} \bar{D}^2 + \eta'_f \bar{D} + m_f \right) \psi + \\ & + |\hat{D}\varphi|^2 + \frac{\eta_s}{\Lambda_L^2} |\bar{D}^2\varphi|^2 + \eta'_s |\bar{D}\varphi|^2 + m_s^2 |\varphi|^2 + \frac{\lambda_4 \bar{g}^2}{4} |\varphi|^4 \\ & + \frac{\bar{g}^2}{4\Lambda_L^2} \left[\frac{\lambda_6}{9} \bar{g}^2 |\varphi|^6 + \lambda_4^{(3)} |\varphi|^2 |\bar{D}\varphi|^2 + \lambda_4^{(2)} |\varphi^\dagger \bar{D}\varphi|^2 + \lambda_4^{(1)} \left((\varphi^\dagger \bar{D}\varphi)^2 + \text{h.c.} \right) \right] \\ & + \frac{Y \bar{g}^2}{\Lambda_L} |\varphi|^2 \bar{\psi} \psi + \frac{Y' \bar{g}^2}{\Lambda_L} (\bar{\psi} \varphi) (\varphi^\dagger \psi) + \frac{\tau_f g}{\Lambda_L} i \bar{F}_{\bar{\mu}\bar{\nu}}^a (\bar{\psi} T^a \sigma_{\bar{\mu}\bar{\nu}} \psi) \\ & + \frac{g}{\Lambda_L^2} \left[\tau_s \bar{F}_{\bar{\mu}\bar{\nu}}^a ((D_{\bar{\mu}}\varphi)^\dagger T^a D_{\bar{\nu}}\varphi) + \tau'_s g |\varphi|^2 \bar{F}^2 \right]. \end{aligned} \quad (5.3)$$

where $\mathcal{L}_{1/\alpha}$ is given in (5.1). For simplicity, we have assumed $U(1)$ charge assignments that forbid terms containing $\bar{\psi}^c \psi$.

We see that the list of new vertices contains also scalar self-interactions of type φ^6 , φ^4 -vertices with spatial derivatives, Pauli terms and several other types of vertices that are not renormalizable in the framework of the usual power counting.

The couplings λ_6 , $\lambda_4^{(i)}$, Y , Y' , τ_f , τ_s and τ'_s are weightless. Since $\kappa_i > 0$ their beta functions vanish identically. Following the arguments explained around formula (3.10), the couplings that are not generated back by renormalization can be consistently switched off, which produces simplified renormalizable models. The simplest one reads

$$\begin{aligned} & \frac{1}{2} F_{\bar{\mu}\bar{\nu}} \eta(\bar{Y}) F_{\bar{\mu}\bar{\nu}} + \frac{1}{4} F_{\bar{\mu}\bar{\nu}} \tau(\bar{Y}) F_{\bar{\mu}\bar{\nu}} + \bar{\psi} \left(\hat{D} + \frac{\eta_f}{\Lambda_L} \bar{D}^2 + \eta'_f \bar{D} + m_f \right) \psi + \\ & + |\hat{D}\varphi|^2 + \frac{\eta_s}{\Lambda_L^2} |\bar{D}^2\varphi|^2 + \eta'_s |\bar{D}\varphi|^2 + m_s^2 |\varphi|^2 + \frac{\lambda_4 \bar{g}^2}{4} |\varphi|^4, \end{aligned}$$

which can be cast in $1/\alpha$ form. Again, because n is even the theory may contain spurious subdivergences.

Theories with four fermion vertices Four fermion interactions $\bar{g}_2^2 \bar{\psi}^2 \psi^2$ are renormalizable if and only if

$$\kappa_2 \leq 1 - \frac{\bar{d}}{2},$$

which can happen only for $\bar{d} \leq 2$, therefore $4 \leq d \leq n + 1$. In four dimensions the simplest case is $n = 3$, $\bar{d} = 2$, $\kappa_2 = 0$. We can still choose $\bar{g}_1 = g$, and, in the presence of scalar fields, $\kappa_3 = 1/2$. Then the model satisfies (2.21) and (4.18), so it is free of spurious subdivergences.

Turning scalar fields off and choosing $G = SU(N)$, with a Dirac fermion in the fundamental representation, the lagrangian is the sum of the pure gauge terms $\mathcal{L}_Q + \mathcal{L}_I$ plus the fermion kinetic terms, some Pauli-type terms and the four fermion vertices. Precisely,

$$\begin{aligned}
 \mathcal{L} = & \mathcal{L}_Q + \mathcal{L}_I + \bar{\psi} \left(\hat{\mathcal{D}} + \sum_{i=0}^2 \frac{\eta_{if}}{\Lambda_L^{2-i}} \bar{\mathcal{D}}^{3-i} + m_f \right) \psi + \frac{g}{\Lambda_L^2} \bar{F}_{\bar{\mu}\bar{\nu}}^a \left[\tau_f (\bar{\psi} T^a \gamma_{\bar{\mu}} \overleftrightarrow{D}_{\bar{\nu}} \psi) + \tau'_f i \bar{D}_{\bar{\nu}} (\bar{\psi} T^a \gamma_{\bar{\mu}} \psi) \right] \\
 & + \frac{\tau_f'' g}{\Lambda_L} i \bar{F}_{\bar{\mu}\bar{\nu}}^a (\bar{\psi} T^a \sigma_{\bar{\mu}\bar{\nu}} \psi) + \frac{1}{\Lambda_L^2} [\lambda_1 (\bar{\psi} \psi)^2 + \lambda_2 (\bar{\psi} \gamma_5 \psi)^2 + \lambda_3 (\bar{\psi} \gamma_{\bar{\mu}} \psi)^2 + \lambda_4 (\bar{\psi} \gamma_{\bar{\mu}} \psi)^2] \\
 & + \frac{1}{\Lambda_L^2} [\lambda_5 (\bar{\psi} \gamma_{\bar{\mu}} \gamma_5 \psi)^2 + \lambda_6 (\bar{\psi} \gamma_{\bar{\mu}} \gamma_5 \psi)^2 + \lambda_7 (\bar{\psi} \sigma_{\bar{\mu}\bar{\nu}} \psi)^2 + \lambda_8 (\bar{\psi} \sigma_{\bar{\mu}\bar{\nu}} \psi)^2]. \tag{5.4}
 \end{aligned}$$

We have

$$\begin{aligned}
 \mathcal{L}_I = & \frac{g\lambda_3}{\Lambda_L^2} f_{abc} \bar{F}_{\bar{\mu}\bar{\nu}}^a \bar{F}_{\bar{\mu}\bar{\rho}}^b \bar{F}_{\bar{\nu}\bar{\rho}}^c + \frac{g\lambda_3'}{\Lambda_L^4} \bar{D}^2 \bar{F}^2 \bar{F} + \frac{g\lambda_3''}{\Lambda_L^4} \hat{D} \bar{D} \bar{F} \bar{F}^2 + \frac{g\lambda_3'''}{\Lambda_L^4} \hat{D}^2 \bar{F}^3 + \frac{g^2 \lambda_4}{\Lambda_L^4} \bar{F}^2 \bar{F}^2 \\
 & + \frac{g}{\Lambda_L^2} \sum_{m=0}^3 \lambda_m^{(1)} \frac{\bar{D}^{2m}}{\Lambda_L^{2m}} \bar{F}^3 + \frac{g^2}{\Lambda_L^4} \sum_{m=0}^2 \lambda_m^{(2)} \frac{\bar{D}^{2m}}{\Lambda_L^{2m}} \bar{F}^4 + \frac{g^3}{\Lambda_L^6} \sum_{m=0}^1 \lambda_m^{(3)} \frac{\bar{D}^{2m}}{\Lambda_L^{2m}} \bar{F}^5 + \frac{g^4}{\Lambda_L^8} \lambda^{(4)} \bar{F}^6.
 \end{aligned}$$

It is straightforward to check that the $\hat{\partial}$ -structures of the theories listed so far agrees with the results of section 3.

The model (5.4) is fully consistent. In particular, it is free of spurious subdivergences. It is straightforward to include scalar fields and two scalar-two fermion interactions.

Abelian strictly renormalizable theories We conclude with the analysis of a peculiar class of strictly renormalizable theories. The quadratic part \mathcal{L}_Q of the lagrangian must have

$$\eta(\bar{\Upsilon}) = \eta_0 \bar{\Upsilon}^{n-1}, \quad \tau(\bar{\Upsilon}) = \tau_0 \bar{\Upsilon}^{2(n-1)}, \quad \xi(\bar{\Upsilon}) = \xi_0 \bar{\Upsilon}^{n-2}.$$

For convenience we can choose a strictly-renormalizable gauge fixing, with $\zeta(\bar{v}) = \zeta_0 \bar{v}^{n-1}$. The IR analysis of Feynman diagrams is still dominated by the weighted power counting, however $\eta(0) = \tau(0) = 0$, so the gauge-field propagator contains additional denominators $\sim 1/\bar{k}^{2(n-1)}$ in the $\langle \bar{A} \bar{A} \rangle$ -sector. The loop integrals over k and the loop sub-integrals over \bar{k} are IR divergent unless

$$\bar{d} > 4 - \frac{2}{n}, \quad \bar{d} > 2(n-1), \tag{5.5}$$

respectively. The latter condition and $n \geq 2$ imply $\bar{d} \geq 3$. We have also to require (3.11) and (3.14), and check the absence of spurious subdivergences.

If $\bar{d}=4$ the gauge coupling itself is strictly-renormalizable and the theory can be cast in a $1/\alpha$ form. This case, considered in paper I, is not guaranteed to be free of spurious subdivergences. On the other hand, if $\bar{d}<4$ the theory can be strictly-renormalizable only if it is Abelian and contains vertices constructed with the field strength and its derivatives. In four dimensions no strictly renormalizable theory with $\hat{d}=1$ exists, since \bar{d} is smaller than 4, and (2.21) and (3.13) imply $n < 5/3$. Thus, we have to consider higher dimensional theories. The conditions (2.21) and (5.5) give $d \geq 3n$, but $\bar{d}<4$ gives also $d \leq 3n$, so we must have $d = 3n \geq 6$. However, it is easy to check that the six-dimensional theory with $n = 2$, $\bar{d} = 5$, $\bar{d} = 7/2$, is trivial, since no strictly renormalizable interaction can be constructed. Then we have the nine dimensional theory with $n = 3$, $\bar{d} = 8$, $\bar{d} = 11/3$, and lagrangian

$$\mathcal{L} = \mathcal{L}_Q + \frac{\lambda}{\Lambda_L^{20}} \bar{D}^2 \bar{F}^6.$$

However, since this theory is odd-dimensional, at present we cannot guarantee that it is free of spurious subdivergences.

6 More general Lorentz violations

So far we have broken the Lorentz group $O(1, d-1)$ into the product of two subfactors $O(1, \hat{d}-1) \times O(\bar{d})$, which means, for $\hat{d}=1$, that we have preserved time reversal, parity and rotational invariance. It is of course possible to break also such symmetries, but that breaking is not going to affect the results of our present investigation. The structure of the theory with respect to the weighted power counting is unmodified as long as each space coordinate has the same weight.

A more general possibility is to break the Lorentz group into the product of more subfactors, so that different space coordinates may have different weights. Invariance under spatial rotations is necessarily lost. To cover the most general case, we can break the spacetime manifold M into a submanifold \hat{M} of dimension \hat{d} , containing time, and ℓ space submanifolds \bar{M}_i of dimensions \bar{d}_i , $i = 1, \dots, \ell$:

$$M = \hat{M} \times \prod_{i=1}^{\ell} \bar{M}_i. \quad (6.1)$$

Denote the space derivatives of the i th space subsector by $\bar{\partial}_i$ and assume that they have weights $1/n_i$. We can assume also $n_1 < n_2 \dots < n_\ell$. Then the weighted dimension \bar{d} , which is also the weight of the momentum-space integration measure $d^d p$, is equal to

$$\bar{d} = \hat{d} + \sum_{i=1}^{\ell} \frac{\bar{d}_i}{n_i}.$$

Again, \bar{d} can be at most 4. If the super-renormalizable subsector is non-trivial d must at least be equal to 4, otherwise Feynman diagrams can have IR divergences.

The weight of a scalar field φ is still $\bar{d}/2 - 1$, because its kinetic term must contain $(\hat{\partial}\varphi)^2$. Similarly, the weight of a fermion is $\bar{d}/2 - 1/2$ and the weight of the \hat{A} -component of the gauge field is $\bar{d}/2 - 1$. Since $\bar{\partial}_i \hat{A}$ and $\hat{\partial} \bar{A}_i$ belong to the same field-strength component, the weight of \bar{A}_i is $\bar{d}/2 - 2 + 1/n_i$.

Every argument of this paper can be generalized straightforwardly to the breaking (6.1), except for the analysis of spurious subdivergences, which is a more delicate issue. The conditions $\hat{d} = 1$, $d = \text{even}$, combined with suitable other restrictions, are still sufficient to ensure that no spurious subdivergences occur in the \hat{k} -subintegrals. Now, however, the propagator behaves irregularly also when $\bar{k}_i \rightarrow \infty$ for any $i < \ell$.

Consider for example a three-factor splitting. The quadratic part of the lagrangian is a quadratic form in the momenta \hat{k} , \bar{k}_1 and \bar{k}_2 , and contains appropriate polynomial functions of \bar{k}_1^2 , or \bar{k}_2^2 , or both. In particular, the \bar{A}_2 -quadratic term has the form, in momentum space,

$$\bar{A}_{2\mu}(-k)Q_{22}(k)\bar{A}_{2\mu}(k) + \bar{A}_{2\mu}(-k)\bar{k}_{2\mu}Q'_{22}(k)\bar{k}_{2\nu}\bar{A}_{2\nu}(k),$$

where $Q_{22}(k)$ is a polynomial of weight $4 - 2/n_2$. The propagator $\langle \bar{A}_2 \bar{A}_2 \rangle$ reads

$$\langle \bar{A}_2(k)\bar{A}_2(-k) \rangle = Q_{22}^{-1}(k)\bar{\delta}_2 + P_{22}(k)\bar{k}_2\bar{k}_2,$$

for some unspecified function $P_{22}(k)$. The weight of Q_{22} cannot be saturated just by \bar{k}_1 , because $Q_{22} \sim (\bar{k}_1^2)^X$ would give

$$X = 2n_1 - \frac{n_1}{n_2},$$

which is not integer. Consequently the propagator $\langle \bar{A}_2 \bar{A}_2 \rangle$ cannot behave regularly in the limit $\bar{k}_1 \rightarrow \infty$ with \hat{k} and \bar{k}_2 fixed, so the \bar{k}_1 -subintegrals may contain spurious subdivergences of new types. If we assume $\bar{d}_1 = 1$ (in addition to $\hat{d} = 1$) then the \bar{k}_1 -integrals do not have spurious divergences, as explained in section 4. However, this is not enough, because the \hat{k} - \bar{k}_1 -subintegrals themselves, which are two dimensional, can generate spurious subdivergences. In this case the arguments of section 4 do not apply. We do not know at present if the problem of spurious subdivergences can be solved in general. Our present results seem to suggest that the unique consistent spacetime splitting in the one into space and time.

Before concluding this section it is worth to emphasize that the models to which our proofs of renormalizability, or absence of spurious subdivergences, do not apply cannot be completely excluded. Some of them might work because of unexpected cancellations, which can occur because of symmetries (e.g. supersymmetry) or peculiar types of expansions or resummations (e.g. large N).

7 Conclusions

In this paper we have completed the program of constructing and classifying the Lorentz violating gauge theories that are renormalizable by weighted power counting. The theories contain higher space derivatives, but no higher time derivatives. We have shown that it is possible to renormalize vertices that are non-renormalizable in the usual power counting framework, such as the two scalar-two fermion interactions and the four fermion interactions. We have studied the time-derivative structure of the theories and the absence of spurious subdivergences in detail. Spacetime is split into space and time.

We recall that once Lorentz symmetry is violated at high energies, its low energy recovery is not automatic, because renormalization makes the low-energy parameters run independently. One possibility is that the Lorentz invariant surface is RG stable (see [8] and [9]). Otherwise, a suitable fine-tuning must be advocated.

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Appendix A: Propagators and dispersion relations

After integrating B^a out, the gauge-field quadratic terms are contained in

$$\mathcal{L}_Q + \frac{1}{2\lambda}(\mathcal{G}^a)^2, \quad (\text{A.1})$$

which gives the propagator

$$\langle A(k) A(-k) \rangle = \begin{pmatrix} \langle \hat{A}\hat{A} \rangle & \langle \hat{A}\bar{A} \rangle \\ \langle \bar{A}\hat{A} \rangle & \langle \bar{A}\bar{A} \rangle \end{pmatrix} = \begin{pmatrix} u\hat{\delta} + s\hat{k}\hat{k} & r\hat{k}\bar{k} \\ r\bar{k}\hat{k} & v\bar{\delta} + t\bar{k}\bar{k} \end{pmatrix}, \quad (\text{A.2})$$

with

$$u = \frac{1}{D(1, \eta)}, \quad s = \frac{\lambda}{D^2(1, \zeta)} + \frac{-\hat{k}^2 + \zeta \left(\frac{\zeta}{\eta} - 2 \right) \bar{k}^2}{D(1, \eta)D^2(1, \zeta)}, \quad r = \frac{\lambda - \frac{\zeta}{\eta}}{D^2(1, \zeta)},$$

$$v = \frac{1}{D(\tilde{\eta}, \tau)}, \quad t = \frac{\lambda}{D^2(1, \zeta)} + \frac{\left(\frac{\tilde{\tau}}{\eta} - 2\zeta \right) \hat{k}^2 - \zeta^2 \bar{k}^2}{D(\tilde{\eta}, \tau)D^2(1, \zeta)},$$

where

$$D(x, y) \equiv x\hat{k}^2 + y\bar{k}^2, \quad \tilde{\eta} = \eta + \frac{\bar{k}^2}{\Lambda_L^2}\xi, \quad \tilde{\tau} = \tau + \frac{\hat{k}^2}{\Lambda_L^2}\xi,$$

and now η , τ , ξ and ζ , as well as x and y , are functions of \bar{k}^2/Λ_L^2 . The ghost propagator is

$$\frac{1}{D(1, \zeta)}. \quad (\text{A.3})$$

In the Feynman gauge $\lambda = 1$, $\zeta = \eta$ we have

$$u = \frac{1}{D(1, \eta)}, \quad s = r = 0, \quad v = \frac{1}{D(\tilde{\eta}, \tau)}, \quad t = \frac{\tilde{\tau} - \eta^2}{\eta D(\tilde{\eta}, \tau) D(1, \eta)}. \quad (\text{A.4})$$

The physical degrees of freedom can be read in the Coulomb gauge $\bar{\partial} \cdot \bar{A}^a = 0$, which can be reached taking the limit $\zeta \rightarrow \infty$ in (A.2) and (A.3). In such a gauge the ghosts are non-propagating, the \hat{A} -sector propagates $\hat{d} - 1$ physical degrees of freedom with energies

$$E = \sqrt{\hat{\mathbf{k}}^2 + \bar{k}^2 \eta (\bar{k}^2 / \Lambda_L^2)}$$

and the \bar{A} -sector propagates $\bar{d} - 1$ physical degrees of freedom with energies

$$E = \sqrt{\hat{\mathbf{k}}^2 + \bar{k}^2 \frac{\tau (\bar{k}^2 / \Lambda_L^2)}{\tilde{\eta} (\bar{k}^2 / \Lambda_L^2)}}.$$

Appendix B: Renormalizability to all orders

In this appendix we study the renormalizability of Lorentz violating gauge theories to all orders, using the Batalin-Vilkovisky formalism [10]. For simplicity we concentrate on pure gauge theories and use the minimal subtraction scheme and the dimensional-regularization technique. Note that the functional integration measure is automatically BRST invariant.

The fields are collectively denoted by $\Phi^i = (A_\mu^a, \bar{C}^a, C^a, B^a)$. Add BRST sources $K_i = (K_a^\mu, K_{\bar{C}}^a, K_C^a, K_B^a)$ for every field Φ^i and extend the action (2.18) as

$$\Sigma(\Phi, K) = \mathcal{S}(\Phi) - \int d^d x \left[(s A_\mu^a) K_a^\mu + (s \bar{C}^a) K_{\bar{C}}^a + (s C^a) K_C^a + (s B^a) K_B^a \right], \quad (\text{B.1})$$

Define the antiparenthesis

$$(X, Y) = \int d^d x \left\{ \frac{\delta_r X}{\delta \Phi^i(x)} \frac{\delta_l Y}{\delta K_i(x)} - \frac{\delta_r X}{\delta K_i(x)} \frac{\delta_l Y}{\delta \Phi^i(x)} \right\}. \quad (\text{B.2})$$

BRST invariance is generalized to the identity

$$(\Sigma, \Sigma) = 0, \quad (\text{B.3})$$

which is a straightforward consequence of (B.1), the gauge invariance of \mathcal{S}_0 and the nilpotency of s . Define also the generalized BRST operator

$$\sigma X \equiv (\Sigma, X), \quad (\text{B.4})$$

which is nilpotent ($\sigma^2 = 0$), because of the identity (B.3).

The generating functionals Z , W and Γ are defined, in the Euclidean framework, as

$$Z[J, K] = \int \mathcal{D}\Phi \exp \left(-\Sigma(\Phi, K) + \int \Phi^i J_i \right) = e^{W[J, K]}, \quad (\text{B.5})$$

$$\Gamma[\Phi_\Gamma, K] = -W[J, K] + \int \Phi_\Gamma^i J_i, \quad \text{where} \quad \Phi_\Gamma^i = \frac{\delta_r W[J, K]}{\delta J_i}.$$

Below we often suppress the subscript Γ in Φ_Γ . Performing a change of variables

$$\Phi' = \Phi + \theta s\Phi, \quad (\text{B.6})$$

in the functional integral (B.5), θ being a constant anticommuting parameter, and using the identity (B.3), we find

$$(\Gamma, \Gamma) = 0. \quad (\text{B.7})$$

A canonical transformation of fields and sources is defined as a transformation that preserves the antiparenthesis. It is generated by a functional $\mathcal{F}(\Phi, K')$ and reads

$$\Phi^{i'} = \frac{\delta \mathcal{F}}{\delta K_i'}, \quad K_i = \frac{\delta \mathcal{F}}{\delta \Phi^i}.$$

As usual, renormalizability is proved inductively. The inductive assumption is that up to the n -th loop included the divergences can be removed redefining the physical parameters α_i contained in \mathcal{S}_0 and performing a canonical transformation of the fields and the BRST sources. Call Σ_n and $\Gamma^{(n)}$ the action and generating functional renormalized up to the n -th loop included. The inductive assumption ensures that Σ_n and $\Gamma^{(n)}$ satisfy (B.3) and (B.7), respectively.

Locality and (B.7) imply that the $(n+1)$ -loop divergences $\Gamma_{n+1 \text{ div}}^{(n)}$ of $\Gamma^{(n)}$ are local and σ -closed, namely $\sigma \Gamma_{n+1 \text{ div}}^{(n)} = 0$. We have to find the most general solution to this cohomological condition. In Lorentz invariant theories the problem has been solved for local functionals with arbitrary ghost number [11]. Since Lorentz invariance does not play a major role in those proofs, we conjecture that the Lorentz invariant result generalizes naturally to Lorentz violating theories, namely that $\Gamma_{n+1 \text{ div}}^{(n)}$ can be decomposed as

$$\Gamma_{n+1 \text{ div}}^{(n)} = \int d^d x (\mathcal{G}_n(A) + \sigma \mathcal{R}_n), \quad (\text{B.8})$$

where $\mathcal{G}_n(A)$ is gauge-invariant.

The functional \mathcal{G}_n is local, gauge-invariant, constructed with A and its derivatives, and has weight \bar{d} . Since, by assumption, \mathcal{S}_0 contains the full set of such terms, \mathcal{G}_n can be reabsorbed renormalizing the physical couplings α_i contained in \mathcal{S}_0 . We denote these renormalization constants by Z_{α_i} . On the other hand, the counterterms $\sigma \mathcal{R}_n$ are reabsorbed by the canonical transformation

$$\mathcal{F}_n(\Phi, K') = \int d^d x \sum_i \Phi^i K_i' - \mathcal{R}_n(\Phi, K'). \quad (\text{B.9})$$

Concluding, the $(n+1)$ -loop divergences are renormalized redefining the physical couplings α_i and performing a canonical transformation on the fields and the BRST sources. Such operations preserve the identities (B.3) and (B.7), which are therefore promoted to all orders.

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