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RENORMALIZATION OF QUANTUM GRAVITY COUPLED WITH MATTER IN THREE DIMENSIONS

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Abstract

In three spacetime dimensions, where no graviton propagates, pure gravity is known to be finite. It is natural to inquire whether finiteness survives the coupling with matter. Standard arguments ensure that there exists a subtraction scheme where no Lorentz-Chern-Simons term is generated by radiative corrections, but are not sufficiently powerful to ensure finiteness. Therefore, it is necessary to perform an explicit (two-loop) computation in a specific model. I consider quantum gravity coupled with Chern-Simons $U(1)$ gauge theory and massless fermions and show that renormalization originates four-fermion divergent vertices at the second loop order. I conclude that quantum gravity coupled with matter, as it stands, is not finite in three spacetime dimensions.

1 Introduction

Gravity is not power-counting renormalizable in dimensions greater than two. It is known [1] that pure gravity in four-dimensions is finite to the first loop order and that one-loop finiteness is spoiled by the coupling with matter. Moreorer, four-dimensional gravity is not finite to the second loop order [2], even in the absence of matter.

In three dimensions there is no propagating graviton and pure gravity

$$\frac{1}{2\kappa}\int\sqrt{g}R(x)\ \mathrm{d}^3x \tag{1.1}$$

is known to be finite to all orders [3]. A quick proof is based on the observation that the counterterms vanish using the field equations of (1.1) and therefore can be reabsorbed by means of field redefinitions. Indeed, in three dimensions the Weyl tensor is identically zero and so the Riemann tensor is a linear combination of the Ricci tensor and the scalar curvature:

$$R_{\mu\nu\rho\sigma}=g_{\mu\rho}R_{\nu\sigma}-g_{\mu\sigma}R_{\nu\rho}-g_{\nu\rho}R_{\mu\sigma}+g_{\nu\sigma}R_{\mu\rho}-\frac{R}{2}g_{\mu\rho}g_{\nu\sigma}+\frac{R}{2}g_{\mu\sigma}g_{\nu\rho}. \tag{1.2}$$

Every counterterm is proportional to $R_{\mu\nu}$ or R, apart from the Lorentz-Chern-Simons term

$$\int\varepsilon^{\mu\nu\rho}\left(\omega_\mu^a\partial_\nu\omega_\rho^a+\frac{1}{3}\omega_\mu^a\omega_\nu^b\omega_\rho^c\varepsilon^{abc}\right), \tag{1.3}$$

which does not appear by parity invariance. By dimensional counting, the counterterms are actually quadratic, at least, in $R_{\mu\nu}$-R and therefore can be reabsorbed by means of covariant field redefinitions, with no renormalization of the Newton constant κ.

It is natural to inquire whether finiteness survives the coupling with matter in three dimensions. The renormalization of the theory has chances to be non-trivial, even if no graviton propagates. If the theory is finite, renormalization requires only field redefinitions, but no running of the coupling constants. If the theory is not finite, then renormalization generates infinitely many new coupling constants, as in four dimensions. In this paper I study these issues.

First I analyze non-renormalization properties and standard arguments about finiteness. I prove that there exists a subtraction scheme where no Lorentz-Chern-Simons term is generated by radiative corrections. This ensures that gravity is not driven to the theory known as “topologically massive gravity” [4]. However, the standard non-renormalization arguments are not sufficiently powerful to ensure finiteness, because higher-dimensioned operators can be generated by renormalization. To decide whether finiteness survives the coupling with matter or not, it is necessary to perform an explicit computation in a specific model. I consider quantum gravity coupled with Chern-Simons $U(1)$ gauge theory and massless fermions. This model is a good laboratory to explore ideas about finiteness and renormalizability beyond power counting. I show that renormalization originates a four-fermion divergent vertex

$$\mathcal{L}_{\mathrm{div}}=\frac{5\kappa g^4n_f}{384\pi^2\varepsilon}\frac{e}{4}(\overline{\psi}\gamma^a\psi)^2 \tag{1.4}$$

at the second order in the loop expansion and first order in the Newton constant κ. The result (1.4) is written up to subleading corrections in $1/n_f$. I conclude that quantum gravity coupled with matter, as it stands, is not finite in three spacetime dimensions.

The computation is two-loop, because in three dimensions every theory is finite to the first loop order. By symmetric integration, an odd-dimensional theory has no one-loop logarithmic divergence. Moreover, the power-like divergences are scheme artifacts (they are automatically absent using the dimensional-regularization technique) and have no effects on the renormalization group. So, the problem of finiteness starts at two loops in three dimensions.

At the classical level, the properties of gravity coupled with matter in three dimensions have been widely studied, starting from ref. [5]. At the quantum level, there have been studies on quantum gravity of point particles [6], quantum cosmology and black-hole quantum mechanics [7], topologically massive gravity [4], gravitating topological matter [8], de Sitter quantum gravity [9], loop quantum gravity [10], dynamically triangulated quantum gravity [11] and many other subjects.

In flat space, the renormalization of 2+1 dimensional quantum field theory has been studied at the perturbative level [12, 13, 14] and in the large N expansion [15, 16, 17, 18, 19]. Besides the finiteness of pure gravity in three dimensions [3], there have been studies on the renormalizability of quantum gravity near two dimensions [20]. The renormalization of 2+1 quantum gravity coupled with matter has attracted less attention, so far. The interest of this research is that it can shed some light on the properties of renormalization beyond power-counting.

The paper is organized as follows. In section 2 I recall the properties of Chern-Simons $U(1)$ gauge theory with matter in flat space. In section 3 I couple it with gravity. In section 4 I prove that no Lorentz-Chern-Simons term is induced by renormalization. In section 5 I introduce the two-loop computations of this paper, the organization of counterterms and the calculational technique. In section 6 I collect the results about four-fermion vertices induced by gravity. Section 7 contains the conclusions. In the appendix I collect useful formulas and some remarks about the difficulties of the dimensional regularization of the Chern-Simons term.

2 Chern-Simons $U(1)$ gauge theory with massless fermions

In this section I recall some properties of Chern-Simons $U(1)$ gauge theory with massless fermions in flat space and fix the notation. I work in the Euclidean framework. The lagrangian reads

$$\mathcal{L}_{\text{cl}} = \overline{\psi} D\!\!\!\!/\,\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho, \tag{2.1}$$

where $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative in flat space. This theory is conformal, since the beta function of g vanishes [14], but the anomalous dimension of ψ is different from zero.

Precisely, (2.1) is a one-parameter family of conformal field theories, parametrized by g. I use two-component complex spinors and consider n_f copies of them. The Dirac matrices are Hermitean and such that $\gamma_\mu^T = -\gamma_2\gamma_\mu\gamma_2$, where T means transpose. The "time" coordinate is x_3.

Discrete symmetries. The parity, charge-conjugation and "time"-reversal transformations are

$$
\begin{array}{llll}
P_1: & x_\mu \to (-x_1, x_2, x_3), & \psi \to \gamma_1\psi, & \overline{\psi} \to -\overline{\psi}\gamma_1, \\
 & & A_\mu \to (-A_1, A_2, A_3) & g^2 \to -g^2. \\
C: & x_\mu \to x_\mu, \quad \psi \to \gamma_2\left(\overline{\psi}\right)^T, & \overline{\psi} \to -\psi^T\gamma_2, & A_\mu \to -A_\mu\ . \\
T: & x_\mu \to (x_1, x_2, -x_3), & \psi \to \gamma_3\psi, & \overline{\psi} \to -\overline{\psi}\gamma_3, \\
 & & A_\mu \to (A_1, A_2, -A_3), & g^2 \to -g^2,
\end{array}
$$

where $\overline{\psi} = \psi^\dagger\gamma_3$. Only C and CPT are true symmetries, since P and T change the sign of g^2.

Regularization. The ordinary dimensional-regularization technique is not convenient for the theory (2.1), because of the difficulties related to the ε tensor and the trace of an odd product of gamma matrices. Some observations on this issue are collected in the appendix. Nevertheless, for the purpose of computing divergent parts of two-loop graphs, where only simple poles appear, it is consistent to ignore this problem and work with the dimensional technique. This reduces the effort and simplifies the algebra. Instead, it is necessary to use an alternative regularization technique to prove properties valid to all orders in the perturbative expansion. A standard choice is to modify the gauge-field propagator with higher-derivative terms in a gauge-invariant way and match the fermion loops with loops of Pauli-Villars fields. This can be achieved with a regularized lagrangian

$$
\mathcal{L}_{\rm B} = \overline{\psi}_{\rm B} D\!\!\!\!/_{\rm B}\psi_{\rm B} + \frac{1}{2g_{\rm B}^2}\varepsilon^{\mu\nu\rho}F_{{\rm B}\mu\nu}\left(1 - \frac{\Box}{\Lambda^2}\right)A_{{\rm B}\rho}, \qquad (2.2)
$$

and a regularized functional integration measure

$$
[{\rm d}\overline{\psi}][{\rm d}\psi][{\rm d}A]\prod_j \det\left(D\!\!\!\!/_{\rm B} + M_j\right)^{c_j}, \qquad \sum_j c_j = -1, \qquad \sum_j c_j M_j^p = 0, \qquad (2.3)
$$

where $p = 1, 2, \ldots$ and the M_j have to tend to infinity. The determinants in (2.3) come from integrating out the Pauli-Villars fields. The superscripts B mean bare. Finally, I identify

$$
\sum_j c_j \ln M_j/\mu = -\ln \Lambda/\mu, \qquad \sum_j c_j M_j \ln M_j/\mu = b\Lambda, \qquad (2.4)
$$

b being an unspecified numerical factor. Here μ denotes the renormalization scale, but the conditions (2.3) ensure that the identifications (2.4) are μ-independent, and therefore consistent with

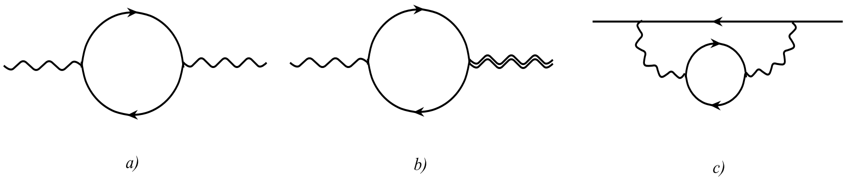

Figure 1: Simplest diagrams with the fermion bubble

renormalization-group invariance. It is also consistent to set $b = 0$, to kill the linear divergence by default.

The regularized gauge-fixing terms are

$$\mathcal{L}_{\mathrm{gf}} = \frac{1}{2\lambda g^2}(\partial_\mu A_\mu)\left(1 - \frac{\Box}{\Lambda^2}\right)(\partial_\nu A_\nu) + \overline{C}\Box\left(1 - \frac{\Box}{\Lambda^2}\right)C.$$

The ghosts decouple, as usual.

Renormalization. The renormalized lagrangian reads

$$\mathcal{L}_{\mathrm{R}} = Z_\psi\overline{\psi}\not{D}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\left(1 - \frac{\Box}{\Lambda^2}\right)A_\rho + \Lambda\delta Z_\Lambda Z_\psi\overline{\psi}\psi$$

and the renormalization constants have expansions

$$Z_\psi = 1 + \sum_{n=1}^{\infty} a_n(g,\lambda)\ (\ln\Lambda/\mu)^n$$

etc., where μ denotes the subtraction point, in the minimal subtraction scheme. Standard Ward identities, combined with the properties of the Chern-Simons term, ensure that the beta functions vanish, $\beta_g = \beta_\lambda = 0$ [14], and there is no need to insert renormalization constants for the gauge field, $Z_A = Z_g = Z_\lambda = 1$.

The perturbative results of this paper are written in the formalism of the dimensional-regularization technique and are easily converted to the cut-off regularization technique defined above replacing $1/\varepsilon$ with $\ln\Lambda^2/\mu^2$ and understanding that power-like divergences are subtracted in the conformal scheme (i.e. the scheme that preserves conformal invariance at the quantum level).

The lowest-order values of the fermion renormalization constant and anomalous dimension are given by the graph $c)$ of Fig. 1. Up to subleading corrections in $1/n_f$ their values are

$$Z_\psi = 1 - \frac{g^4 n_f}{384\pi^2\varepsilon}, \qquad \gamma_\psi = \frac{1}{2}\frac{\mathrm{d}\ln Z_\psi}{\mathrm{d}\ln\mu} = \frac{g^4 n_f}{384\pi^2}.$$

3 Gravity coupled with Chern-Simons $U(1)$ gauge theory and matter

The conventions for covariant derivatives $\mathcal{D}_\mu$, curvature R^a, torsion $\mathcal{D}e^a$ and spin connection ω_μ^a are given in the appendix. The lagrangian of gravity coupled with Chern-Simons $U(1)$ gauge theory and massless fermions reads

$$\mathcal{L} = \frac{1}{2\kappa} eR + e\overline{\psi}\not{\mathcal{D}}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho, \tag{3.1}$$

where $e = \sqrt{g}$. The constant κ has dimensionality -1 in units of mass and serves as an expansion parameter for the irrelevant couplings. The perturbative expansion is a double expansion in powers of g and κE, where E is the energy scale.

The gravitational field is defined expanding the dreibein e_μ^a around flat space:

$$e_\mu^a = \delta_\mu^a + \phi_\mu^a, \qquad\qquad \omega_\mu^a = \varepsilon^{abc}\partial^b\phi_\mu^c + \mathcal{O}(\phi^2),$$

and choosing the symmetric gauge $\phi_{\mu a} = \phi_{a\mu}$. It is convenient to gauge-fix both gravity and the $U(1)$ gauge field in flat space. The symmetric gauge is algebraic and so can be imposed directly. The gauge-fixing sector of the theory is therefore

$$\mathcal{L}_{\rm gf} = \frac{1}{2\alpha\kappa}(\partial_\mu\phi_{\mu\nu})^2 + \frac{1}{2\lambda g^2}(\partial_\mu A_\mu)^2 + \mathcal{L}_{\rm ghost}.$$

The gauge parameters λ and α are kept throughout the calculations, because a powerful way to check the results is to check the gauge-fixing independence of various quantities. The ghost part of the gauge-fixing lagrangian is derived in detail in the appendix. The ghosts do not contribute to the quantities calculated in this paper (this is proven in section 5).

Regularization. The theory (3.1) is power-counting non-renormalizable, and therefore, up to miraculous cancellations (that the results of this paper exclude), divergences can be removed introducing infinitely many new coupling constants, multiplying all possible irrelevant operators. The vertices of the complete theory can contain arbitrarily many derivatives and the regularized propagators should tend to zero faster than any power at large momenta. The most convenient cut-off regularization framework for the coupled theory is a Slavnov higher-derivative regularization where propagators are exponentially corrected. For example, the Chern-Simons field is regularized with

$$\frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\exp\left(-\frac{\Box}{\Lambda^2}\right)A_\rho + \text{non-minimal}. \tag{3.2}$$

The D'Alembertian is the covariant one and non-minimal terms have to be fixed to ensure that the integrated regularized Chern-Simons term is gauge invariant. To the order $1/\Lambda^2$ we need to add

$$-\frac{1}{2g^2\Lambda^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}R_{\rho\alpha}A^\alpha.$$

It is immediate to prove that there exist appropriate non-minimal terms to all orders in $1/\Lambda^2$. Similar operations can be used to introduce appropriate exponentials in the graviton and fermion propagators. However, the exponentials regularize only the superficial divergences of diagrams with more than one loop. One-loop divergences and subdivergences have to be regularized apart, for example with the gauge-invariant Pauli-Villars method of Fadeev and Slavnov [21].

The existence of a manifestly gauge invariant regularization ensures the absence of gauge anomalies to all orders in perturbation theory.

Result. The four-fermion divergent vertex (1.4) is generated at the second order in the loop expansion and first order in the κ-expansion, up to subleading corrections in $1/n_f$. Therefore, finiteness of three-dimensional gravity does not survive the coupling with matter. To renormalize (1.4), it is first necessary to add new vertices and coupling constants to the theory (3.1),

$$\mathcal{L} = \frac{1}{2\kappa}eR + e\overline{\psi}\not{D}\psi + \frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + \kappa\lambda_1\frac{e}{4}(\overline{\psi}\psi)^2 + \kappa\lambda_2\frac{e}{4}(\overline{\psi}\gamma^a\psi)^2 + \mathcal{O}(\kappa^2). \qquad (3.3)$$

Then, it is necessary to renormalize the new couplings by means of suitable renormalization constants, $\lambda_{1,2\mathrm{B}} = \lambda_{1,2}Z_{1,2}$, and redefine the fields. The field redefinitions have the form

$$A_{\mu\mathrm{B}} = A_\mu + \mathcal{O}(\kappa), \qquad \psi_\mathrm{B} = Z_\psi^{1/2}\psi + \mathcal{O}(\kappa), \qquad e^a_{\mu\mathrm{B}} = e^a_\mu + \mathcal{O}(\kappa).$$

As in every non-renormalizable theory, the number of couplings is expected to grow indefinitely with the order of the perturbative expansion in κ. Thus (3.3), as a fundamental theory, is not physically predictive. Of course, it is still predictive as an effective field theory.

4 Absence of the Lorentz-Chern-Simons term

The theory (3.1) is parity violating. A priori, renormalization might generate a Lorentz-Chern-Simons counterterm (1.3). Now I prove that this does not happen. Basically, this term is finite and therefore it is possible to set its renormalized coupling to zero consistently.

The Lorentz-Chern-Simons term has dimensionality 3 in units of mass, so the contributions to its renormalization must be $O(\kappa^0)$. Two types of diagrams can contribute: *i)* diagrams with internal gravitons and gravitational ghosts; *ii)* diagrams with no internal gravitons nor gravitational ghosts.

By dimensional counting, the diagrams of type *i)* can only be one-loop. Indeed, higher loop diagrams with internal gravitons and/or gravitational ghosts contribute to the renormalization of lagrangian terms with dimensionality greater than 3. One-loop diagrams, on the other hand, have no logarithmic divergence in three dimensions.

The diagrams of type *ii)* can be studied in external gravity, using properties of the trace anomaly. For concreteness, I consider the case of Chern-Simons $U(1)$ gauge theory with massless fermions, but the generalization of the proof is immediate.

It is useful to include the Lorentz-Chern-Simons term in the renormalized lagrangian of the theory embedded in external gravity. In general, the Lorentz-Chern-Simons term has to be multiplied by a coupling constant ζ plus a counterterm Δ_ζ,

$$\mathcal{L}_{\rm R}=eZ_\psi\overline{\psi}\not{D}\psi+\frac{1}{2g^2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho+(\zeta+\Delta_\zeta)\ \varepsilon^{\mu\nu\rho}\left(\omega_\mu^a\partial_\nu\omega_\rho^a+\frac{1}{3}\omega_\mu^a\omega_\nu^b\omega_\rho^c\varepsilon^{abc}\right)+\mathcal{L}_{\rm gf}.$$

The regularizing terms are not written explicitly. Obviously, Z_ψ and Δ_ζ depend only on g and the gauge-fixing parameter λ, but not on ζ, because the Lorentz-Chern-Simons term is just the identity operator, in external gravity. Now I prove that $\Delta_\zeta=0$.

The stress tensor is expressed as a functional derivative of the action,

$$T_\mu^\nu(x)=\frac{e^{\nu a}(x)}{2e(x)}\frac{\delta S}{\delta e^{\mu a}(x)}+\frac{e_\mu^a(x)g^{\nu\rho}(x)}{2e(x)}\frac{\delta S}{\delta e^{\rho a}(x)}. \tag{4.1}$$

In the differentiation, the gauge-fixing and ghost terms can be ignored, since they add gauge-exact contributions, which do not affect the physical correlation functions. Because of the non-local regularization (3.2), the functional differentiation of (4.1) is involved. It is convenient to focus the attention on the integrated stress tensor

$$\int e(x)T_\mu^\nu(x)\ {\rm d}^3x,$$

and in particular the integrated trace

$$\int e(x)\Theta(x)\ {\rm d}^3x=\int e(x)T_\mu^\mu(x)\ {\rm d}^3x. \tag{4.2}$$

Inside (4.2), the differentiation (4.1) simplifies considerably. For example, it is possible to treat the spin connection, Kristoffel symbols and curvatures as constants, because the functional differentiation of objects such as $\partial_\mu g_{\nu\rho}$ and $\partial_\mu e_\nu^a$ produces total derivatives, which are killed by the space-time integration of (4.2). In practice, the operation (4.2) reduces to a constant Weyl rescaling. Since the theory depends on a unique scale, at the bare level, namely the cut-off Λ, the result is easily proved to be

$$\int e(x)\Theta(x)\ {\rm d}^3x=-2\int eZ_\psi\overline{\psi}\not{D}\psi-\Lambda\left.\frac{\partial S}{\partial\Lambda}\right|_{\rm B}, \tag{4.3}$$

where the subscript means that the bare fields and coupling constants are kept fixed in the Λ-differentiation. Since the renormalization constants depend only on $\ln\Lambda/\mu$, the expression (4.3) can be easily converted into a renormalized equivalent,

$$\int e(x)\Theta(x)\ {\rm d}^3x=-2\int eZ_\psi\overline{\psi}\not{D}\psi-\Lambda\frac{\partial S}{\partial\Lambda}-\mu\frac{\partial S}{\partial\mu}. \tag{4.4}$$

Now, consider a convergent correlation function

$$G(x_1\cdots x_n,y_1\cdots y_m,z_1\cdots z_m)=\langle A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\ \overline{\psi}(y_1)\cdots\overline{\psi}(y_m)\ \psi(z_1)\cdots\psi(z_m)\rangle \tag{4.5}$$

The $\Lambda\partial/\partial\Lambda$ derivative of a renormalized correlation function is zero in the $\Lambda\rightarrow\infty$ limit, by definition. Such derivative is equal to the insertion of $-\Lambda\partial S/\partial\Lambda$, so the first term of (4.4) can be ignored when the integral of Θ is inserted inside these correlation functions. The result is

$$\int e(x)\Theta(x)\ \mathrm{d}^3x=-2\int eZ_\psi\overline{\psi}\not{D}\psi-\mu\frac{\partial S}{\partial\mu}.$$

Since the theory is conformal in flat space, the couplings do not run, apart possibly from ζ, and so the partial $\ln\mu$ derivatives of the renormalization constants can be replaced with total $\ln\mu$ derivatives, up to terms proportional to the ζ beta function $\beta_\zeta=-\mathrm{d}\Delta_\zeta/\mathrm{d}\ln\mu$. The result is

$$\int e(x)\Theta(x)\ \mathrm{d}^3x=-2(1+\gamma_\psi)\int eZ_\psi\overline{\psi}\not{D}\psi+\beta_\zeta\int\ \varepsilon^{\mu\nu\rho}\left(\omega_\mu^a\partial_\nu\omega_\rho^a+\frac{1}{3}\omega_\mu^a\omega_\nu^b\omega_\rho^c\varepsilon^{abc}\right).$$

The integral signs can be removed up to total derivatives. The only ambiguity is a term $\partial_\mu(\overline{\psi}\gamma^\mu\psi)$, which however cannot appear, because it violates the symmetry under charge conjugation (see the appendix).

The result is

$$\Theta(x)=-2(1+\gamma_\psi)[\mathrm{E}_\psi]+\beta_\zeta e^{-1}\ \varepsilon^{\mu\nu\rho}\left(\omega_\mu^a\partial_\nu\omega_\rho^a+\frac{1}{3}\omega_\mu^a\omega_\nu^b\omega_\rho^c\varepsilon^{abc}\right) \tag{4.6}$$

where

$$[\mathrm{E}_\psi]=\frac{1}{2}Z_\psi\overline{\psi}\overleftrightarrow{\not{D}}\psi=\frac{1}{2}e^{-1}\left(\overline{\psi}\frac{\delta_l S}{\delta\overline{\psi}}+\frac{\delta_r S}{\delta\psi}\psi\right)$$

and δ_l, δ_r denote the left and right functional derivatives, respectively. The operator $[\mathrm{E}_\psi](x)$ is proportional to the fermion field equation and therefore is finite. An immediate proof is that inserting $[\mathrm{E}_\psi](x)$ in the correlation function (4.5) simply multiplies it by

$$\frac{1}{2e(x)}\sum_{i=1}^{m}[\delta(x-y_i)+\delta(x-z_i)]. \tag{4.7}$$

This result is standard and follows from a functional integration by parts.

Moreover, the second term of (4.6) should simply not be there, because the unintegrated Lorentz-Chern-Simons term is not Lorentz invariant, while Θ is. Therefore, ζ does not run:

$$\beta_\zeta=0,\qquad \Delta_\zeta=\text{constant}. \tag{4.8}$$

The constant can be moved inside ζ, so it is safe to write $\Delta_\zeta=0$.

Having proved that the renormalized coupling ζ does not run, it is meaningful to set it to zero. This means that the subtraction scheme can be adapted in such a way that the Lorentz-Chern-Simons term is absent at each order of the perturbative expansion. It is worth mentioning that if the Lorentz-Chern-Simons term is treated within the minimal subtraction scheme (or any generic scheme), finite contributions can survive and have to be removed by hand. These facts have been recently confirmed by a number of explicit two-loop computations [22].

To conclude, there exists a modified subtraction scheme where the finite part of the Lorentz-Chern-Simons term is identically zero. This is important, because the theory with $\zeta \neq 0$, known as "topologically massive gravity" [4], is physically inequivalent to the theory with $\zeta = 0$. In the rest of the paper I focus on the theory with $\zeta = 0$. On the other hand, it is easy to prove that a small nonzero ζ does not change the two-loop results of the next sections and does not affect the conclusion that quantum gravity coupled with matter, as it stands, is not finite in three dimensions.

The arguments of this section are completely general and apply to every theory of matter coupled with gravity. The generalization is straightforward.

Observe that Lorentz invariance is crucial in the derivation. The point is that the unintegrated trace operator $\Theta(x)$ is Lorentz invariant, but the unintegrated Lorentz-Chern-Simons term is not. A similar argument proves that the beta function of the Chern-Simons coupling g is zero [14]: the gauge invariance of Θ and the gauge non-invariance of the unintegrated $U(1)$ Chern-Simons term are not compatible with a running of g. Instead, the invariance under diffeomorphisms is not helpful in this kind of reasonings, since the unintegrated operator $\Theta(x)$ is not invariant under diffeomorphisms.

The second crucial point is the possibility to reduce to the theory in external gravity. This is a lucky situation. The aguments based on the trace of the energy momentum tensor cannot be applied if gravity is dynamical, where the "energy momentum tensor" (by which I mean the derivative (4.1) of the action with respect to the metric) vanishes identicaly using the field equations. Other definitions of the stress tensor for quantized gravity are more tricky to use.

Finally, it is known that pure gravity can be related to a Chern-Simons theory in three dimensions [3]. However, Witten's arguments for finiteness are based on the possibility to express the action in a form that does not contain the inverse dreibein e_a^μ, nor the inverse metric tensor $g^{\mu\nu}$. This is impossible if gravity is coupled with propagating matter.

These remarks explain why gravity coupled with matter can be not finite despite the fact that there exists no graviton.

5 Two-loop calculations

In this section I describe the general setting of the two-loop computations.

Four-fermion vertices. I focus on the irrelevant terms of dimensionality four in units of mass, which are

$$e\overline{\psi}\not{\mathcal{D}}^2\psi, \qquad eF_{\mu\nu}F^{\mu\nu}, \qquad \varepsilon^{\mu\nu\rho}e^a_\rho F_{\mu\nu}\overline{\psi}\gamma^a\psi, \qquad e(\overline{\psi}\psi)^2, \qquad e(\overline{\psi}\gamma^a\psi)^2. \tag{5.1}$$

Only the last two terms are independent, as I now prove.

Considering the presence of irrelevant terms, necessary for renormalization, the most general field equations have the form

$$\not{\mathcal{D}}\psi = \mathcal{O}(\kappa), \qquad\qquad F_{\mu\nu} + \frac{ig^2}{2}e\varepsilon_{\mu\nu\rho}e^{\rho a}\overline{\psi}\gamma^a\psi = \mathcal{O}(\kappa), \tag{5.2}$$

$$\frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{8}e^a_\mu\overline{\psi}\gamma^a\overleftrightarrow{\mathcal{D}}_\nu\psi + \frac{1}{8}e^a_\nu\overline{\psi}\gamma^a\overleftrightarrow{\mathcal{D}}_\mu\psi - \frac{1}{4}g_{\mu\nu}\overline{\psi}\overleftrightarrow{\not{\mathcal{D}}}\psi = \mathcal{O}(\kappa). \tag{5.3}$$

The first counterterm of (5.1) vanishes using the fermion field equation, up to higher orders in κ, so it can be removed by means of a field redefinition. The second and third counterterms in (5.1) are equal to the forth of (5.1) up to terms proportional to the field equation of the gauge field (5.2) and terms of dimensionality greater than four. Finally, it is immediate to prove, using Fierz identities, that the independent four-fermion vertices are precisely the ones listed in (5.1).

The second term in (5.1) is an ordinary gauge-field kinetic term. It has to be removed with a field redefinition of the form

$$A_\mu \rightarrow A_\mu + a\kappa\ e\varepsilon_{\mu\nu\rho}F^{\nu\rho} + b\kappa\ e^a_\mu\overline{\psi}\gamma^a\psi, \tag{5.4}$$

where a and b are numerical coefficients. A theory with a propagating gauge-field is physically inequivalent to (3.1)-(3.3). Moreover, power-counting has to be reconsidered and the calculations have to be repeated using the complete gauge field propagator. Here I stick to the theory (3.1)-(3.3).

For the calculations of this paper, the use of field equations to simplify the counterterms amounts in practice to the replacements

$$R_{\mu\nu\rho\sigma} \rightarrow 0, \qquad F_{\mu\nu} \rightarrow -\frac{ig^2}{2}e\varepsilon_{\mu\nu\rho}e^{\rho a}\overline{\psi}\gamma^a\psi, \qquad \not{D}\psi \rightarrow 0. \tag{5.5}$$

Collecting these observations, the two-loop counterterms have the form

$$\mathcal{L}^{grav}_{\text{counter}} = c\frac{\kappa}{4}e(\overline{\psi}\psi)^2 + d\frac{\kappa}{4}e(\overline{\psi}\gamma^a\psi)^2 + \mathcal{O}(\kappa^2), \tag{5.6}$$

and the values of the numerical coefficients c and d have to be determined with an explicit computation. Since the one-loop diagrams are convergent in three dimensions, subdivergences are absent at two loops and the divergent parts are simple poles $1/\varepsilon$ or simple logs $\ln\Lambda^2/\mu^2$.

Reduction of the number of diagrams. Observe that the counterterms (5.6) are necessarily polynomial in the number n_f of fermions. It is immediate to check the at the second loop order they are at most linear in n_f. A quadratic contribution in n_f would come from two fermion loops. Two fermion loops can be connected only by a four fermion vertex, otherwise the diagram is either not one-particle irreducible or not two-loop. Then, however, the diagram factorizes into the product of two one-loop diagrams, which are convergent in three dimensions.

The number of two-loop diagrams contributing to the four-fermion vertices is high. It is convenient to concentrate on the contributions proportional to n_f, given by the diagrams that contain one fermion loop. Fermion loops with two external legs are shown in Fig. 1 and appear frequently as subdiagrams of the two-loop diagrams. Fermion loops with three gauge-field legs or one graviton leg and two gauge-field legs can also appear inside the two-loop diagrams.

As anticipated in section 3, the ghosts do not contribute to the results of this paper. Indeed, the relevant Feynman diagrams do not have external ghost legs (diagrams with external ghost legs affect only the gauge-trivial sector of the theory). Diagrams with internal ghost legs giving linear contributions in n_f must have a ghost loop and a fermion loop. Arguing as above, these diagrams factorize into the product of two one-loop subdiagrams and therefore converge.

The one-loop self-energy of the gauge field is given by Fig. 1-a) and is equal to

$$-\frac{n_f}{16}\frac{1}{(k^2)^{(1+\varepsilon)/2}}(\delta_{\mu\nu}k^2 - k_\mu k_\nu).$$

The ε-dependence in the power of k is kept, because it affects the pole parts of the two-loop diagrams that contain the fermion bubble as a subdiagram.

Obvious considerations based on spin conservation imply that the graviton-gauge-field self-energy of Fig. 1-b) is identically zero. This fact can be immediately checked with an explicit calculation.

Calculations. The divergent parts of the diagrams can be evaluated with the techniques that follow. First, the diagrams are contracted with external momenta, Dirac matrices, Kronecker tensors and ε tensors in all possible ways, and traced in spinor indices. Then, the results of these operations are differentiated a sufficient number of times with respect to the external momenta, to arrive at dimensionless integrals, and the external momenta are set to zero. Scalar products of internal momenta in the numerators are converted into sums of squares, using for example

$$p \cdot q = \frac{1}{2}\left[p^2 + q^2 - (p-q)^2\right].$$

After a number of such algebraic manipulations, the calculation is reduced to a set of integrals of the form

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{d}^D q}{(2\pi)^D}\frac{1}{[p^2]^a[q^2]^b[(p-q)^2]^c} \quad (5.7)$$

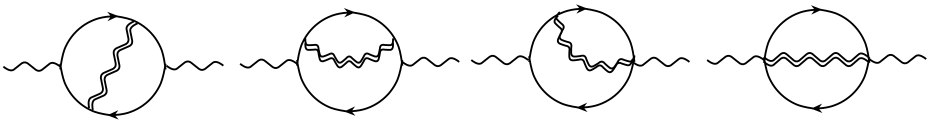

Figure 2: Two-loop self-energy of the gauge field with an internal graviton

where $D = 3 - \varepsilon$ and a, b, c are integers such that $a + b + c = 3$. It is convenient to imagine that the fermions have a mass, to avoid IR divergences at zero external momenta, and in some diagrams it is also useful to give fictitious masses to the $U(1)$ field and the graviton.

The unique non-trivial contributions comes from the two-loop "master" integral

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{p^2 q^2 (p-q)^2} = \frac{1}{32\pi^2 \varepsilon} + \text{finite part.} \tag{5.8}$$

The other integrals (5.7) are convergent. Indeed, if a, b, c are not all equal to one, then at least one of them is zero or negative, so there are only two denominators. Integrals with two denominators factorize, eventually after a translation, into the product of two integrals of the form

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{p_{\mu_1} \cdots p_{\mu_n}}{[p^2]^m}$$

which are convergent. Also the integral (5.8) can be reduced to the product of two integrals, using the technique of partial integration [23]

$$0 = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\partial}{\partial p_\mu} \frac{p_\mu}{p^2 q^2 (p-q)^2},$$

but this operation factorizes a $1/\varepsilon$.

The manipulations described so far are reversible, in the sense that it is possible to reconstruct the structure of the divergent parts of the diagrams, using the fact that they are local in the external momenta.

6 Four-fermion vertices induced by gravity

The counterterms of dimensionality 4 induced by gravity are proportional to the Newton constant κ and can therefore be computed at $\lambda_1 = \lambda_2 = 0$.

Three classes of diagrams contribute: the gauge-field two-point function, the fermion-gauge-field three-point function and the fermion four-point function. The divergent diagrams contain one internal graviton leg, one fermionic loop and zero, one or two internal gauge-field legs, respectively. There is no contribution to the fermion self-energy, since the potentially relevant diagrams contain the subdiagram of Fig. 1 b). In the figures, diagrams are depicted up to permutations of external legs and reversions of the fermion arrows. It is not necessary to compute diagrams with two external fermion legs plus two external gauge-field legs, because they do not give new gauge-invariant contributions. To the order $\mathcal{O}(\kappa)$ it is not even necessary to consider diagrams with external graviton legs. Such diagrams either contribute to the gauge-trivial sector of the lagrangian or factorize a curvature tensor. Then, using (1.2) and the graviton field equation of (5.2), they can be converted into $\mathcal{O}(\kappa^2)$ counterterms.

Gravitational contribution to the self-energy of the gauge field. The graphs that have an internal graviton leg and contribute to the two-loop self-energy of the gauge field are shown in Fig. 2. The counterterms associated with these graphs sum up to

$$\mathcal{L}_{\text{counter-1}} = -\frac{5n_f\kappa\alpha}{384\pi^2\varepsilon} eF_{\mu\nu}F^{\mu\nu}. \tag{6.1}$$

Observe that (6.1) is proportional to the gauge-fixing parameter α and should therefore be cancelled by some other contribution (see below).

Gravitational corrections to the fermion-gauge-field vertex. The divergent parts of these graphs, which are shown in the first half of Fig. 3, are subtracted by the counterterm

$$\mathcal{L}_{\text{counter-2}} = -\frac{i\kappa g^2 n_f}{768\pi^2\varepsilon}(3+10\alpha)\,\varepsilon^{\mu\nu\rho}e^a_\rho F_{\mu\nu}\overline{\psi}\gamma^a\psi. \tag{6.2}$$

This is a Pauli term, but using the field equations (5.5) it can be converted into a four-fermion vertex.

Gravitational contribution to the fermion four-point function. The four-fermion counverterms that cancel the poles of the graphs shown in the second half of Fig. 3 are

$$\mathcal{L}_{\text{counter-3}} = \frac{\kappa g^4 n_f}{384\pi^2\varepsilon}(1+10\alpha)\;\frac{e}{4}(\overline{\psi}\gamma^a\psi)^2. \tag{6.3}$$

Summing the three contributions (6.1)-(6.3) and using the substitutions (5.5), the α-dependence drops out and we obtain

$$\mathcal{L}^{grav}_{\text{counter}} = \mathcal{L}_{\text{counter-1}} + \mathcal{L}_{\text{counter-2}} + \mathcal{L}_{\text{counter-3}} = -\frac{5\kappa g^4 n_f}{384\pi^2\varepsilon}\frac{e}{4}(\overline{\psi}\gamma^a\psi)^2$$

plus terms proportional to the field equations. The field redefinition that reabsorbs the terms proportional to the field equations reads

$$A_\mu \to A_\mu - \frac{5n_f\alpha g^2\kappa}{768\pi^2\varepsilon}\;e\varepsilon_{\mu\nu\rho}F^{\nu\rho} - \frac{in_f g^4\kappa(3+5\alpha)}{768\pi^2\varepsilon}\;e^a_\mu\overline{\psi}\gamma^a\psi.$$

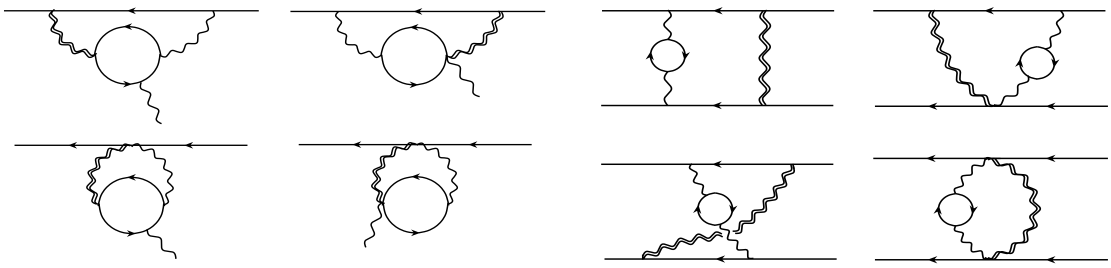

Figure 3: Two-loop gravitational corrections to the fermion-gauge-field vertex and to the four-fermion vertices

7 Conclusions

In this paper I have studied the renormalization of three-dimensional quantum gravity coupled with matter. Using standard arguments it is possible to show that the Lorentz-Chern-Simons term is not renormalized and therefore there exists a subtraction scheme where it is identically absent. Instead, irrelevant counterterms cannot be excluded *a priori*. I have performed a two-loop computation in a concrete model, gravity coupled with Chern-Simons $U(1)$ gauge theory and massless fermions, to show that it is not finite. The calculation can be simplified in various ways, but involves a considerable number of diagrams. A good source of checks is the gauge-fixing independence of the final result. A four-fermion counterterm

$$-\frac{5\kappa g^4 n_f}{384\pi^2\varepsilon}\frac{e}{4}(\overline{\psi}\gamma^a\psi)^2$$

is turned on by renormalization. Therefore finiteness is violated at the second order in the loop expansion and first order in the κ expansion.

Acknowledgement. I am grateful to P. Menotti for drawing my attention to references on 2+1 quantum gravity.

8 Appendix

In this appendix I recall some basic formulas, useful to fix the notation and simplify the analysis of the graphs. I also comment on the difficulties to treat the Chern-Simons form in the context of the dimensional-regularization technique.

Curved-space conventions. Torsion, curvatures, covariant derivatives and connections are

$$\mathcal{D}e^a = \mathrm{d}e^a + \varepsilon^{abc}\omega^b e^c = 0, \qquad R^a = \mathrm{d}\omega^a + \frac{1}{2}\varepsilon^{abc}\omega^b\omega^c,$$

$$
\begin{aligned}
\mathcal{D}_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho, \qquad\qquad \Gamma^\rho_{\mu\nu} = e^{\rho a}\partial_\mu e^a_\nu + \varepsilon^{abc}\omega^a_\mu e^b_\nu e^{\rho c}, \\
\omega^a_\mu &= \varepsilon^{abc}\left(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu\right) e^{\nu c} - \frac{1}{4} e^a_\mu \varepsilon^{bcd}\left(\partial_\rho e^b_\nu - \partial_\nu e^b_\rho\right) e^{\nu c} e^{\rho d}, \\
\mathcal{D}_\mu \psi &= \partial_\mu \psi - \frac{i}{2}\omega^a_\mu \gamma^a \psi + i A_\mu \psi.
\end{aligned}
$$

The Ricci tensor and scalar curvature are defined as $R_{\mu\nu} = R^{ab}_{\mu\rho} e^{\rho b} e^a_\nu$, $R = R_{\mu\nu} g^{\mu\nu}$, where

$$
R^{ab} = \varepsilon^{abc} R^c = \frac{1}{2} R^{ab}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \qquad R^\mu{}_{\nu\rho\sigma} = \partial_\sigma \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\sigma} - \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} + \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma},
$$

and of course $g_{\mu\nu} = e^a_\mu e^a_\nu$.

Ghosts. The ghosts are: C for $U(1)$, C^μ for diffeomorphisms and $C^{ab} = -C^{ba}$ for Lorentz rotations. Indices are raised and lowered with δ^a_μ. The BRST variations of the fields read

$$
\begin{aligned}
sA_\mu &= \partial_\mu C - A_\rho \partial_\mu C^\rho - C^\rho \partial_\rho A_\mu, \qquad\qquad se^a_\mu = -e^a_\rho \partial_\mu C^\rho - C^\rho \partial_\rho e^a_\mu - C^{ab} e^b_\mu, \\
sC &= -C^\rho \partial_\rho C, \qquad sC^{ab} = -C^{ac}C^{cb} - C^\rho \partial_\rho C^{ab}, \qquad sC^\rho = -C^\sigma \partial_\sigma C^\rho.
\end{aligned}
$$

It is necessary to introduce antighosts $\overline{C}$, $\overline{C}^a$ and $\overline{C}^{\mu a} = -\delta^\mu_b \delta^a_\nu \overline{C}^{\nu b}$. The ghost lagrangian reads

$$
\mathcal{L}_{\text{ghost}} = \partial_\mu \overline{C}(\partial_\mu C - A_\rho \partial_\mu C^\rho - C^\rho \partial_\rho A_\mu) + (\overline{C}^{\mu a} - \partial_\mu \overline{C}^a)(e^a_\rho \partial_\mu C^\rho + C^\rho \partial_\rho e^a_\mu + C^{ab} e^b_\mu).
$$

Contracted indices may appear both as subscripts or superscripts in Euclidean flat space. We have two ghost sectors: $U(1)$ and gravitational (diffeomorphisms plus Lorentz symmetry). The two sectors have a diagonal quadratic lagrangian, but mix due to a vertex of the form $\overline{C}AC^\rho$.

Propagators. The gauge-field propagator is

$$
\langle A_\mu(k)\ A_\nu(-k)\rangle_{\text{free}} = -\frac{i}{2} g^2 \varepsilon_{\mu\nu\rho} \frac{k_\rho}{k^2} + g^2 \lambda \frac{k_\mu k_\nu}{k^4}.
$$

The graviton propagator reads

$$
\begin{aligned}
\langle \phi_{\mu\nu}(p)\ \phi_{\rho\sigma}(-p)\rangle_{\text{free}} &= \frac{\kappa}{2}\frac{1}{p^2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - 2\delta_{\mu\nu}\delta_{\rho\sigma}\right) + \frac{\kappa}{p^4}\left(\delta_{\mu\nu} p_\rho p_\sigma + p_\mu p_\nu \delta_{\rho\sigma}\right) + \\
&\quad + \left(\alpha - \frac{1}{2}\right)\frac{\kappa}{p^4}\left(\delta_{\mu\rho} p_\nu p_\sigma + \delta_{\nu\rho} p_\mu p_\sigma + \delta_{\mu\sigma} p_\nu p_\rho + \delta_{\nu\sigma} p_\mu p_\rho\right) - 3\alpha\kappa \frac{p_\mu p_\nu p_\rho p_\sigma}{p^6}.
\end{aligned}
$$

Difficulties of the dimensional-regularization technique in curved space. Here I collect some observations about the difficulties to define a consistent dimensional regularization for the Chern-Simons term in curved space. If the theory contains two-component spinors, it is possible to define the tensor

$$
E^{abc} = -\frac{i}{2}\mathrm{tr}[\gamma^a \gamma^b \gamma^c],
$$

where γ^a are the dimensionally continued Pauli matrices. If the trace is assumed to be cyclic, the E tensor is set to zero by the dimensional regularization [24]. However, if the theory contains two-component fermions, the $D = 3$ limit of E^{abc} should be the ordinary ε tensor. In curved space the situation worsens. Since the Pauli matrices are constant and covariantly constant, so is the E tensor, assuming that it exists: $\partial_\mu E^{abc} = \mathcal{D}_\mu E^{abc} = 0$. The Bianchi identity following from these equations is

$$R^{ad}_{\mu\nu} E^{dbc} + R^{cd}_{\mu\nu} E^{dab} + R^{bd}_{\mu\nu} E^{dca} = 0. \tag{8.1}$$

To define the propagator of the $U(1)$ gauge field, it would be useful to have an "inverse" of the E tensor, for example an $\underline{E}$ tensor satisfying

$$E^{abc} \underline{E}_{mnp} = \frac{1}{3!} \delta^{abc}_{mnp}. \tag{8.2}$$

However, contracting (8.1) with the $\underline{E}$ tensor it is immediate to obtain

$$(D-3) R^{ab}_{\mu\nu} = 0, \tag{8.3}$$

which implies that either the dimension of spacetime is exactly equal to 3 or the spacetime is flat.

Moreover, an identity similar to (8.1) holds with $R^{ad}_{\mu\nu}$ replaced by $\underline{E}^{adm}$. This follows immediately from the definition (8.2). Then,

$$(D-3) \underline{E}^{abc} = 0.$$

This implies that the $\underline{E}$ tensor does not exist in D dimensions.

An alternative approach is to split the D dimensional spacetime into the tensor product of a three-dimensional spacetime and a $(-\varepsilon)$-dimensional spacetime, as is commonly done in four dimensions to define the matrix γ_5 and the tensor $\varepsilon_{\mu\nu\rho\sigma}$ [24]. Let $\mu, \overline{\mu}, \widetilde{\mu}$ denote the D-dimensional, three-dimensional and $(-\varepsilon)$-dimensional spacetime indices, respectively. The kinetic lagrangian of the $U(1)$ field lives in the three-dimensional subspace. The $(-\varepsilon)$-dimensional component $A_{\widetilde{\mu}}$ of the $U(1)$ gauge-field appears only in the Dirac term and thus has no kinetic term. A way to treat a situation like this can be found in ref. [18], using the large-n_f expansion. The missing kinetic term is provided by the fermion bubble, which is leading in the large-n_f expansion. Alternatively, it is possible to add $F^2_{\mu\nu}$ multiplied by $1/\Lambda$, where Λ is a further cut-off, that is sent to infinity after $\varepsilon \to 0$.

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