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# CONSISTENT IRRELEVANT DEFORMATIONS OF INTERACTING CONFORMAL FIELD THEORIES

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**Abstract**

I show that under certain conditions it is possible to define consistent irrelevant deformations of interacting conformal field theories. The deformations are finite or have a unique running scale ("quasi-finite"). They are made of an infinite number of lagrangian terms and a finite number of independent parameters that renormalize coherently. The coefficients of the irrelevant terms are determined imposing that the beta functions of the dimensionless combinations of couplings vanish ("quasi-finiteness equations"). The expansion in powers of the energy is meaningful for energies much smaller than an effective Planck mass. Multiple deformations can be considered also. I study the general conditions to have non-trivial solutions. As an example, I construct the Pauli deformation of the IR fixed point of massless non-Abelian Yang-Mills theory with $N_c$ colors and $N_f \lesssim 11N_c/2$ flavors and compute the couplings of the term $F^3$ and the four-fermion vertices. Another interesting application is the construction of finite chiral irrelevant deformations of N=2 and N=4 superconformal field theories. The results of this paper suggest that power-counting non-renormalizable theories might play a role in the description of fundamental physics.

## 1 Introduction

Certain power-counting non-renormalizable theories can be quantized successfully, for example the four-fermion models in three spacetime dimensions [1] in the large N expansion. Using the procedure of ref. [2] it is possible to renormalize quantum gravity coupled with matter in three spacetime dimensions as a finite theory. A theory that is not power-counting renormalizable does not necessarily violate fundamental physical principles and cannot be discarded *a priori*. At present, it is not clear why some theories can be quantized and other cannot, which theories are meaningful and which are meaningless. In this paper I present results that are expected to shed some light on this problem.

In four-dimensions pure gravity is finite to the first loop order [3], but finiteness is spoiled by the presence of matter. Moreover, gravity is not finite to the second loop order [4] even in the absence of matter. In three dimensions the situation is different. In ref. [2] I have formulated a quantization procedure to construct finite theories of quantum gravity coupled with matter in three dimensions, under the assumption that the matter sector satisfies certain restrictions. Those ideas do not generalize immediately to four-dimensional quantum gravity, but admit a number of other interesting four-dimensional extensions. In this paper I explore a class of such applications, namely the construction of consistent irrelevant deformations of interacting conformal field theories.

In general, it is not known how to define *one* irrelevant deformation, because as soon as one irrelevant term is added to the lagrangian, renormalization turns on infinitely many other terms, multiplied by independent couplings. This spoils predictivity at the level of fundamental field theory (but not at the level of effective field theory). The goal of this paper is precisely to disentangle the irrelevant deformations from one another. This result makes it possible to study "one" irrelevant deformation, or "two" irrelevant deformations, etc., or all of them together, which is the usual situation. A single irrelevant deformation is made of an infinite series of lagrangian terms that renormalize coherently, with a unique renormalization constant, associated with a dimensionful coupling (the scale). In this sense, the so-deformed theory is still physically predictive as a fundamental field theory, although it is not power-counting renormalizable.

The irrelevant couplings are responsible of the non-polynomial structure of the renormalized lagrangian. The beta functions and renormalization constants of the irrelevant couplings are polynomial in the irrelevant couplings themselves. So, on the one side non-renormalizable theories are complicated, on the other side they are extremely simple. For this reason, it is possible to work with them.

Consider a conformal field theory $\mathcal{C}$ and its irrelevant deformations. If $\lambda$ is the coupling constant that multiplies the irrelevant term $\mathrm{O}_\lambda$, the beta function of $\lambda$ has the form [2]

$$\beta_\lambda = \lambda\gamma_\lambda + \delta_\lambda. \tag{1.1}$$

Here $\gamma$ is the anomalous dimension of $\mathrm{O}_\lambda$ and depends only on the marginal couplings of $\mathcal{C}$. Instead, $\delta$ does not depend on $\lambda$ and depends polynomially on a finite number of other irrelevant couplings.

A structure as simple as (1.1) suggests that in a number of cases it is possible to solve the finiteness equations $\beta_\lambda = 0$. A set of non-trivial solutions has been studied in [2]. However, in various situations, the finiteness equations admit only the trivial solution $\lambda = 0$, which is just the conformal theory $\mathcal{C}$. To construct non-trivial deformations in these cases, it is possible to define "quasi-finite" theories, i.e. theories that have a unique runnning parameter, the scale.

Taking appropriate combinations of dimensionful couplings, it is always possible to organize the set of couplings of a theory into a unique dimensionful parameter, the scale $\kappa$, with conventional dimensionality $-1$, plus dimensionless couplings. A *quasi-finite* theory is a theory whose dimensionless couplings have vanishing beta functions. The scale is free to run. If the scale does not run, the theory is *finite*. If there is no scale, the theory is *conformal*. In this paper I construct finite and quasi-finite consistent irrelevant deformations of interacting conformal field theories.

Known examples of quasi-finite theories are the mass deformations of conformal field theories. Consider for example N=4 supersymmetric Yang-Mills theory in four dimensions (for an introduction to supersymmetry in the language of superfields, see for example [5]) and denote the gauge coupling with $g$. Supersymmetry can be softly broken to N=0 giving masses $m$ to the scalar fields $\varphi$, for example. After this breaking, the beta function $\beta_g$ remains zero, because, by dimensional considerations, it cannot depend on $m$. On the other hand, the mass operator $\overline{\varphi}\varphi$ is not finite (its anomalous dimension is non-vanishing at $g \neq 0$; see for example [6]). This implies that the scale $m$ does run. Therefore, the deformed theory is quasi-finite. The quasi-finite theories constructed in this paper are a counterpart, in the irrelevant sector, of the relevant deformations of conformal field theories.

On the other hand, N=4 supersymmetry can be broken to N=1 with a chiral mass deformation. This deformation is finite, because of a well-known non-renormalization theorem. In this paper I construct also finite chiral irrelevant deformations of superconformal field theories (section 6).

Now I sketch the construction of quasi-finite irrelevant deformations. The set of couplings can be conveniently split into an energy scale $1/\widetilde{\kappa}$ and dimensionless ratios $g_i$. The beta functions $\beta_i$ of the dimensionless couplings $g_i$ cannot depend on $\widetilde{\kappa}$. The beta function of $\widetilde{\kappa}$ is equal to $\widetilde{\kappa}$ times a function of the $g_i$s. Then, it is consistent to solve the *quasi-finiteness equations*

$$\beta_i = 0, \qquad \frac{\mathrm{d}\widetilde{\kappa}}{\mathrm{d}\ln\mu} = \beta_{\widetilde{\kappa}}. \tag{1.2}$$

The solutions of the quasi-finiteness equations are, in general, non-trivial and contain a unique arbitrary parameter besides the marginal couplings of $\mathcal{C}$, namely the value of $\widetilde{\kappa}$ at some reference energy $\overline{\mu}$. It is also correct to view $1/\widetilde{\kappa}(\overline{\mu})$ as the definition of the unit of mass.

In the paper, after developing the general approach, I consider a concrete model, the Pauli deformation of the IR fixed point of massless non-Abelian Yang-Mills theory with $N_c$ colors and $N_f \lesssim 11N_c/2$ flavors. I study the self-renormalization of the Pauli term and the structure of the Pauli deformation to the order $\mathcal{O}(\kappa^2)$ included, which is made of the irrelevant terms of dimensionality 6 (four-fermion vertices and $F^3$). I solve the quasi-finiteness equations and show that the solutions relate in a unique way the couplings of $F^3$ and the four-fermion vertices to the coupling of the Pauli term. One-loop calculations are sufficient for these studies.

Multiple deformations can be defined also, where various dimensionful parameters run independently. Multiple deformations are not sums or superpositions of simple deformations.

The paper is organized as follows. In section 2 I present the general theory of finite and quasi-finite irrelevant deformations and study conditions to have non-trivial solutions. In sections 3, 4 and 5 I study the Pauli deformation of the IR fixed point of Yang-Mills theory coupled with matter. I solve the quasi-finiteness equations to the second order in $\kappa$ and first order in the loop expansion. In section 6 I construct the finite chiral irrelevant deformations of N=2 and N=4 superconformal field theories. Section 7 contains the conclusions. The appendix collects a number of useful identities and the field equations.



## 2 Consistent irrelevant deformations

Consider the set of irrelevant deformations of a conformal field theory $\mathcal{C}$ of interacting fields $\varphi$. The classical lagrangian in $d$ dimensions has the form

$$\mathcal{L}_{cl}[\varphi] = \mathcal{L}_{\mathcal{C}}[\varphi, \alpha] + \sum_i \kappa^i \sum_{I=1}^{N_i} \lambda_{iI} \mathcal{O}_{iI}(\varphi). \tag{2.1}$$

The $\mathcal{O}_{iI}$ are a basis of (gauge-invariant) local lagrangian terms with canonical dimensionalities $d+i$ in units of mass. The index $i$ denotes the "level" of $\mathcal{O}_i$ (irrelevant operators have positive levels, marginal operators have level 0 and relevant operators have negative levels) and can be integer or half-integer. The $\lambda_{iI}$ denote a complete set of essential couplings, labelled by their level $i$ plus an index $I$ that distinguishes the couplings of the same level. The essential couplings are the couplings that multiply a basis of lagrangian terms that cannot be renormalized away or into one another by means of field redefinitions [7].

The constant $\kappa$ is an auxiliary quantity with dimensionality $-1$ in units of mass. Every $\lambda$ is dimensionless. I assume that the theory does not contain masses and superrenormalizable parameters (positive-level couplings). Parameters with positive dimensionalites in units of mass form dimensionless quantities when they are multiplied by suitable powers of the irrelevant couplings. These dimensionless combinations are responsible for unnecessary complicacies, both at the theoretical and practical levels, because the beta functions do not depend polynomially on them.

The redundancy of the constant $\kappa$ is exhibited by the invariance of (2.1) under the scale symmetry

$$\lambda_{iI} \rightarrow \Omega^{-i}\lambda_{iI}, \qquad \kappa \rightarrow \Omega\kappa. \tag{2.2}$$

**Structure of the beta functions.** The beta function of $\lambda_{iI}$ transforms as $\lambda_{iI}$ under the scale symmetry (2.2) and therefore its structure is

$$\beta_{iI} = \sum_{\{n_{jJ}^{iI}\}} f_{\{n_{jJ}^{iI}\}}(\alpha) \prod_{j\leq i} \prod_{J=1}^{N_j} (\lambda_{jJ})^{n_{jJ}^{iI}}, \tag{2.3}$$

where $f_{\{n_{jJ}^{iI}\}}(\alpha)$ are functions of the marginal couplings of $\mathcal{C}$ and the sum is performed over the sets $\{n_{jJ}^{iI}\}$ of non-negative integers $n_{jJ}^{iI}$ such that

$$\sum_{j\leq i} j \sum_{J=1}^{N_j} n_{jJ}^{iI} = i. \tag{2.4}$$

The constant $\kappa$ is the only dimensionful parameter in the theory and does not appear in the beta functions.

Due to (2.4), only a finite set of numbers $n_{jJ}^{iI}$ can be greater than zero. This implies that the beta functions depend on the irrelevant couplings in a polynomial way. Special sets $\{n_{jJ}^{iI}\}$ satisfying (2.4) are those where $n_{jJ}^{iI}$ is equal to one for $j = i$ and some index $J$, zero otherwise. It is useful to isolate this contribution from the rest, obtaining

$$\beta_{iI} = \sum_{J=1}^{N_i} \gamma_i^{IJ}(\alpha)\,\lambda_{iJ} + \delta_{iI}, \qquad\qquad \delta_{iI} = \sum_{\{m_{jJ}^{iI}\}} f_{\{m_{jJ}^{iI}\}}(\alpha) \prod_{j<i} \prod_{J=1}^{N_j} (\lambda_{jJ})^{m_{jJ}^{iI}}. \tag{2.5}$$

Now the sum is performed over the sets $\{m_{jJ}^{iI}\}$ of non-negative integers such that

$$\sum_{j<i} j \sum_{J=1}^{N_j} m_{jJ}^{iI} = i. \tag{2.6}$$

The functions $\gamma_i^{IJ}(\alpha)$ are the entries of the matrix $\gamma_i(\alpha)$ of anomalous dimensions of the operators $\mathcal{O}_{iI}$ of level $i$. The second term of (2.5) collects the contributions of the operators $\mathcal{O}_{jJ}$ of levels $j < i$. Observe that (2.6) implies

$$\sum_{j<i} \sum_{J=1}^{N_j} m_{jJ}^{iI} \geq 2, \tag{2.7}$$

which means that the beta function of $\lambda_i$ is at least quadratic in the irrelevant couplings with $j < i$. *A fortiori*, the $\delta_{iI}$s vanish when all of the $\lambda_{iI}$s vanish. Indeed, at $\lambda_{iI} = 0$ the theory reduces to $\mathcal{L}_{\mathcal{C}}[\varphi, \alpha]$, which is finite by assumption.

**Deformation of level $\ell$.** Let $\gamma_i$ denote the matrix having entries $\gamma_i^{IJ}(\alpha)$. The deformation of level $\ell$ is defined as follows. First, set

$$\lambda_{jJ} = 0 \quad \text{for} \quad j \neq n\ell, \ n \text{ integer}.$$

Using (2.5), this implies $\delta_{jJ} = \beta_{jJ} = 0$ for $j \neq n\ell$ and $\delta_{\ell I} = 0$. The equation

$$\beta_{\ell I} = \frac{\mathrm{d}\lambda_{\ell I}}{\mathrm{d}\ln\mu} = \sum_{J=1}^{N_\ell} \gamma_\ell^{IJ}(\alpha)\,\lambda_{\ell J}$$

is solved by

$$\lambda_{\ell I}(\mu) = \sum_{J=1}^{N_\ell} \exp\left(\gamma_\ell \ln\mu/\overline{\mu}\right)^{IJ} \lambda_{\ell J}(\overline{\mu}).$$

The solution contains $N_\ell$ arbitrary parameters, which are the values of the couplings $\lambda_{\ell I}$ at some reference scale $\overline{\mu}$.

It is convenient to consider one arbitrary parameter at a time. The matrix $\gamma_\ell$ is real but in general its characteristic roots are complex. For the moment I assume that $\gamma_\ell$ has at least one real characteristic root, $r_\ell$, with multiplicity one. Let a tilde denote vectors and matrices in a basis in which the matrix $\gamma_\ell$ has Jordan canonical form $\widetilde{\gamma}_\ell$ with $\widetilde{\gamma}_\ell^{11} = r_\ell$ (see for example [8]). Finally, let $\widetilde{\lambda}_{\ell I}(\overline{\mu}) = (\overline{\lambda}_\ell, 0, \ldots 0)$. Then

$$\widetilde{\lambda}_\ell(\mu) = \exp\left(r_\ell \ln\mu/\overline{\mu}\right)\overline{\lambda}_\ell. \tag{2.8}$$

For $n > 1$, I write

$$\lambda_{n\ell I} = A_{n\ell I}\widetilde{\lambda}_\ell^n. \tag{2.9}$$

The coefficients $A$ are scale invariant, i.e. invariant under the scale symmetry (2.2). Quasi-finiteness is the requirement that the beta functions of scale-invariant quantities vanish. If the $\ln\mu$ derivatives of both sides of (2.9) are equated, the quasi-finiteness equations read

$$\sum_{J=1}^{N_\ell} \widetilde{\gamma}_{n\ell}^{IJ} A_{n\ell J}\widetilde{\lambda}_\ell^n + \delta_{n\ell I} = n r_\ell A_{n\ell I}\widetilde{\lambda}_\ell^n. \tag{2.10}$$

This is a system of equations in the unknowns $A$. The solution can be worked out inductively in $n$. Assuming that the systems (2.10) have been solved for $n = 2, \ldots m-1$, and the solutions have the form (2.9), then formula (2.5) ensures that the $\delta_{m\ell I}$s are equal to known numbers times $\widetilde{\lambda}_\ell^m$. The $m^{th}$ system of equations (2.10) can be solved if the matrix

$$\widehat{\gamma}_{m\ell} \equiv \gamma_{m\ell} - m r_\ell \mathbf{1} \tag{2.11}$$

is invertible, where $\mathbf{1}$ denotes the identity matrix. For real $r_\ell$ the invertibility of $\widehat{\gamma}_{m\ell}$ holds if and only if no characteristic root of the matrix $\gamma_{m\ell}$ is equal to $m$ times $r_\ell$.

If the matrices $\widehat{\gamma}_{m\ell}$ are not invertible, solutions exist if suitable entries of the vector $\delta_{m\ell}$ in (2.10) vanish. In some cases a symmetry can ensure that certain irrelevant terms have $\delta$

identically zero. Then the system (2.10) can always be solved. Operators with $\delta$ identically zero are called *protected.* Examples of protected operators are the chiral operators in four-dimensional supersymmetric theories [5], which are discussed in detail in section 6. At this stage of the discussion, it is convenient to isolate the protected operators from the rest and concentrate the search for solutions of the quasi-finiteness equations in the remaining subclass of irrelevant terms. For simplicity, it is also convenient to set the couplings of the protected operators to zero. Indeed, it is always possible to turn those couplings on at a later stage. This operation is studied in section 6 and defines the protected irrelevant deformations. In the rest of this section, I assume that the protected operators are dropped from (2.1) and that the $\lambda_i$s refer only to the unprotected irrelevant operators, unless otherwise specified.

So, leaving the protected operators aside, the requirement that should be satisfied for the existence of a consistent quasi-finite deformation of level $\ell$, associated with the characteristic root $r_\ell$ of the matrix $\gamma_\ell$, is the invertibility of the matrices (2.11) for $m > 1$. Then the theory described by the lagrangian

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{C}}[\varphi, \alpha] + \widetilde{\kappa}\ \widetilde{\mathcal{O}}_\ell(\varphi) + \sum_{n=1}^{\infty} \widetilde{\kappa}^n \sum_{I=1}^{N_{n\ell}} A_{n\ell I}(\alpha)\ \mathcal{O}_{n\ell I}(\varphi), \tag{2.12}$$

where $\widetilde{\kappa} = \kappa^\ell \widetilde{\lambda}_\ell$, is quasi-finite. The coefficients $A_{n\ell I}(\alpha)$ are uniquely specified functions of the marginal couplings $\alpha$ of $\mathcal{C}$, determined solving the system of equations (2.10). The independent parameters of (2.12) are $\alpha$ and $\widetilde{\kappa}$. The theory (2.12) is renormalized redefining the fields and the scale $\widetilde{\kappa}$, while the marginal couplings $\alpha$ and the coefficients $A_{n\ell I}(\alpha)$ are unrenormalized. The scale $\widetilde{\kappa}$ is the unique parameter of the theory that can run. The power-like divergences do not contribute to the RG equations and so can be subtracted as they come, without adding new independent couplings. The number $\ell$ is called *lowest level* of the deformation, the term $\widetilde{\mathcal{O}}_\ell$ is the lowest-level operator of the deformation and the sum in (2.12) is called *queue* of the deformation.

Now I now discuss the meaning of (2.11) and the existence of solutions.

**Existence of solutions.** Neglecting, for pedagogical purposes, the renormalization mixing for a moment, i.e. assuming that the indices $I, J$ can have only one value, the condition (2.11) reads

$$\gamma_{m\ell} \neq m r_\ell, \tag{2.13}$$

for $m > 1$, and says that the anomalous dimension of the irrelevant term of level $m\ell$ should not be equal to $m$ times the anomalous dimension of the lowest-level operator.

The meaning of the apparently obscure condition (2.13) is actually simple, as I now explain. The considerations that follow are not claimed to be rigorous, but purely illustrative. Consider for concreteness a theory containing scalar fields $\varphi$ in $d = 4$ dimensions and restrict the attention to operators without derivatives. Then

$$\mathrm{O}_{m\ell} \sim \varphi^{d+m\ell} = \varphi^d \left(\varphi^\ell\right)^m, \qquad \mathrm{O}_\ell \sim \varphi^{d+\ell} = \varphi^d \left(\varphi^\ell\right). \tag{2.14}$$

The operator $\mathrm{O}_{m\ell}$ is obtained sticking $m$ factors $\varphi^\ell$ to the operator $\varphi^d$. The operator $\varphi^d$ has level zero: it can be thought as a marginal deformation of $\mathcal{C}$ and considered finite. Now, in renormalization theory, when operators are multiplied together at the same point in spacetime, it is not sufficient to renormalize the factors to renormalize the product, but it is necessary to introduce a further renormalization constant for the product. The condition (2.13) says in practice that the renormalization constant for the product should be non-trivial. Common experience with renormalization suggests that whenever a quantity can diverge (because it is not protected by symmetries, power-counting, etc.), it generically does diverge. So, excluding miraculous cancellations, it is reasonable to assume that the products of operators have non-trivial renormalization constants and therefore that the matrices (2.11) are invertible.

When this is true, it is possible to define the consistent irrelevant deformations (2.12) of the conformal field theory $\mathcal{C}$. Observe that, in any case, the invertibility or non-invertibility of the matrices (2.11) is a property of the conformal theory $\mathcal{C}$, so it is possible to say which irrelevant deformations are allowed from the sole knowledge of $\mathcal{C}$, before actually deforming the theory.

There might exist situations in which the condition (2.13) is valid up to, say, $m=N$ and violated for $m>N$. Then, the number of parameters necessary for the renormalization of this deformation remains constant up to the order $\kappa^{N\ell}$. At the order $\kappa^{(N+1)\ell}$ the system (2.10) cannot be solved and (2.9) cannot be imposed. This means that new independent (running) parameters must be added at the level $(N+1)\ell$. This is a particular case of "multiple" deformation, in the sense explained below. The deformation remains predictive at the level of fundamental field theory if the total number of independent free parameters remains finite.



**Multiple deformations.** It is possible to construct also multiple deformations, of levels $\ell_1,\cdots\ell_k$. These deformations have more parameters that run independently. For the moment, I still assume that the relevant characteristic roots $r_{\ell j}$ are real with multiplicity one. Moreover, I assume that the integers $\ell_1,\cdots\ell_k$ are relatively prime. Formula (2.9) generalizes to

$$\lambda_{iI}=\sum_{\{n\}}A_{iI}^{n_1\cdots n_k}\widetilde{\lambda}_{\ell_1}^{n_1}\cdots\widetilde{\lambda}_{\ell_k}^{n_k},\qquad \sum_{j=1}^{k}n_j\ell_j=i,\quad n_j\geq 0. \tag{2.15}$$

The couplings that cannot be written in this form are set to zero. The couplings $\widetilde{\lambda}_{n_j}$ run as

$$\widetilde{\lambda}_{\ell_j}(\mu)=\exp\left(r_{\ell_j}\ln\mu/\overline{\mu}\right)\overline{\lambda}_{\ell_j}.$$

Quasi-finiteness demands that the scale-invariant quantities $A$ have vanishing beta functions. The system of equations (2.10) generalizes to

$$\sum_{J=1}^{N_j}\sum_{\{n\}}\gamma_i^{IJ}A_{iJ}^{n_1\cdots n_k}\widetilde{\lambda}_{\ell_1}^{n_1}\cdots\widetilde{\lambda}_{\ell_k}^{n_k}+\delta_{iI}=\sum_{\{n\}}A_{iI}^{n_1\cdots n_k}\widetilde{\lambda}_{\ell_1}^{n_1}\cdots\widetilde{\lambda}_{\ell_k}^{n_k}\sum_{j=1}^{k}n_j\overline{\gamma}_{\ell j}. \tag{2.16}$$

Proceeding inductively in the level $i$, formula (1.1) shows that $\delta_{iI}$ can be expressed as

$$\delta_{iI}=\sum_{\{n\}}\delta_{iI}^{n_1\cdots n_k}\widetilde{\lambda}_{\ell_1}^{n_1}\cdots\widetilde{\lambda}_{\ell_k}^{n_k},$$

where $\delta_{iI}^{n_1\cdots n_k}$ are numbers determined solving the systems (2.16) for levels $j < i$. The system (2.16) can be split into a set of equations

$$\sum_{J=1}^{N_j}\gamma_i^{IJ}A_{iJ}^{n_1\cdots n_k} - A_{iI}^{n_1\cdots n_k}\sum_{j=1}^{k}n_j r_{\ell j} = -\delta_{iI}^{n_1\cdots n_k}.$$

It is immediate to see that the number of unknowns is equal to the number of equations. There exists a unique solution if no characteristic root of the matrix $\gamma_i$ is equal to

$$\sum_{j=1}^{k}n_j r_{\ell j}, \qquad \text{with } n_j \text{ non-negative integers such that } \sum_{j=1}^{k}n_j = i. \tag{2.17}$$

This requirement has an interpretation similar to the one of (2.14): an operator of level $i$ can be written in many ways as the product of an operator of level 0 and operators of positive levels $j < i$, but in no case the anomalous dimension of the product should be equal to the sum of the anomalous dimensions of the factors.

When the characteristic roots of the matrix $\gamma_\ell$ are complex or have multiplicity greater than one, or when the levels $\ell_1, \cdots \ell_k$ are not relatively prime, the derivation generalizes straightforwardly, as well as the requirement (2.17), but the formulas become technically heavier. These generalizations do not teach anything new and are left to the reader.

**Sufficient condition for the existence of a perturbative expansion**. If $\mathcal{C}$ is a family of conformal field theories that becomes free when some marginal coupling $\alpha$ tends to zero, then the irrelevant deformation (2.12) might not admit a smooth $\alpha \to 0$ limit, due to the denominator appearing when the matrix (2.11) is inverted. However, if the anomalous dimensions of the irrelevant couplings satisfy a certain boundedness condition, it is possible to keep $\alpha$ small, but different from zero, and have a meaningful perturbative expansion in powers of an effective $\kappa_{\text{eff}}$.

Assume that there exist $\alpha$-independent numbers $c_n$ and a $\eta > 0$ such that

$$\left|(\widehat{\gamma}_{n\ell}^{-1})^{IJ}\right| < \frac{c_n}{\eta} \tag{2.18}$$

for every $n$. The quantity $\eta$ is a function of $\alpha$ (and $\ell$) and tends to zero when $\alpha$ tends to zero. Then, it is possible to prove that the solutions of (2.10) behave not worse than

$$|A_{n\ell I}| \sim \widetilde{c}_n \frac{1}{\eta^{(n-1)\ell}}, \tag{2.19}$$

where $\widetilde{c}_n$ are numbers and depend on the $c_n$s. The behavior (2.19) can be proved inductively in $n$. Indeed, if (2.19) is true for $n < m$, then (2.5), (2.10) and (2.7) imply that (2.19) is also true for $n = m$.

Under the assumption (2.18), let us compare the behaviors of the irrelevant terms of dimensionality $d + n\ell$ versus the behaviors of the marginal terms of $\mathcal{C}$, as functions of the energy scale $E$ of a physical process. The ratio between these two types of contributions behaves not worse than

$$a_n \eta^\ell \left(\frac{\widetilde{\kappa}^{1/\ell} E}{\eta}\right)^{n\ell},$$

$a_n$ being calculable numbers that take care also of the $c_n$s. The perturbative expansion in powers of the energy is meaningful for energies $E$ much smaller than the "effective Planck mass"

$$\frac{1}{\kappa_{\text{eff}}} \equiv \frac{\eta}{\widetilde{\kappa}^{1/\ell}}. \tag{2.20}$$

This up to the behavior of the numerical factors $a_n$, which cannot be predicted before solving the theory.

In conclusion, to have consistent irrelevant deformations almost all of the matrices (2.11) should be invertible and there must exist a $\eta > 0$ satisfying (2.18). "Almost all" means all but a finite number. The restrictions concern only the renormalizable subsector $\mathcal{C}$ of the theory and can be studied before turning the irrelevant deformation on. In the free-field limit the effective Planck mass $1/\kappa_{\text{eff}}$ tends to zero and the expansion in powers of the energy has zero convergence radius. This is why the free field theories do not admit consistent irrelevant deformations in the approach of this paper. This is also the reason why the method of this paper and ref. [2] cannot be used to quantize gravity in four dimensions.

## 3 Pauli deformation of Yang-Mills theory coupled with fermions

In this and the next two sections, I apply the general approach of the previous section to a concrete model, the Pauli deformation of an interacting conformal field theory made of fermions and gauge fields. I study the levels 1 and 2 of the deformation. The Pauli term has dimensionality 5 and is multiplied by a running parameter $\kappa$. In section 4 I study the self-renormalization of this term. In section 5 I solve the quasi-finiteness equations and compute the values of the couplings that multiply the irrelevant terms of dimensionality 6 (four-fermion terms and $F^3$). I work in the Euclidean framework and use the dimensional-regularization technique.

The renormalized lagrangian of non-Abelian Yang-Mills theory coupled massless fermions is

$$\mathcal{L} = \frac{\mu^{-\varepsilon}}{4g^2 Z_g^2}(\mathcal{F}^a_{\mu\nu})^2 + \overline{\Psi}^I_i \not{\mathcal{D}}_{ij} \Psi^I_j, \tag{3.1}$$

where $\mathcal{A}^a_\mu = Z_A^{1/2} A^a_\mu$, $\Psi^I_i = Z_\psi^{1/2} \psi^I_i$ and $\mathcal{F}^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu$, $\mathcal{D}^{ij}_\mu \Psi^I_j = \partial_\mu \Psi^I_i + \mathcal{A}^a_\mu T^a_{ij} \Psi^I_j$, The index $I = 1, \ldots N_f$ is a flavor index. I assume that the fermions are in the fundamental representation. Details about the notation are given in the appendix.

The gauge-fixing part reads

$$\mathcal{L}_{gf} = \frac{\mu^{-\varepsilon}}{2\alpha g^2}(\partial_\mu A^a_\mu)^2 + Z_C \overline{C}^a \partial_\mu \mathcal{D}^{ab}_\mu C^b, \tag{3.2}$$

where $\mathcal{D}^{ab}_{\mu}C^b = \partial_\mu C^a + f^{abc}\mathcal{A}^b_\mu C^c$.

The renormalization constants are, to the first loop order (see for example [9])

$$Z_\psi = 1 - \frac{g^2\alpha}{8\pi^2\varepsilon}\frac{N_c^2-1}{2N_c} \equiv 1+\delta Z_\psi, \qquad Z_A = 1 - \frac{g^2 N_c}{16\pi^2\varepsilon}(3+\alpha) \equiv 1+\delta Z_A,$$
$$Z_C = 1 + \frac{g^2 N_c}{32\pi^2\varepsilon}(3-\alpha), \qquad Z_g = 1 - \frac{g^2}{48\pi^2\varepsilon}(11N_c - 2N_f)\,. \tag{3.3}$$

**IR interacting fixed point.** To identify the IR fixed point, it is necessary to write the beta function of $g$ to the second loop order. In the limit where $N_c$ and $N_f$ are large, $g$ is small, but $g^2N_c$ and $N_f/N_c$ are fixed, and such that $N_f/N_c \lesssim 11/2$ , the beta function reads [9]

$$\frac{\beta_g}{g} = -\frac{\Delta}{3}\frac{g^2N_c}{16\pi^2} + \frac{25}{2}\left(\frac{g^2N_c}{16\pi^2}\right)^2 + \sum_{n=3}^{\infty} c_n \left(\frac{g^2N_c}{16\pi^2}\right)^n, \tag{3.4}$$

where $\Delta \equiv 11 - 2N_f/N_c \ll 1$ and the $c_n$s are numerical coefficients. In the limit just described, the theory is asymptotically free and the first two contributions of the beta function have opposite signs. Moreover, the first contribution is arbitrarily small. This ensures that, expanding in powers of $\Delta$, the beta function has a second zero for

$$\frac{g_*^2N_c}{16\pi^2} = \frac{2}{75}\Delta + \mathcal{O}(\Delta^2). \tag{3.5}$$

This zero defines a non-trivial conformal field theory.

The purpose of this section and sections 4-5 is to construct the Pauli deformation of this interacting conformal field theory. I concentrate on the irrelevant terms of levels 1 and 2.

**Irrelevant terms of level 1.** There is only one irrelevant term of level 1, the Pauli term

$$\mathcal{L}_{\text{Pauli}} = \kappa\lambda Z_\lambda\ \mathcal{F}^a_{\mu\nu}\ \overline{\Psi}^I_i T^a_{ij}\sigma_{\mu\nu}\Psi^I_j, \tag{3.6}$$

where $\sigma_{\mu\nu} = -i[\gamma_\mu,\gamma_\nu]/2$. The terms

$$\kappa\overline{\Psi}\not{\mathcal{D}}\not{\mathcal{D}}\Psi, \qquad \kappa\overline{\Psi}\mathcal{D}^2\Psi$$

can be converted to (3.6) up to $\mathcal{O}(\kappa^2)$ using the field equations (see the appendix).

**Irrelevant terms of level 2.** The classification of the irrelevant terms of level 2, instead, is more involved. First, there is a unique term of level 2 that does not contain fermion fields. This is the $F^3$-term

$$\mathcal{L}_{F^3} = \frac{\kappa^2\mu^{-\varepsilon}}{6!}\zeta Z_\zeta\ f^{abc}\mathcal{F}^a_{\mu\nu}\mathcal{F}^b_{\nu\rho}\mathcal{F}^c_{\rho\mu}. \tag{3.7}$$

Other terms that do not contain fermions, such as

$$\kappa^2\mathcal{F}^a_{\mu\nu}\mathcal{D}^2\mathcal{F}^a_{\mu\nu}, \qquad \kappa^2(\mathcal{D}_\rho\mathcal{F}^a_{\rho\mu})^2,$$

can be converted to (3.7) plus four-fermion terms using the Bianchi identity and the field equations, up to $\mathcal{O}(\kappa^3)$.

The independent four-fermion vertices are ten, precisely

$$\begin{aligned}&\mathrm{S}=(\overline{\Psi}_i^I\Psi_i^I)^2, \qquad \mathrm{P}=(\overline{\Psi}_i^I\gamma_5\Psi_i^I)^2, \qquad \mathrm{V}=(\overline{\Psi}_i^I\gamma_\mu\Psi_i^I)^2, \qquad \mathrm{A}=(\overline{\Psi}_i^I\gamma_5\gamma_\mu\Psi_i^I)^2,\\ &\mathrm{T}=(\overline{\Psi}_i^I\sigma_{\mu\nu}\Psi_i^I)^2, \qquad \mathrm{S}'=(\overline{\Psi}_i^I\Psi_j^I)(\overline{\Psi}_j^I\Psi_i^I), \qquad \mathrm{P}'=(\overline{\Psi}_i^I\gamma_5\Psi_j^I)(\overline{\Psi}_j^I\gamma_5\Psi_i^I),\\ &\mathrm{V}'=(\overline{\Psi}_i^I\gamma_\mu\Psi_i^I)^2, \qquad \mathrm{A}'=(\overline{\Psi}_i^I\gamma_5\gamma_\mu\Psi_i^I)^2, \qquad \mathrm{T}'=(\overline{\Psi}_i^I\sigma_{\mu\nu}\Psi_j^I)(\overline{\Psi}_j^I\sigma_{\mu\nu}\Psi_i^I).\end{aligned} \tag{3.8}$$

The proof that these ten vertices are a basis is done using Fierz identities. Every fermion has three indices: Lorentz, gauge and flavor. We have to study the contractions of

$$\overline{\Psi}_i^{I\alpha}\Psi_j^{I\beta}\overline{\Psi}_k^{J\gamma}\Psi_l^{J\delta}.$$

The Clifford algebra contains 5 elements $\Gamma^A$ (scalar, pseudoscalar, vector, pseudovector and tensor). Parity and Lorentz invariance impose that only the pairings $\Gamma^A\times\Gamma^A$ (i.e. $1\times 1$, $\gamma_5\times\gamma_5$, $\gamma_\mu\times\gamma_\mu$, etc.) are allowed. The matrices $\Gamma^A$ can contract the Lorentz indices in two ways ($\alpha$ and $\gamma$ or $\alpha$ and $\delta$), but Fierz identities relate these two contractions. Consequently, the Lorentz indices can be contracted in 5 independent ways. Finally, the gauge indices can be contracted in two ways: $i$ with $j$ or $i$ with $l$. So, in total there exist 10 independent contractions, the ones of (3.8). This proves the statement.

The four-fermion lagrangian is written as

$$\begin{aligned}\mathcal{L}_{4\mathrm{F}}=\frac{\kappa^2\mu^\varepsilon}{4}&\left[\xi_1 Z_{\xi_1}\ \mathrm{S}+\xi_2 Z_{\xi_2}\ \mathrm{P}+\xi_3 Z_{\xi_3}\ \mathrm{S}'+\xi_4 Z_{\xi_4}\ \mathrm{P}'+\lambda_1 Z_{\lambda_1}\ \mathrm{V}+\lambda_2 Z_{\lambda_2}\ \mathrm{A}+\right.\\ &\left.+\lambda_3 Z_{\lambda_3}\ \mathrm{V}'+\lambda_4 Z_{\lambda_4}\ \mathrm{A}'+\eta_1 Z_{\eta_1}\ \mathrm{T}+\eta_2 Z_{\eta_2}\ \mathrm{T}'\right].\end{aligned} \tag{3.9}$$

There exist no other independent parity-invariant lagrangian terms of level 2. The terms containing two fermions and gauge fields, such as

$$\mathcal{F}_{\mu\nu}^a\ (\mathcal{D}_\mu\overline{\Psi}^I)T^a\gamma_\nu^I\Psi, \qquad\qquad \varepsilon_{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}^a\ (\mathcal{D}_\mu\overline{\Psi}^I)T^a\gamma_\nu\gamma_5^I\Psi,$$

are not independent. Using the field equations and Bianchi identities, they can be converted into four-fermion terms, up to total derivatives and $\mathcal{O}(\kappa^3)$. The proof is given in the appendix, see formulas (8.6) and (8.7).

**Pauli deformation.** The Pauli deformation of the theory (3.1) is described by the lagrangian

$$\mathcal{L}=\frac{\mu^{-\varepsilon}}{4g^2Z_g^2}(\mathcal{F}_{\mu\nu}^a)^2+\overline{\Psi}_i^I \not{D}_{ij}\Psi_j^I+\mathcal{L}_{\mathrm{Pauli}}+\mathcal{L}_{F^3}+\mathcal{L}_{4\mathrm{F}}+\mathcal{O}(\kappa^3). \tag{3.10}$$

The couplings of levels $>1$ have to be determined iteratively as explained in section 2. This is illustrated in section 5 for level 2.

The field redefinitions have the form $\mathcal{A}_\mu^a=Z_A^{1/2}A_\mu^a+\mathcal{O}(\kappa)$, $\Psi_i^I=Z_\psi^{1/2}\psi_i^I+\mathcal{O}(\kappa)$. The Pauli vertex is the lowest-level operator of the deformation (3.10). The four-fermion vertices, the $F^3$ term and the $\mathcal{O}(\kappa^3)$-terms are the queue of the deformation.

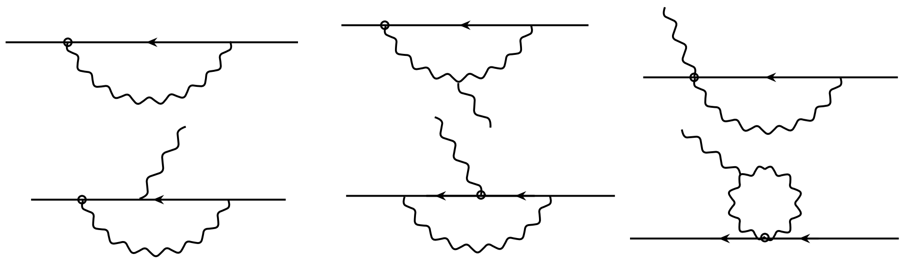

Figure 1: Self-renormalization of the Pauli term



In the next two sections I perform a complete one-loop calculation up to $\mathcal{O}(\kappa^3)$ excluded. The calculation can be divided in two steps. The first step is the renormalization of the Pauli coupling (section 4). This determines the coherent running of the Pauli deformation (3.10), including the queue. At this level it is necessary to work out also the field redefinitions explicitly, because they can be important for the $\mathcal{O}(\kappa^2)$-calculations. The second step (section 5) is the renormalization of the first terms of the queue (level 2), that is to say the $F^3$-vertex and the four-fermion vertices. This calculation can be divided itself into two parts, the self-renormalization of the level-2 vertices and their generation from two Pauli insertions. Using gauge invariance it is possible to reduce the number of diagrams. Counterterms with external legs $\overline{\psi}$-$\psi$, $A$-$\overline{\psi}$-$\psi$, $\overline{\psi}$-$\psi$-$\overline{\psi}$-$\psi$, $A$-$A$ and $A$-$A$-$A$ need to be calculated explicitly, but counterterms of the form $A$-$A$-$\overline{\psi}$-$\psi$ and $A$-$A$-$A$-$A$, for example, are related by gauge invariance to the previous ones.

# 4 Renormalization of the Pauli coupling

The non-trivial graphs containing one insertion of the Pauli vertex are shown in Fig. 1. Using the identity (8.4), the associated counterterms are

$$\Delta\mathcal{L}_{\text{Pauli}} = \frac{3ig^2\kappa\lambda}{8\pi^2\varepsilon}\frac{N_c^2-1}{N_c}\overline{\psi}D\!\!\!/^2\psi + \frac{g^2\kappa\lambda}{64\pi^2\varepsilon N_c}\left[N_c^2(1-6\alpha)+4(\alpha-5)\right]\overline{\psi}T^a\sigma_{\mu\nu}\psi\ F^a_{\mu\nu} + \\ + \frac{3ig^2\kappa\lambda N_c}{32\pi^2\varepsilon}\left(\overline{\psi}T^a\psi\ \partial_\mu A^a_\mu + 2\partial_\mu\overline{\psi}T^a\psi\ A^a_\mu\right) + \mathcal{O}(\overline{\psi}A^2\psi). \tag{4.1}$$

Using (8.5), the terms proportional to the $\mathcal{O}(\kappa^0)$-field equations can be rabsorbed by means of the field redefinitions

$$\Psi^I_i = Z_\psi^{1/2}\psi^I_i + \frac{3ig^2\kappa\lambda}{16\pi^2\varepsilon}\left(\frac{N_c^2-1}{N_c}D\!\!\!/_{ij}\psi^I_j - \frac{N_c}{2}A\!\!\!/^a T^a_{ij}\psi^I_j + \mathcal{O}(A^2\psi)\right) + \mathcal{O}(\kappa^2), \\ \mathcal{A}^a_\mu = Z_A^{1/2}A^a_\mu + \mathcal{O}(\kappa^2). \tag{4.2}$$

Observe that the fermion field redefinition is non-covariant. Isolating the contributions of these field redefinitions inside (4.1) the remaining counterterms are

$$\frac{g^2\kappa\lambda}{32\pi^2 N_c\varepsilon}\left[-N_c^2(1+3\alpha)+2(\alpha-5)\right]\overline{\psi}T^a\sigma_{\mu\nu}\psi\ F^a_{\mu\nu}.$$

The dependence on the gauge-fixing parameter $\alpha$ drops out factorizing $Z_\psi Z_A^{1/2}$ in front of the Pauli term. Finally, the net renormalization constant of the Pauli coupling $\lambda$ is

$$Z_\lambda = 1+\frac{g^2(N_c^2-5)}{16\pi^2 N_c\varepsilon}+\mathcal{O}(g^4).$$

This gives the beta function

$$\beta_\lambda=\frac{g^2\lambda(N_c^2-5)}{16\pi^2 N_c}\sim\frac{2}{75}\lambda\Delta$$

and the running behavior

$$\lambda(\Lambda)=\lambda(\mu)\left(\frac{\Lambda}{\mu}\right)^{2\Delta/75}.$$

The Pauli coupling is IR free, but this fact is not necessary for the consistency of the perturbative expansion. The reason is that perturbation theory can only generate logarithmic corrections, while the irrelevant deformations contain powers of the energy. At energy $E$ the behavior of the Pauli term versus the behavior of, say, the term $F^2$ is

$$\sim\lambda(\mu)\left(\frac{E}{\mu}\right)^{2\Delta/75}(\kappa E). \tag{4.3}$$

When the energy is small with respect to $1/\kappa$ and $\mu$ (it is possible to choose $\mu\sim 1/\kappa$ without loss of generality) the behavior (4.3) is compatible with the perturbative expansion in powers of the energy if $\Delta$ is greater than $-75/2$.

Observe that the gauge field has no $\mathcal{O}(\kappa)$-field redefinition. This is good, because it ensures that the gauge-fixing sector of the theory is unmodified to this order. It is easy to check that no one-loop divergent graph with external ghost legs and one Pauli vertex can be constructed.

Moreover, the form of the $\mathcal{O}(\kappa)$ corrections to the field redefinitions, shown in (4.2), ensures that these corrections can be ignored to order $\mathcal{O}(\kappa^2)$, because they do not contribute to the renormalization of the essential couplings. Indeed, they produce $\mathcal{O}(\kappa^2)$-contributions that are either proportional to the $\mathcal{O}(\kappa^0)$ fermion field equations or have the form $A$-$A$-$\overline{\psi}$-$\psi$.

## 5 The Pauli deformation to order $\mathcal{O}(\kappa^2)$

In this section I study the level-2 terms of the queue of the Pauli deformation. First I compute the relevant Feynman diagrams and then solve the quasi-finiteness equations that determine the values of the couplings multiplying the $F^3$ term and the four-fermion vertices of (3.10).

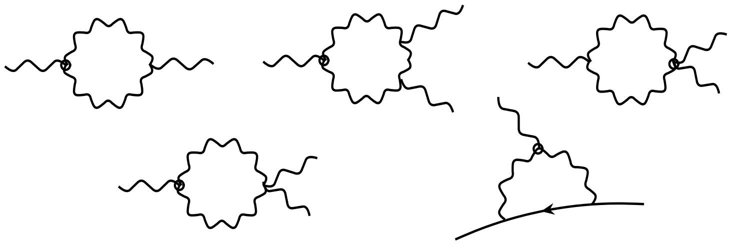

Figure 2: Renormalization of the $F^3$ vertex

**Renormalization of the $F^3$ term.** The diagrams containing one insertion of the $F^3$-vertex are depicted in Fig. 2. The counterterms for these graphs are

$$\Delta\mathcal{L}_{F^3} = -\frac{g^2\kappa^2 N_c \zeta \mu^{-\varepsilon}}{16\pi^2\varepsilon}\left(D^{ab}_{\mu}F^b_{\mu\nu}\right)^2 + \frac{3(5-\alpha)g^2 N_c}{32\pi^2\varepsilon}\frac{\zeta\kappa^2\mu^{-\varepsilon}}{6!}\ f^{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu} + \frac{g^4\kappa^2 N_c\zeta}{16\pi^2\varepsilon}\overline{\psi}T^a\gamma_\nu\psi\ D^{ab}_{\mu}F^b_{\mu\nu} \tag{5.1}$$

plus terms proportional to $\partial_\mu A^a_\mu$ (which can be subtracted with a redefinition of the gauge fixing), total derivatives, terms of the form $A$-$A$-$A$-$A$ and $A$-$A$-$\overline{\psi}$-$\psi$ and terms proportional to the field equations (8.3).

Using the field equations, the first and third terms of (5.1) mutually cancel. Finally, isolating the contribution

$$\frac{3}{2}\delta Z_A\frac{\zeta\kappa^2\mu^{-\varepsilon}}{6!}\ f^{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu}$$

of the wave-function renormalization constants, the $\alpha$-dependence drops out, as it should be, and the net result contributing to $Z_\zeta$ is

$$\Delta\mathcal{L}_{F^3\text{-net}} = \frac{3g^2 N_c}{4\pi^2\varepsilon}\frac{\zeta\kappa^2\mu^{-\varepsilon}}{6!}\ f^{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu}. \tag{5.2}$$

The renormalization constant $Z_\zeta$ receives contributions also from graphs containing two insertions of the Pauli vertex (see below).

**Renormalization of the four-fermion terms.** The divergent graphs containing a four-fermion vertex are shown in Fig. 3. The counterterms

$$\Delta\mathcal{L}_{\rm 4F} = 2\delta Z_\psi Z_\psi^{-2}Z_\zeta^{-1}\mathcal{L}_{\rm 4F} + \Delta\mathcal{L}_{\rm vf} + \Delta\mathcal{L}_{\rm 4f}, \tag{5.3}$$

can be split in three parts: the contributions associated with the wave-function renormalization constant $Z_\psi$, which reabsorb every gauge-fixing dependence, the vertex counterterm

$$\Delta\mathcal{L}_{\rm vf} = -\frac{\kappa^2}{48\pi^2\varepsilon}\left(2\lambda_1 + 2\lambda_2 + 4N_f\lambda_3 - \xi_1 + \xi_2\right)\overline{\psi}T^a\gamma_\nu\psi\ D^{ab}_{\mu}F^b_{\mu\nu} \tag{5.4}$$

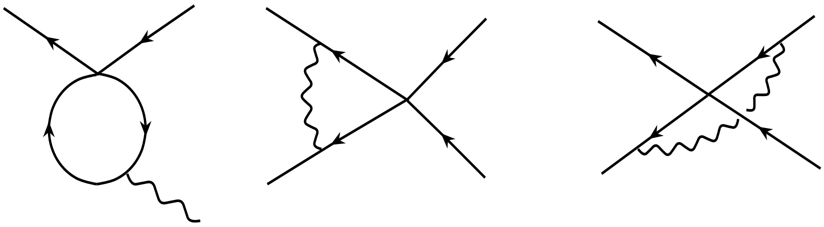

Figure 3: Self-renormalization of the four-fermion vertices

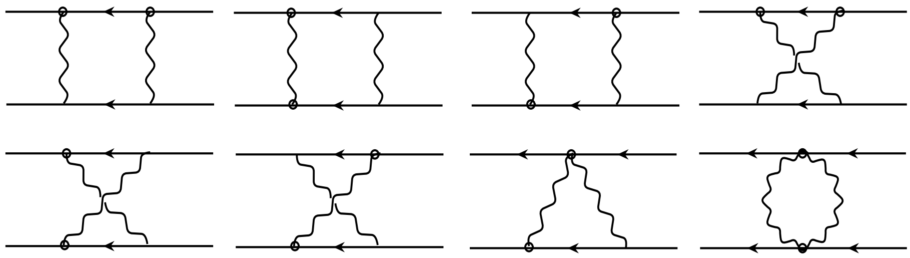

Figure 4: Four-fermion renormalization to order $\lambda^2$

and the four-fermion counterterms

$$\begin{aligned}\Delta\mathcal{L}_{4\mathrm{f}} = &-\frac{g^2}{32\pi^2\varepsilon N_c}\frac{\kappa^2\mu^\varepsilon}{4}\big[12\left(\xi_1(N_c^2-1)+2\eta_2 N_c+\xi_3 N_c-4\eta_1\right)\mathrm{S}+\\ &+12\left(\xi_2(N_c^2-1)+2\eta_2 N_c+\xi_4 N_c-4\eta_1\right)\mathrm{P}+12\left(2\eta_2(N_c^2-2)+4\eta_1 N_c-\xi_3\right)\mathrm{S}'+\\ &+12\left(2\eta_2(N_c^2-2)+4\eta_1 N_c-\xi_4\right)\mathrm{P}'+6\left(2\lambda_2-\lambda_3 N_c-\lambda_4 N_c\right)\mathrm{V}+\\ &+6\left(2\lambda_1-\lambda_3 N_c-\lambda_4 N_c\right)\mathrm{A}+6\left(\lambda_3 N_c^2-2\lambda_2 N_c-\lambda_4(N_c^2-2)\right)\mathrm{V}'+\\ &+6\left(\lambda_4 N_c^2-2\lambda_1 N_c-\lambda_3(N_c^2-2)\right)\mathrm{A}'+\\ &+\left((\xi_3+\xi_4)N_c-2(\xi_1+\xi_2)-12\eta_2 N_c-4\eta_1(N_c^2-1)\right)\mathrm{T}+\\ &+\left((\xi_3+\xi_4)(N_c^2-2)+2N_c(\xi_1+\xi_2)+4\eta_2(2N_c^2+1)\right)\mathrm{T}'\big]\,. \end{aligned} \tag{5.5}$$

Using the field equations and the identity (8.1), the vertex counterterm (5.4) is converted into the sum of two four-fermion terms:

$$\Delta\mathcal{L}_{\mathrm{vf}} \to -\frac{g^2\kappa^2\mu^\varepsilon}{96\pi^2\varepsilon N_c}\left(2\lambda_1+2\lambda_2+4N_f\lambda_3-\xi_1+\xi_2\right)\left(\mathrm{V}-N_c\mathrm{V}'\right). \tag{5.6}$$

**Generation of $F^3$ and four-fermion terms from two Pauli insertions.** There remain to study the contributions of type $\delta$ in (1.1). These are due to the graphs containing two insertions of Pauli vertices. The graphs are grouped into two sets, depicted in Figs. 4 and 5.

The contributions of the graphs of Fig. 4 are

$$\Delta\mathcal{L}_{4\text{F-}\lambda^2} = \frac{g^4\lambda^2}{8\pi^2\varepsilon N_c^2}\frac{\kappa^2\mu^\varepsilon}{4}\left[-24(N_c^2+2)\text{S} - 24N_c(N_c^2-4)\text{S}' + 3N_c^2\text{V} + 6(N_c^2+2)\text{A} + \right.$$
$$\left. -3N_c^3\text{V}' + 6N_c(N_c^2-4)\text{A}' + N_c^2\text{T} - N_c^3\text{T}'\right]. \tag{5.7}$$

The contributions of the graphs of Fig. 5 are

$$\Delta\mathcal{L}_{F^3\text{-}\lambda^2} = -\frac{\lambda^2 N_f\kappa^2\mu^{-\varepsilon}}{12\pi^2\varepsilon}\left(D_\mu^{ab}F_{\mu\nu}^b\right)^2 - \frac{\lambda^2 N_f}{4\pi^2\varepsilon}\frac{\kappa^2\mu^{-\varepsilon}}{6!}\ f^{abc}F_{\mu\nu}^a F_{\nu\rho}^b F_{\rho\mu}^c +$$
$$+\frac{g^2\lambda^2\kappa^2}{48\pi^2\varepsilon N_c}\left[(23N_c^2+10)\overline{\psi}T^a\gamma_\nu\psi\ D_\mu^{ab}F_{\mu\nu}^b + 12(N_c^2-1)F_{\mu\nu}^a\ D_\mu\overline{\psi}T^a\gamma_\nu\psi + \right.$$
$$\left. + 12N_c^2\ \varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}^a\ D_\rho\overline{\psi}T^a\gamma_\sigma\gamma_5\psi\right] \tag{5.8}$$

plus terms proportional to $\partial_\mu A_\mu^a$, total derivatives, terms of the form $A$-$A$-$A$-$A$ and $A$-$A$-$\overline{\psi}$-$\psi$ and terms proportional to the field equations. Using the identities (8.6) and (8.7) and the field equations, the counterterm (5.8) can be re-written as

$$\Delta\mathcal{L}_{F^3\text{-}\lambda^2} \to \frac{\lambda^2 g^4\kappa^2\mu^\varepsilon}{96\pi^2\varepsilon N_c^2}\left(5N_c^2+16-4N_fN_c\right)\left(\text{V}-N_c\text{V}'\right) - \frac{\lambda^2 N_f}{4\pi^2\varepsilon}\frac{\kappa^2\mu^{-\varepsilon}}{6!}\ f^{abc}F_{\mu\nu}^a F_{\nu\rho}^b F_{\rho\mu}^c. \tag{5.9}$$

**The Pauli deformation to order $\mathcal{O}(\kappa^2)$.** Recapitulating, the $\mathcal{O}(\kappa^2)$ counterterms, cleaned of the contributions due to the wave-function renormalization constants and the terms proportional to the gauge-fixing and the field equations, are the sum of (5.2), (5.5), (5.6), (5.7) and (5.9):

$$\Delta\mathcal{L}_{\text{net}} = \Delta\mathcal{L}_{F^3\text{-net}} + \Delta\mathcal{L}_{4\text{f}} + \Delta\mathcal{L}_{\text{vf}} + \Delta\mathcal{L}_{4\text{F-}\lambda^2} + \Delta\mathcal{L}_{F^3\text{-}\lambda^2}.$$

It is convenient to start from the $F^3$-terms, which can be easily isolated from the rest. The $\zeta$-renormalization constant is

$$\zeta Z_\zeta = \zeta\left(1+\frac{3g^2N_c}{4\pi^2\varepsilon}\right) - \frac{\lambda^2 N_f}{4\pi^2\varepsilon}.$$

The quasi-finiteness equations relate $\zeta$ to $\lambda$ in such a way that the scale-invariant combination

$$u \equiv \frac{\zeta}{\lambda^2}$$

has vanishing beta function. This means also

$$\zeta Z_\zeta = \zeta Z_\lambda^2.$$

The result is

$$\zeta = \frac{11\lambda^2}{5g_*^2} = \frac{165}{2}\frac{1}{\Delta}\left(\frac{\lambda^2 N_c}{16\pi^2}\right). \tag{5.10}$$

The beta function of $\zeta$ is

$$\beta_\zeta = \frac{3g_*^2\zeta N_c}{4\pi^2} - \frac{\lambda^2 N_f}{4\pi^2} = 2\frac{\zeta}{\lambda}\beta_\lambda,$$

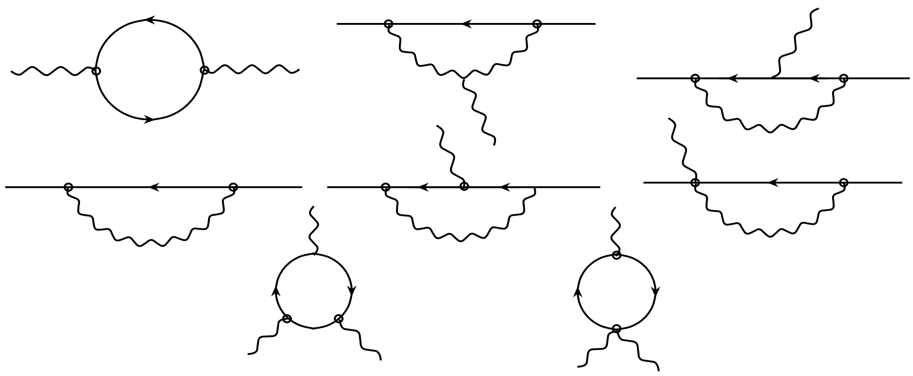

Figure 5: Renormalization of two- and three-point functions to order $\lambda^2$

so that $\beta_u = 0$.

The value (5.10) is large, because of the $\Delta$ in the denominator. This means that the perturbative expansion in powers of the energy is meaningful if the energy is much smaller than the effective Planck scale

$$M_{P\text{eff}} = \frac{1}{\kappa_{\text{eff}}} = \frac{\Delta}{\kappa}. \tag{5.11}$$

The other factor in (5.10) can be taken of order one, if $\lambda^2(\mu)N_c$ is kept fixed in the large $N_c, N_f$ limit.

Repeating the same calculation for the four-fermion counterterms, the result is

$$\frac{\xi_1 N_c}{16\pi^2} = \frac{16\Delta}{225 N_c}\lambda^2, \qquad \frac{\xi_2 N_c}{16\pi^2} = -\frac{56\Delta}{225 N_c}\lambda^2, \qquad \frac{\xi_3 N_c}{16\pi^2} = -\frac{92\Delta}{225}\lambda^2, \qquad \frac{\xi_4 N_c}{16\pi^2} = \frac{52\Delta}{225}\lambda^2,$$
$$\frac{\lambda_1 N_c}{16\pi^2} = \frac{544\Delta}{3225 N_c}\lambda^2, \qquad \frac{\lambda_2 N_c}{16\pi^2} = \frac{8\Delta}{43 N_c}\lambda^2, \qquad \frac{\lambda_3 N_c}{16\pi^2} = -\frac{94\Delta}{3225}\lambda^2, \qquad \frac{\lambda_4 N_c}{16\pi^2} = \frac{2\Delta}{43}\lambda^2,$$
$$\frac{\eta_1 N_c}{16\pi^2} = -\frac{N_c\Delta}{45}\lambda^2, \qquad \frac{\eta_2 N_c}{16\pi^2} = \frac{4\Delta}{675}\lambda^2.$$

Here no denominator contains $\Delta$, so (5.11) is not modified.

## 6 Applications to supersymmetric theories

In this section I prove that the non-renormalization theorem for the chiral operators in supersymmetric theories does not preclude the existence of solutions of the quasi-finiteness equations. I construct finite and quasi-finite chiral irrelevant deformations.

Consider a supersymmetric theory in four dimensions, formulated using N=1 superfields. A well-known non-renormalization theorem states that no chiral counterterm appears in the renormalization of the theory. A chiral counterterm is a term that cannot be written as the integral in $\mathrm{d}^4\theta$ of a local superfield. Consider for example the chiral lagrangian term

$$C_k = Y_k \int \Phi^k \ \mathrm{d}^2\theta, \tag{6.12}$$

where $Y_k$ is a coupling constant and $\Phi$ is an elementary chiral superfield. This operator has level $k-3$. The non-renormalization theorem implies that

$$Y_k \int \Phi^k \ \mathrm{d}^2\theta = Y_k Z_k Z_\Phi^{k/2} \int \Phi^k \ \mathrm{d}^2\theta, \qquad \text{or } Z_k Z_\Phi^{k/2} = 1, \ \text{i.e. } \gamma_{(k)} = k\gamma_\Phi. \tag{6.13}$$

Thus, the anomalous dimension $\gamma_{(k)}$ of a chiral operator of the form (6.12) is $k$ times the anomalous dimension of the elementary field $\Phi$. Moreover, the equation $Y_{k\mathrm{B}} = Y_k Z_k = Y_k Z_\Phi^{-k/2}$ implies that the beta function is

$$\beta_{(k)} = kY_k\gamma_\Phi = Y_k\gamma_{(k)}.$$

This means that the quantity $\delta$ of formula (1.1) vanishes, i.e. that the chiral operators are protected, in the sense explained in section 2.

Now, consider an interacting superconformal field theory. Examples are the IR fixed points of certain UV-free theories (see [10] for a collection of properties of these conformal theories). Turn on the deformation $C_k$, which has level $\ell = k-3$. The requirement (2.13) for the existence of solutions of the quasi-finiteness equations reads

$$\gamma_{(nk-3n+3)} \neq n\gamma_{(k)}$$

for every integer $n > 1$. Using (6.13) this condition becomes

$$(nk - 3n + 3)\gamma_\Phi \neq kn\gamma_\Phi,$$

which is always true if $\gamma_\Phi \neq 0$. The value of $\gamma_\Phi$ is known in various models, using the NSVZ exact beta function [11]. Typically, $\gamma_\Phi \neq 0$ in N=1 superconformal field theories, but there exist finite N=2 and N=4 supersymmetric theories where $\gamma_\Phi = 0$. Since, however, the chiral operators are protected, the fact that (2.13) is violated is not a problem for the construction of consistent irrelevant deformations. I consider the cases $\gamma_\Phi \neq 0$ and $\gamma_\Phi = 0$ separately.

I recall from section 2 that an irrelevant deformation (2.12) is made of a lowest-level operator $\mathcal{O}_\ell$ and a queue. The lowest-level term is multiplied by an arbitrary parameter $\widetilde{\kappa}$ that can run according to (2.8). The queue consists of infinitely many lagrangian terms, whose couplings are determined by the quasi-finiteness equations. I call *chiral* a deformation whose lowest-level term $\mathcal{O}_\ell$ is chiral. Now, repeat the construction of section 2 for a chiral deformation. If $\gamma_\Phi \neq 0$, the parameter $\widetilde{\kappa}$ still runs. Moreover, the queue does not contain other chiral terms, because their coefficients are set to zero by the quasi-finiteness equations.

Now, consider a case where $\gamma_\Phi = 0$, for example N=4 supersymmetric Yang-Mills theory. In the formalism of N=1 superfields, the theory contains a vector multiplet and three chiral multiplets $\Phi^i$. The fields $\Phi^i$ have zero anomalous dimensions and therefore the chiral operators, for example

$$\int Y_{i_1\cdots i_n}\Phi^{i_1}\cdots\Phi^{i_n}\ \mathrm{d}^2\theta,$$

are finite and protected. (Here $Y_{i_1\cdots i_n}$ is a constant tensor.) Consider a chiral irrelevant deformation of N=4 supersymmetric Yang-Mills theory. Since the anomalous dimension of the lowest-level operator is zero by assumption, the running (2.8) is trivial, i.e. $\widetilde{\kappa}$ is an arbitrary finite parameter. The deformation is therefore finite. Since chiral operators are protected, the coefficients of the other chiral terms in the queue can be consistently set to zero. If the queue of a chiral deformation contains no other chiral term, the chiral deformation is *simple*. If other chiral terms appear in the queue (multiplied by arbitrary parameters), then it is a *multiple* chiral deformation. The queue contains also non-chiral terms. For these, the condition (2.13) is non-trivial. Since the lowest-level term is finite ($r_\ell = 0$), the condition (2.13) for the existence of the deformation states that the non-chiral operators of levels $n\ell$ should have non-vanishing anomalous dimensions. This is generically true. Then, the quasi-finiteness equations (which are actually *finiteness* equations, in this particular case) admit a solution. The perturbative expansion in powers of the energy is well-defined if the quantity $\eta$ defined in eq. (2.18) is strictly positive. Since the anomalous dimensions of non-chiral operators are generically non-trivial already at the first loop order, $\eta$ is typically of order $g^2$, where $g$ is some gauge coupling, and the "effective Planck mass" $M_{P\mathrm{eff}} = 1/\kappa_{\mathrm{eff}}$ is typically of order $\sim g^2 M_P = g^2/\kappa$.

In conclusion, I have proved that the requirement (2.13) for the existence of quasi-finite irrelevant deformations is not in contradiction with the non-renormalization theorem of supersymmetric theories. The construction of quasi-finite irrelevant deformations of N=1 superconformal field theories proceeds as in the absence of supersymmetry, while N=2 and N=4 finite theories admit also finite chiral irrelevant deformations.

Observe that the finite chiral irrelevant deformations of superconformal field theories are also good examples of finite four-dimensional power-counting non-renormalizable theories. They can be seen as particular applications of the construction elaborated in ref. [2]. Their renormalization requires only field redefinitions, but no coupling redefinition.

## 7 Conclusions

Using the strategy of this paper, it is possible to construct consistent irrelevant deformations of interacting conformal field theories. These deformations have a scale, which multiplies the lowest-level irrelevant term, a finite number of coupling constants, and a queue made of infinitely many lagrangian terms. A finite number of renormalization constants, plus field redefinitions, are sufficient to remove the divergences. The scale can run (quasi-finite deformations) or not (finite deformations). If the scale runs, the queue of the deformation runs coherently with the lowest-level term. In certain families of supersymmetric theories it is possible to construct chiral finite deformations.

The perturbative expansion is meaningful for energies much smaller than an effective Planck mass. The effective Planck mass becomes small when the interaction of the renormalizable sector of the theory becomes weak.

Generalizations are possible. For example, in some cases it is possible to construct irrelevant deformations of running power-counting renormalizable theories. However, this issue is technically more tricky and is left for a future publication.

Although the ideas of this paper do not apply directly to quantum gravity, a more general framework where they do might exist. Then, the quantization of gravity might be possible only thanks to the existence of other interacting matter. The effects of quantum gravity might show up at energies some orders of magnitude smaller than the Planck mass, depending on the strength of the interaction in the matter sector.

The results of this paper and ref. [2] suggest that power-counting non-renormalizable theories are candidate to play a relevant role in the description of fundamental physics. Certainly, the problem of predictivity of fundamental field theory needs to be carefully reconsidered in the light of these results.

## 8 Appendix. Useful identities

In this appendix I collect useful identities, fields equations and certain manipulations that are helpful to identify the irreducible set of irrelevant terms, and to simplify the counterterms for the study of the Pauli deformation of the IR fixed point of massless non-Abelian Yang-Mills theory with $N_c$ colors and $N_f \lesssim 11N_c/2$ flavors.

**Notation and identities for gauge group and representations.** The notation is such that $[T^a, T^b] = f^{abc}T^c$, $(f^{abc})^* = f^{abc}$, $(T^a)^\dagger = -T^a$. Useful identities are

$$\mathrm{tr}[T^aT^b] = -\frac{1}{2}\delta^{ab}, \qquad T^a_{ij}T^a_{kl} = -\frac{1}{2}\delta_{il}\delta_{jk} + \frac{1}{2N_c}\delta_{ij}\delta_{kl}. \tag{8.1}$$

**Useful identities for the Dirac algebra.** The products between the elements 1, $\gamma_5 = \varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma/4!$, $\gamma_\mu$, $\gamma_5\gamma_\mu$, $\sigma_{\mu\nu} = -i[\gamma_\mu, \gamma_\nu]/2$ of the Clifford algebra are immediate or can be read from the following table:

$$\sigma_{\mu\nu}\gamma_\rho = i\delta_{\mu\rho}\gamma_\nu - i\delta_{\nu\rho}\gamma_\mu + i\varepsilon_{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5, \qquad \gamma_\rho\sigma_{\mu\nu} = -i\delta_{\mu\rho}\gamma_\nu + i\delta_{\nu\rho}\gamma_\mu + i\varepsilon_{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5,$$
$$\sigma_{\mu\nu}\gamma_5 = -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}, \qquad \gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\sigma_{\mu\nu},$$
$$\sigma_{\mu\nu}\sigma_{\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} - \varepsilon_{\mu\nu\alpha\beta}\gamma_5 - i(\delta_{\nu\alpha}\sigma_{\mu\beta} - \delta_{\nu\beta}\sigma_{\mu\alpha} + \delta_{\mu\beta}\sigma_{\nu\alpha} - \delta_{\mu\alpha}\sigma_{\nu\beta}).$$

**Field equations.** The $\mathcal{O}(\kappa)$-field equations read

$$\not{D}_{ij}\Psi^I_j + \kappa\lambda Z_\lambda T^a_{ij}\sigma_{\mu\nu}\Psi^I_j\mathcal{F}^a_{\mu\nu} + \mathcal{O}(\kappa^2) = 0, \tag{8.2}$$

$$\frac{\mu^{-\varepsilon}}{g^2Z_g^2}\mathcal{D}^{ab}_\nu\mathcal{F}^b_{\mu\nu} + \overline{\Psi}^I_i\gamma_\mu T^a_{ij}\Psi^I_j + 2\kappa\lambda Z_\lambda\mathcal{D}^{ab}_\nu(\overline{\Psi}^I_i\sigma_{\mu\nu}T^a_{ij}\Psi^I_j) + \mathcal{O}(\kappa^2) = 0. \tag{8.3}$$

**Identities for terms of level 1.** Here I write some identities that are useful for the simplification of the $\mathcal{O}(\kappa)$ counterterms. The first one is obvious

$$\int \overline{\psi} D^2 \psi = \int \overline{\psi} \not{D}^2 \psi - \frac{i}{2} \int \overline{\psi} T^a \sigma_{\mu\nu} \psi \ F^a_{\mu\nu}. \tag{8.4}$$

The second one is obtained dropping the $\square$-terms on each side:

$$\int \overline{\psi} T^a \psi \ \partial_\mu A^a_\mu + 2 \int (\partial_\mu \overline{\psi}) T^a \psi \ A^a_\mu = - \int A^a_\mu \ \overline{\psi} \left( \gamma_\mu T^a \not{D} - \overleftarrow{\not{D}} T^a \gamma_\mu \right) \psi + \frac{i}{2} \int \overline{\psi} T^a \sigma_{\mu\nu} \psi \ F^a_{\mu\nu}, \tag{8.5}$$

up to terms $A$-$A$-$\overline{\psi}$-$\psi$. The integral appears to allow partial integrations and drop total derivatives. It is understood that the derivative covered with a left arrow acts only on $\overline{\psi}$.

The identities (8.4) and (8.5) are useful to express certain counterterms as sums of objects proportional to the $\mathcal{O}(\kappa^0)$-field equations plus the Pauli term.

**Identities for terms of level 2.** A similar work has to be done at $\mathcal{O}(\kappa^2)$. It is sufficient to work out identities up to terms of the form $A$-$A$-$\overline{\psi}$-$\psi$, which are unnecessary for the computations of the paper. Moreover, in the manipulations of the $\mathcal{O}(\kappa^2)$ counterterms, the terms proportional to the $\mathcal{O}(\kappa^0)$-field equations can be safely dropped. This operation is denoted with an arrow.

With these conventions, it is easy to show that

$$0 \leftarrow \int F^a_{\mu\nu} \ \overline{\psi} \left( T^a \sigma_{\mu\nu} \not{D} + \overleftarrow{\not{D}} \sigma_{\mu\nu} T^a \right) \psi = 2i \int F^a_{\mu\nu} \ \overline{\psi} T^a \gamma_\nu \left( \partial_\mu - \overleftarrow{\partial_\mu} \right) \psi.$$

To derive this identity it is sufficient to partially integrate, use the Bianchi identity and ignore the terms of the form $A$-$A$-$\overline{\psi}$-$\psi$.

Furthermore, partially integrating and using the gauge-field equations, we have also

$$2i \int F^a_{\mu\nu} \ \overline{\psi} T^a \gamma_\nu \left( \partial_\mu + \overleftarrow{\partial_\mu} \right) \psi \rightarrow -2ig^2 \int \left( \overline{\psi} T^a \gamma_\mu \psi \right)^2 .$$

Combining the two, we get

$$\int F^a_{\mu\nu} \ (D_\mu \overline{\psi}) T^a \gamma_\nu \psi \rightarrow -\frac{1}{2} g^2 \int \left( \overline{\psi} T^a \gamma_\mu \psi \right)^2 , \tag{8.6}$$

up to $\mathcal{O}(\kappa)$, terms $A$-$A$-$\overline{\psi}$-$\psi$ and terms proportional to the $\mathcal{O}(\kappa^0)$-field equations.

Similarly, we have

$$0 \leftarrow \int F^a_{\mu\nu} \ \overline{\psi} \left( T^a \sigma_{\mu\nu} \not{D} - \overleftarrow{\not{D}} \sigma_{\mu\nu} T^a \right) \psi = i \int \varepsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} \ \overline{\psi} T^a \gamma_\sigma \gamma_5 \left( \partial_\rho - \overleftarrow{\partial_\rho} \right) \psi +$$
$$-2i \int D^{ab}_\mu F^b_{\mu\nu} \ \overline{\psi} T^a \gamma_\nu \psi \rightarrow i \int \varepsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} \ \overline{\psi} T^a \gamma_\sigma \gamma_5 \left( \partial_\rho - \overleftarrow{\partial_\rho} \right) \psi - 2ig^2 \int \left( \overline{\psi} T^a \gamma_\mu \psi \right)^2 .$$

Combining this formula with a simple consequence of the Bianchi identity, namely

$$0 = i \int \varepsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} \ \overline{\psi} T^a \gamma_\sigma \gamma_5 \left( \partial_\rho + \overleftarrow{\partial_\rho} \right) \psi,$$

we obtain

$$\int \varepsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}\ (D_\rho\overline{\psi}) T^a \gamma_\sigma \gamma_5 \psi \to -g^2 \int \left(\overline{\psi} T^a \gamma_\mu \psi\right)^2 , \tag{8.7}$$

up to $\mathcal{O}(\kappa)$, terms $A$-$A$-$\overline{\psi}$-$\psi$ and terms proportional to the $\mathcal{O}(\kappa^0)$-field equations.

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