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Renormalization Of A Class Of Non-Renormalizable Theories

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Abstract

Certain power-counting non-renormalizable theories, including the most general self-interacting scalar fields in four and three dimensions and fermions in two dimensions, have a simplified renormalization structure. For example, in four-dimensional scalar theories, $2n$ derivatives of the fields, $n > 1$, do not appear before the nth loop. A new kind of expansion can be defined to treat functions of the fields (but not of their derivatives) non-perturbatively. I study the conditions under which these theories can be consistently renormalized with a reduced, eventually finite, set of independent couplings. I find that in common models the number of couplings sporadically grows together with the order of the expansion, but the growth is slow and a reasonably small number of couplings is sufficient to make predictions up to very high orders. Various examples are solved explicitly at one and two loops.

1 Introduction

The divergences of power-counting non-renormalizable theories are commonly subtracted away introducing infinitely many independent couplings in the theory. In this naive formulation, non-renormalizable theories can be used only as effective field theories. Although effective field theories are good for all practical purposes, they do not have much power to suggest new physics beyond them.

Some features of the Standard Model (such as its large number of independent parameters) and certain experimental facts (such as the neutrino masses) point towards a more fundamental theory. Probably, LHC is going to tell us whether a fundamental Higgs scalar exists or not. If there exists no right-handed neutrino or no Higgs scalar, it is necessary to consider theories that include power-counting non-renormalizable interactions and study their predictive power. The investigation of non-renormalizable interactions is interesting also for its potential applications to quantum gravity, although quantum gravity has other problems beyond that of renormalization. Some physicists consider quantum field theory inadequate or limited under several respects, and look for approaches beyond quantum field theory. It is certainly useful to know what can be said on these issues within quantum field theory, before abandoning it. Probably this investigation can also suggest the best directions to go beyond quantum field theory, and clarify if it is really necessary to do so.

The present understanding of non-renormalizable theories is still imperfect. An improved formulation could be very useful. Some steps in this direction have been made in ref.s [1, 2], where finite and quasi-finite irrelevant deformations of interacting conformal field theories have been constructed. These non-renormalizable theories contain an infinite number of lagrangian terms, but only a finite number of independent couplings, and are renormalized by means of field redefinitions, plus eventually renormalization constants for the independent couplings. The constructions are perturbative, which ensures calculability. The purpose of this paper is to study the properties of another noticeable class of non-renormalizable theories, to make progress towards the inclusion of more realistic models, hopefully useful for physics beyond the Standard Model and maybe quantum gravity.

The first observation of this paper is that certain theories have a simplified renormalization structure, even if they contain power-counting non-renormalizable interactions. Using this property it is possible to treat functions of the fields (but not of their derivatives) non-perturbatively. Non-analytic dependencies are allowed also.

More precisely, there is a relation between the loop expansion and the expansion in the derivatives of the fields. For example, in four-dimensional scalar theories, terms with $2n$ derivatives of the fields, $n > 1$, do not appear before the n^{th} loop. Schematically, the renormalizable lagrangian has the form

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + {\sum_{n=0}^{\infty}}' \hbar^n [\partial^{2n}] V_n(\varphi, \hbar), \tag{1.1}$$

where $[\partial^{2n}]$ is a symbolic notation to denote $2n$ variously distributed derivatives of the fields φ, contracted in all possible ways. The prime over the sum means that only inequivalent contractions are considered. Equivalent contractions are those that differ by total derivatives and field redefinitions.

The functions $V_n(\varphi, \hbar)$ need not be polynomial, nor analytic, and can be treated non-perturbatively, working directly on the quantum action $\Gamma[\varphi]$, instead of working on correlation functions. The background field method [3] is pratically convenient for this purpose.

The quantum action Γ can be expanded in powers

$$\left(\frac{E}{M_{P\text{eff}}}\right)^m \left(\frac{\alpha M_{P\text{eff}}^2}{E^2}\right)^n, \tag{1.2}$$

where $M_{P\text{eff}}$ is an "effective Planck mass", E is the energy and α collectively denotes the marginal couplings of the theory. Every quantity is calculable with a finite number of steps at each order of this expansion, which means for energies E in the range $\alpha M_{P\text{eff}}^2 \ll E^2 \ll M_{P\text{eff}}^2$. It is not necessary to assume that the field φ is small with respect to $M_{P\text{eff}}$, but just that its momentum $p \sim E$ is. More common expansions are meaningful in the presence of a mass gap, but the expansion (1.2) is meaningful also in the absence of a mass gap.

These tools can be used to study the predictive power of the theories (1.1), in particular the conditions under which the divergences can be renormalized with a reduced, eventually finite, set of independent couplings. Start from a classical potential, e.g. $V_{\text{cl}}(\varphi, 0) = \lambda_0\varphi^4/4! + \lambda_2\varphi^6/6!$ and look for a completion (2.3) that is stable under renormalization and possibly depends only on the couplings contained in V_{cl}. Imposing consistency with renormalization and other simple requirements differential or integro-differential equations for the functions V_n are obtained. The V_n-equations determine a "corrected classical potential" $V_0(\varphi, 0)$ from the "initial conditions" $V_{\text{cl}}(\varphi, 0)$, and the functions $V_n(\varphi, \hbar)$s, $n \geq 0$, from $V_0(\varphi, 0)$. The non-perturbative solutions are regular even where the expansions in powers of the fields do not converge.

I find that, in general, the initial conditions V_{cl} are not sufficient to solve the V_n-equations unambiguously and new independent couplings need to be introduced along the way. When V_{cl} is analytic, the number of new couplings grows sporadically with the order of the perturbative expansion, but the growth is so slow that a reasonably small number of couplings is sufficient to make predictions up to very high orders. In the end the complete lagrangian (1.1) contains infinitely many couplings, although each $V_n(\varphi, \hbar)$ depends on finitely many. Instead, a class of non-analytic theories, e.g. those defined by $V_{\text{cl}}(\varphi, 0) = \lambda_0\varphi^4/4! + \lambda_r\varphi^{r+4}/(r+4)!$ with r^2 irrational, are renormalized with a finite number of independent couplings, typically just those contained in V_{cl}.

The study of quantum field theory beyond power counting has attracted interest for decades, motivated by low-energy QCD, quantum gravity and the search for new physics beyond the Standard Model. Some non-renormalizable theories can be given sense thanks to *ad hoc* procedures, such as the large N expansion used to construct three-dimensional four-fermion theories [4] and sigma models [5]. A more general proposal for non-renormalizable theories is Weinberg's asymptotic safety [6], where the theory is assumed to have an interacting fixed point in the ultraviolet, with a finite-dimensional critical surface. The RG flow tends to the fixed point only if the irrelevant couplings are appropriately fine-tuned. This fine-tuning leaves a finite number of arbitrary parameters. It is difficult to prove the existence of an interacting UV fixed point with the usual perturbative techniques, so alternative methods have been used, such as the lattice and the exact renormalization-group (ERG) approach. Asymptotic safety in the ERG framework has been recently studied for gravity [8] and the Higgs sector of the Standard Model [9].

The main difference between *a*) the ERG asymptotic-safety approach and *b*) the approach of this paper and ref.s [1, 2] is that *a*) is essentially non-perturbative and relies on the existence of a UV fixed point, while *b*) is perturbative or partially non-perturbative and does not assume the existence of a UV fixed point. In approach *a*) the attention is focused on the overall scale dependence, so the RG flow is defined varying the overall energy with respect to the dimensionful couplings and the dynamical scale μ. Instead, in approach *b*) the attention is focused on the role of divergences, so the RG flow is defined varying the renormalization point μ with respect to the the overall energy and the dimensionful couplings. In particular, the finite non-renormalizable theories of [1, 2] are fixed points of the RG flow in the *b*)-sense, but not fixed points of the RG flow in the *a*)-sense, since they contain an explicit scale, the Planck mass. The two approaches *a*) and *b*) pursue similar objectives starting from different assumptions and often using rather different techniques, so it would be extremely interesting to understand how they can be merged, to gain the maximum of information from both. For example, the partially non-perturbative techniques developed here might be useful to search for a UV fixed point in the *b*)-sense in some cases.

Techniques to reduce the number of independent couplings consistently with renormalization have been first studied by Zimmermann [10, 11] in the realm of power-counting renormalizable theories. The reduction of couplings in non-renormalizable theories has rather different properties, that deserve to be studied apart. Section 4 contains the knowledge that is useful for the purposes of this paper, while a systematic treatment of this issue is left to separate publications.

An analysis of non-renormalizable theories beyond power counting was performed by Atance and Cortes in ref.s [12, 13] within the usual expansion in powers of the fields and their derivatives, by Kubo and Nunami [14] and Halpern and Huang [15] using the Wilsonian approach. Here, besides the progress made in the directions outlined above, some claims of ref. [12] are corrected, with particular attention to the number of independent couplings, that in general grows sporadically together with the order of the perturbative expansion.

The issues treated here are technically involved. To rationalize the presentation I first derive the general properties and later give results of explicit computations in various models, at one and two loops, using the dimensional-regularization technique in the Euclidean framework. Such models do not have immediate realistic applications, but the investigation of their renormalization properties is instructive, because they provide a laboratory and a fruitful arena to test ideas that might inspire the research about physics beyond the Standard Model and the renormalization of quantum gravity.

The paper is organized as follows. In section 2 I prove the renormalizability of lagrangian structures such as (1.1) and study various properties of those, in particular their simplified dependence on the marginal and irrelevant couplings. In section 3 I derive the perturbative expansion that is compatible with (1.1). Section 4 contains the main properties of the reduction of couplings in non-renormalizable theories. In section 5 I solve the $\varphi^4+\varphi^6$ theory in four dimensions, at one and two loops. In section 6 I consider the $\varphi^4+\varphi^{m+4}$ theory in four dimensions and study under which conditions the divergences can be renormalized with a reduced, eventually finite, number of independent couplings to all orders in the perturbative expansion. In section 7 I study a case where a new coupling appears already at the tree level: the $\varphi^4+\varphi^5$ theory in four dimensions. Section 8 contains the conclusions.

2 Structure of the renormalized lagrangian

In this section I prove that certain theories have simplified renormalization structures and study how they depend on the marginal and irrelevant couplings. This class of theories includes scalar fields in four and three spacetime dimensions, fermions in two dimensions and scalar-fermion theories coupled in a peculiar way. The rationalized lagrangian of scalar fields in four dimensions is written schematically in equation (1.1). Here $2n$ derivatives of the fields, $n>1$, do not appear before the n^{th} loop. The functions $V_n(\varphi,\hbar)$s are for the moment generic functions, that renormalize with the rules worked out below. In three-dimensional scalar theories and two-dimensional fermion theories n derivatives of the fields do not appear before the n^{th} loop, for $n>2$ and $n>1$, respectively. In coupled scalar-fermion theories some non-derivative vertices, such as the Yukawa coupling, are multiplied by powers of $\hbar$.

The results of this section do not ensure that the divergences are removed with a finite number of independent couplings. Nevertheless, they classify the divergences in a convenient way, that is useful both for applications to effective field theory and to study the conditions under which a reduced, eventually finite, set of independent couplings are sufficient to renormalize the divergences.

Renormalization structure. Consider a generic non-renormalizable theory, described by a lagrangian $\mathcal{L}$. Let $\mathcal{V}$ denote the set of its vertices. Typically, these are infinitely many. Let $\widetilde{\mathcal{L}}$ denote a "reduction" of the theory, namely a lagrangian that contains only some finite subset $\widetilde{\mathcal{V}}$ of vertices. In general, the reduced theory $\widetilde{\mathcal{L}}$ is not stable under renormalization, which means that not all counterterms generated by the Feynman diagrams constructed with the vertices of the subset $\widetilde{\mathcal{V}}$ are contained in $\widetilde{\mathcal{V}}$. Then, new vertices have to be added to $\widetilde{\mathcal{L}}$, multiplied by independent couplings. Call the extended lagrangian $\widetilde{\mathcal{L}}_{\text{ext}}$ and the extended set of vertices $\widetilde{\mathcal{V}}_{\text{ext}}$. The renormalization constants of the couplings contained in $\widetilde{\mathcal{L}}_{\text{ext}} - \widetilde{\mathcal{L}}$ cancel the new divergences, those of type $\widetilde{\mathcal{V}}_{\text{ext}} \backslash \widetilde{\mathcal{V}}$. Typically $\widetilde{\mathcal{L}}_{\text{ext}}$ is itself unstable, and the extension has to be iterated, possibly infinitely many times. This iteration defines a theory $\mathcal{L}_s$ that is certainly stable, but in general contains all possible vertices (so it coincides with $\mathcal{L}$).

In this iterative procedure, the new vertices are normally introduced at the tree level. Then the stable theory $\mathcal{L}_s$ contains all possible vertices already at the tree level. However, in some models it is possible to rationalize the renormalization structure, in the following sense. Assume that the Feynman diagrams constructed with the vertices of the reduced lagrangian $\widetilde{\mathcal{L}}$ generate a divergence proportional to a certain vertex v only at order $\hbar^n$. Then this divergence is local and a simple pole [16], so it has the form

$$\frac{1}{\varepsilon}\hbar^n v \tag{2.1}$$

where $\varepsilon = D - d$, D the physical spacetime dimension and d is its continuation in dimensional regularization. Include a vertex $\hbar^{n-1}v$ in the extended theory $\widetilde{\mathcal{L}}_{\text{ext}}$ and cancel the divergence (2.1) with the $\mathcal{O}(\hbar)$-contribution to the renormalization constant Z_v of v. If also the extended theory does not generate (2.1) before the order $\hbar^n$, then the extension is stable. In this case, it is unnecessary to introduce the vertex v at orders lower than $\hbar^{n-1}$. It is said that the coupling v does not appear before the order $\hbar^{n-1}$.

In this section I study the models that admit this kind of stable renormalization structure. In the rest of the paper I use this reduction as a tool to study a further, more important, reduction, namely the reduction of couplings.

More precisely, suppose that the theory contains vertices that do not contribute at the tree level, because they are multiplied by powers of $\hbar$. Call $\mathcal{L}_n$ the lagrangian truncated up to the order $\hbar^n$ included and let $\mathcal{V}_n$ denote the set of vertices of $\mathcal{L}_n$. The set $\{\mathcal{V}_n\}$ is a *renormalization structure*. Obviously, $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$. Let $\mathcal{C}_{n+1}$ denote the counterterms of order up to $\hbar^{n+1}$. The counterterms $\mathcal{C}_{n+1}$ are originated by Feynman diagrams constructed with the vertices $\mathcal{V}_n$. Observe that the $\hbar$-powers that multiply a diagram are partially due to the number of loops and partially due to the $\hbar$-powers that multiply the vertices. The structure $\{\mathcal{V}_n\}$ is stable under renormalization if $\mathcal{C}_{n+1} \subseteq \mathcal{V}_n$ for every n.

In the example (1.1) $\mathcal{V}_n$ contains vertices with at most $2n$ derivatives and an arbitrary number of legs. Below I prove that the structure (1.1) is stable under renormalization.

Consider a classical theory of scalar fields interacting by means of the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + V(\varphi) \tag{2.2}$$

in four spacetime dimensions. For the moment I am interested only in the UV divergences of the quantum theory, and their renormalization. Then it is consistent to treat the mass term, if present, as a (two-leg) vertex of V, the propagator being just $1/k^2$. To avoid IR problems in the intermediate steps, it is convenient to calculate the UV divergences of Feynman diagrams with a deformed propagator $1/(k^2+\delta^2)$ and let δ tend to zero at the end. The limit is smooth, since the divergent parts of Feynman diagrams depend polynomially on δ. The tadpoles are loops with a single vertex and vanish identically. However, note that loops with at least two vertices are not tadpoles (even if one of the vertices is a two-leg "mass" vertex) and do give a non-trivial divergent contribution, which can be calculated at $\delta \neq 0$.

Now I prove that, taking the quantum corrections into account, the complete renormalizable lagrangian has the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + {\sum_{n=0}^{\infty}}' \hbar^n [\partial^{2n}] V_n(\varphi, \hbar). \tag{2.3}$$

Obviously, there is a finite number of independent ways to distribute and contract the partial derivatives for every n and the symbolic notation understands also the sum over them. The prime over the sum means that terms differing by total derivatives are equivalent and that contractions factorizing a $\Box\varphi$ are ignored. Indeed, the terms proportional to $\Box\varphi$ can be converted into terms of other types by means of a field redefinition. In particular, note that all terms with $n=1$ in (2.3) belong to such class, so they do not contribute to the sum.

The tree lagrangian coincides with (2.2), where $V(\varphi) = V_0(\varphi, 0)$. The functions $V_n(\varphi, \hbar)$ are power series in $\hbar$ and can be understood as field-dependent "coupling constants" of the theory. In particular, the renormalization is achieved by means of a field redefinition

$$\varphi \to \phi(\varphi) = \varphi + \hbar \sum_{n=0}^{\infty} \hbar^n [\partial^{2n}] F_n(\varphi, \hbar, \varepsilon), \qquad F_n(\varphi, \hbar, \varepsilon) = \sum_{m=1}^{\infty} \frac{\hbar^m}{\varepsilon^{P'(m)}} F_{n,m}(\varphi, \varepsilon), \tag{2.4}$$

plus redefinitions

$$V_n(\varphi, \hbar) \to V_{\mathrm{R}n}(\phi, \hbar, \varepsilon) = V_n(\phi, \hbar) + \sum_{m=1}^{\infty} \frac{\hbar^m}{\varepsilon^{P(m)}} V_{n,m}(\phi, \varepsilon), \tag{2.5}$$

where $P(m) \leq m$, $P'(m) \leq m$ and $V_{n,m}(\phi, \varepsilon)$, $F_{n,m}(\varphi, \varepsilon)$ are analytic functions of ε. The renormalized lagrangian reads

$$\mathcal{L}_{\mathrm{R}}(\varphi) = \frac{1}{2}(\partial_\mu \phi)^2 + {\sum_{n=0}^{\infty}}' \hbar^n [\partial^{2n}] V_{\mathrm{R}n}(\phi, \hbar, \varepsilon), \tag{2.6}$$

or, equivalently,

$$\mathcal{L}_{\mathrm{R}}(\varphi) = \frac{1}{2}(\partial_\mu\varphi)^2 + \sum_{n=0}^{\infty} \hbar^n [\partial^{2n}] \widetilde{V}_{\mathrm{R}n}(\varphi, \hbar, \varepsilon), \tag{2.7}$$

where now the sum includes contractions of indices factorizing $\Box\varphi$, up to total derivatives, and the functions $\widetilde{V}_{\mathrm{R}n}$ have the same structure as the functions $V_{\mathrm{R}n}$ of (2.5). All functions of $\hbar$ written in this paper are meant to be power series in $\hbar$.

With the exception of the free kinetic term, each finite term of (2.3), (2.6) and (2.7) satisfies the key inequality

$$\omega_\partial \leq 2\omega_\hbar, \tag{2.8}$$

where ω_∂ is the number of derivatives and $\omega_\hbar$ is the power of $\hbar$. All counterterms satisfy

$$\omega_\partial \leq 2(\omega_\hbar - 1), \tag{2.9}$$

while the counterterms contained in the field redefinition (2.4) satisfy

$$\omega_\partial \leq 2(\omega_\hbar - 2). \tag{2.10}$$

Observe that a field redefinition that fulfills (2.10) preserves (2.8) and (2.9).

Proof of renormalizability. Now I prove that the lagrangian (2.3) is renormalizable in the form (2.6).

Let G denote a Feynman diagram with L loops, V vertices, I internal legs and v_n vertices with $2n$ derivatives. The superficial degree of divergence is

$$\omega(G) = -2I + 4L + 2\sum_n n v_n. \tag{2.11}$$

Using the formula $I = L + V - 1$, that relates the number of internal legs to the number of loops and the number of vertices, (2.11) becomes

$$\omega(G) = 2\left(L + \sum_n n v_n - V + 1\right). \tag{2.12}$$

Formula (2.3) shows that every derivative vertex carries a power of $\hbar$. The total $\hbar$-power of the diagram G is denoted with L_e and called "effective number of loops", to distinguish it from the true number of loops L. Clearly, L_e is equal to L plus the powers of $\hbar$ attached to the vertices. Each vertex with $2n$ derivatives carries at least n powers of $\hbar$. Write

$$L_e = L + \sum_n n v_n + \Delta, \qquad \Delta \geq 0. \tag{2.13}$$

Then

$$\omega(G) = 2(L_e + 1 - \Delta - V) \leq 2(L_e - V + 1). \tag{2.14}$$

The last inequality shows that the degree of divergence decreases when the number of vertices increases, therefore the number of divergent diagrams of order $\hbar^{L_e}$ is finite. Since the mass term is treated as a vertex, the tadpoles vanish, so the number V of vertices can be restricted to $V \geq 2$, which implies

$$\omega(G) \leq 2(L_e - 1). \tag{2.15}$$

In particular, the counterterms satisfy (2.9).

Now, proceed inductively in L_e. Assume that the theory is renormalized by (2.6) and (2.4) up to the order $\hbar^{L_e - 1}$ included. Then the theorem of locality of counterterms [16] ensures that the $\mathcal{O}(\hbar^{L_e})$-counterterms are local. Thus the counterterms due to the diagram G have the form

$$\frac{1}{\varepsilon^{P(G)}}\hbar^{L_e}[\partial^{\omega(G)}]H_G(\varphi,\varepsilon) = \hbar^{\omega(G)/2}[\partial^{\omega(G)}]\frac{\hbar^{L_e-\omega(G)/2}}{\varepsilon^{P(G)}}H_G(\varphi,\varepsilon),$$

where $P(G)$ is the maximal pole of the diagram and $H_G(\varphi,\varepsilon)$ is a certain function of φ, analytic in ε. The theorem proved in the appendix ensures that $P(G) \leq V - 1$. Then, associating a factor $\hbar^{\omega(G)/2}$ with the derivatives as in (2.3), the remaining power of $\hbar$ satisfies

$$L_e - \frac{1}{2}\omega(G) = V - 1 + \Delta \geq V - 1 \geq P(G),$$

namely the counterterms have the form specified by (2.4), (2.5) and (2.6).

In total, using (2.15), the $\mathcal{O}(\hbar^{L_e})$-counterterms read

$$\Delta\mathcal{L}_{L_e} = \hbar^{L_e}\sum_{k=0}^{L_e-1}\frac{1}{\varepsilon^{p(k)}}[\partial^{2k}]H_{L_e,k}(\varphi,\varepsilon), \tag{2.16}$$

where $p(k) \leq L_e - k$ and $H_{L_e,k}(\varphi,\varepsilon)$ are certain analytic functions of ε.

Consider the counterterms that factorize a $\Box\varphi$, which have the form

$$\Delta_1\mathcal{L}_{L_e} = \hbar^{L_e}\Box\varphi\sum_{k=1}^{L_e-1}\frac{1}{\varepsilon^{p(k)}}[\partial^{2(k-1)}]K_{L_e,k}(\varphi,\varepsilon).$$

Such counterterms can be reabsorbed into a field redefinition

$$\delta\varphi = -\hbar^{L_e}\sum_{k=1}^{L_e-1}\frac{1}{\varepsilon^{p(k)}}[\partial^{2(k-1)}]K_{L_e,k}(\varphi,\varepsilon) \tag{2.17}$$

of type (2.4), since it satisfies (2.10) and $p(k) \leq L_e - k$. The divergent terms generated by this field redefinition contain all orders in $\hbar$, starting from $\hbar^{L_e}$. They respect the inequality (2.9) and the structure (2.7):

$$\mathcal{L}_{\mathrm{R}}[\phi(\varphi)+\delta\varphi] - \mathcal{L}_{\mathrm{R}}[\phi(\varphi)] = -\delta\varphi\Box\phi - \frac{1}{2}\delta\varphi\Box\delta\varphi + \sum_{j=1}^{\infty}\frac{1}{j!}(\delta\varphi)^j\frac{\partial^j}{\partial\phi^j}{\sum_{n=0}^{\infty}}'\hbar^n[\partial^{2n}]V_{\mathrm{R}n}(\phi,\hbar,\varepsilon). \tag{2.18}$$

The orders higher than $\hbar^{L_e}$ can be ignored at this step of the inductive procedure. In the subsequent steps they will be subtracted away just as they come (i.e. as "diagrams" with no true loop, $L=0$), because they are local, divergent and satisfy (2.9),(2.15), as the counterterms with $L>0$. The counterterms of order $\hbar^{L_e}$ contained on the left-hand side of (2.18) are

$$\delta\mathcal{L}\equiv\Delta_1\mathcal{L}_{L_e}+\frac{\partial V_{\mathrm{R}0}}{\partial\varphi}\delta\varphi. \tag{2.19}$$

It is immediate to prove that the difference $\Delta\mathcal{L}_{L_e}-\delta\mathcal{L}$ has the same form as (2.16),

$$\hbar^{L_e}\sum_{k=0}^{L_e-1}\frac{1}{\varepsilon^{p(k)}}[\partial^{2k}]H'_{L_e,k}(\varphi,\varepsilon), \tag{2.20}$$

for certain modified functions H', but carries fewer factors $\Box\varphi$ than (2.16). In general (2.20) is not in the final irreducible form, yet. Other terms factorising $\Box\varphi$ can be brought by the last term of (2.19), due to factors $\Box\varphi$ inside $\delta\varphi$ (2.17), or, equivalently, factors $(\Box\varphi)^n$, $n>1$, in (2.16). Obviously, factors $(\Box\varphi)^n$ in (2.16) generate factors $(\Box\varphi)^{n-1}$ in (2.4) and therefore (2.20). Thus, repeating the field-redefinition procedure (2.16)→(2.20) a finite number of times, the surviving $\mathcal{O}(\hbar^{L_e})$-counterterms acquire the irreducible form

$$\hbar^{L_e}\sum_{k=0}^{L_e-1}{}'\frac{1}{\varepsilon^{p(k)}}[\partial^{2k}]\widetilde{H}_{L_e,k}(\varphi,\varepsilon). \tag{2.21}$$

The irreducible counterterms (2.21) are reabsorbed into redefinitions

$$V_{\mathrm{R}n}(\phi,\hbar,\varepsilon)\rightarrow V_{\mathrm{R}n}(\phi,\hbar,\varepsilon)+\frac{\hbar^{L_e-n}}{\varepsilon^{p(n)}}\widetilde{H}_{L_e,n}(\phi,\varepsilon),\qquad \text{for } n=0 \text{ and } 1<n<L_e, \tag{2.22}$$

which have the form (2.5). This concludes the proof that the structure (2.6) is preserved by renormalization.

Dependence on the marginal and irrelevant couplings. Now, suppose that the theory (2.2) has no relevant coupling, a marginal coupling α and irrelevant couplings λ. Let M_P denote a reference scale for the irrelevant couplings, which I call the "Planck mass". Now I prove that the renormalization structure (2.3) is compatible with the following α-M_P-$\hbar$-dependence:

$$V_n(\alpha,\varphi,M_P,\hbar)\equiv\alpha M_P^{4-2n}W_n(\hbar\alpha,\chi),\qquad W_n(\hbar\alpha,\chi)=\sum_{k=0}^{\infty}(\hbar\alpha)^kW_{n,k}(\chi), \tag{2.23}$$

where $\chi=\varphi/M_P$. Then the renormalization structure reads

$$\mathcal{L}=\frac{M_P^2}{2}(\partial_\mu\chi)^2+\sum_{n=0}^{\infty}{}'\hbar^n\alpha M_P^{4-2n}[\partial^{2n}]W_n(\hbar\alpha,\chi), \tag{2.24}$$

whose terms satisfy

$$\omega_\partial + 2(\omega_\alpha - \omega_\hbar) = 2, \tag{2.25}$$

and, with the exception of the free kinetic term, $\omega_\alpha \geq 1$, ω_α denoting the power of α. The renormalized lagrangian is obtained from (2.24) with the replacements

$$W_n(\hbar\alpha, \chi) \rightarrow W_{\mathrm{R}n}(\hbar\alpha, \chi, \varepsilon) = W_n(\hbar\alpha, \chi) + \sum_{m=1}^{\infty} \frac{(\hbar\alpha)^m}{\varepsilon^{P(m)}} W_{n,m}(\chi, \varepsilon), \tag{2.26}$$

and

$$\chi \rightarrow \chi + \hbar\alpha \sum_{n=0}^{\infty} \hbar^n M_P^{-2n} [\partial^{2n}] F_n(\hbar\alpha, \chi, \varepsilon), \qquad F_n(\hbar\alpha, \chi, \varepsilon) = \sum_{m=1}^{\infty} \frac{(\hbar\alpha)^m}{\varepsilon^{P'(m)}} F_{n,m}(\chi, \varepsilon), \tag{2.27}$$

with $P(m) \leq m$, $P'(m) \leq m$ and $W_{n,m}$, $F_{n,m}$ analytic functions of ε. The divergent contributions of (2.26) satisfy (2.25) with $\omega_\alpha \geq 2$, while those of (2.27) satisfy $\omega_\partial + 2(\omega_\alpha - \omega_\hbar) = 0$ with $\omega_\alpha \geq 2$. Obviously, a lagrangian that satisfies (2.25) is preserved by a field redefinition (2.27).

Consider a diagram G of order $\hbar^{L_e}$ with L loops, I internal legs and $w_{n,k}$ vertices of type $W_{n,k}$. Let $v_n = \sum_k w_{n,k}$, $V = \sum_n v_n$. The total power of $\hbar$ attached to the vertices is

$$L_e - L = \sum_{n,k} (k+n) w_{n,k} \tag{2.28}$$

and the total power of α multiplying the diagram is, using the definition (2.13),

$$\sum_{n,k} (k+1) w_{n,k} = \sum_{n,k} (k+n-n) w_{n,k} + V = L_e - L - \sum_n n v_n + V = V + \Delta. \tag{2.29}$$

The structure of the counterterms is therefore

$$\Delta\mathcal{L}_{L_e} = \hbar^{L_e} \sum_G \frac{1}{\varepsilon^{P(G)}} \alpha^{V+\Delta} M_P^{4V-2\sum_n n v_n - 2I} [\partial^{\omega(G)}] H_G(\chi, \varepsilon),$$

with $P(G) \leq V-1$. The power M_P^{-2I} is due to the propagators $\hbar/(p^2 M_P^2)$ of the rescaled fields χ. Using (2.12) and (2.29) we have

$$\Delta\mathcal{L}_{L_e} = \sum_G \hbar^{N(G)} \alpha M_P^{4-2N(G)} \frac{(\hbar\alpha)^{L_e - N(G)}}{\varepsilon^{P(G)}} [\partial^{2N(G)}] H_G(\chi, \varepsilon), \tag{2.30}$$

where $N(G) = L + \sum_n n v_n - V + 1 = \omega(G)/2$. Observe that, from (2.15), $L_e - N(G) \geq 1$, so (2.30) satisfies (2.25) with $\omega_\alpha \geq 2$. Moreover, $P(G) \leq L_e - N(G)$, in agreement with (2.26). The terms proportional to the field equations are reabsorbed with field redefinitions of the form (2.27). After the field redefinitions the renormalization of $W_{N(G)}$ reads

$$W_{N(G)}(\hbar\alpha, \chi) \rightarrow W_{N(G)}(\hbar\alpha, \chi) + \frac{(\hbar\alpha)^{L_e - N(G)}}{\varepsilon^{P(G)}} \widetilde{H}_G(\chi, \varepsilon).$$

Thus the structure (2.23)-(2.24) is preserved by renormalization.

Extensions to other theories. The renormalization structure (2.3) is peculiar of scalar field theories in four dimensions. No similar renormalization structure works in higher dimensions, but the results proved so far do admit generalizations to theories in lower dimensions and to other four-dimensional theories.

For example, it is immediate to prove that scalar field theories in three spacetime dimensions admit the renormalization structure

$$\mathcal{L}_{D=3}=\frac{1}{2}(\partial_\mu\varphi)^2+{\sum_{n=0}^{\infty}}'\hbar^{2n}[\partial^{2n}]V_n(\varphi,\hbar). \tag{2.31}$$

Indeed, formula (2.11) is corrected replacing $4L$ with $3L$, so

$$\omega(G)=L+2\sum_n nv_n-2V+2.$$

Moreover from (2.31) the effective number of loops satisfies

$$L_e\geq L+2\sum_n nv_n,$$

whence

$$\omega(G)\leq L_e-2V+2.$$

Finally $V\geq 2$ implies

$$\omega(G)\leq L_e-2. \tag{2.32}$$

The inequality (2.32) is the right inequality to prove that the structure (2.31) is preserved by renormalization, proceeding as before. Only even Ls have non-trivial logarithmic divergences in odd dimensions, so L_e is even. After appropriate field redefinitions the irreducible counterterms of order $\hbar^{L_e}$ have the form

$$\hbar^{L_e}{\sum_{k=0}^{L_e/2-1}}'\frac{1}{\varepsilon^{p(k)}}[\partial^{2k}]\widetilde{H}_{L_e,k}(\varphi,\varepsilon),$$

with $p(k)\leq L_e/2-k$ and are renormalized with the replacements

$$V_n(\varphi,\hbar,\varepsilon)\rightarrow V_n(\varphi,\hbar,\varepsilon)+\frac{\hbar^{L_e-2n}}{\varepsilon^{p(n)}}\widetilde{H}_{L_e,n}(\varphi,\varepsilon),\qquad \text{for } n=0 \text{ and } 1<n<L_e/2,$$

thus preserving (2.31). Finally, (2.24) is replaced with

$$\mathcal{L}=\frac{M_P}{2}(\partial_\mu\chi)^2+{\sum_{n=0}^{\infty}}'\hbar^{2n}\alpha^2M_P^{3-2n}[\partial^{2n}]W_n(\hbar\alpha,\chi),\qquad W_n(\hbar\alpha,\chi)=\sum_{k=0}^{\infty}(\hbar\alpha)^{2k}W_{n,k}(\chi),$$

with $\chi=\varphi/M_P^{1/2}$.

Similar conclusions hold for fermionic theories of the form

$$\mathcal{L}_{D=2} = \overline{\psi}\partial\!\!\!/\psi + V(\overline{\psi}, \psi, \lambda) \tag{2.33}$$

in two spacetime dimensions. If the number of fermions is finite, a non-derivative potential $V(\overline{\psi}, \psi, \lambda)$ necessarily contains a finite number of terms, and so it is polynomial. However, the construction of this section applies also when the number of fermions is infinite (which is useful to study the large N limit, for example), in which case $V(\overline{\psi}, \psi, \lambda)$ is non-polynomial. The complete renormalizable lagrangian has the form

$$\mathcal{L}_R = \overline{\psi}\partial\!\!\!/\psi + \sum_{n=0}^{\infty}{}' \hbar^n [\partial^n] V_n(\overline{\psi}, \psi, \lambda). \tag{2.34}$$

Coupled models with simplified renormalization structures exist also. An example is the scalar-fermion theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \overline{\psi}\partial\!\!\!/\psi + \sum_{n,p=0}^{\infty}{}' \hbar^{(n+p)/2} [\partial^n] (\overline{\psi}\psi)^p V_{np}(\varphi, \hbar) \tag{2.35}$$

in four dimensions. Observe that the Yukawa interactions has a factor $\hbar^{1/2}$. These developments and the study of applications to physics beyond the Standard Model are left to a separate investigation.

3 Perturbative expansion for the quantum action

In this section I discuss expansions compatible with the renormalization structure (2.3),(2.24). In a non-renormalizable theory it is generically necessary to perform a double expansion: the loop expansion and the expansion in powers $(E/M_P)^n$ of the energy E, where M_P is a reference scale for the irrelevant interactions. The expansion in powers of the energy is an expansion in the dimensionality of the irrelevant operators, which means in general an expansion in powers of the fields and their derivatives. The loop expansion can be traded for the expansion in powers of the marginal couplings, denoted collectively with α, which is "orthogonal" to the expansion in powers of the energy. Summarizing, the ordinary expansion for a non-renormalizable theory is in powers of α and E/M_P, so it is necessary to assume that

$$\alpha \ll 1, \qquad E \ll M_P, \tag{3.1}$$

and $\varphi \sim E$, $\partial\varphi \sim E^2$, etc. An expansion of this type applies also to the theories studied here, once the functions V_n are expanded in powers of φ. However, when the V_ns are treated non-perturbatively, new kinds of expansions apply.

Using the background field method [3] the Feynman diagrams are well defined when the action is non-polynomial or non-analytic in the fields, and it is not expanded in powers of the fields, but only in their derivatives. Indeed, expanding a generic function $V(\varphi)$ around a background φ_0, $\varphi = \varphi_0 + h$,

$$V(\varphi) = V(\varphi_0) + hV'(\varphi_0) + \frac{1}{2}h^2V''(\varphi_0) + \cdots,$$

each vertex has an integer number of h-legs, which are the internal legs of the diagrams. The external "legs" need not be an integer number. They are the functions $V'(\varphi_0), V''(\varphi_0)\ldots$ When $V(\varphi)$ is not analytic the vertices are always infinitely many, and contain also negative powers of φ. Therefore the natural framework to study non-analytic theories is non-renormalizable quantum field theory.

The renormalization structure (2.3),(2.24) can be used in two contexts: i) in the realm of effective field theory, where the functions V_n depend on infinitely many independent couplings; ii) in the realm of fundamental field theory, where the number of independent couplings can be consistently reduced. In the next section I prove that in case ii) (2.24) is replaced by

$$\mathcal{L} = \frac{M_{P\mathrm{eff}}^2}{2}(\partial_\mu\chi)^2 + {\sum_{n=0}^{\infty}}' \hbar^n \alpha M_{P\mathrm{eff}}^{4-2n}[\partial^{2n}]W_n(\hbar\alpha,\chi), \tag{3.2}$$

where $M_{P\mathrm{eff}}$ is a "dressed" Planck mass and now $\chi = \varphi/M_{P\mathrm{eff}}$.

I now prove that in both cases i) and ii) there exists a meaningful perturbative expansion such that the physical quantities can be calculated algorithmically with a finite number of steps at each order. I work with the case ii) for definiteness, case i) being easily obtained from ii) with the replacements $M_{P\mathrm{eff}} \to M_P$, $\chi \to \varphi/M_P$. For the time being I study a theory that does not contain relevant parameters, so in particular there are no masses.

Consider the contributions $\Delta\Gamma_G = \int \delta\gamma_G$ to the quantum action $\Gamma = \int \gamma$ due to the renormalized diagrams G of order L_e in $\hbar$, with V vertices and L loops. Since there are no masses, the dependence of $\delta\gamma_G$ on the overall energy E is of the form $\delta\gamma_G(E,\mu) = E^{\omega(G)}\delta\widetilde{\gamma}_G(E/\mu)$, where μ is the subtraction point. The μ-dependence of $\delta\widetilde{\gamma}_G(E/\mu)$ is only logarithmic, the maximal power of $\ln\mu$ being bounded by L_e. Moreover, because of (3.2), the power of α multiplying the diagram is given by (2.29). Collecting these pieces of information, the α-E-$M_{P\mathrm{eff}}$ dependence of $\delta\gamma_G$ is

$$\delta\gamma_G = M_{P\mathrm{eff}}^4 \alpha^{V+\Delta}(M_{P\mathrm{eff}}/E)^{-\omega(G)} \sum_{k\leq L_e} c_k \ln^\kappa(E/\mu), \tag{3.3}$$

where c_k are coefficients that depend on the dimensionless ratios among the external momenta. The logarithms are reabsorbed into the running couplings, including α and $M_{P\mathrm{eff}}$, and into the running field. It is convenient to choose the subtraction point μ in the same range as the energy E (see below), because then the logarithms are of order one. For simplicity, I keep the symbols

α and $M_{P\text{eff}}$ for the running constants. Using (2.14), we have

$$\alpha^{V+\Delta}\left(\frac{M_{P\text{eff}}}{E}\right)^{-\omega(G)}=\left(\frac{E}{M_{P\text{eff}}}\right)^{2(L_e+1)}\left(\frac{\alpha M_{P\text{eff}}^2}{E^2}\right)^{V+\Delta}.$$

Now, the number of diagrams with given L_e and V is finite. Indeed, (2.28) implies $w_{n,k}=0$ for $n+k$ sufficiently large, so finitely many types of vertices can be used to build the diagrams. Moreover, $w_{n,k}\leq V$ and $I=L+V-1\leq L_e+V-1$, so the number of vertices of each type and the number of internal legs are finite. Since $0\leq\Delta\leq L_e$, also the number of diagrams with given L_e and $V+\Delta$ is finite. Thus if $E\ll M_{P\text{eff}}$ and $\alpha M_{P\text{eff}}^2\ll E^2$ every quantity is computable with a finite number of steps at each order of the expansion, both in the renormalization structure (2.3) and in the quantum action Γ.

Summarizing, in the absence of a mass gap, the expansion, at the level of counterterms and in the quantum action Γ, is meaningful for energies contained in the range

$$\alpha M_{P\text{eff}}^2\ll E^2\ll M_{P\text{eff}}^2, \tag{3.4}$$

while the field φ is arbitrary, in particular it can be of order $M_{P\text{eff}}$. The marginal couplings α have to be small, so the renormalizable subsector is weakly coupled. Observe that (3.4) implies (3.1) (with $M_P\rightarrow M_{P\text{eff}}$), but not viceversa.

In the presence of a mass gap m some cases need to be distinguished. If $m^2\lesssim\alpha M_{P\text{eff}}^2$ then the mass gap can in practice be neglected and everything works as before. In this case it is meaningful to expand in powers of m/E. If $\alpha M_{P\text{eff}}^2\ll m^2\ll M_{P\text{eff}}^2$ it is meaningful to consider energies up to $E^2\ll M_{P\text{eff}}^2$. For energies $E^2\ll m^2$ it is possible to consider the expansion in E/m and the behavior of $\delta\gamma_G$ is certainly bounded by (3.3) with $E\rightarrow m$. Finally, when there is no mass gap, but some particles are massive, it is sufficient to assume $\alpha M_{P\text{eff}}^2\ll E^2\ll M_{P\text{eff}}^2$, $\alpha M_{P\text{eff}}^2\ll m^2\ll M_{P\text{eff}}^2$.

4 Reduction of couplings

The rationalized renormalization structure (2.3),(2.24) is a remarkable restriction on non-renormalizable theories, but still contains infinitely many independent couplings. In this section I derive general conditions to renormalize the divergences with a reduced, eventually finite, number of independent couplings. It is sufficient to focus the attention on the logarithmic divergences, setting the power-like divergences aside. Indeed, the power-like divergences are RG invariant and can be unambiguously subtracted away just as they come, without introducing new independent couplings. The logarithmic divergences can be studied at the level of the renormalization group, because the logarithms of the subtraction point μ are in one-to-one correspondence with the logarithms of the cut-off Λ.

The arguments of this section apply to the theories of the form (2.3), but immediate generalizations work also with the most general non-renormalizable theory (see [7] for details). For definiteness, I work in even d spacetime dimensions. The generalization to odd dimensions is straightforward, keeping in mind that, since the diagrams with an odd number of loops have no logarithmic divergence, in the existence conditions derived below the one-loop coefficients of the beta functions and anomalous dimensions are replaced by two-loop coefficients, and so on. For the moment I set $\hbar = 1$.

Consider a strictly renormalizable theory $\mathcal{R}$ with marginal couplings α and no relevant coupling. Let O_λ denote a basis of "essential", local, symmetric, scalar, canonically irrelevant operators constructed with the fields of $\mathcal{R}$ and their derivatives. The essential operators are the equivalence classes of operators that differ by total derivatives and terms proportional to the field equations [6]. Total derivatives are trivial in perturbation theory, while terms proportional to the field equations can be renormalized away by means of field redefinitions, so they do not affect the beta functions of the physical couplings. Finally, the operators O_λ are Lorentz scalars and have to be "symmetric", that is to say invariant under the non-anomalous symmetries of the theory, up to total derivatives.

I assume that each O_λ has a definite canonical dimensionality d_λ in units of mass. The irrelevant terms can be ordered according to their "level" $\ell = d_\lambda - d$. It is understood that in general each level contains several (but anyway finitely many) operators, which can mix under renormalization. For the moment I do not distinguish operators of the same level. I collectively denote the irrelevant couplings of level n with λ_n. On dimensional grounds, the general form of the beta function β_n is (see also [1, 2])

$$\beta_n = \lambda_n \gamma_n(\alpha) + \delta_n, \tag{4.1}$$

where δ_n depends only on λ_p with $p < n$ and α, while γ_n is the anomalous dimension of O_{λ_n}, calculated in the undeformed theory $\mathcal{R}$. Observe that a simple structure such as (4.1) follows from the assumption that no relevant parameters are around. Define the marginal coupling α so that the lowest α-orders read

$$\beta_\alpha = \alpha^2 \beta_0^{(1)} + \mathcal{O}(\alpha^3), \qquad \beta_\ell = \lambda_\ell \left(\alpha \gamma_\ell^{(1)} + \mathcal{O}(\alpha^2)\right), \tag{4.2}$$

$$\beta_n = \lambda_n \left(\alpha \gamma_n^{(1)} + \mathcal{O}(\alpha^2)\right) + \delta_n(\lambda, \alpha), \qquad n > \ell. \tag{4.3}$$

Consider an irrelevant deformation with "head" λ_ℓ of level ℓ. The head of the deformation is the irrelevant term with the lowest dimensionality. The "queue" of the deformation is made of the irrelevant terms that have dimensionalities greater than ℓ, whose couplings are not independent, but functions of α and λ_ℓ:

$$\mathcal{L}_{\mathrm{cl}}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, \alpha] + \lambda_\ell \mathcal{O}_\ell(\varphi) + \sum_{n=2}^{\infty} \lambda_{n\ell}(\lambda_\ell, \alpha) \mathcal{O}_{n\ell}(\varphi). \tag{4.4}$$

By dimensional arguments, the queue of the deformation is made only of terms whose levels are integer multiples of ℓ. All other λ_ns are set to zero.

The goal is to find a "reduction of couplings", namely a set of functions $\lambda_{n\ell}(\lambda_\ell, \alpha)$, $n > 1$, such that the theory is renormalized by means of field redefinitions plus renormalization constants for α and λ_ℓ. Dimensional arguments ensure that the functions $\lambda_{n\ell}(\lambda_\ell, \alpha)$ have the form

$$\lambda_{n\ell} = f_n(\alpha)\lambda_\ell^n \tag{4.5}$$

and consequently the form of the beta functions (4.1) is

$$\beta_{n\ell} = \lambda_\ell^n \left[f_n(\alpha)\gamma_{n\ell}(\alpha) + \delta_n(f, \alpha) \right], \tag{4.6}$$

where $\delta_n(f, \alpha)$ depends only on f_k with $k < n$. Formulas (4.5) and (4.6) hold also for $n = 1$, with $f_1 = 1$ and $\delta_1 = 0$. Differentiating the functions (4.5) with respect to the dynamical scale μ and using the definitions of beta functions, the RG consistency equations

$$f_n'(\alpha)\beta_\alpha - f_n(\alpha)\left[\gamma_{n\ell}(\alpha) - n\gamma_\ell(\alpha)\right] = \delta_n(f, \alpha) \tag{4.7}$$

are obtained. Now I show that these equations are necessary and sufficient conditions to renormalize the logarithmic divergences of the theory by means of renormalization constants just for λ_ℓ and α, plus field redefinitions.

Write the bare couplings $\lambda_{n\ell}(\Lambda)$ and $\alpha(\Lambda)$ in terms of their renormalization constants $Z_{n\ell}$ and Z_α in the minimal subtraction scheme, where $\lambda_{n\ell}$ and α are the renormalized couplings at the subtraction point μ:

$$\lambda_{n\ell}(\Lambda) = \lambda_{n\ell} Z_{n\ell}(\lambda, \alpha, \ln \Lambda/\mu), \qquad \alpha(\Lambda) = \alpha Z_\alpha(\alpha, \ln \Lambda/\mu).$$

Now, assume that $\lambda_{n\ell}$ satisfy (4.5) and (4.7). The RG consistency conditions (4.7) imply that the reduction relations have the same form at every energy scale, in particular μ and Λ. Consequently,

$$\lambda_{n\ell} Z_{n\ell} = \lambda_{n\ell}(\Lambda) = f_n(\alpha(\Lambda))\lambda_\ell^n(\Lambda) = \lambda_\ell^n Z_\ell^n f_n(\alpha Z_\alpha), \qquad n > 1. \tag{4.8}$$

Thus the couplings $\lambda_{n\ell}$, $n > 1$, can be renormalized just attaching renormalization constants to λ_ℓ and α.

However, no true reduction of couplings is achieved simply solving the RG consistency conditions (4.7). Indeed, (4.7) are differential equations for the unknown functions $f_n(\alpha)$, $n > 1$. The solutions depend on arbitrary constants ξ. From the point of view of the renormalization, the arbitrary constants ξ are finite parameters, namely $Z_\xi \equiv 1$. The equations (4.7) and the arguments leading to (4.8) are simply a rearrangement of renormalization, with no true gain, because the number of renormalization constants is reduced at the price of introducing new constants ξ_n.

To remove the ξ-ambiguity contained in the solutions of (4.7) and achieve a true reduction of couplings, extra assumptions have to be made.

A similar problem appears in the context of renormalizable theories, where Zimmermann proposed to eliminate the ξ-arbitrariness requiring that the reduction relations be analytic [10]. However, in the realm of non-renormalizable theories analyticity is a too restrictive requirement: negative powers of the marginal coupling α appear [1, 2] and are reabsorbed into the effective Planck mass $M_{P\text{eff}}$. Thus the correct requirement is meromorphy.

For example, in the beta function $\beta_{2\ell}$ the coupling $\lambda_{2\ell}$ is multiplied by $\gamma_{2\ell} = \mathcal{O}(\alpha)$, thus $f_2(\alpha)$ has the form

$$f_2(\alpha) = \frac{1}{\alpha} \sum_{k=0}^{\infty} d_{2,k} \alpha^k. \tag{4.9}$$

Inserting the ansatz (4.9) into (4.7), using (4.2) and solving for $d_{2,k}$ recursively in k, it is easy to check that the coefficients $d_{2,k}$ have the form

$$d_{2,k} = \frac{P_{2,k}}{\prod_{j=1}^{k+1} \left(\gamma_{2\ell}^{(1)} - 2\gamma_{\ell}^{(1)} + (2-j)\beta_0^{(1)} \right)}. \tag{4.10}$$

The numerator $P_{2,k}$ depends polynomially on the coefficients of the beta functions $\beta_{n\ell}, \beta_\alpha$ and in general does not vanish when one of the factors appearing in the denominator vanishes. Thus, assuming that $\beta_0^{(1)} \neq 0$, the meromorphic solution (4.9) is meaningful, and unique, when the quantity

$$r_{2,\ell} \equiv \frac{\gamma_{2\ell}^{(1)} - 2\gamma_{\ell}^{(1)}}{\beta_0^{(1)}} + 1 \tag{4.11}$$

is not a natural number.

Now I prove that, if $\beta_0^{(1)} \neq 0$ and

$$r_{n,\ell} \equiv \frac{\gamma_{n\ell}^{(1)} - n\gamma_{\ell}^{(1)}}{\beta_0^{(1)}} + n - 1 \notin \mathbb{N}, \qquad n > 1, \tag{4.12}$$

there exists a unique meromorphic reduction such that the couplings $\lambda_{k\ell}$ behave at worst as [1, 2]

$$\lambda_{k\ell} \sim c_k \frac{\lambda_\ell^k}{\alpha^{k-1}}, \tag{4.13}$$

for small α, where c_k are constants. This result is proved by induction. It is certainly true for $k = 1$ and follows from (4.9) for $k = 2$. Assume that it is true for $k < n$. Since δ_n depends on the λs with lower levels, we have, by dimensional analysis,

$$\delta_{n\ell} \sim \sum_{\{n_k\}} \prod_{k<n} {\lambda_{k\ell}}^{n_k} \left(1 + \mathcal{O}(\alpha)\right),$$

where the sum is made over the sets $\{n_k\}$ of non-negative integers n_k such that $\sum_{k<n} kn_k = n$. Moreover, $m \equiv \sum_{k<n} n_k \geq 2$, since δ_n is at least quadratic. Therefore

$$\delta_{n\ell} \sim \lambda_\ell^n \sum_{\{n_k\}} \prod_{k<n} \left(\frac{1}{\alpha^{k-1}}\right)^{n_k} = \lambda_\ell^n \sum_{\{n_k\}} \frac{1}{\alpha^{n-m}} \lesssim \lambda_\ell^n \frac{1}{\alpha^{n-2}}. \tag{4.14}$$

The general form of the meromorphic reduction relation is

$$f_n(\alpha) = \frac{1}{\alpha^p} \sum_{k=0}^{\infty} d_{n,k}\alpha^k. \tag{4.15}$$

Inserting this ansatz into (4.7), using (4.15) and solving for $d_{n,k}$ recursively in k it is immediate to find that $p = n-1$ and the coefficients $d_{n,k}$ have expressions

$$d_{n,k} = \frac{P_{n,k}}{\prod_{j=1}^{k+1}\left(\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)} + (n-j)\beta_0^{(1)}\right)}, \tag{4.16}$$

where $P_{n,k}$ depends polynomially on the coefficients of the beta functions and on $d_{m,k}$, $m < n$. In general the numerator $P_{n,k}$ does not vanish when the denominator vanishes. Thus (4.13) is inductively proved for arbitrary n.

In conclusion, when the invertibility conditions (4.12) are fulfilled, there exists a unique meromorphic reduction with the behavior (4.13).

The quantities $r_{n,\ell}$ depend only on one-loop coefficients, yet the conditions (4.12) determine the existence of the meromorphic reduction to all orders. Moreover, the $r_{n,\ell}$s are just rational numbers and it is not unfrequent that they coincide with natural numbers for some ns, so sometimes the invertibility conditions are violated.

Violations of the invertibility conditions. Suppose that $r_{\overline{n},\ell}$ is a natural number $\overline{k}$ for some $\overline{n}$. Then, (4.16) shows that the reduction fails at the $\overline{k}$th order in α. This problem is avoided introducing a new independent parameter $\overline{\lambda}_{\overline{n}\ell}$ at order $\overline{k}$ in front of $\mathcal{O}_{\overline{n}\ell}$, writing

$$\lambda_{\overline{n}\ell} = \frac{1}{\alpha^{\overline{n}-1}} \left[\lambda_\ell^{\overline{n}} \sum_{j=0}^{\overline{k}-1} d_{\overline{n},j}\alpha^j + \alpha^{\overline{k}}\overline{\lambda}_{\overline{n}\ell}\right], \tag{4.17}$$

where $d_{\overline{n},j}$, $j < \overline{k}$ are calculated as above. The beta function of $\overline{\lambda}_{\overline{n}\ell}$ has the form

$$\overline{\beta}_{\overline{n}\ell} = \overline{\gamma}_{\overline{n}\ell}(\alpha)\,\overline{\lambda}_{\overline{n}\ell} + \overline{\delta}_{\overline{n}\ell}(\alpha, \lambda_{m<\overline{n}}), \qquad \overline{\gamma}_{\overline{n}\ell}(\alpha) = \overline{n}\gamma_\ell^{(1)}\alpha + \mathcal{O}(\alpha^2), \qquad \overline{\delta}_{\overline{n}\ell} = \lambda_\ell^{\overline{n}}\mathcal{O}(\alpha).$$

The one-loop coefficient of $\overline{\gamma}_{\overline{n}\ell}$ is derived from $r_{\overline{n},\ell} = \overline{k}$.

The new parameter $\overline{\lambda}_{\overline{n}\ell}$ modifies the reduction relations also for $n > \overline{n}$. Taking into account that $\overline{\lambda}_{\overline{n}\ell}$ contributes only from order $\overline{k}$, the modified reduction relations read

$$\lambda_{n\ell} = \frac{1}{\alpha^{n-1}} \sum_{q=0}^{[n/\overline{n}]} \alpha^{\overline{k}q} a_{n\ell}^{(q)}(\alpha)\lambda_\ell^{n-\overline{n}q}\overline{\lambda}_{\overline{n}\ell}^{q}, \qquad n > \overline{n}, \tag{4.18}$$

where [ ] denotes the integral part and the coefficients $a_{n\ell}^{(q)}$ are power series in α. Inserting (4.18) in (4.6) the coefficients $a_{n\ell}^{(q)}$ are worked out recursively, from $q = [n/\overline{n}]$ to $q = 0$, term-by-term in the α-expansion. The existence conditions for $a_{nm}^{(q)}$, namely

$$r_{n,\ell,q} = r_{n,\ell} - \overline{k}q \notin \mathbb{N}, \tag{4.19}$$

do not add further restrictions, because they are already contained in (4.12).

When another invertibility condition (4.12), $n > \overline{n}$, is violated, the story repeats. A new parameter $\overline{\lambda}_{n\ell}$ is introduced at order $\alpha^{r_{n,\ell}}$. If several conditions (4.19), for different values of q, are violated at the same time, all relevant monomials of (4.18) are reabsorbed into the same new parameter $\overline{\lambda}_{n\ell}$.

Effects of the renormalization mixing. Consider the renormalization mixing, calculated in the undeformed theory $\mathcal{R}$, among operators with the same dimensionality $n\ell$ in units of mass, $n \geq 1$. Distinguish the mixing operators with indices $I, J, \ldots$. If ℓ is the level of the deformation, denote the inequivalent operators of level ℓ with O_ℓ^I, the coefficient-matrix of their lowest-order anomalous dimensions with $\gamma_\ell^{(1)\ IJ}$, an eigenvalue of $\gamma_\ell^{(1)\ IJ}$ with $\gamma_\ell^{(1)}$ and the corresponding eigenvector with d_0^I. Denote the operators of the queue with $\mathrm{O}_{n\ell}^I$, $n > 1$, and their couplings with $\lambda_{n\ell}^I$. The beta functions read

$$\beta_{n\ell}^I = \sum_J \gamma_{n\ell}^{IJ}(\alpha)\lambda_{n\ell}^J + \delta_{n\ell}^I,$$

where $\delta_{n\ell}^I$ depends only on $\lambda_{m\ell}^I$ with $m < n$ and α. Introduce an auxiliary coupling λ_ℓ of level ℓ, with beta function $\beta_{\lambda_\ell} = \gamma_\ell^{(1)}\alpha\lambda_\ell$. The beta function of λ_ℓ can be chosen to be one-loop exact with an approprite scheme choice (any other choice being equivalent to a redefinition $\lambda_\ell \to h(\alpha)\lambda_\ell$, with $h(\alpha)$ analytic in α, $h(0) = 1$). The reduction relations have the form

$$\lambda_{n\ell}^I = f_n^I(\alpha)\lambda_\ell^n, \qquad n \geq 1,$$

where $f_n^I(\alpha) = \mathcal{O}(\alpha^{1-n})$. If k is a natural number, the existence conditions are that the matrices

$$r_{n,k,\ell}^{IJ} = \gamma_{n\ell}^{(1)\ IJ} - n\gamma_\ell^{(1)}\delta^{IJ} + (n - 1 - k)\beta_0^{(1)}\delta^{IJ} \tag{4.20}$$

be invertible for $n > 1$, $k \geq 0$ and for $n = 1$, $k > 0$. If the invertibility conditions are fulfilled, the solution is uniquely determined in terms of d_0^I. The head of the deformation is $\sum_I \mathrm{O}_\ell^I \lambda_\ell^I$.

A lowest-order non-trivial renormalization mixing makes the existence conditions much easier to fulfill. Indeed, the entries of the matrices $\gamma_{n\ell}^{(1)\ IJ}$ are rational numbers divided by $\pi^{d/2}$. Non-trivial (i.e. non-triangular) matrices with rational entries have in general irrational, or possibly complex, eigenvalues. The invertibility of $r_{n,k,\ell}^{IJ}$ requires that certain linear combinations of generically irrational or complex numbers do not coincide with natural numbers. The violations of this requirement are much rarer than the violations of (4.12).

In the theories considered here, which have renormalization structures such as (1.1), the matrices $r_{n,k,\ell}^{IJ}$ are block-triangular. More precisely, at the lowest order the renormalization mixing calculated in the undeformed theory $\mathcal{R}$ can be non-triangular only among operators that have the same number of derivatives. Write

$$\mathrm{O}_{p,q}=[\partial^{2p}]\varphi^{q}, \tag{4.21}$$

where, as usual, derivatives are distributed and contracted in all possible ways, up to total derivatives and terms factorizing $\Box\varphi$. Each $\mathrm{O}_{p,q}$ is a set of operators $\mathrm{O}_{p,q}^{I}$, where I labels the independent contractions of derivatives. Now I prove that at the lowest-order the matrix $\gamma_{p,q|p',q'}^{I|J}$ of anomalous dimensions (calculated in the undeformed theory $\mathcal{R}$, that is to say the four-dimensional φ^4 theory) vanishes for $p<p'$. To simplify the notation, call $\gamma_{pp'}^{(1)}$ the lowest-order coefficients of this matrix and collectively denote with $\mathrm{O}_p=[\partial^{2p}]F_p(\varphi)$ the set of operators $\mathrm{O}_{p,q}^{I}$ that have $2p$ derivatives. Study the lowest-order counterterms of type O_p generated by the diagrams that contain one insertion of $\mathrm{O}_{p'}$. This amounts to set $v_0=1$, $v_{p'}=1$, $v_n=0$ for $n\neq p'$, $n\neq 0$ and $L\geq 1$ in (2.12), so

$$p=\frac{1}{2}\omega(G)=p'+L-1\geq p'.$$

Thus $\gamma_{pp'}^{(1)}=0$ for $p<p'$, which proves the claimed block-triangular structure of the matrix of lowest-order anomalous dimensions.

Obviously, there is only one operator (4.21) with given q and no derivative, that is $\varphi^{n\ell+4}$, so the first diagonal block of (4.20) is a one-by-one matrix and coincides with $(r_{n,\ell}-k)\beta_0^{(1)}$, $k\in\mathbb{N}$: a necessary condition for the invertibility of the matrices (4.20) is still $r_{n,\ell}\notin\mathbb{N}$. The eigenvector d_0^I corresponds to the operator $\varphi^{\ell+4}$, which is the head of the deformation, and $\gamma_\ell^{(1)}$ is the coefficient of its one-loop anomalous dimension. Operators (4.21) with $q=3$ are either relevant ($p=0$) or factorize a $\Box\varphi$. Operators with $p=1$ are trivial. There is a unique operator with $p=2$ and given q, e.g.

$$\varphi^{q-4}\left[(\partial_\mu\varphi)^2\right]^2, \tag{4.22}$$

which provides invertibility conditions similar to (4.12). Instead, starting from $p=3$ there are at least two inequivalent operators for every q, e.g. for $p=3$

$$\varphi^{q-6}\left[(\partial_\mu\varphi)^2\right]^3, \qquad \varphi^{q-4}(\partial_\mu\varphi)^2(\partial_\nu\partial_\rho\varphi)^2, \tag{4.23}$$

so the diagonal blocks with $p>2$ provide less restrictive invertibility conditions. If some matrix $r_{n,k,\ell}^{IJ}$ exceptionally has a null eigenvector, the new parameter is introduced with a procedure similar to the one explained in formulas (4.17) and (4.18).

Summarizing, neither the extra parameters $\overline{\lambda}_{\overline{n}\ell}$ nor the renormalization mixing modify (4.12). The structure of the invertibility conditions is always (4.12) or (4.20).

In the next sections I apply these considerations to the theory (2.3), at one and two loops, and show that in common models (analytic potentials) the existence conditions are violated in an infinity of cases, namely there exist infinitely many $\overline{n}$s such that $r_{\overline{n},\ell} = \overline{k}(\overline{n}) \in \mathbb{N}$. The final theory contains infinitely many independent couplings, but each new coupling is introduced at an order $\overline{k}(\overline{n})$ in α that typically grows with $\overline{n}$, so it is possible to make calculations up to high orders using a relatively small number of couplings. I identify also the cases where the existence conditions do not admit any violations.

Role of the effective Planck mass. Finally, I prove (3.2). Start from the renormalization structure (1.1) in four dimensions. Restore the $\hbar$-dependence and expand

$$\hbar^p V_p(\varphi, \hbar) = \sum_{q\geq 4} \lambda_{p,q}\varphi^q. \tag{4.24}$$

Consider an irrelevant deformation of level ℓ. The non-vanishing irrelevant couplings are such that $n_{p,q} \equiv (2p + q - 4)/\ell$ is integer ≥ 1. Decompose the beta functions in the usual form, $\beta_{p,q} = \lambda_{p,q}\gamma_{p,q}(\hbar\alpha) + \delta_{p,q}(\lambda_{p',q'}, \alpha, \hbar)$, ignoring for the moment the renormalization-mixing, where $\delta_{p,q}$ depends only on the couplings $\lambda_{p',q'}$ with $n_{p',q'} < n_{p,q}$.

Precisely, $\delta_{p,q}$ receives contributions form the diagrams G such that $\omega(G)/2 = p$. If v_0 is the number of marginal vertices and $v_{p',q'}$ is the number of irrelevant vertices of type p', q', $\delta_{p,q}$ has the structure

$$\delta_{p,q} \sim \sum_G \hbar^L \alpha^{v_0} \prod_{n_{p',q'}<n_{p,q}} \lambda_{p',q'}^{v_{p',q'}}, \tag{4.25}$$

where $v_{p',q'}$ are such that $\sum_{n_{p',q'}<n_{p,q}} v_{p',q'}n_{p',q'} = n_{p,q}$, while $\sum_{n_{p',q'}<n_{p,q}} v_{p',q'} = V - v_0$ is the total number of irrelevant vertices. The condition $\omega(G)/2 = p$ gives, using (2.12), $L + \sum_{n_{p',q'}<n_{p,q}} p'v_{p',q'} - V + 1 = p$.

If the invertibility conditions are fulfilled, the solution to the reduction equations is $\lambda_{p,q} = f_{p,q}(\alpha, \hbar)\lambda_\ell^{n_{p,q}}$, with

$$f_{p,q}(\alpha, \hbar) = \frac{\hbar^p}{\alpha^{n_{p,q}-1}} \sum_{k=0}^{\infty} (\hbar\alpha)^k d_{p,q,k}, \tag{4.26}$$

$d_{p,q,k}$ being uniquely determined numerical coefficients. Indeed, assuming inductively that (4.26) holds for p, q such that $n_{p,q} < n_{\overline{p},\overline{q}}$, (4.25) implies

$$\delta_{\overline{p},\overline{q}} \sim \sum_G \hbar^L \alpha^{v_0} \lambda_\ell^{n_{\overline{p},\overline{q}}} \prod_{n_{p,q}<n_{\overline{p},\overline{q}}} \left(\frac{\hbar^p}{\alpha^{n_{p,q}-1}}\right)^{v_{p,q}} (1 + \mathcal{O}(\hbar\alpha)),$$

the diagrams G being such $\omega(G)/2 = \overline{p}$. Therefore,

$$\delta_{\overline{p},\overline{q}} \sim \hbar^{\overline{p}} \frac{\lambda_\ell^{n_{\overline{p},\overline{q}}}}{\alpha^{n_{\overline{p},\overline{q}}-1}} \sum_G (\hbar\alpha)^{V-1}(1 + \mathcal{O}(\hbar\alpha)).$$

Recalling that $\lambda_{p,q} \sim \delta_{p,q}/(\hbar\alpha)$ and $V \geq 2$, the result (4.26) follows for $\overline{p}, \overline{q}$.

It is easy to show that the result holds also when the renormalization-mixing is taken into account. This check is left to the reader.

Finally, inserting (4.26) into (4.24) and (2.3), the complete lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + {\sum_{p=0}^{\infty}}' \hbar^p \sum_{n\geq 2p/\ell} \frac{\lambda_\ell^n}{\alpha^{n-1}} w_{p,n}(\hbar\alpha)[\partial^{2p}]\varphi^{n\ell-2p+4}, \tag{4.27}$$

where $w_{n,p}(\hbar\alpha)$ are power series in $\hbar\alpha$. The negative powers of α are reabsorbed into the effective Planck mass $M_{P\text{eff}}$. Defining

$$\lambda_\ell = \frac{1}{M_P^\ell}, \qquad M_{P\text{eff}}^\ell = M_P^\ell \alpha, \qquad \chi = \frac{\varphi}{M_{P\text{eff}}}, \qquad W_n(\alpha, \chi, \hbar) = \sum_{s\geq 2n/\ell} w_{n,s}(\hbar\alpha)\chi^{s\ell-2n+4}, \tag{4.28}$$

(4.27) becomes (3.2).

Due to (4.17) and (4.18), the introduction of new parameters $\overline{\lambda}_{\overline{n}\ell}$ does not change this structure: it is sufficient to define $\overline{\lambda}_{\overline{n}\ell\text{eff}} = \alpha^{-\overline{n}}\overline{\lambda}_{\overline{n}\ell}$ and take α small at fixed $M_{P\text{eff}}$ and $\overline{\lambda}_{\overline{n}\ell\text{eff}}$.

The divergences of the reduced theory (3.2) are reabsorbed into renormalization constants for the independent couplings, α, $M_{P\text{eff}}$ and eventually $\overline{\lambda}_{\overline{n}\ell\text{eff}}$, plus field redefinitions. It is easy to show that the form of the field redefinition is the same as (2.27) with $M_P \rightarrow M_{P\text{eff}}$ and $\chi \rightarrow \varphi/M_{P\text{eff}}$. I leave the proof to the reader.

5 The $\varphi^4 + \varphi^6$ theory in four dimensions

Now I study the theory

$$\mathcal{L}_{\text{cl}} = \frac{1}{2}(\partial_\mu\varphi)^2 + V_{\text{cl}}(\varphi, \lambda), \qquad V_{\text{cl}}(\varphi, \lambda) = \frac{\lambda_0}{4!}\varphi^4 + \frac{\lambda_2}{6!}\varphi^6, \tag{5.1}$$

in four dimensions, using the tools developed in the previous sections. The theory (5.1) is classically meaningful, but as a quantum field theory it is not consistent, since the φ^6 interaction is unstable under renormalization. It is necessary to look for a completion of type (2.3) that is stable under renormalization, such that the divergences are removed only with field redefinitions and renormalization constants for λ_0 and λ_2. In this section I study this completion and one and two loops, non-perturbatively in φ.

Structure of the lagrangian. The bare lagrangian has generically the form

$$\mathcal{L}_{\text{B}} = \frac{1}{2}(\partial_\mu\varphi_{\text{B}})^2 + {\sum_{n=0}^{\infty}}' [\partial^{2n}]V_{\text{B}n}(\varphi_{\text{B}}, \lambda_{\text{B}}, \varepsilon). \tag{5.2}$$

The information (2.3) that for $n > 1$, $2n$ derivatives of the fields do not appear before the nth loop will be used later. The functions $V_{\mathrm{B}n}(\varphi_{\mathrm{B}}, \lambda_{\mathrm{B}}, \varepsilon)$ can depend on ε, but only analytically, since the divergences are reabsorbed inside φ_{B} and λ_{B}, by definition. The ε-dependence of (5.2) is important for the reduction of couplings discussed below, since the ε-powers of (5.2) can simplify divergences and give finite contributions.

Now I study the constraints on $\mathcal{L}_{\mathrm{B}}$ due to renormalization stability, using the dimensional-regularization technique, where $d = 4 - \varepsilon$ is the continued dimension. The dimensionalities of objects are denoted by means of square brackets, so

$$[\varphi_{\mathrm{B}}] = 1 - \frac{\varepsilon}{2}, \qquad [\lambda_{0\mathrm{B}}] = \varepsilon, \qquad [\lambda_{2\mathrm{B}}] = -2 + 2\varepsilon.$$

With the three dimensionful quantities $\lambda_{0\mathrm{B}}, \lambda_{2\mathrm{B}}$ and φ_{B} it is possible to construct two dimensionless combinations

$$\frac{\varphi_{\mathrm{B}}^2 \lambda_{2\mathrm{B}}}{\lambda_{0\mathrm{B}}}, \qquad \lambda_{0\mathrm{B}}^{2(\varepsilon-1)} \lambda_{2\mathrm{B}}^{-\varepsilon}. \tag{5.3}$$

One of them, however, does not depend on φ_{B} and has ε-dependent exponents, so it can generate logarithms and fractional powers of the couplings in the limit $d \to 4$. Since the reduction is meromorphic, the functions $V_{\mathrm{B}n}(\varphi_{\mathrm{B}}, \lambda_{\mathrm{B}}, \varepsilon)$ can depend only on the first quantity of (5.3). Assume for simplicity that λ_0 and λ_2 are positive. Defining

$$\chi_{\mathrm{B}} = \varphi_{\mathrm{B}} \sqrt{\frac{\lambda_{2\mathrm{B}}}{\lambda_{0\mathrm{B}}}}$$

and matching the dimensionalities at $\varepsilon \neq 0$, the functions $V_{\mathrm{B}n}$ have the form

$$V_{\mathrm{B}n}(\varphi_{\mathrm{B}}, \lambda_{\mathrm{B}}, \varepsilon) = \frac{\lambda_{2\mathrm{B}}^{n-2}}{\lambda_{0\mathrm{B}}^{2n-3}} W_{\mathrm{B}n}(\chi_{\mathrm{B}}, \varepsilon). \tag{5.4}$$

Again, the functions $W_{\mathrm{B}n}(\chi_{\mathrm{B}}, \varepsilon)$ can depend on ε only analytically, since the divergences are reabsorbed inside $\chi_{\mathrm{B}}, \lambda_{\mathrm{B}}$.

Now I show that the property (2.6) that for $n > 1$, $2n$ derivatives of the fields do not appear in the renormalized lagrangian before the nth loop is equivalent to $W_{\mathrm{B}n} = \mathcal{O}(\varepsilon^n)$. Write $W_{\mathrm{B}n}(\chi_{\mathrm{B}}, \varepsilon) = \varepsilon^n \overline{W}_{\mathrm{B}n}(\chi_{\mathrm{B}}, \varepsilon)$, with $\overline{W}_{\mathrm{B}n}(\chi_{\mathrm{B}}, \varepsilon)$ analytic in ε. The bare lagrangian (5.2) reads

$$\mathcal{L}_{\mathrm{B}} = \frac{\lambda_{0\mathrm{B}}}{2\lambda_{2\mathrm{B}}} (\partial_\mu \chi_{\mathrm{B}})^2 + \sum_{n=0}^{\infty}{}' \varepsilon^n \frac{\lambda_{2\mathrm{B}}^{n-2}}{\lambda_{0\mathrm{B}}^{2n-3}} [\partial^{2n}] \overline{W}_{\mathrm{B}n}(\chi_{\mathrm{B}}, \varepsilon). \tag{5.5}$$

At the tree level bare and renormalized quantities coincide, so (5.5) corresponds to the tree lagrangian

$$\mathcal{L} = \frac{\lambda_0}{2\lambda_2} \mu^{-\varepsilon} (\partial_\mu \chi)^2 + \mu^{-\varepsilon} \sum_{n=0}^{\infty}{}' \varepsilon^n \frac{\lambda_2^{n-2}}{\lambda_0^{2n-3}} [\partial^{2n}] \overline{W}_n(\chi, \varepsilon), \tag{5.6}$$

where $\chi = \varphi\mu^{\varepsilon/2}\sqrt{\lambda_2/\lambda_0}$ and I have suppressed the subscript B in $\overline{W}_n$. Defining

$$\lambda_2 = \frac{1}{M_P^2}, \qquad M_{P\text{eff}}^2 = M_P^2\lambda_0, \qquad \chi = \frac{\varphi\mu^{\varepsilon/2}}{M_{P\text{eff}}}, \tag{5.7}$$

the classical lagrangian (5.6) becomes

$$\mathcal{L} = \frac{1}{2}\mu^{-\varepsilon}M_{P\text{eff}}^2(\partial_\mu\chi)^2 + \mu^{-\varepsilon}\sum_{n=0}^{\infty}{}' \frac{M_{P\text{eff}}^{4-2n}}{\lambda_0^{n-1}}\varepsilon^n[\partial^{2n}]\overline{W}_n(\chi,\varepsilon). \tag{5.8}$$

I prove that (5.8) is equivalent to (3.2), the negative powers of λ_0 being effectively cancelled by the powers of ε.

The lagrangian (5.8) contains evanescent vertices, which can be traded for higher-order local vertices. Consider an evanescent bare local operator $E_{\rm B}$. Being unsubtracted, its contribution is not negligible, because the evanescence can simplify divergences and give finite contributions. According to the theory of evanescent operators [16], the renormalized operator $[E_{\rm B}]$ is equal to the sum of non-evanescent renormalized local operators $[{\rm O_R}]$ plus truly evanescent local operators $[E_{\rm R}]$,

$$[E_{\rm B}] = [E_{\rm R}] + [{\rm O_R}].$$

"Truly" evanescent operators are those that give vanishing contributions in the physical limit ($\varepsilon \to 0$) of the 1PI action Γ and can be consistently neglected. The finite operator $[{\rm O_R}]$ is obtained simplifying evanescences with divergences. This means that $[{\rm O_R}]$ is higher-order in the expansion. Its order can be computed as follows.

Consider a diagram G with L loops and v_n vertices of type $\overline{W}_n$, constructed with (5.8). Write $V = \sum_n v_v$ and $\mathcal{E} = \sum_n nv_n$. Assume that G has overall divergences (the subdivergences being treated inductively in the usual way), namely $L + \mathcal{E} - V + 1 \geq 0$ (see (2.12)). Look for the finite contributions that are generated when the divergences of G are simplified by the powers of ε attached to the vertices. Counting the povers of λ_0 attached to the vertices, the desired contributions are local and have the form

$$\hbar^{L+\mathcal{E}-V+1}\lambda_0(\hbar\lambda_0)^{V-1-\mathcal{E}}M_{P\text{eff}}^{2-2L-2\mathcal{E}+2V}[\partial^{2(L+\mathcal{E}-V+1)}]H(\chi), \tag{5.9}$$

The theorem of the appendix ensures that the maximal ε-pole of G is $\leq V-1$. The evanescences attached to the vertices are $\geq \mathcal{E}$, because of (5.8). They can simplify and give finite contributions only if $V - 1 - \mathcal{E} \geq 0$. Therefore the finite terms (5.9) are of type (3.2). Similarly, it is easy to check that the divergent contributions have the form specified by (2.26)-(2.27) with $\alpha \to \lambda_0$.

The evanescent vertices, together with appropriate finite and divergent terms, can be reorganized in objects of type $[E_{\rm R}]$ and neglected. The remaining finite vertices, together with appropriate divergent terms, can be organized in objects of type $[{\rm O_R}]$ and kept. The remaining divergent contributions are the net counterterms that need to be subtracted away. Thus (5.8) generates a renormalized lagrangian of the form

$$\mathcal{L}_{\mathrm{R}} = \frac{\mu^{-\varepsilon}}{2} M_{P\mathrm{eff}}^2 (\partial_\mu \chi)^2 + \sum_{n=0}^{\infty} \mu^{-\varepsilon} \hbar^n \lambda_0 M_{P\mathrm{eff}}^{4-2n} [\partial^{2n}] \widetilde{W}_{\mathrm{R}n}(\hbar\lambda_0, \chi, \varepsilon), \tag{5.10}$$

that can be converted into

$$\mathcal{L}_{\mathrm{R}} = \frac{\mu^{-\varepsilon}}{2} M_{P\mathrm{eff}}^2 (\partial_\mu \widetilde{\chi})^2 + {\sum_{n=0}^{\infty}}' \mu^{-\varepsilon} \hbar^n \lambda_0 M_{P\mathrm{eff}}^{4-2n} [\partial^{2n}] W_{\mathrm{R}n}(\hbar\lambda_0, \widetilde{\chi}, \varepsilon). \tag{5.11}$$

with a field redefinition of the form

$$\widetilde{\chi} = \chi + \hbar\lambda_0 \sum_{n=0}^{\infty} \hbar^n M_{P\mathrm{eff}}^{-2n} [\partial^{2n}] Q_n(\hbar\lambda_0, \chi, \varepsilon). \tag{5.12}$$

The functions $\widetilde{W}_{\mathrm{R}n}$ and $W_{\mathrm{R}n}$ in (5.10) and (5.11) have the form (2.26) with $\alpha \to \lambda_0$, while the function Q_n of (5.12) has an expression similar to the F_n of (2.27), with the only difference that in Q_n the sum starts from $m = 0$, to include finite field redefinitions.

The argument just given provides another proof of (3.2). The theory is singular in the limit $\lambda_0 \to 0$ at fixed M_P and trivial in the limit $\lambda_0 \to 0$ at fixed $M_{P\mathrm{eff}}$, where the entire interacting sector disappears. These properties are analogous to those found in ref.s [1, 2], to which the reader is referred for comparison.

5.1 Calculation of the corrected classical potential

I call "corrected" classical potential $W(\chi)$ the complete tree-level potential. In the case of the $\varphi^4 + \varphi^6$ theory, where

$$W(\chi) = \frac{\chi^4}{4!} + \frac{\chi^6}{6!} + \mathcal{O}(\chi^8), \tag{5.13}$$

the $\mathcal{O}(\chi^8)$-corrections are determined self-consistently in the way that I now describe, using one-loop results.

Differential equation for the corrected classical potential. Start from the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + V(\varphi, \lambda_0 \mu^\varepsilon, \lambda_2 \mu^{2\varepsilon}, \varepsilon), \tag{5.14}$$

where V is analytic in ε, but can contain evanescent terms. Using the background-field method, expand φ as $\varphi_0 + h$ inside (5.14), where φ_0 is the background field and h is the quantum fluctuation. For one-loop calculations it is sufficient to expand up to $\mathcal{O}(h^2)$. The order $\mathcal{O}(h)$ can be dropped, since it does not contribute to the 1PI generating functional $\Gamma[\varphi_0]$. Thus the one-loop divergences are encoded in the lagrangian

$$\mathcal{L}_h = \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2} h^2 K, \qquad \text{where } K = V''(\varphi_0, \lambda_0 \mu^\varepsilon, \lambda_2 \mu^{2\varepsilon}, \varepsilon), \tag{5.15}$$

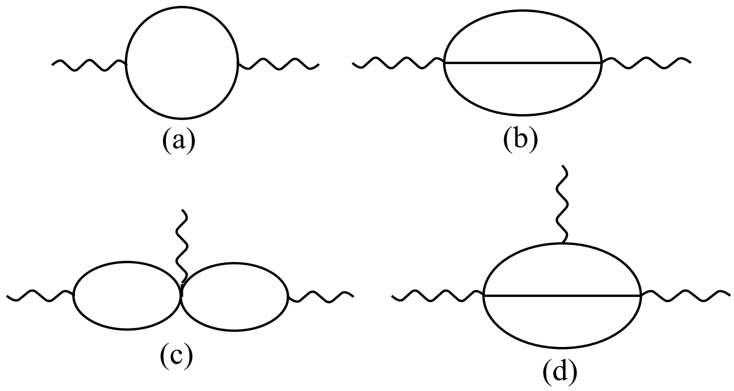

Figure 1: Relevant Feynman diagrams

and are given by diagram (a) of Fig.1. Here the prime denotes differentiation with respect to φ. The counterterm is easily evaluated:

$$\Delta\mathcal{L}_h = \frac{\hbar\mu^{-\varepsilon}}{32\varepsilon\pi^2}K^2,$$

so the one-loop renormalized lagrangian reads

$$\mathcal{L}_{\text{R one-loop}} = \frac{1}{2}(\partial_\mu\varphi)^2 + V(\varphi,\lambda_0\mu^\varepsilon,\lambda_2\mu^{2\varepsilon},\varepsilon) + \frac{\hbar\mu^{-\varepsilon}}{32\varepsilon\pi^2}\left[V''(\varphi,\lambda_0\mu^\varepsilon,\lambda_2\mu^{2\varepsilon},\varepsilon)\right]^2. \tag{5.16}$$

Consistency with renormalization demands

$$\mathcal{L}_\text{R} = \mathcal{L}_\text{B} = \frac{1}{2}(\partial_\mu\varphi_\text{B})^2 + V_\text{B}(\varphi_\text{B},\lambda_\text{B},\varepsilon). \tag{5.17}$$

The bare and renormalized potentials have expansions:

$$V_\text{B}(\varphi_\text{B},\lambda_\text{B},\varepsilon) = \frac{\lambda_{0\text{B}}^3}{\lambda_{2\text{B}}^2}W_\text{B}(\chi_\text{B},\varepsilon), \qquad V(\varphi,\lambda,\varepsilon) = \mu^{-\varepsilon}\frac{\lambda_0^3}{\lambda_2^2}\sum_{k=0}^{\infty}(\hbar\lambda_0)^k W_{0,k}(\chi,\varepsilon). \tag{5.18}$$

Both $W_\text{B}(\chi_\text{B},\varepsilon)$ and $W_{0,k}(\chi,\varepsilon)$ are analytic in ε. Comparing the bare and renormalized lagrangians it follows that $\varphi_\text{B} = \varphi + \mathcal{O}(\hbar^2)$. The corrected classical potential is the function $W_{0,0}(\chi,0) \equiv W(\chi)$.

The problem (5.17) is solved matching the bare and renormalized lagrangians. Write $\lambda_{0\text{B}} = \lambda_0\mu^\varepsilon Z_0$, $\lambda_{2\text{B}} = \lambda_2\mu^{2\varepsilon}Z_2$, with $Z_0 = 1 + a\hbar\lambda_0$, $Z_2 = 1 + b\hbar\lambda_0$. Then the matching gives immediately $W_{0,0}(\chi,\varepsilon) = W_\text{B}(\chi,\varepsilon)$ and

$$W_{0,1}(\chi,\varepsilon) = -\frac{1}{32\varepsilon\pi^2}\left[\left(W''_{0,0}(\chi,\varepsilon)\right)^2 - 32\pi^2(3a-2b)W_{0,0}(\chi,\varepsilon) + 16\pi^2(a-b)\chi W'_{0,0}(\chi,\varepsilon)\right], \tag{5.19}$$

where now the prime denotes differentiation with respect to χ. Since $W_{0,1}(\chi,\varepsilon)$ is analytic in ε the pole in (5.19) cancels out. This gives the equation

$$(3a-2b)W+\frac{\chi}{2}(-a+b)W'=\frac{(W'')^2}{32\varepsilon\pi^2}. \tag{5.20}$$

Matching the powers χ^4 and χ^6, the one-loop renormalization constants

$$Z_0=1+\frac{3\hbar\lambda_0}{16\varepsilon\pi^2},\qquad Z_2=1+\frac{15\hbar\lambda_0}{16\varepsilon\pi^2},$$

are obtained, so the final equation for W reads

$$(W'')^2+42W-12\chi W'=0. \tag{5.21}$$

The power-counting solution is $W(\chi)=\chi^4/4!$. Beyond power-counting, the solution is uniquely specified by the initial conditions (5.13).

Study of the corrected classical potential. The non-linear differential equation (5.21) is not of known types. Some exact properties of the solution can be easily derived. Since W is positive in the neighborhood of $\chi=0$, let $[0,p)$ denote the maximal interval containing $\chi=0$ where $W>0$. The equation (5.21) implies that $W'>0$ in such an interval, so $p=\infty$: W is everywhere positive and monotonically increasing. We have the two inequalities

$$12\chi W'>(W'')^2,\qquad \frac{W'}{W}\geq\frac{7}{2\chi}. \tag{5.22}$$

Integrating (5.22) between δ and $\chi>\delta$ we obtain the estimates

$$\frac{W(\delta)}{\delta^{7/2}}\,\chi^{7/2}\leq W(\chi)<\frac{\chi^4}{3}+\frac{8c}{5\sqrt{3}}\chi^{5/2}+c^2\chi+b,$$

where b and c are constants.

Finally, inserting a power-like behavior $W(\chi)\sim\beta\chi^\alpha$ into the equation (5.21) for $\chi\rightarrow\infty$ it is immediate to prove that the possible power-like behaviors at infinity are

$$W(\chi)\sim\beta\chi^{7/2},\qquad W(\chi)\sim\frac{\chi^4}{4!},$$

with β arbitrary.

Summarizing, the behavior of W is regular, with a unique minimum in the origin, and very strongly restricted. In particular, W does not increase faster than χ^4 at infinity, even if it is originated by a φ^6 classical potential! Going through the arguments just derived, it is easy to see that the same conclusions hold when $\lambda_0>0$, $\lambda_2<0$: an unstable classical potential is turned into a stable corrected classical potential.

The equation (5.21) can be studied around $\chi \sim 0$ with a series expansion. The recursion relations for the coefficients of the expansion are quadratic, see (6.5). I have calculated the first 400 coefficients using Mathematica and checked empirically that the radius of convergence is about 1. The first few terms of the expansion are

$$W(\chi)=\frac{\chi^4}{4!}+\frac{\chi^6}{6!}-\frac{\chi^8}{1152}+\frac{7\,\chi^{10}}{20736}-\frac{203\,\chi^{12}}{1244160}+\frac{211\,\chi^{14}}{2488320}-\frac{28553\,\chi^{16}}{597196800}+\mathcal{O}(\chi^{18}).$$

For $\chi \gtrsim 1$ the expansion badly diverges. With a Range-Kutta method it is easy to numerically extend the solution beyond $\chi \sim 1$. I have arrived at $\chi \sim 2.8$.

Thus, although the solution of (5.21) is regular everywhere, the series expansion in powers of the fields diverges for large χ. This proves that the non-perturbative approach of this paper is useful in concrete cases.

It should be remarked that the corrected classical potential $W(\chi)$ is not a purely "classical" potential, because it knows about the quantum theory: it can be determined only using one-loop knowledge. On the other hand, $W(\chi)$ is not the quantum potential either, since the quantum potential is contained in the 1PI generating functional Γ. Summarizing, the quantization prescription for non-renormalizable theories contains one step more than usual: the starting point is a classical lagrangian, such as (5.1), which is generically unstable with respect to renormalization; the first step is to determine a corrected classical lagrangian that is stable under renormalization, such as (5.11); the last step is to compute the quantum action Γ.

5.2 Two-loop calculations

The two-loop relevant Feynman diagrams are (b), (c) and (d) of Fig. 1 and produce the counterterms

$$\Delta\mathcal{L}_{\mathrm{R\ two\text{-}loop}}=\frac{\hbar^2\mu^{-2\varepsilon}}{2\varepsilon(4\pi)^4}\left[-\frac{1}{12}(\partial_\mu V''')^2+\left(\frac{1}{\varepsilon}-\frac{1}{2}\right)V''(V''')^2+\frac{1}{\varepsilon}(V'')^2V''''\right], \tag{5.23}$$

that have to be added to the one-loop renormalized lagrangian $\mathcal{L}_{\mathrm{R\ one\text{-}loop}}$ of (5.16). The first term of (5.23) can be converted into a contribution to the potential by means of a field redefinition. For this purpose, define the bare field

$$\varphi_{\mathrm{B}}=\varphi-\frac{\hbar^2\mu^{-2\varepsilon}}{24\varepsilon(4\pi)^4}\int_0^{\varphi}\left[V''''(\varphi')\right]^2\mathrm{d}\varphi'+\mathcal{O}(\hbar^3).$$

Consistency with renormalization is expressed by the equation

$$\mathcal{L}_{\mathrm{R}}=\mathcal{L}_{\mathrm{R\ one\text{-}loop}}+\Delta\mathcal{L}_{\mathrm{R\ two\text{-}loop}}=\mathcal{L}_{\mathrm{B}}=\frac{1}{2}(\partial_\mu\varphi_{\mathrm{B}})^2+V_{\mathrm{B}}(\varphi_{\mathrm{B}},\lambda_{\mathrm{B}},\varepsilon), \tag{5.24}$$

where the bare and renormalized potentials have the expansions (5.18). The function $W_{0,1}(\chi,0)\equiv Y(\chi)$ is the one-loop contribution to the corrected classical potential and its initial condition is

$Y(\chi) = \mathcal{O}(\chi^8)$. Indeed, the powers of φ^4 and φ^6 in $V(\varphi, \lambda, 0)$ are multiplied by the independent couplings λ_0 and λ_2, so no $\hbar$-corrections are necessary in front of φ^4 and φ^6. This means that $W_{0,n}(\chi, 0) = \mathcal{O}(\chi^8)$ for $n > 1$.

The problem (5.24) is solved matching the bare and renormalized lagrangians. This matching gives immediately the functions $W_{0,i}(\chi, \varepsilon)$, $i = 0, 1, 2$, in terms of $W_{\rm B}(\chi_{\rm B}, \varepsilon)$. Then, using the fact that $W_{0,i}$ are analytic in ε, differential or integro-differential equations for $W(\chi)$ and $Y(\chi)$ are obtained. Matching the powers χ^4 and χ^6 in such equations the two-loop renormalization constants

$$Z_0 = 1 + \frac{3\lambda_0}{16\varepsilon\pi^2} + \left(\frac{9}{\varepsilon^2} - \frac{17}{6\varepsilon}\right)\frac{\lambda_0^2}{(4\pi)^4}, \qquad Z_2 = 1 + \frac{15\lambda_0}{16\varepsilon\pi^2} + \left(\frac{135}{\varepsilon^2} - \frac{427}{12\varepsilon}\right)\frac{\lambda_0^2}{(4\pi)^4}, \qquad (5.25)$$

are obtained. At two loops, the procedure just outlined gives two equations, one for the double pole and one for the simple pole. The equation due to the double pole is equivalent to (5.21). Indeed, the one-loop simple pole and the two-loop double pole are related to each other by the renormalization group [16]. The equation given by the simple two-loop pole is

$$18Y - 6\chi Y' + W''Y'' = \frac{1}{192\pi^2}\left(1504W - 393\chi W' + 6W''(W''')^3 - W'\int_0^\chi \left[W''''(\chi')\right]^2 \mathrm{d}\chi'\right)$$

and has a unique solution with the initial condition $Y(\chi) = \mathcal{O}(\chi^8)$, once $W(\chi)$ is known. The first few coefficients of the solution are

$$Y(\chi) = \frac{\chi^8}{46080\pi^2}\left[43 - \frac{401}{30}\chi^2 + \frac{883}{72}\chi^4 - \frac{201497}{23760}\chi^6 + \mathcal{O}(\chi^8).\right]$$

5.3 Inclusion of the mass term

With a mass term,

$$\mathcal{L}_{\rm cl} = \frac{1}{2}(\partial_\mu\varphi)^2 + V_{\rm cl}(\varphi, \lambda), \qquad V_{\rm cl}(\varphi, \lambda) = \frac{m^2}{2}\varphi^2 + \frac{\lambda_0}{4!}\varphi^4 + \frac{\lambda_2}{6!}\varphi^6,$$

the dimensionless combinations are

$$\frac{\varphi_{\rm B}^2\lambda_{2\rm B}}{\lambda_{0\rm B}}, \qquad m_{\rm B}^2\frac{\lambda_{2\rm B}}{\lambda_{0\rm B}^2}, \qquad \lambda_{0\rm B}^{2(\varepsilon-1)}\lambda_{2\rm B}^{-\varepsilon}.$$

The corrected classical potential can depend only on the first two. Defining $\tau = m^2\lambda_2/\lambda_0^2$, we have

$$V(\varphi, \lambda) = \frac{\lambda_0^3}{\lambda_2^2}W(\chi, \tau), \qquad W(\chi, \tau) = \tau\frac{\chi^2}{2!} + \frac{\chi^4}{4!} + \frac{\chi^6}{6!} + \mathcal{O}(\chi^8).$$

Proceeding with the methods explained in the previous subsections, it is possible to derive a partial differential equation for $W(\chi, \tau)$, with coefficients related to the one-loop renormalization constants. The equations can be solved perturbatively in τ, starting from the solution $W(\chi)$ found before. The result is

$$W(\chi,\tau)=W(\chi)+\tau\left(\frac{\chi^2}{2!}-\frac{\chi^8}{288}+\frac{7\,\chi^{10}}{4320}-\frac{791\,\chi^{12}}{518400}+\mathcal{O}(\chi^{14})\right)+$$
$$\tau^2\left(-\frac{17\,\chi^8}{684}+\frac{106\,\chi^{10}}{12825}+\mathcal{O}(\chi^{12})\right)+\tau^3\left(-\frac{41437\,\chi^8}{198360}+\mathcal{O}(\chi^{10})\right)+\mathcal{O}(\tau^4\chi^8).$$

The renormalization constants of the couplings are

$$m_{\mathrm{B}}^2=m^2\left(1+\frac{\lambda_0}{16\varepsilon\pi^2}\right),\qquad \lambda_{0\mathrm{B}}=\lambda_0\left(1+\frac{(\tau+3)\lambda_0}{16\varepsilon\pi^2}\right),$$
$$\lambda_{2\mathrm{B}}=\lambda_2\left[1+\left(15-35\tau-140\tau^2-\frac{19040\,\tau^3}{19}+\mathcal{O}(\tau^4)\right)\frac{\lambda_0}{16\varepsilon\pi^2}\right].$$

6 The $\varphi^4+\varphi^{m+4}$ theory in $D=4$: conditions for a meaningful reduction of couplings

In this section I consider a more general class of irrelevant deformations, those with

$$V_{\mathrm{cl}}(\varphi,\lambda)=\frac{\lambda_0}{4!}\varphi^4+\frac{\lambda_m}{(m+4)!}\varphi^{m+4}, \tag{6.1}$$

where m is not necessarily integer, but positive. The dimensionality of $\lambda_{m\mathrm{B}}$ is

$$[\lambda_{m\,\mathrm{B}}]=-m+\frac{\varepsilon}{2}(m+2),$$

so the only acceptable dimensionless combination is

$$\frac{\varphi_{\mathrm{B}}^m\lambda_{m\,\mathrm{B}}}{\lambda_{0\mathrm{B}}}\equiv\chi_{\mathrm{B}}^m,$$

and the corrected classical potential has the form

$$V(\varphi,\lambda)=\left(\frac{\lambda_0^{m+4}}{\lambda_m^4}\right)^{1/m}W(\chi). \tag{6.2}$$

Defining $\lambda_{m\mathrm{B}}=\lambda_m\mu^{(m+2)\varepsilon/2}Z_m$, $Z_m=1+b_m\hbar\lambda_0$, and using the strategy described in the previous section, the differential equation

$$\frac{(m+4)a-4b_m}{m}W+\frac{b_m-a}{m}\chi W'=\frac{(W'')^2}{32\varepsilon\pi^2}$$

is obtained. The values of a and b_m are calculated matching the orders χ^4 and χ^m of the solution. This gives the renormalization constant of λ_0 and the one of λ_m:

$$Z_m=1+\frac{(m+4)(m+3)}{32\varepsilon\pi^2}\hbar\lambda_0. \tag{6.3}$$

The final equation for W reads

$$m(W'')^2+2(m+4)(2m+3)W-(m+6)(m+1)\chi W'=0, \qquad W(\chi)=\frac{\chi^4}{4!}+\frac{\chi^{m+4}}{(m+4)!}+\mathcal{O}(\chi^8).$$

The study of the solution proceeds as before. It is easily found that W is positive, monotonically increasing and its large-χ behavior is bounded by

$$\beta\chi^{\frac{2(m+4)(2m+3)}{(m+6)(m+1)}}\lesssim W(\chi)\lesssim\frac{\chi^4}{4!}.$$

Expand $W(\chi)$ in powers of χ,

$$W(\chi)=\sum_{n=0}^{\infty}c_{nm}\chi^{nm+4}, \qquad c_0=\frac{1}{4!}, \qquad c_m=\frac{1}{(m+4)!}. \tag{6.4}$$

Observe that the expansion of W is consistent with the fact that only non-negative integer powers of λ_m and integer powers of λ_0 appear in $V(\varphi,\lambda)$, see (6.2). The coefficients c_{nm}, $n>1$, are determined by the recursion relations

$$c_{nm}=-\frac{1}{(n-1)(nm^2-6)}\sum_{k=1}^{n-1}c_{km}c_{(n-k)m}(km+3)(km+4)\left((n-k)m+3\right)\left((n-k)m+4\right). \tag{6.5}$$

The first few terms of the expansion are

$$\begin{aligned}W(\chi)=&\frac{\chi^4}{4!}+\frac{\chi^{m+4}}{(m+4)!}-\frac{1}{2}\frac{\chi^{2m+4}}{(m^2-3)[(m+2)!]^2}+\frac{1}{3}\frac{(m+2)(2m+3)\ \chi^{3m+4}}{(m^2-2)(m^2-3)[(m+2)!]^3}+\\&-\frac{1}{6}\frac{(m+2)(2m+3)(8m^4+21m^3-8m^2-56m-36)\ \chi^{4m+4}}{(m^2-2)(2m^2-3)(m^2-3)^2[(m+2)!]^4}+\mathcal{O}(\chi^{5m+4}).\end{aligned}$$

Reduction of couplings. I show that formula (6.5) solves the one-loop RG-consistency equations (4.7). Write

$$V(\varphi,\lambda)=\sum_{n=0}^{\infty}\frac{\lambda_{nm}}{(nm+4)!}\varphi^{nm+4}. \tag{6.6}$$

The reduction relations to this order can be read from (6.2) and (6.4):

$$\frac{\lambda_{nm}}{(nm+4)!}=a_{nm}(\hbar\lambda_0)\frac{\lambda_m^n}{\lambda_0^{n-1}},$$

with $a_{nm}(0)=c_{nm}$. Reading the one-loop anomalous dimensions from (6.3),

$$\gamma_{nm}^{(1)}=\frac{(nm+4)(nm+3)}{32\pi^2},$$

and using $\beta_0^{(1)} = 3/(16\pi^2)$, the existence conditions (4.12) become

$$r_{n,m} = \frac{1}{6}(n-1)(nm^2-6) \notin \mathbb{N}. \qquad (6.7)$$

Consider first the case of integer m, or, more generally, rational m^2. It is clear from (6.7) that infinitely many ns violate the existence condition. Each violation implies the introduction of a new parameter at order $r_{n,m} \in \mathbb{N}$ in λ_0. In particular, the corrected classical potential is uniquely determined if $r_{n,m}$ is not zero, in agreement with (6.5).

For $m = 2$, which is the theory $\varphi^4 + \varphi^6$, no value $n > 1$ gives $r_{n,2} = 0, 1$, so no new parameter appears at the tree and one-loop levels, in agreement with the results of the previous section, which gave unique functions $W(\chi)$ and $Y(\chi)$. The first integer values of $r_{n,2}$ are 2,5,15,22,40... The first new parameter appears at two loops.

In section 4 it was proved that the introduction of new parameters and the renormalization mixing do not modify (6.7). At the lowest order the renormalization mixing (calculated in the undeformed theory $\mathcal{R}$) can be non-triangular only among operators that have the same number of derivatives. Consider the operators $\mathrm{O}_{p,q}$ of (4.21) with fixed p and arbitrary q. Simple combinatorics shows that the one-loop anomalous dimensions, therefore also $r_{n,\ell}$ and $r_{n,k,\ell}^{IJ}$, grow quadratically with the number q of legs. Therefore, the four-derivative operators (4.22), which have a trivial one-loop renormalization mixing, provide invertibility conditions similar to (6.7). Their violations are expected to be infinitely many, but the new parameters are introduced at orders that grow with q. Operators with more than four derivatives, e.g. (4.23), have a non-trivial one-loop renormalization mixing, so their invertibility conditions are violated much more rarely, again at orders growing with q.

In conclusion, the complete theory is expected to work precisely as illustrated by the one-loop results. The number of new couplings grows together with the order of the expansion. Every finite order is described by a finite number of independent couplings. A relatively small number of couplings is sufficient to make physical predictions up to very high orders. A reliable estimate of the number of new couplings and the orders at which they appear can be obtained counting the violations of (6.7), so calculations up to the fortieth order in the theory $\varphi^4 + \varphi^6$ with operators of arbitrarily high dimensionalities are expected to require about ten independent couplings. Non-renormalizable theories formulated in the naive way contain infinitely many independent couplings already at the tree level, thus the simplification and the gain in predictive power are enormous.

The conditions (6.7) are always fulfilled when m^2 is irrational. Also the invertibility conditions associated with the four-derivative operators (4.22) are quadratic in m, so they are expected to be violated at most by a finite set of qs. Thus, non-analytic theories with irrational m^2 are expected to be renormalizable with a finite number of independent couplings.

Summarizing, in general potentials with irrational m^2 can be quantized consistently with a finite number of independent couplings, while rational potentials contain a number of independent couplings that grows sporadically with the order of the perturbative expansion. In the next section I consider the theory $\varphi^4+\varphi^5$ in four dimensions, where a new independent coupling appears in the corrected classical potential.

7 The $\varphi^4+\varphi^5$ theory in $D=4$: appearance of new parameters

The theory $\varphi^4+\varphi^5$ in four dimensions has $m=1$ and $r_{6,1}=0$. The equation for $W(\chi)$, which reads

$$(W'')^2+50W-14\chi W'=0, \tag{7.1}$$

does not admit a power series solution, because the formula (6.5) has a singular denominator for $n=6$, $m=1$. A new independent coupling $\overline{\lambda}_6$ has to be introduced in front of φ^{10}. It is convenient to define a dimensionless running parameter ξ. Its beta function is calculated from (5.16) with the usual procedure:

$$\overline{\lambda}_6=\xi\frac{\lambda_1^6}{\lambda_0^5},\qquad \beta_\xi=\frac{5915}{972}\frac{\hbar\lambda_0}{32\pi^2}.$$

The modified equation for $W(\chi,\xi)$ reads

$$(W'')^2+50W-14\chi W'=\frac{5915}{972}\frac{\partial W}{\partial\xi}, \tag{7.2}$$

whose solution has an expansion

$$\begin{aligned}W(\chi,\xi)=&\frac{\chi^4}{24}+\frac{\chi^5}{120}+\frac{\chi^6}{144}+\frac{5}{432}\chi^7+\frac{355}{10368}\chi^8+\frac{6545}{31104}\chi^9+\xi\,\chi^{10}-\frac{450275}{69984}\chi^{11}\\&-5\,\xi\,\chi^{11}+\frac{67399525}{3359232}\chi^{12}+\frac{125}{12}\xi\,\chi^{12}+\mathcal{O}(\chi^{13}).\end{aligned} \tag{7.3}$$

Once the new parameter ξ is introduced, the solution $W(\chi,\xi)$ of (7.2) is uniquely determined.

For completeness, I report an alternative way to solve the problem, which does not modify equation (7.1). The solution of (7.1) contains a new finite parameter ζ ($\beta_\zeta=0$), but is not analytic in the fields:

$$W(\chi,\zeta)=W(\chi)|_9-\frac{1183}{972}\chi^{10}\ln(\zeta\,\chi)-\frac{502327}{69984}\chi^{11}+\frac{5915}{972}\chi^{11}\ln(\zeta\,\chi)+\mathcal{O}(\chi^{12}),$$

where $W(\chi)|_9$ are the terms χ^4-χ^9 of (7.3).

8 Conclusions

Are power-counting renormalizable theories more "fundamental" than the other theories? In the light of the results obtained so far, the reasons usually advocated to privilege the power-counting renormalizable theories over the non-renormalizable ones appear to be weak. Predictivity and calculability are not peculiar features of renormalizable theories. Here and in ref.s [1, 2] I have shown that calculability belongs to a much larger class of models, that include also power-counting non-renormalizable theories, which can be formulated in a perturbative framework, often with a finite or reduced set of independent couplings, without assuming knowledge about the ultraviolet limit. The so-formulated theories have a non-trivial predictive content. Therefore, if power-counting renormalizable theories are viewed as fundamental, then there are several other theories that have to be considered equally fundamental. On the other hand, our present notion of fundamental theory might still be imperfect. The research pursued here might contribute to find the right definition.

The techniques to reduce the number of independent couplings consistently with renormalization are useful tools to classify the non-renormalizable interactions, although they cannot select which ones are switched on and off in nature. The key issue is the number of independent couplings that are necessary to subtract the divergences. Some insight in this problem is provided by the construction of finite and quasi-finite non-renormalizable theories [1, 2], which are irrelevant deformations of interacting conformal field theories. Their divergences are renormalized away just with field redefinitions (finiteness), plus, eventually, a finite number of independent renormalization constants (quasi-finiteness). The non-renormalizable theories studied in this paper are irrelevant deformations of running renormalizable theories. They admit a simplified renormalization structure, where, for example, terms with $2n$ derivatives of the fields, $n > 1$, do not appear before the n^{th} loop. The dependence on the fields, although not on their derivatives, can be treated non-perturbatively. A suitable perturbative expansion can be defined, such that each order of the quantum action is calculable with a finite number of steps. I have studied the predictive power of these theories, in particular under which conditions the divergences are renormalized with a reduced, eventually finite, number of independent couplings. In the simplest (analytic) non-renormalizable theories the number of parameters grows sporadically with the order of the expansion. It becomes infinite in the complete theory, but the growth is so slow that a reasonably small number of parameters is sufficient to make predictions up to very high orders. Most non-analytic theories can be renormalized with a finite number of couplings in a strict sense.

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A Appendix: a theorem on the maximal logarithmic divergences of diagrams

Here I prove a general result that is used in the paper. Recall that the UV divergences are calculated treating the mass term, if present, as a (two-leg) vertex. The propagator is just $1/k^2$, so tadpoles vanish.

Theorem. The maximal pole of a diagram with V vertices and L loops is at most of order $m(V-1,L) \equiv \min(V-1,L)$.

Proof. I prove the statement inductively in V and, for fixed V, inductively in L. The diagrams with $V=1$ and arbitrary L are tadpoles, which vanish identically and therefore satisfy the theorem. Suppose that the statement is true for $V<\overline{V}$, $\overline{V}>1$, and arbitrary L. Consider diagrams with $\overline{V}$ vertices. Clearly for $L=1$ the maximal divergence is $1/\varepsilon$, so the theorem is satisfied. Proceed inductively in L, i.e. suppose that the theorem is satisfied by the diagrams with $\overline{V}$ vertices and $L<\overline{L}$ loops, and consider the diagrams $G_{\overline{V},\overline{L}}$ that have $\overline{V}$ vertices and $\overline{L}$ loops. If $G_{\overline{V},\overline{L}}$ has no subdivergence, its divergence is at most a simple pole. Higher-order poles are related to the subdivergences of $G_{\overline{V},\overline{L}}$ and can be classified replacing the subdiagrams with their counterterms. Consider the subdiagrams $\gamma_{v,l}$ of $G_{\overline{V},\overline{L}}$, with l loops and v vertices. Clearly, $1\leq l<\overline{L}$ and $1\leq v\leq \overline{V}$. By the inductive hypothesis, the maximal divergence of $\gamma_{v,l}$ is a pole of order $m(v-1,l)$. Contract the subdiagram $\gamma_{v,l}$ to a point and multiply by $1/\varepsilon^{m(v-1,l)}$. A diagram with $\overline{V}-v+1\leq \overline{V}$ vertices and $\overline{L}-l<\overline{L}$ loops is obtained, whose maximal divergence, taking into account of the factor $1/\varepsilon^{m(v-1,l)}$, is at most a pole of order $m(v-1,l)+m(\overline{V}-v,\overline{L}-l)$. The inequality

$$m(v-1,l)+m(\overline{V}-v,\overline{L}-l)\leq m(\overline{V}-1,\overline{L}),$$

which can be derived case-by-case, proves that the maximal divergence of $G_{\overline{V},\overline{L}}$ associated with $\gamma_{v,l}$ satisfies the theorem. Since this is true for every subdiagram $\gamma_{v,l}$, the theorem follows for $G_{\overline{V},\overline{L}}$. By induction, the theorem follows for every diagram.

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