

SUM RULES FOR TRACE ANOMALIES AND IRREVERSIBILITY OF THE RENORMALIZATION-GROUP FLOW

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I review my explanation of the irreversibility of the renormalization-group flow in even dimensions greater than two and address new investigations and tests.

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The purpose of my talk is to review exact results for the trace anomalies c and a in the critical limits of supersymmetric theories, which provide non-perturbative evidence that the renormalization-group (RG) flow is irreversible, and then derive universal sum rules for c , a and a' in even dimensions, formulate a theory of the irreversibility of the RG flow, recapitulate the arguments and the evidence in favor of this theory, and address new investigations and tests of the predictions of this theory, analytical and on the lattice.

I consider the most general renormalizable quantum field theory, such that the RG flow interpolates between UV and IR conformal fixed points. The theory is embedded in external gravity. Let $\Gamma[g_{\mu\nu}]$ denote the quantum action in the gravitational background. The trace anomaly is given by the derivative of Γ with respect to the conformal factor of the metric:

$$\Theta = - \left. \frac{\delta\Gamma[g_{\mu\nu}e^{2\phi}]}{\delta\phi} \right|_{\phi=0}. \quad (1)$$

At criticality, the trace anomaly has the form

$$\Theta = \frac{1}{120} \frac{1}{(4\pi)^2} \left[c W^2 - \frac{a}{4} G + \frac{2}{3} a' \square R \right], \quad (2)$$

and the constants c , a and a' are called central charges. Here W is the Weyl tensor ($W^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$) and $G = 4R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 16R_{\mu\nu}R^{\mu\nu} + 4R^2$ is the Euler density. In a free-field theory of n_s real scalars, n_f Dirac fermions and n_v vectors, the values of c and a are

$$c = n_s + 6n_f + 12n_v, \quad a = \frac{1}{3} (n_s + 11n_f + 62n_v). \quad (3)$$

The quantity a' , instead, has an ambiguity. The quantum action is defined up to the addition of arbitrary, finite local terms. The scalars of dimension four constructed with the curvature tensors and their covariant derivatives are W^2 , G , $\square R$ and R^2 . $\int \sqrt{g}W^2$ gives zero in (1). $\int \sqrt{g}\square R$

trivially vanishes. $\sqrt{g}G$ is a total derivative in four dimensions and so $\int \sqrt{g}G$ does not contribute. There remains $\int \sqrt{g}R^2$. Using (1) we see that

$$\Gamma \rightarrow \Gamma + \frac{1}{2160} \frac{\delta a'}{(4\pi)^2} \int \sqrt{g}R^2 \quad \Rightarrow \quad \Theta \rightarrow \Theta + \frac{1}{180} \frac{\delta a'}{(4\pi)^2} \square R \quad \Rightarrow \quad a' \rightarrow a' + \delta a'. \quad (4)$$

The ambiguity $\delta a'$ is RG invariant, since the local terms must be finite. Therefore, $\delta a'$ cancels out in differences such as $\Delta a' = a'_{\text{UV}} - a'_{\text{IR}}$. Nevertheless, while Δc and Δa are flow invariants, i.e. they do not depend on the particular flow connecting two fixed points, $\Delta a'$ is not a flow invariant [1].

At intermediate energies, the coefficients c , a and a' depend on the coupling constant α (and the subtraction scheme). For example, in (massless) QED or QCD, the trace anomaly operator equation has the form

$$\Theta = \frac{1}{120} \frac{1}{(4\pi)^2} \left[c(\alpha) W^2 - \frac{1}{4} a(\alpha) G + \frac{2}{3} a'(\alpha) \square R + \beta(\alpha) h(\alpha) R^2 \right] - \frac{1}{4} \beta(\alpha) F^2,$$

where $\beta(\alpha) = d \ln \alpha / d \ln \mu$, $h(\alpha)$ in an unspecified, regular function of α and F is the field strength of the gauge field. In flat space, $\Theta = -\beta(\alpha) F^2 / 4$.

The central charges are universally normalized as follows: c is normalized to be 1 for the free real scalar field; the relative normalization of c and a is fixed in such a way that the subclass of renormalization-group flows with $\Delta c = \Delta a$ is special in a sense to be specified below; the relative normalization of a' and a is fixed in such a way that the subclass of renormalization-group flows with $\Delta a' = \Delta a$ is special in another sense to be explained below.

Exact results for c and a in interacting conformal field theories are known [2, 3]. For example, $N=1$ $SU(N_c)$ supersymmetric QCD with N_f massless quark flavors in the conformal window $3N_c/2 \leq N_f \leq 3N_c$, has an interacting fixed point in the infrared, and

$$c_{\text{IR}} = \frac{15}{2} \left(7N_c^2 - 2 - 9 \frac{N_c^4}{N_f^2} \right), \quad a_{\text{IR}} = \frac{45}{2} \left(2N_c^2 - 1 - 3 \frac{N_c^4}{N_f^2} \right).$$

The IR values of the central charges c and a are derived combining the Adler-Bardeen theorem with the relation between the trace and axial anomalies, provided by supersymmetry. Due to the Adler-Bardeen theorem, a certain class of axial anomalies is one-loop exact [4]. The relation between the trace and axial anomalies is such that, in most supersymmetric theories, exact results can be obtained for the trace anomaly also, and one-loop calculations are enough to extract c_{IR} and a_{IR} .

Using the UV values (3), we obtain the differences

$$\begin{aligned} \Delta c &= c_{\text{UV}} - c_{\text{IR}} = \frac{5}{2} N_c N_f \left(1 - 3 \frac{N_c}{N_f} \right) \left(4 - 3 \frac{N_c}{N_f} - 9 \frac{N_c^2}{N_f^2} \right), \\ \Delta a &= a_{\text{UV}} - a_{\text{IR}} = \frac{5}{2} N_c N_f \left(1 - 3 \frac{N_c}{N_f} \right)^2 \left(2 + 3 \frac{N_c}{N_f} \right). \end{aligned}$$

It is immediate to verify that, in the conformal window, $a_{\text{UV}} \geq a_{\text{IR}} \geq 0$ and $c_{\text{IR}} \geq 0$, while Δc can be either positive or negative. The inequalities $a_{\text{UV}} \geq a_{\text{IR}} \geq 0$ and $c_{\text{IR}} \geq 0$ are satisfied in all the models studied in [2, 3].

The property $a_{UV} \geq a_{IR} \geq 0$ means that there exists a positive function a which always decreases along the RG flow, from the UV to the IR. This property is called irreversibility of the RG flow. In the rest of the talk I present my theory of the irreversibility of the RG flow.

Universal sum rules for the trace anomalies c , a and a' can be derived [5]. For simplicity, we take a conformally flat metric $g_{\mu\nu} = \delta_{\mu\nu} e^{2\phi}$, which is enough to study the correlation functions containing insertions of Θ , since ϕ is the external source coupled to the operator Θ . The Θ -correlators have complicated expressions, at intermediate energies, but tend to universal, simple limits at criticality, which contain just two local structures, multiplied by a and a' , and can be derived taking ϕ -derivatives of (2). The sum rules for $\Delta a = a_{UV} - a_{IR}$ and $\Delta a' = a'_{UV} - a'_{IR}$ are direct consequences of this. Studying the critical limits of the Θ -correlators in the sense of distributions, we arrive at [5]

$$\begin{aligned} \Delta a' &= \frac{5\pi^2}{2} \int d^4x |x|^4 \langle \tilde{\Theta}(x) \tilde{\Theta}(0) \rangle, \\ \Delta a - \Delta a' &= \frac{5\pi^2}{2} \int d^4x d^4y x^2 y^2 \left\{ \langle \tilde{\Theta}(x) \tilde{\Theta}(y) \tilde{\Theta}(0) \rangle \right. \\ &\quad \left. + \left\langle \frac{\tilde{\delta}\tilde{\Theta}(x)}{\tilde{\delta}\phi(y)} \tilde{\Theta}(0) \right\rangle + 2 \left\langle \frac{\tilde{\delta}\tilde{\Theta}(x)}{\tilde{\delta}\phi(0)} \tilde{\Theta}(y) \right\rangle \right\}. \end{aligned} \quad (5)$$

The difference $\Delta a - \Delta a'$ can be expressed in many equivalent ways [5]. Here I have reported the simplest formula. Sum rules for Δc can be derived also, but they are more complicated.

In formulas (5), the tildes mean that certain terms proportional to the field equations have been eliminated. Denoting with φ the dynamical fields of the theory, with conformal weight h , the notation $\tilde{\delta}/\tilde{\delta}\phi$ stands for the derivative with respect to ϕ at constant $\tilde{\varphi} \equiv \varphi e^{h\phi}$. In particular, $\tilde{\Theta} = -\tilde{\delta}S/\tilde{\delta}\phi$, where S denotes the action. It is understood that in (5) ϕ is set to zero, after taking the ϕ -derivatives of $\tilde{\Theta}$.

Osterwalder-Schrader (OS) positivity [6] can be applied to the sum rule for $\Delta a'$, and implies $\Delta a' \geq 0$. This is not precisely what we want, since a' is ill-defined at criticality by the addition of an arbitrary constant and $\Delta a'$ is not a flow invariant. OS positivity is ineffective on the sum rules for Δa , so that it is not immediate to derive $\Delta a \geq 0$ from (5).

In [7, 8], I have proposed a solution to this puzzle, which I now recall. An immediate observation is that the problem is solved in the class of flows which satisfy

$$\Delta a \geq f \Delta a', \quad (6)$$

where f is a non-negative numerical factor. We would like to understand if this class of theories is sufficiently interesting. The method of [2, 3] to compute Δa and Δc in supersymmetric theories does not allow us to compute $\Delta a'$, so we have to search for new arguments. It is helpful to treat, at a time, the case of generic even dimension $d = 2n$, so as to include, in particular, two-dimensional quantum field theory, which satisfies Zamolodchikov's c -theorem [9]. The trace anomaly at criticality in even dimensions contains three types of terms constructed with the curvature tensors and their covariant derivatives: i) terms \mathcal{W}_i , $i = 0, 1, \dots, I$, such that $\sqrt{g}\mathcal{W}_i$

are conformally invariant; *ii*) the Euler density

$$G_d = (-1)^n \varepsilon_{\mu_1 \nu_1 \dots \mu_n \nu_n} \varepsilon^{\alpha_1 \beta_1 \dots \alpha_n \beta_n} \prod_{i=1}^n R_{\alpha_i \beta_i}^{\mu_i \nu_i};$$

iii) covariant total derivatives \mathcal{D}_j , $j = 0, 1, \dots, J$, having the form $\nabla_\alpha J^\alpha$, J^α denoting a covariant current. The coefficients multiplying these terms in the trace anomaly are denoted with c_d^i , a_d and $a_d^{j'}$, respectively. We write $c_d = c_d^0$ and $a_d' = a_d^{0'}$. We have

$$\begin{aligned} \Theta_{d=2}^* &= \frac{1}{24\pi} c_2 R, & \Theta_{d=4}^* &= \frac{1}{120} \frac{1}{(4\pi)^2} \left[c_4 W^2 - \frac{a_4}{4} G_4 + \frac{2}{3} a_4' \square R \right], \\ \Theta_{d=2n}^* &= \frac{n!}{(4\pi)^n (d+1)!} \left[\frac{c_d (d-2)}{4(d-3)} \mathcal{W}_0 + \sum_{i=1}^I c_d^i \mathcal{W}_i - \frac{2^{1-n}}{d} \left(a_d G_d + \sum_{j=0}^J a_d^{j'} \mathcal{D}_j \right) \right]. \end{aligned} \quad (7)$$

Here c_2 is a_2 . \mathcal{W}_0 is the unique term of the form $W \square^{n-2} W + \dots$ such that $\sqrt{g} \mathcal{W}_0$ is conformally invariant, where the dots denote cubic terms in the curvature tensors. We choose a basis in which the \mathcal{W}_i s with $i = 1, \dots, I$ are at least cubic in the curvature tensors. I have separated \mathcal{W}_0 from the other \mathcal{W}_i s, because its coefficient c_d is also the coefficient of the stress-tensor two-point function (see [10] for details) and is normalized so that for free fields (n_s real scalars, n_f Dirac fermions and n_v $(n-1)$ -forms) it reads

$$c_d = n_s + 2^{n-1} (d-1) n_f + \frac{d!}{2[(n-1)!]^2} n_v. \quad (8)$$

The covariant total derivative term

$$\mathcal{D}_0 = -\frac{2^n d}{2(d-1)} \square^{n-1} R,$$

needs to be singled out among the \mathcal{D}_j s, since it is the unique term linear in the curvature tensors. We choose a basis such that all the \mathcal{D}_j s, $j > 0$, are at least quadratic in the curvature tensors. Then, on conformally flat metrics, \mathcal{D}_0 contains the unique term linear in ϕ . Proceeding as for the first formula of (5), the sum rule

$$\Delta a_d' = \frac{\pi^n (d+1)}{n!} \int d^d x |x|^d \langle \Theta(x) \Theta(0) \rangle \quad (9)$$

is derived [10], so that $\Delta a_d' \geq 0$ in arbitrary even dimensions.

The covariant total derivatives \mathcal{D}_j are in one-to-one correspondence with the arbitrary finite local terms that can be added to Γ , and the coefficients $a_d^{j'}$ have ambiguities similar to the ambiguity (4) of the coefficient a' in four dimensions. In [8] the \mathcal{W}_i s and the \mathcal{D}_j s are studied explicitly in $d = 6$ and the \mathcal{D}_j s in $d = 8$.

We first observe that, in two dimensions, on conformally flat metrics, the trace anomaly at criticality is linear in the conformal factor,

$$\sqrt{g} \Theta_{d=2}^* = \frac{1}{24\pi} c_2 \sqrt{g} R = -\frac{1}{12\pi} c_2 \square \phi. \quad (10)$$

It can be shown that Zamolodchikov's c -theorem follows directly from this fact. Indeed, in two dimensions c_2 plays also the role of a'_2 , for which a sum rule similar to the first of (5) can be derived [11], implying $\Delta c_2 \geq 0$. It is interesting to investigate under which conditions $\sqrt{g}\Theta^*$ is linear in ϕ on conformally flat metrics in arbitrary even dimension. We then discover [8] that there exists a "pondered" modification of the Euler density

$$\tilde{G}_d = G_d - \frac{2^n d}{2(d-1)} f_d^0 \square^{n-1} R + \sum_{j=1}^J f_d^j \mathcal{D}_j = G_d + \nabla_\alpha J_d^\alpha, \quad (11)$$

such that on conformally flat metrics

$$\sqrt{g}\tilde{G}_d = 2^n d f_d^0 \square^n \phi.$$

To fix the weight f_d^0 , we compute the Euler characteristics of a d -dimensional sphere S^d (equal to 2) with \tilde{G}_d and G_d , and match the results [8]. I have distributed appropriate numerical factors, so far, such that the outcome of this calculation is $f_d^0 = 1$. This procedure fixes the relative normalization of a and a' . More involved calculations are necessary to fix the other f_d^j s. In [8] complete expressions of \tilde{G}_d are written in $d = 6$ and $d = 8$.

On conformally flat metrics, the terms \mathcal{W}_i drop and the trace anomaly has the form

$$\Theta_{d=2n}^* = -N_d a_d^* \tilde{G}_d - N_d \sum_{j=0}^J \left(a_d^{j*'} - a_d^* f_d^j \right) \mathcal{D}_j, \quad (12)$$

where the overall (positive) numerical factor N_d is still unspecified. The convention for the relative normalization of c with respect to a and a' will be fixed later (in ref.s [8, 10] it was $N_d = 1$). In eq. (12) the stars refer to the values at criticality.

Using the RG-invariant ambiguities $\delta a'$ of the a' s, and the existence of \tilde{G}_d (11), we can arrange the a' s so that at one critical point, say the UV, the trace anomaly is linear in ϕ on conformally flat metrics, namely it has the form

$$\sqrt{g}\Theta_{d=2n}^{\text{UV}} = -N_d a_d^{\text{UV}} \sqrt{g}\tilde{G}_d = -2^n d N_d a_d^{\text{UV}} \square^n \phi, \quad (13)$$

precisely as in (10). This means, in particular, $a_d^{\text{UV}} = a_d^{\text{UV}}$. Once this choice is made, however, the a' s are fixed at all energies. In particular, it is not obvious that the trace anomaly has the form (13) also at the IR fixed point, i.e.

$$\sqrt{g}\Theta_{d=2n}^{\text{IR}} = -N_d a_d^{\text{IR}} \sqrt{g}\tilde{G}_d = -2^n d N_d a_d^{\text{IR}} \square^n \phi, \quad (14)$$

nor that $a_d^{\text{IR}} = a_d^{\text{IR}}$. Let us assume that this happens. Then, we have the equality in (6) with $f = 1$. The $\Delta a'_d$ -sum rule (9) is promoted to a Δa_d -sum rule and the RG flow is irreversible. Since the normalization factor N_d is still unfixed, we find

$$\Delta a_d = \Delta a'_d = \frac{1}{2^{3n-1} d \Gamma(d+1) N_d} \int d^d x |x|^d \langle \Theta(x) \Theta(0) \rangle. \quad (15)$$

We have to understand when a_d and a'_d can be identified at both critical points of the RG flow, i.e. when $a_d^{\text{UV}} = a_d^{\text{UV}}$ implies $a_d^{\text{IR}} = a_d^{\text{IR}}$. I begin listing a few facts.

– Explicit calculations show that free massive fields have $\Delta_{a_d} \neq \Delta_{a'_d}$. For example, the free massive scalar and the free massive Dirac fermion in $d = 4$ have

$$\Delta_{a_{\text{scal}}} = \frac{1}{3}, \quad \Delta_{a'_{\text{scal}}} = \Delta_{c_{\text{scal}}} = 1, \quad \Delta_{a_{\text{ferm}}} = \frac{11}{3}, \quad \Delta_{a'_{\text{ferm}}} = \Delta_{c_{\text{ferm}}} = 6.$$

We see that $\Delta_{a'}$ and Δ_c coincide in these cases. We will have more to see about this coincidence later.

– In perturbation theory, $\Delta a - \Delta a' = 0$ to some non-trivial loop orders in classically conformal theories in $d = 4$ [7] and $d = 6$ [8].

It is therefore necessary to distinguish two main classes of quantum field theories:

i) the classically conformal theories. They contain only marginal deformations, that is to say no dimensionful parameter other than the dynamically generated scale, which I denote with μ . Massless QCD belongs to this class, as well as the conformal windows of [2]. The RG flow is “pure”, i.e. not contaminated by the effects of classical scales.

ii) theories which are not conformal at the classical level, because they contain relevant deformations, e.g. masses.

I claim that $\Delta a - \Delta a'$ is exactly zero in classically conformal theories, i.e. that the pure RG flow is irreversible and satisfies (15). In [7, 8] I showed that this property can be derived from the following statement about the dependence of the quantum action Γ on the conformal factor:

S. *The quantum action for the conformal factor $\Gamma[\phi]$ is bounded from below (in the Euclidean framework) at all energies if it is bounded from below at some energy.*

The proof that **S** implies $\Delta a = \Delta a'$ in classically conformal theories can be found in [7] and is not repeated here for reasons of space.

S is suggested by the following considerations:

– A statement like **S** holds for the dependence of Γ on the dynamical fields. This is the requirement that the quantum action have a minimum in the space of physical fields.

– A statement like **S** does not hold, in general, for the dependence of Γ on external sources, because divergences generate arbitrarily negative terms. These terms can be normally reabsorbed in the running coupling constants, but when the sources are external there are no such constants (or need to be introduced anew).

– The unique known case in which Γ is convergent even in the presence of external sources is when the external source is the conformal factor ϕ and the theory is classically conformal, because Θ is an evanescent operator. The rigorous derivation of the convergence of $\Gamma[\phi]$ at arbitrary energies is in [5]. We conclude that **S** is expected to hold for $\Gamma[\phi]$ in classically conformal theories.

Let us summarize the evidence in favor of this theory of the irreversibility of the RG flow. It explains the distinction between the two classes *i)* and *ii)* mentioned above: $\Gamma[\phi]$ is not convergent in the presence of masses. It predicts the existence of the “pondered” Euler density \tilde{G}_d (11) and explains its physical meaning. The predictions of this theory agree with the perturbative calculations made so far, to four loops in $d = 4$ and $d = 6$. There exists a physical argument for the validity of this theory to all orders. It predicts and explains the existence of a remarkable class of theories, having $c = a$ (see below), singled out also by independent arguments. I conclude with the discussion of this last point.

In two dimensions, there is no distinction between the classes *i)* and *ii)*. This suggests that in even dimensions greater than two there exists a subclass of flows where there is no distinction

$i)$ – $ii)$ either. This subclass, in particular, should have $\Delta a_d = \Delta a'_d$ and contain also theories with masses and classical scales.

To figure out how this subclass looks like, let us study the simplest properties of massive flows. In particular, if we compute the integral of (15) for free massive scalars and fermions in arbitrary dimension, we discover that the ratio between the values of the integral for scalars and fermions equals the ratio between the scalar- and fermion-values of c_d in (8):

$$\int d^d x |x|^d \langle \Theta(x) \Theta(0) \rangle = \frac{\Delta c_d n!}{\pi^n (d+1)}, \quad \text{i.e.} \quad \Delta c_d = \frac{2^{3n-1}}{n!} \pi^n d (d+1)! N_d \Delta a'_d.$$

This fact is known to hold by explicit computation, but a complete understanding is still lacking: see [1] for more details. We conclude that there exists a class of classically nonconformal theories where Δc_d is related to $\Delta a'_d$. We can fix the normalization factor N_d such that this class has precisely $\Delta c_d = \Delta a'_d$, i.e.

$$N_d = \frac{n!}{2^{3n-1} \pi^n d (d+1)!}.$$

Recapitulating, we have identified two main classes of flows, so far:

- 1) the flows which have $\Delta a_d = \Delta a'_d$;
- 2) the flows which have $\Delta c_d = \Delta a'_d$.

We have seen that the theories $i)$ satisfy 1) and that the flows 2) contain theories of $ii)$. Therefore, we can argue that the subclass of flows where classical conformality is violated but the equality $\Delta a_d = \Delta a'_d$ still holds, is the class of flows with $\Delta c_d = \Delta a_d$. In particular, we have $\Delta c_d = \Delta a_d$ when both the UV and IR fixed points have $c_d = a_d$. If we plug this relation in the expression (7) of the trace anomaly at criticality, we single out the combination

$$\mathcal{G}_d = G_d - \frac{2^{n-3} d(d-2)}{d-3} \mathcal{W}_0 + \sum_{i=1}^I h_d^j \mathcal{W}_i, \quad (16)$$

where h_d^j are yet-unspecified numerical factors. The conformal field theories with $c_d = a_d$ have a trace anomaly proportional to \mathcal{G}_d up to the \mathcal{D}_j s. The combination \mathcal{G}_d is also known from arguments completely independent of our considerations about the irreversibility of the RG flow. First, in $d = 4$ we have

$$\mathcal{G}_4 = G_4 - 4W^2 = -8R_{\mu\nu}^2 + \frac{8}{3}R^2.$$

The square of the Riemann tensor drops out in the difference. An example of theory with $c = a$ in $d = 4$ is $N = 4$ supersymmetric Yang-Mills theory, whose peculiarity is well-known. The combination (16) in six dimensions was pointed out in [12] and in arbitrary even dimensions by the authors of [13]. Actually, the construction of [12, 13] fixes uniquely also the h_d^i s. Finally, the structure of \mathcal{G}_d is uniquely characterized as pointed out in [10], namely \mathcal{G}_d has the form

$$\mathcal{G}_d = R_{\mu\nu} \mathcal{T}_d^{\mu\nu\rho\sigma} R_{\rho\sigma},$$

where $\mathcal{T}_d^{\mu\nu\rho\sigma}$ is a local covariant differential operator of dimension $d - 4$, constructed with the curvature tensors, their covariant derivatives, and the covariant-derivative operator ∇ , acting both on the left and on the right.

To conclude, I have formulated a theory of the irreversibility of the RG flow which leads to the claim that

$$\Delta a_d = \frac{\pi^n (d+1)}{n!} \int d^d x |x|^d \langle \Theta(x) \Theta(0) \rangle \quad (17)$$

in the classically conformal quantum field theories and in the flows with $\Delta c_d = \Delta a_d$. Irreversibility ($\Delta a_d \geq 0$) is then a consequence of reflection positivity. The sum rules (5) and (17) apply also to non-unitary flows [1, 14]. An appealing graphical interpretation of formula (17) is that Δa_d is proportional to the scheme-invariant area of the graph of the beta function between the fixed points [7].

As a byproduct, I have found a number of other results which are important steps towards the classification of all conformal theories and renormalization-group flows in arbitrary even dimensions greater than two.

Further checks of the predictions of my theory can be made using the universal sum rules (5). One possibility is to test the equality of Δa and $\Delta a'$ to the fifth or higher loop orders in four-dimensional classically conformal theories, using the second sum rule of (5). Work is in progress in this direction. Another possibility is to study the formulas (5) and (17) on the lattice. This can provide knowledge about the low-energy limit of QCD. For example, the existence of a mass gap in QCD could be tested computing the difference $\Delta a = a_{UV} - a_{IR}$ using (5) and knowing that a_{UV} is given by (3), while $a_{IR} = 0$. In massless QCD, it should be possible to test the expected, non-trivial value of a_{IR} , proportional to the number of pions, and so have an indirect check that quarks and gluons generate (the right number of) pions at low energies.

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