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FINITENESS OF QUANTUM GRAVITY COUPLED WITH MATTER IN THREE SPACETIME DIMENSIONS

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Abstract

As it stands, quantum gravity coupled with matter in three spacetime dimensions is not finite. In this paper I show that an algorithmic procedure that makes it finite exists, under certain conditions. To achieve this result, gravity is coupled with an interacting conformal field theory $\mathcal{C}$. The Newton constant and the marginal parameters of $\mathcal{C}$ are taken as independent couplings. The values of the other irrelevant couplings are determined iteratively in the loop- and energy-expansions, imposing that their beta functions vanish. The finiteness equations are solvable thanks to the following properties: the beta functions of the irrelevant couplings have a simple structure; the irrelevant terms made with the Riemann tensor can be reabsorbed by means of field redefinitions; the other irrelevant terms have, generically, non-vanishing anomalous dimensions. The perturbative expansion is governed by an effective Planck mass that takes care of the interactions in the matter sector. As an example, I study gravity coupled with Chern-Simons $U(1)$ gauge theory with massless fermions, solve the finiteness equations and determine the four-fermion couplings to two-loop order. The construction of this paper does not immediately apply to four-dimensional quantum gravity.

1 Introduction

Gravity is not power-counting renormalizable. This might mean that quantum field theory is inadequate to quantize gravity or, more conservatively, that power-counting renormalizability is not an essential feature of the theories that describe nature. At the theoretical level, there exist power-counting non-renormalizable theories that can be quantized successfully, such as the four-fermion models in three spacetime dimensions [1] in the large N expansion. Moreover, a theory that is not power-counting renormalizable does not necessarily violate fundamental physical principles and so it cannot be discarded *a priori*.

In four-dimensions, 't Hooft and Veltman showed that pure gravity is finite to one-loop order [2], but finiteness is spoiled by the coupling with matter. Goroff and Sagnotti showed that gravity is not finite to two-loop order [3], even in the absence of matter. These results depressed the hopes to find a finite theory of quantum gravity.

To some extent, the problem of finiteness is simpler in three spacetime dimensions. In odd dimensions every theory is finite to one-loop order, because there are no logarithmic one-loop divergences. So, the problem starts from two loops. Moreover, pure gravity in three dimensions,

$$S = \frac{1}{2\kappa} \int \sqrt{g} R, \tag{1.1}$$

propagates no graviton and is finite to all orders [4]. Indeed, since the Weyl tensor vanishes, the Riemann tensor is a linear combination of the Ricci tensor and the scalar curvature. This ensures that all possible counterterms can be reabsorbed by means of field redefinitions.

The issue of finiteness is non-trivial in three dimensions if gravity is coupled with matter. In [5] I have proved that renormalization generates counterterms with dimensionality greater than three, in general infinitely many. I recall here the main results of that paper:

1) the Lorentz-Chern-Simons term

$$\int \varepsilon^{\mu\nu\rho} \left(\omega_\mu^a \partial_\nu \omega_\rho^a + \frac{1}{3} \omega_\mu^a \omega_\nu^b \omega_\rho^c \varepsilon^{abc} \right), \tag{1.2}$$

is not induced by renormalization, so there exists a subtraction scheme where it is absent at each order of the perturbative expansion, if it is absent at the classical level. This property can be proved combining a power-counting analysis of the complete theory with properties of the trace anomaly of the matter sector embedded in external gravity. It is important that the Lorentz-Chern-Simons term is not turned on by renormalization, because three-dimensional gravity with a Lorentz-Chern-Simons term, known as "topologically massive gravity" [6], is physically inequivalent to the theory without it.

2) I have then considered a specific model, gravity coupled with Chern-Simons $U(1)$ gauge theory and massless fermions and proved by explicit computation that a four-fermion counterterm

is induced by radiative corrections to the second order in the loop expansion and first order in the κ expansion, namely

$$-\frac{5\kappa g^4 n_f}{384\pi^2\varepsilon}\frac{e}{4}(\overline{\psi}\gamma^a\psi)^2. \tag{1.3}$$

The result (1.3) is written up to subleading corrections in $1/n_f$, where n_f is the number of complex two-component spinors. This counterexample is sufficient to conclude that, *as it stands,* quantum gravity coupled with matter in three spacetime dimensions is not finite.

The purpose of this paper is to show that, under certain conditions, quantum gravity coupled with matter in three spacetime dimensions can be quantized in a unique way as a finite theory.

A sketch of the idea is as follows. Gravity is coupled with an interacting conformal field theory $\mathcal{C}$, subject to some restrictions. If λ denote an irrelevant coupling, i.e. the coupling multiplying an irrelevant lagrangian term O_λ, then the beta function of λ a has a simple structure. In particular, it is linear in λ:

$$\beta_\lambda = \lambda\gamma_\lambda + \delta_\lambda. \tag{1.4}$$

Here γ_λ is the anomalous dimension of O_λ, which depends only on the marginal couplings of $\mathcal{C}$, but not on the irrelevant couplings of the complete theory. Instead, δ_λ depends on the marginal couplings $\mathcal{C}$ plus a finite number of irrelevant couplings, but not on λ itself. The formula (1.4) is written in symbolic form. A precise treatment is presented in the next section.

The *finiteness equations* $\beta_\lambda = 0$ can be solved if γ_λ is nonzero or γ_λ and δ_λ are simultaneously zero. I show that, generically, in three dimensions the finiteness equations admit a solution, thanks to the properties of three-dimensional spacetime, in particular the absence of a propagating graviton. The Newton constant and the marginal couplings of $\mathcal{C}$ are taken as independent couplings of the theory coupled with gravity. The values of the other irrelevant couplings are uniquely determined solving the finiteness equations. This can be done perturbatively.

The perturbative expansion in powers of the energy is valid for energies much smaller than an effective Planck constant, obtained multiplying the Planck mass by a factor that depends only on the matter subsector $\mathcal{C}$.

After working out the general principles of this approach to finiteness, I illustrate the quantization mechanism in the case of gravity coupled with Chern-Simons $U(1)$ gauge theory and massless fermions. I solve the finiteness conditions to the second order in the loop expansion, first order in the κ expansion, and leading order in the $1/n_f$ expansion. The solution uniquely determines the values of the couplings multiplying the four-fermion vertices.

The paper is organized as follows. In section 2 I present the idea in the most general terms, so that it can be applied, in principle, to every non-renormalizable theory. Moreover, I study the conditions for finiteness (structure of the beta functions of the irrelevant couplings, existence of solutions to the finiteness equations, etc.). In section 3 I consider quantum gravity coupled with matter in three dimensions and show that the finiteness equations admit generically one solution. In section 4 I introduce the model studied explicitly in the rest of the paper. I recall the regularization technique, some renormalization properties, and the four-fermion divergent vertex calculated in ref. [5]. In section 5 I report the results concerning the two-loop self-renormalization of the four-fermion vertices. In section 6 I solve the finiteness equations and determine the values of the irrelevant couplings that multiply the four-fermion vertices. The solution is contained in formulas (6.4) and (8.1). In section 7 I briefly discuss some obstacles that prevent a straightforward generalization of the approach of this paper to quantum gravity in four dimensions. Section 8 collects the conclusions and the appendix contains some notation.

2 Structure of the beta functions of the irrelevant couplings and solutions of the finiteness equations

I consider a generic power-counting non-renormalizable theory of interacting fields φ in d dimensions, having a classical lagrangian of the form

$$\mathcal{L}_{cl}[\varphi] = \mathcal{L}_0[\varphi, \alpha] + \sum_i \kappa^i \sum_{I=1}^{N_i} \lambda_{iI} \mathcal{O}_{iI}(\varphi). \tag{2.1}$$

The first piece, $\mathcal{L}_0$, denotes the power-counting renormalizable sector of the theory, with couplings α. The theory $\mathcal{L}_0$ is assumed to be finite. For example, in the case of three-dimensional quantum gravity coupled with matter, $\mathcal{L}_0$ is the sum of the free spin-2 kinetic term and the lagrangian of a conformal field theory $\mathcal{C}$, which I call the *matter sector* of the theory.

The objects $\mathcal{O}_{iI}$ are a basis of (gauge-invariant) local lagrangian terms with canonical dimensionalities $d + i$ in units of mass. The index i denotes the "level" of $\mathcal{O}_i$ (irrelevant operators have positive levels, marginal operators have level 0 and relevant operators have negative levels) and can be a non-negative integer or a half-integer. The λ_{iI} denote a complete set of essential couplings, labelled by their level i plus an index I that distinguishes the couplings of the same level (subject, in general, to renormalization mixing). The essential couplings are the couplings that multiply a basis of lagrangian terms that cannot be renormalized away or into one another by means of field redefinitions [7].

The parameter κ is an auxiliary constant with dimensionality -1 in units of mass. Every λ is dimensionless. For simplicity, I assume also that the theory (2.1) does not contain masses, the cosmological constant and super-renormalizable parameters (couplings with strictly positive dimensionalities in units of mass), because they form dimensionless quantities when they are multiplied by suitable powers of the irrelevant couplings. The beta functions can depend nonpolynomially on such dimensionless combinations, which adds unnecessary complications to the treatment.

The redundancy of the constant κ is exhibited by the invariance of (2.1) under the scale symmetry

$$\lambda_{iI} \rightarrow \Omega^{-i}\lambda_{iI}, \qquad \kappa \rightarrow \Omega\kappa. \tag{2.2}$$

Structure of the beta functions. The beta function of λ_{iI} transforms like λ_{iI} under the scale symmetry (2.2) and cannot contain negative powers of the λs. Therefore, the structure of β_{iI} is

$$\beta_{iI} = \sum_{\{n_{jJ}^{iI}\}} f_{\{n_{jJ}^{iI}\}}(\alpha) \prod_{j\leq i} \prod_{J=1}^{N_j} (\lambda_{jJ})^{n_{jJ}^{iI}}, \tag{2.3}$$

where the $f_{\{n_{jJ}^{iI}\}}(\alpha)$s are functions of the marginal couplings and the sum is performed over the sets $\{n_{jJ}^{iI}\}$ of non-negative integers n_{jJ}^{iI} such that

$$\sum_{j\leq i} j \sum_{J=1}^{N_j} n_{jJ}^{iI} = i. \tag{2.4}$$

The constant κ, which is, by assumption, the only dimensionful parameter in the theory, does not appear in the beta functions.

Due to (2.4), only a finite set of numbers n_{jJ}^{iI} can be greater than zero. This ensures that the beta functions depend on the irrelevant couplings in a polynomial way. Special sets $\{n_{jJ}^{iI}\}$ satisfying (2.4) are those where n_{jJ}^{iI} is equal to one for $j = i$ and some index J, zero otherwise. It is useful to isolate this contribution from the rest, obtaining

$$\beta_{iI} = \sum_{J=1}^{N_i} \gamma_i^{IJ}(\alpha)\, \lambda_{iJ} + \delta_{iI}, \qquad \delta_{iI} = \sum_{\{m_{jJ}^{iI}\}} f_{\{m_{jJ}^{iI}\}}(\alpha) \prod_{j<i} \prod_{J=1}^{N_j} (\lambda_{jJ})^{m_{jJ}^{iI}}. \tag{2.5}$$

Now the sum is performed over the sets $\{m_{jJ}^{iI}\}$ of non-negative integers such that

$$\sum_{j<i} j \sum_{J=1}^{N_j} m_{jJ}^{iI} = i. \tag{2.6}$$

The functions $\gamma_i^{IJ}(\alpha)$ are the entries of the matrix $\gamma_i(\alpha)$ of anomalous dimensions of the operators $\mathcal{O}_{iI}$ of level i. The second term of (2.5) collects the contributions of the operators $\mathcal{O}_{jJ}$ of levels $j < i$. Observe that (2.6) implies

$$\sum_{j<i} \sum_{J=1}^{N_j} m_{jJ}^{iI} \geq 2, \tag{2.7}$$

which means that the beta function of λ_i is at least quadratic in the irrelevant couplings with $j < i$. *A fortiori*, the δ_{iI}s vanish when all of the λ_{iI}s vanish. Indeed, at $\lambda_{iI} = 0$ the theory reduces to $\mathcal{L}_0[\varphi, \alpha]$, which is finite by assumption. So, $\lambda_{iI} = 0\ \forall i, I$ must be a trivial solution of the finiteness equations.

Finiteness equations. The finiteness equations are the conditions $\beta_{iI} = 0$ for every i and I, namely

$$\sum_{J=1}^{N_i} \gamma_i^{IJ}(\alpha)\, \lambda_{iJ} = -\delta_{iI}. \tag{2.8}$$

If γ_i denotes the $N_i \times N_i$ matrix having entries $\gamma_i^{IJ}(\alpha)$, let $(\gamma_i|\delta_i)$ denote the $N_i \times (N_i + 1)$ matrix obtained adding the column δ_{iI} to γ_i. The equation $\beta_{iI} = 0$ admits solutions if and only if the ranks of the matrices γ_i and $(\gamma_i|\delta_i)$ are equal. Writing $\text{rank}(\gamma_i) = \text{rank}(\gamma_i|\delta_i) = n_i \leq N_i$, then the solution of $\beta_{iI} = 0$ contains $N_i - n_i$ free parameters.

Simple situations in which (2.8) admits solutions are those in which the matrix γ_i is invertible, or, if it is not invertible, suitable entries of the vector δ_i vanish. In some cases a symmetry ensures that certain irrelevant operators have δ identically zero. I call these lagrangian terms *protected.* The beta functions of the protected operators can be set to zero in a straightforward way. If a protected operator is finite, i.e. its anomalous dimension vanishes, then its coupling λ remains unconstrained. Examples of protected operators are the chiral operators in four-dimensional supersymmetric theories [8]. The anomalous dimensions of the chiral operators are generically different from zero in N=1 supersymmetric theories, but they can vanish in families of finite N=2 and N=4 theories. These cases are not of primary interest for the investigation of this paper. I briefly come back to this issue in the next section, but more details can be found in ref. [9].

It is convenient to isolate the protected operators from the rest and concentrate the search for solutions of the finiteness equations in the remaining subclass of irrelevant terms. For simplicity, it is also convenient to set the couplings of the protected operators to zero even if their anomalous dimensions vanish. Indeed, it is always possible to turn those couplings on at a later stage. This operation is studied in [9] and defines a protected finite irrelevant deformation. In the rest of this section, I assume that the protected operators are dropped from (2.1) and that the λ_is refer only to the unprotected irrelevant operators, unless otherwise specified.

Finite solutions. Suppose that there exists an integer or a half-integer $\ell > 0$ such that the matrices $\gamma_{n\ell}$ are invertible for every $n > 1$ and $n_\ell = \text{rank}(\gamma_\ell) < N_\ell$. Then the finiteness equations (2.8) admit a non-trivial solution with $N_\ell - n_\ell$ free parameters.

If $\lambda_{\ell I}$ denote the solutions of the equations

$$\sum_{J=1}^{N_\ell} \gamma_\ell^{IJ}(\alpha)\, \lambda_{\ell J} = 0, \tag{2.9}$$

let

$$\lambda_{jJ}=0 \qquad\qquad \text{for every} \quad j\neq n\ell,\quad n=\text{integer}, \tag{2.10}$$

$$\lambda_{n\ell I}=-\sum_{J=1}^{N_{n\ell}}(\gamma_{n\ell}^{-1})^{IJ}\delta_{n\ell J} \qquad\qquad \text{for every } n>1. \tag{2.11}$$

The solutions of (2.9) contain $N_\ell - n_\ell$ free parameters, by assumption. Now, formula (2.5), with the condition (2.6), and (2.10) imply $\delta_{jJ}=0$ for every $j\neq n\ell$. This ensures that the finiteness equations $\beta_{jJ}=0$ are trivially satisfied for $j\neq n\ell$. Moreover, formula (2.5) implies also $\delta_{\ell I}=0$, and therefore the $\lambda_{\ell I}$s solve $\beta_{\ell I}=0$, i.e. the finiteness equations (2.8) for $i=\ell$. Finally, the existence of the solutions (2.11) is ensured by the invertibility of the matrices $\gamma_{n\ell}$ for $n>1$. The $\delta_{n\ell I}$s for $n>1$ are determined recursively as functions of $\lambda_{\ell I}$ and α, using formula (2.5).

Summarizing, the theory described by the lagrangian

$$\mathcal{L}[\varphi]=\mathcal{L}_0[\varphi,\alpha]+\kappa^\ell\sum_{I}^{N_\ell}\lambda_{\ell I}\ \mathcal{O}_{\ell I}(\varphi)-\sum_{n=1}^{\infty}\kappa^{n\ell}\sum_{I,J=1}^{N_{n\ell}}(\gamma_{n\ell}^{-1})^{IJ}\delta_{n\ell J}\ \mathcal{O}_{n\ell I}(\varphi) \tag{2.12}$$

is finite. Its independent couplings are α and the $N_\ell - n_\ell$ free parameters contained in $\lambda_{\ell I}$. The beta functions are identically zero, but in general renormalization demands non-trivial field redefinitions. The power-like divergences do not contribute to the RG equations and so can be subtracted as they come, without adding new independent couplings.

The theory $\mathcal{L}[\varphi]$ is a finite irrelevant deformation of the theory $\mathcal{L}_0[\varphi,\alpha]$. The level ℓ is called *lowest level* of the deformation, while the last sum in (2.12) is called *queue* of the deformation. If $N_\ell = n_\ell$ the solution is trivial (all of the λs vanish) and coincides with $\mathcal{L}_0[\varphi,\alpha]$, which is finite by assumption.

The inclusion of protected operators in the solution (2.9-2.11) is straightforward, since it is sufficient to set their couplings to zero. As remarked above, if some protected operators are finite, it is possible to consider more general solutions that contain one extra independent parameter for each finite protected operator [9].

Sufficient conditions for the existence of a perturbative expansion. If $\mathcal{C}$ is a family of conformal field theories that become free when some marginal parameter g tends to zero, then the theory coupled with gravity might not admit a smooth $g\rightarrow 0$ limit, due to the inverse matrices that appear in formula (2.11). However, if the anomalous dimensions of the irrelevant couplings satisfy a certain boundedness condition, it is possible to keep g small, but different from zero, and have a meaningful perturbative expansion in powers of g and $\kappa_{\text{eff}}E$, where E is the energy scale and κ_{eff} is an effective inverse Planck mass that depends on g. Basically, the absolute values of the anomalous dimensions of the unprotected irrelevant operators should admit a strictly positive bound from below.

The first non-vanishing irrelevant couplings are the $\lambda_{\ell I}$s, namely the solutions of (2.9), some of which can have arbitrary values. Let $\lambda_\ell = \max_I |\lambda_{\ell I}|$. Assume that there exists a $\eta > 0$, depending on ℓ and g, and non-vanishing g-independent numbers c_n, such that

$$\left|(\gamma_{n\ell}^{-1})^{IJ}\right| < \frac{c_n}{\eta} \tag{2.13}$$

when $g \sim 0$, for every $n > 1$ and every I, J. The quantity η generically tends to zero when g tends to zero. Observe that perturbation theory ensures that $\gamma_{n\ell}^{IJ}(\alpha)$ and $\delta_{n\ell I}$ have a smooth limit when $g \to 0$ at λ fixed.

Under the assumption (2.13), when $g \to 0$ the solutions (2.11) behave not worse than

$$|\lambda_{n\ell I}| \sim \widetilde{c}_n \frac{\lambda_\ell^n}{\eta^{\ell(n-1)}}. \tag{2.14}$$

for other g-independent numbers $\widetilde{c}_n$, constructed with the c_ns. The behavior (2.14) can be proved inductively in n. Indeed, if (2.14) is true for $n < m$, then (2.5), (2.7) and (2.11) immediately imply that it is also true for $n = m$.

Let us compare the behavior of an irrelevant term of dimensionality $d + n\ell$ with the behavior of the marginal terms of $\mathcal{C}$, as functions of the energy scale E of a process. The ratio between these two types of contributions behaves not worse than

$$a_n \eta^\ell \left(\frac{\lambda_\ell^{1/\ell} \kappa E}{\eta}\right)^{n\ell},$$

a_n being calculable numbers, that depend on the c_ns of (2.13). The perturbative expansion in powers of κ (equivalently, in powers of the energy) is meaningful for energies E much smaller than the effective Planck mass

$$\frac{1}{\kappa_{\text{eff}}} \equiv \frac{\eta}{\kappa \lambda_\ell^{1/\ell}}. \tag{2.15}$$

This up to the behavior of the numerical factors a_n, which cannot be predicted unless the theory is solved. The constant λ_ℓ can be set to one without loss of generality, since it always appears in the combination $\kappa^\ell \lambda_\ell$.

In conclusion, the condition to have a consistent non-trivial finite irrelevant deformation is that there exists a lowest level ℓ such that

$$0 < \ell < \infty, \qquad n_\ell < N_\ell, \qquad \eta_\ell > 0. \tag{2.16}$$

I have emphasized that η can depend on ℓ. Observe that the conditions (2.16) concern only the renormalizable subsector $\mathcal{L}_0[\varphi, \alpha]$ of the theory, and can be studied before turning the irrelevant deformation on.

3 Application to quantum gravity in three dimensions

The discussion of the previous section was completely general. Applied, for example, to quantum gravity in four dimensions, it shows that it is not possible to make it finite in a simple way, because (2.16) does not hold ($\eta_\ell = 0$ for every lowest level ℓ). I come back to this at the end of this section. Other types of four-dimensional applications can be thought, as shown for example in [9].

A situation where (2.16) does hold is the case of gravity coupled with matter in three spacetime dimensions, with $\ell = 1$ and $\eta \sim \alpha$, α denoting some marginal coupling of $\mathcal{C}$. In this section I discuss three-dimensional quantum gravity in general terms. In the rest of the paper I consider an explicit model in detail.

I assume that the $\kappa \to 0$ limit $\mathcal{L}_0[\varphi, \alpha]$ is the sum of the free spin-2 kinetic term plus the lagrangian $\mathcal{L}_{\mathcal{C}}[\varphi, \alpha]$ of the matter sector, which I take to be a conformal field theory $\mathcal{C}$. The theory $\mathcal{C}$ is subject to the restrictions (2.16), which I discuss below. The beta functions of the marginal couplings of $\mathcal{C}$ are independent of the irrelevant couplings and determined solely within the conformal field theory $\mathcal{C}$, i.e. at $\kappa = 0$. Since the matter subsector of the theory is conformal, the beta functions of the marginal couplings of $\mathcal{C}$ vanish also when $\kappa \neq 0$.

The Einstein term

$$\frac{1}{2\kappa\overline{\lambda}_1}\sqrt{g}R \tag{3.1}$$

contains the spin-2 kinetic term and an irrelevant deformation of level $i = 1$. The coupled theory can contain other irrelevant terms with $i = 1$, such as four-fermion terms.

I prefer the notation (3.1), keeping $\overline{\lambda}_1$ and κ in the denominator ($\overline{\lambda}_1$ is redundant and can be set to one at the end) and expanding the dreibein around flat space as $e_\mu^a = \delta_\mu^a + \phi_\mu^a$. The formulas of the previous section apply unchanged, because it is easy to prove that in (2.3) only positive powers of $\overline{\lambda}_1$ can appear. Instead, expanding the dreibein around flat space as $e_\mu^a = \delta_\mu^a + \sqrt{\kappa\overline{\lambda}_1}\phi_\mu^a$, to eliminate κ and $\overline{\lambda}_1$ from the denominator of (3.1), the three-graviton vertex is regarded as an irrelevant deformation of level $i = 1/2$.

I assume that the Lorentz-Chern-Simons term (1.2) is absent at the classical level and that the subtraction scheme is such that this term remains absent also at the quantum level [5].

The beta function of $\overline{\lambda}_1$ vanishes identically, because the Einstein term is non-renormalized. The reason is that no denominator $1/\kappa$ can be generated by the Feynman diagrams. This fact implies that the lowest level ℓ is at least equal to 1.

If the conformal field theory $\mathcal{C}$ is interacting and "generic", then it is reasonable to expect that the anomalous dimensions of the irrelevant deformations of $\mathcal{C}$ are non-vanishing. This ensures that $\ell = 1$ satisfies the restriction (2.16). I now discuss this point in detail.

The set of irrelevant terms of the coupled theory can be split into three subsets:

i) the irrelevant terms that belong to the matter sector, i.e. those that have a non-vanishing flat-space limit;
ii) the irrelevant terms that belong to the gravity sector, i.e. those that are constructed with powers of the curvature tensors and their covariant derivatives, but contain no matter fields;
iii) the mixed terms.
It is convenient to analyse the finiteness equations separately within these subsets.

Sufficient condition for a solution. The simplest sufficient condition to have a non-trivial solution is that the following two requirements be satisfied:
a) All of the unprotected irrelevant operators of the conformal field theory $\mathcal{C}$ have non-vanishing anomalous dimensions (this is a restriction on $\mathcal{C}$);
b) The subsets *ii*) and *iii*) are empty, apart from the Einstein term.
Now I study when these requirements can be met.

A necessary condition for *a*) is that $\mathcal{C}$ be interacting, otherwise the irrelevant terms of class *i*) have vanishing anomalous dimensions. In most cases, this restriction is also sufficient to ensure that all of the terms of class *i*) have non-vanishing anomalous dimensions.

Exact results proving the existence (or non-existence) of theories satisfying *a*) are not available, to my knowledge. Nevertheless, common experience with renormalization theory suggests that almost all interacting conformal field theories are expected to satisfy *a*). I make a brief digression to illustrate some aspects of this issue.

Operators that have vanishing anomalous dimensions are called finite. To disprove *a*) it is necessary to exhibit examples of finite unprotected irrelevant operators in flat space. Generically speaking, in renormalization, whenever a quantity can diverge (because it is not protected by symmetries, power-counting, etc.), it does diverge. Therefore, a counter-example can only be the product of a miraculous cancellation. The finite operators known to me represent no obstacle to the solubleness of the finiteness equations, either because they are not irrelevant, or because they are protected.

The simplest finite operators are associated with conserved (and anomalous) currents, and the marginal deformations of $\mathcal{C}$. However, these operators have level zero or negative, so they are not irrelevant.

Examples of irrelevant finite operators of arbitrary positive levels are provided by the chiral operators of N=2 and N=4 superconformal field theories in four dimensions [8]. However, these operators are protected. For concreteness, consider N=4 supersymmetric Yang-Mills theory. In the formalism of N=1 superfields, this theory contains a vector multiplet and three chiral multiplets Φ^i. The fields Φ^i have zero anomalous dimensions and the chiral operators, for example

$$\int Y_{i_1\cdots i_n}\Phi^{i_1}\cdots\Phi^{i_n}\ \mathrm{d}^2\theta,$$

are finite. (Here $Y_{i_1\cdots i_n}$ is a constant tensor.) Because of the non-renormalization theorem [8], the chiral operators have also $\delta = 0$. Therefore, their beta functions vanish identically.

I stress that the anomalous dimensions depend on the marginal couplings of $\mathcal{C}$ and so, in the worst case, if the anomalous dimension of an unprotected irrelevant operator vanishes, it is expected to vanish only for some special values of the marginal couplings α. In this sense, the requirement *a*) can be viewed as a restriction on the conformal field theory $\mathcal{C}$.

In summary, the present knowledge supports the statement that almost all interacting conformal field theories satisfy *a*).

Now it is necessary to discuss the existence of solutions of the finiteness equations in the subsectors *ii*) and *iii*) listed above. Since the lagrangian terms of class *ii*) do not contain matter fields, they are just the identity operator, from the point of view of $\mathcal{C}$, and can be studied embedding $\mathcal{C}$ in external gravity. This means that the anomalous dimensions of the terms of class *ii*) are zero and their finiteness equations cannot be solved in the way described in the previous section. Therefore, the quantization procedure outlined above does not work, unless class *ii*) contains only the Einstein term.

Classes *ii*) and *iii*) are empty, apart from the Einstein term, precisely in three-dimensional quantum gravity. In three dimensions the field equations express the Riemann tensor in terms of the matter fields and so the unique independent lagrangian term of classes *ii*) and *iii*) is the Einstein term.

The Einstein term has $i = 1$. Other irrelevant terms of level 1, belonging to class *i*), can be present (four-fermion vertices, Pauli terms, and so on) and their matrix of anomalous dimensions is in general non-vanishing. It is convenient to decompose the matrix γ_1^{IJ}, $I, J = 1, \ldots N_1$ into

$$(\gamma_1)^{IJ} = \begin{pmatrix} (\widetilde{\gamma}_1)^{\overline{IJ}} & (\gamma_1)^{\overline{I}N_1} \\ 0 & 0 \end{pmatrix}. \qquad (3.2)$$

Here the N_1th value of the indices I, J is conventionally associated with the Einstein term ($\lambda_{1N_1} \equiv \overline{\lambda}_1$). The block $(\widetilde{\gamma}_1)^{\overline{IJ}}$, $\overline{I}, \overline{J} = 1, \ldots N_1 - 1$, denotes the matrix of anomalous dimensions of the irrelevant terms of level 1 belonging to class *i*). The N_1th row of the matrix γ_1 is zero, because the beta function of the Newton constant is identically zero.

Because of the discussion made above, the matrix $\widetilde{\gamma}_1$ can be assumed to be invertible. This ensures that the rank of the matrix γ_1 is equal to $N_1 - 1$ and therefore $\ell = 1$. So, the finiteness equations admit a non-trivial solution with lowest level equal to 1. The coupled theory contains only one arbitrary parameter, the Newton constant, besides the marginal coupligs of $\mathcal{C}$.

Using the decomposition (3.2) the finiteness equations

$$\beta_{1I} = \sum_{J=1}^{N_1} \gamma_1^{IJ}(\alpha)\, \lambda_{1J} = 0,$$

split into

$$\beta_{1N_1} = 0, \qquad \text{and} \qquad \beta_{1\overline{I}} = \sum_{\overline{J}=1}^{N_1-1} \widetilde{\gamma}_1^{\overline{IJ}}(\alpha)\,\lambda_{1\overline{J}} + \widetilde{\delta}_{1\overline{I}} = 0,$$

where $\widetilde{\delta}_{1\overline{I}} = (\gamma_1)^{\overline{I}N_1}\overline{\lambda}_1$. The beta functions of the level-1 operators belonging to the matter sector have the same form as (2.5) and so their solutions have the form (2.11).

Finally, the finite theory of quantum gravity coupled with the conformal field theory $\mathcal{C}$ has lagrangian

$$\mathcal{L}[\varphi] = \frac{1}{2\kappa}\sqrt{g}R + \mathcal{L}_{\mathcal{C}}[\varphi,\alpha] - \kappa \sum_{\overline{I},\overline{J}=1}^{N_1-1} (\widetilde{\gamma}_1^{-1})^{\overline{IJ}}\widetilde{\delta}_{1\overline{J}}\ \mathcal{O}_{1\overline{I}}(\varphi) - \sum_{i=2}^{\infty}\kappa^i \sum_{I,J=1}^{N_i} (\gamma_i^{-1})^{IJ}\delta_{iJ}\ \mathcal{O}_{iI}(\varphi),$$

where $\overline{\lambda}_1$ has been set to 1. Renormalization requires non-trivial field redefinitions, but the coupling constants are non-renormalized.

Existence of a perturbative expansion. In general, the anomalous dimensions of the unprotected irrelevant operators are non-zero already at two-loop order (the one-loop diagrams converge in odd dimensions), so the quantity η of (2.13) is typically of order $\alpha^2 \sim g^4$, where $\alpha \sim g^2$ is a generic marginal coupling of $\mathcal{C}$ (the power is fixed assuming that g multiplies a three-leg vertex, such as $\overline{\psi}A\!\!\!/\psi$) that tends to zero in the free-field limit. The perturbative expansion is meaningful for energies E much smaller than the effective Planck mass

$$M_{P\text{eff}} = \frac{1}{\kappa_{\text{eff}}} = \frac{\eta}{\kappa\overline{\lambda}_1} \sim \alpha M_P. \tag{3.3}$$

In practice, the Planck scale is screened by the interactions of $\mathcal{C}$ and effectively reduced by a factor $1/\eta$. To cross the energy $M_{P\text{eff}}$ it is necessary to resum the perturbative expansion.

In summary, in three dimensions it is possible to define a procedure of quantization in the presence of gravity, when the matter sector has $\eta > 0$. Since this restriction concerns only the matter sector of the theory, it is possible to say which kind of matter can be coupled to gravity before effectively coupling it to gravity. In the next sections I study gravity coupled with Chern-Simons $U(1)$ gauge theory and massless fermions.

The reason why the the procedure described in this paper cannot be applied straightforwardly to quantize four-dimensional gravity is that in four-dimensional gravity the class *ii*) contains infinitely many non-trivial terms, of arbitrarily high levels, and no symmetry protects them, i.e. they have $\gamma = 0$, $\delta \neq 0$ [3]. Therefore, $\eta_\ell = 0$ for every candidate lowest level $\ell < \infty$.

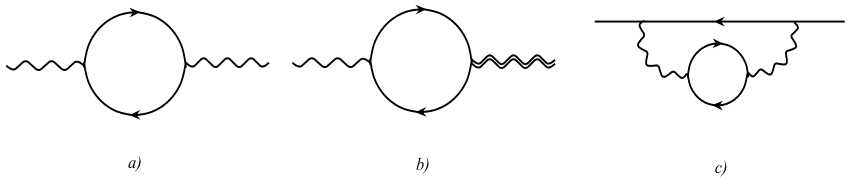

Figure 1: One-loop gauge-field and graviton-gauge-field self-energies

4 Gravity coupled with Chern-Simons $U(1)$ gauge theory with massless fermions

In the rest of the paper I illustrate the quantization procedure defined in the previous sections in a concrete model, namely three-dimensional gravity coupled with Chern-Simons $U(1)$ gauge theory with massless fermions. In this section I recall the basic properties of this theory and the results of [5]. I work in the Euclidean framework.

In flat space, Chern-Simons $U(1)$ gauge theory with massless fermions is described by the lagrangian

$$\mathcal{L}_{\rm cl} = \overline{\psi}\not{D}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho, \tag{4.1}$$

where $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative in flat space. This theory is conformal, because the beta function of g vanishes [10]. The anomalous dimension of ψ is different from zero. I consider n_f copies of complex two-component spinors. The renormalized lagrangian reads

$$\mathcal{L}_{\rm R} = Z_\psi\overline{\psi}\not{D}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho.$$

The lowest-order values of the fermion renormalization constant and anomalous dimension are given by the graph c) of Fig. 1, up to subleading corrections in $1/n_f$:

$$Z_\psi = 1 - \frac{g^4 n_f}{384\pi^2\varepsilon}, \qquad \gamma_\psi = \frac{1}{2}\frac{{\rm d}\ln Z_\psi}{{\rm d}\ln\mu} = \frac{g^4 n_f}{384\pi^2}.$$

This theory is taken as the conformal field theory $\mathcal{C}$ for the coupling with gravity.

Coupling with gravity. The lagrangian is

$$\mathcal{L} = \frac{1}{2\kappa}eR + e\overline{\psi}\not{\mathcal{D}}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + \mathcal{O}(\kappa), \tag{4.2}$$

where $e = \sqrt{g}$. This theory is not finite [5], because a counterterm (1.3) is induced by renormalization to the second order in the loop expansion and first order in the κ expansion. So, it is necessary to include in (4.2) the irrelevant terms generated by renormalization. I focus here on the irrelevant terms of dimensionality four, or level 1, which are

$$\kappa e\overline{\psi}\not{D}^2\psi, \qquad \kappa e F_{\mu\nu}F^{\mu\nu}, \qquad \kappa\varepsilon^{\mu\nu\rho}e^a_\rho F_{\mu\nu}\overline{\psi}\gamma^a\psi, \qquad \kappa e(\overline{\psi}\psi)^2, \qquad \kappa e(\overline{\psi}\gamma^a\psi)^2. \tag{4.3}$$

Only two of these are independent, e.g. the four-fermion vertices [5]. Up to $\mathcal{O}(\kappa^2)$, the complete lagrangian

$$\mathcal{L}_{cl} = \frac{1}{2\kappa}eR + e\overline{\psi}\not{D}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + \frac{\lambda_1\kappa}{4}e(\overline{\psi}\psi)^2 + \frac{\lambda_2\kappa}{4}e(\overline{\psi}\gamma^a\psi)^2 + \mathcal{O}(\kappa^2) \tag{4.4}$$

has the field equations

$$\not{D}\psi + \frac{\lambda_1\kappa}{2}(\overline{\psi}\psi)\psi + \frac{\lambda_2\kappa}{2}(\overline{\psi}\gamma^a\psi)\gamma^a\psi + \mathcal{O}(\kappa^2) = 0, \tag{4.5}$$

$$F_{\mu\nu} + \frac{ig^2}{2}e\varepsilon_{\mu\nu\rho}e^{\rho a}\overline{\psi}\gamma^a\psi + \mathcal{O}(\kappa^2) = 0, \tag{4.6}$$

$$\frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{8}e^a_\mu\overline{\psi}\gamma^a\overleftrightarrow{D}_\nu\psi + \frac{1}{8}e^a_\nu\overline{\psi}\gamma^a\overleftrightarrow{D}_\mu\psi - \frac{1}{4}g_{\mu\nu}\overline{\psi}\overleftrightarrow{\not{D}}\psi + $$
$$-\frac{\lambda_1\kappa}{8}g_{\mu\nu}(\overline{\psi}\psi)^2 - \frac{\lambda_2\kappa}{8}g_{\mu\nu}(\overline{\psi}\gamma^a\psi)^2 + \mathcal{O}(\kappa^2) = 0.$$

Using the fermion field equation (4.5), the first term of the list (4.3) can be converted to $\mathcal{O}(\kappa^2)$. Using the gauge-field equation (4.6) the second and third terms of (4.3) can be converted into the forth term of the same list, up to $\mathcal{O}(\kappa^2)$. So, the Newton constant and the couplings $\lambda_{1,2}$ make a complete set of essential couplings of level 1.

The gravitational field is defined expanding the dreibein e^a_μ around flat space:

$$e^a_\mu = \delta^a_\mu + \phi^a_\mu, \qquad\qquad \omega^a_\mu = \varepsilon^{abc}\partial^b\phi^c_\mu + \mathcal{O}(\phi^2).$$

I choose the symmetric gauge $\phi_{\mu a} = \phi_{a\mu}$.

As remarked in ref. [5], since the divergent parts of the diagrams are polynomial in the number n_f of fermions, it is convenient to concentrate the attention on the contributions proportional to n_f. These are given by the diagrams that contain one fermion loop. At the second loop order the diagrams containing two fermion loops factorize into the product of two one-loop subdiagrams, and are therefore convergent.

The gauge-fixing lagrangian is

$$\mathcal{L}_{\rm gf} = \frac{1}{2\alpha\kappa}(\partial_\mu\phi_{\mu\nu})^2 + \frac{1}{2\lambda g^2}(\partial_\mu A_\mu)^2 + \mathcal{L}_{\rm ghost}.$$

The gauge parameters λ and α are kept throughout the calculations, because gauge-independence provides a powerful check of the calculations. The $U(1)$ field is conveniently gauge-fixed in flat space.

The ghost part of the gauge-fixing lagrangian can be ignored in the calculations of this paper. Indeed, diagrams with external ghost legs do not contribute to the renormalization of the four-fermion vertices, but belong to the gauge-trivial sector of the theory. Instead, diagrams with internal ghosts must have, to the leading order in $1/n_f$, one ghost loop and one fermion loop. These diagrams necessarily factorize into two one-loop subdiagrams and therefore converge.

A convenient regularization technique consists of modifying the propagators with an exponential cut-off:

$$\frac{1}{p^2} \rightarrow \frac{1}{p^2} \exp\left(-\frac{p^2}{\Lambda^2}\right).$$

This can be done in a gauge invariant way to all orders [5]. Instead, the dimensional-regularization technique presents some difficulties, because of the ε tensor appearing in the $U(1)$ Chern-Simons term and because the trace of an odd number of Dirac matrices does not always vanish. Nevertheless, for the purposes of this paper, it is consistent to use the dimensional-regularization framework, since the divergent parts of the two-loop diagrams are made of simple poles $1/\varepsilon$, if $\varepsilon = 3 - D$, and the residues of simple poles can be evaluated directly in three dimensions. The conversion of the results to the cut-off approach is performed by means of the replacement $1/\varepsilon \rightarrow \ln \Lambda^2/\mu^2$ and the power-like divergences are subtracted as they come.

The bare lagrangian reads

$$\frac{\mathcal{L}_{\rm B}}{e_{\rm B}} = \frac{1}{2\kappa} R_{\rm B} + \overline{\psi}_{\rm B} \not{D}_{\rm B} \psi_{\rm B} + \frac{1}{2g^2 e_{\rm B}} \varepsilon^{\mu\nu\rho} F_{{\rm B}\mu\nu} A_{{\rm B}\rho} + \frac{1}{4} \lambda_{1\rm B} \kappa (\overline{\psi}_{\rm B} \psi_{\rm B})^2 + \frac{1}{4} \lambda_{2\rm B} \kappa (\overline{\psi}_{\rm B} \gamma^a \psi_{\rm B})^2 + \mathcal{O}(\kappa^2). \quad (4.7)$$

I have not written the regularizing terms explicitly. The relations between bare and renormalized quantities read

$$\begin{aligned} &\lambda_{1\rm B} = \lambda_1 Z_1, \qquad && \lambda_{2\rm B} = \lambda_2 Z_2, \\ &A_{\mu \rm B} = A_\mu + \mathcal{O}(\kappa), \qquad && \psi_{\rm B} = Z_\psi^{1/2} \psi + \mathcal{O}(\kappa), \qquad e^a_{\mu \rm B} = e^a_\mu + \mathcal{O}(\kappa). \end{aligned} \quad (4.8)$$

Calculations. The calculation of the two-loop counterterms can be divided into two parts: the contributions of type δ in (1.4), which have been computed in ref. [5], and the self-renormalization of the four-fermion terms, that is to say the contributions of type $\lambda\gamma_\lambda$ in (1.4).

The counterterms have to be simplified using the field equations, to separate the renormalization of the essential couplings from the field redefinitions. In the case at hand, this means that the following replacements

$$R_{\mu\nu\rho\sigma} \rightarrow 0, \qquad F_{\mu\nu} \rightarrow -\frac{ig^2}{2} e \varepsilon_{\mu\nu\rho} e^{\rho a} \overline{\psi} \gamma^a \psi, \qquad \not{D}\psi \rightarrow 0, \quad (4.9)$$

are allowed.

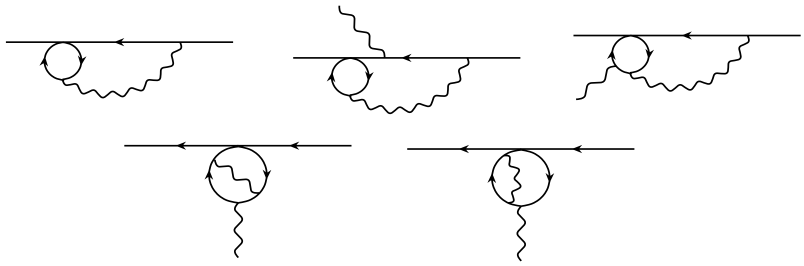

Figure 2: Fermion self-energy and fermion-gauge-field vertex

The one-loop gauge-field self-energy, given by Fig. 1-a, is

$$-\frac{n_f}{16}\frac{1}{(k^2)^{(1+\varepsilon)/2}}(\delta_{\mu\nu}k^2 - k_\mu k_\nu).$$

The graviton-gauge-field self-energy of Fig. 1-b) vanishes by spin conservation. This fact reduces the number of two-loop diagrams.

Counterterms induced by gravity. The results of ref. [5] are that for $\lambda_{1,2} = 0$, at the second order in the loop expansion, first order in κ and leading order in $1/n_f$, renormalization requires the four-fermion counterterm

$$\mathcal{L}^{grav}_{\text{counter}} = -\frac{5\kappa g^4 n_f}{384\pi^2\varepsilon}\frac{e}{4}(\overline{\psi}\gamma^a\psi)^2 \tag{4.10}$$

and the field redefinition

$$A_\mu \to A_\mu - \frac{5n_f\alpha g^2\kappa}{768\pi^2\varepsilon}\ e\varepsilon_{\mu\nu\rho}F^{\nu\rho} - \frac{in_f g^4\kappa(3+5\alpha)}{768\pi^2\varepsilon}\ e^a_\mu\overline{\psi}\gamma^a\psi.$$

Moreover, no Lorentz-Chern-Simons term (1.2) is generated.

5 Self-renormalization of the four-fermion vertices

The counterterms proportional to $\lambda_{1,2}$ can be computed in flat space and are associated with the anomalous dimensions of the four-fermion vertices. The set of diagrams can be split into two subsets: the diagrams that have two external fermions and one or no external gauge field (see Fig. 2); the diagrams that have four external fermions (see Fig. 3). The diagrams are constructed with one four-fermion vertex, one fermion loop and one or two internal gauge-field legs, respectively. The two-loop diagrams with one four-fermion vertex and two external gauge-field legs factorize into products of one-loop subdiagrams and therefore converge.

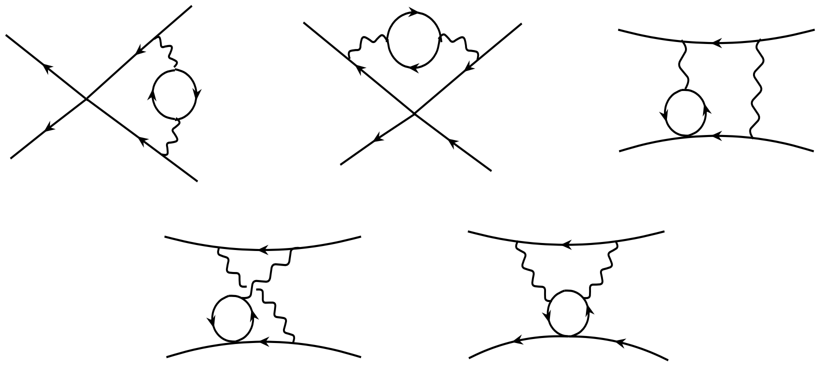

Figure 3: Renormalization of the four-fermion vertices

Fermion self-energy and fermion-gauge-field vertex. The diagrams are shown in Fig. 2. The counterterms sum to

$$\mathcal{L}_{\text{counter-1}} = -\frac{ig^2\lambda_2 n_f\kappa}{192\pi^2\varepsilon}e\overline{\psi}\not{D}\not{D}\not{D}\psi + \frac{ig^2 n_f\kappa}{192\pi^2\varepsilon}(\lambda_1-\lambda_2)\varepsilon^{\mu\nu\rho}e_\rho^a F_{\mu\nu}\overline{\psi}\gamma^a\psi. \tag{5.1}$$

Four-fermion counterterms. The graphs contriguting to these counterterms are shown in Fig. 3 and give

$$\mathcal{L}_{\text{counter-2}} = \frac{g^4 n_f\kappa}{192\pi^2\varepsilon}e\left[(2\lambda_1-\lambda_2)\ \frac{1}{4}(\overline{\psi}\gamma^a\psi)^2 + 3\,(5\lambda_1+6\lambda_2)\ \frac{1}{4}(\overline{\psi}\psi)^2\right]. \tag{5.2}$$

6 Solution of the finiteness equations

It is now time to collect the results of ref. [5] and this paper, solve the finiteness equations, and determine the values of the irrelevant couplings $\lambda_{1,2}$ that multiply the four-fermion vertices.

Totals. The total four-fermion counterterms can be obtained summing (4.10), (5.1) and (5.2) and using the replacements (4.9). The result is

$$\mathcal{L}_{\text{counter}} = \frac{g^4 n_f\kappa}{384\pi^2\varepsilon}e\left[(12\lambda_1-10\lambda_2-5)\ \frac{1}{4}(\overline{\psi}\gamma^a\psi)^2 + 6\,(5\lambda_1+6\lambda_2)\ \frac{1}{4}(\overline{\psi}\psi)^2\right].$$

The renormalization constants of the couplings λ_1 and λ_2 are obtained subtracting the contribution associated with the fermion wave-function renormalization constant. The net counterterm is then

$$\mathcal{L}_{\text{counter-net}} = \frac{g^4 n_f\kappa}{384\pi^2\varepsilon}e\left[(12\lambda_1-8\lambda_2-5)\ \frac{1}{4}(\overline{\psi}\gamma^a\psi)^2 + 4\,(8\lambda_1+9\lambda_2)\ \frac{1}{4}(\overline{\psi}\psi)^2\right]. \tag{6.3}$$

Using (4.7) and (4.8) the bare couplings are

$$\lambda_{1\mathrm{B}} = \lambda_1 + \frac{g^4 n_f\,(8\lambda_1 + 9\lambda_2)}{96\pi^2\varepsilon}, \qquad \lambda_{2\mathrm{B}} = \lambda_2 + \frac{g^4 n_f\,(12\lambda_1 - 8\lambda_2 - 5)}{384\pi^2\varepsilon}.$$

Solution of the finiteness equations. Finiteness demands that the counterterm (6.3) vanishes, whence

$$\lambda_1 = \frac{45}{172}, \qquad \lambda_2 = -\frac{10}{43}. \tag{6.4}$$

In conclusion, the finiteness conditions admit one solution and uniquely determine the values of the four-fermion couplings.

To couplings $\lambda_{1,2}$ turn out to be g-independent. This is due to the fact that γ_λ and δ_λ are of the same order in g. The irrelevant terms belonging to higher levels, however, are expected to have $\delta_\lambda \sim 1$ and so the quantity η defined in (2.13) is expected to behave like $1/g^4$. The effective Planck mass is therefore $\sim g^4/\kappa$.

7 Applications to four dimensions

The quantization procedure defined in sections 2 and 3 is meaningful for those theories that have $\eta > 0$, where η is defined by equation (2.13). I have shown that three-dimensional quantum gravity coupled with a generic interacting conformal field theory has the desired properties. This is not the case of four-dimensional quantum gravity, coupled with matter or not, because every candidate lowest level $\ell < \infty$ has $\eta_\ell = 0$. Indeed, the beta functions of the irrelevant terms made with the Riemann tensor and its derivatives, such as

$$\sqrt{g} R_{\mu\nu}^{\rho\sigma} R_{\alpha\beta}^{\mu\nu} R_{\rho\sigma}^{\alpha\beta} \tag{7.1}$$

have the form (1.4) with $\gamma = 0$ and $\delta \neq 0$ [3]. In three dimensions, a term like (7.1) can be reabsorbed by means of field redefinitions, because there is no graviton, but in four dimensions this is impossible.

I have made a certain number of attempts, not reported here, to try to circumvent the difficulty of four-dimensional gravity. These will be probably collected in a separate publication. It is certainly possible to modify the theory to have non-vanishing γs for the operators (7.1), for example adding a cosmological constant. Then, however, it is not easy to solve the finiteness equations. Moreover, other problems appear in the presence of a cosmological constant in four dimensions. The difficulties might be just technical or hide more conceptual aspects.

It is worth mentioning that even if the ideas of this paper do not extend immediately to quantum gravity in four dimensions, a more general framework where they do might exist, with potentially appealing implications. The quantization of gravity might be possible only in the presence of interacting matter, for example thanks to the existence of QCD. The energy at which the effects of quantum gravity become relevant could be not the Planck mass, but an effective Planck mass that takes care of the presence of matter. If the matter is weakly interacting, the effective Planck mass could be considerably small. The limit in which the interaction of the matter sector is switched off could be singular.

Other types of applications to four dimensions are possible, as shown for example in [9]. Generalization to running theories are possible also, but more tricky.

8 Conclusions

In this paper I have shown that it is possible to give a quantization prescription that ensures, under certain conditions, finiteness of quantum gravity coupled with matter in three spacetime dimensions. The procedure is algorithmic and so it can be implemented perturbatively. Gravity is coupled with an interacting conformal field theory $\mathcal{C}$. The values of the irrelevant couplings, apart from the Newton constant, are determined imposing that their beta functions vanish. The finiteness equations have solutions thanks of the properties of three-dimensional spacetime, in particular the absence of a propagating graviton, and because the unprotected irrelevant operators of $\mathcal{C}$ have, generically, non-vanishing anomalous dimensions. A quantity η, defined by formula (2.13), characterizes the strength of the interactions of the matter subsector. The expansion in powers of the energy is valid for energies much smaller than the effective Planck mass ηM_P.

In a concrete example, I have studied the Chern-Simons $U(1)$ gauge theory with massless fermions coupled with gravity and applied the iterative procedure of sections 2 and 3 to compute the coefficients of the four fermion vertices. The "classical" lagrangian of the finite theory defined by this quantization prescription is

$$\mathcal{L} = \frac{1}{2\kappa} eR + e\overline{\psi}\not{D}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + \frac{45}{172}\frac{\kappa}{4}e(\overline{\psi}\psi)^2 - \frac{10}{43}\frac{\kappa}{4}e(\overline{\psi}\gamma^a\psi)^2 + \mathcal{O}(\kappa^2) \qquad (8.1)$$

and has only two arbitrary parameters: the Chern-Simons coupling g and the Newton constant κ. The action (8.1) is renormalizable as it stands, i.e. without adding new parameters, but just redefining the fields. In this sense, it is finite.

The results of this paper might revive some hopes to find a finite theory of gravitational interactions. Several aspects of the ideas applied here admit generalizations to four dimensions [9]. However, the peculiarity of three dimensions is crucial to have a non-vanishing effective Planck mass in the presence of gravity. Quantum gravity in four dimensions does not fulfil this requirement in a straightforward way. For this reason, the generalization of these ideas to quantum gravity in four dimensions demands further insight.

9 Appendix

Torsion, curvatures, covariant derivatives and connections are

$$\begin{aligned}
\mathcal{D}e^a &= \mathrm{d}e^a - \omega^{ab}e^b = 0, \qquad\qquad R^a = \mathrm{d}\omega^a + \frac{1}{2}\varepsilon^{abc}\omega^b\omega^c, \\
\mathcal{D}_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu}V_\rho, \qquad\qquad \Gamma^\rho_{\mu\nu} = e^{\rho a}\partial_\mu e^a_\nu + \omega^{ab}_\mu e^a_\nu e^{\rho b}, \\
\omega^a_\mu &= \varepsilon^{abc}\left(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu\right)e^{\nu c} - \frac{1}{4}e^a_\mu\varepsilon^{bcd}\left(\partial_\rho e^b_\nu - \partial_\nu e^b_\rho\right)e^{\nu c}e^{\rho d}, \\
\mathcal{D}_\mu\psi &= \partial_\mu\psi - \frac{i}{2}\omega^a_\mu\gamma^a\psi + iA_\mu\psi.
\end{aligned}$$

The Ricci tensor and scalar curvature are defined as $R_{\mu\nu} = R^{ab}_{\mu\rho}e^{\rho b}e^a_\nu$, $R = R_{\mu\nu}g^{\mu\nu}$, where $R^{ab} = \varepsilon^{abc}R^c = R^{ab}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu/2$, $R^\mu{}_{\nu\rho\sigma} = \partial_\sigma\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\sigma} - \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} + \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}$ and of course $g_{\mu\nu} = e^a_\mu e^a_\nu$.

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