01A3 Renorm

IFUP-TH/01-28

A NOTE ON THE IMPROVEMENT AMBIGUITY OF THE STRESS TENSOR AND THE CRITICAL LIMITS OF CORRELATION FUNCTIONS

D. Anselmi
Dipartimento di Fisica, Università di Pisa, via F. Buonarroti 2, 56126 Pisa, Italia

Abstract

I study various properties of the critical limits of correlators containing insertions of conserved and anomalous currents. In particular, I show that the improvement term of the stress tensor can be fixed unambiguously, studying the RG interpolation between the UV and IR limits. The removal of the improvement ambiguity is encoded in a variational principle, which makes use of sum rules for the trace anomalies a and a'. Compatible results follow from the analysis of the RG equations. I perform a number of self-consistency checks and discuss the issues in a large set of theories.

1 Introduction

In a large set of models, the renormalization-group (RG) flow interpolates between well-defined ultraviolet (UV) and infrared (IR) fixed points, the zeros of the beta function. The RG interpolation can be studied comparing the UV and IR limits of a certain class of correlators. Finite operators play a special role in this context, since they define central charges in the conformal limits.

Among the finite operators, noticeable are the conserved currents, in particular the stress tensor $T_{\mu\nu}$. When the theory contains scalar fields φ, there exists an improvement operator

$$\Delta T_{\mu\nu} = (\partial_\mu \partial_\nu - \Box \delta_{\mu\nu})\varphi^2,$$

which mixes with $T_{\mu\nu}$ under renormalization. It is possible to diagonalize this mixing [1, 2], and this makes the improvement term finite as well. There exists a one-parameter family of finite, conserved, spin-2, dimension-4 operators $T_{\mu\nu}(\eta) = T_{\mu\nu} + \eta\Delta T_{\mu\nu}$. At the level of the Lorentz commutator algebra, the operators $T_{\mu\nu}(\eta)$ are equivalent. At the level of the correlation functions and operator-product expansions, they are not. For example, in a conformal field theory, the embedding in external gravity is fixed unambiguously by conformal invariance. This means that there is no improvement arbitrariness in the UV and IR limits. In most models, the RG equations for η extend the removal of the improvement arbitrariness from the critical points to the intermediate energies.

A universal principle for the removal of the improvement ambiguity can be formulated using the sum rules for the trace anomalies a and a' [3]. This is a sort of variational principle [4], which fixes a priviledged value $\bar{\eta}$ for η. We can distinguish two cases.

i) When the improvement term survives in a critical limit (typically, the UV), the value $\bar{\eta}$ determined by the variational principle coincides with the value fixed by conformal invariance at criticality and the RG equations. Matching the stress tensor at intermediate energies with its UV limit removes the η-arbitrariness at all energies.

ii) When the improvement term vanishes at the critical points, all operators $T_{\mu\nu}(\eta)$ are in principle equally acceptable, but the value $\bar{\eta}$ is still priviledged. Specifically, the minimum of $\Delta a'(\bar{\eta})$ over the flow trajectories connecting the same pair of fixed points is equal to Δc in a class of models [1].

There is a universal way to remove the η-arbitrariness and select a unique stress tensor, in accord with all present knowledge.

In this paper I study this issue and other properties of the critical limits of correlators. In section 2 I discuss the properties of the improvement term and list the criteria for the removal of the η-ambiguity. In section 3 I illustrate the statements in a set of gaussian models where calculations can be carreid over to the end. Then, I analyse the RG equations in IR-free and UV-free theories. In all cases the parameter η is fixed uniquely with the rules of section 2. In the appendix I discuss other aspects of the critical limits of correlators containing insertions of finite

[1]This relation is empirically known to hold in massive gaussian models, unitary and not unitary. Nevertheless, a satisfactory theoretical understanding of this relation is still lacking.

and non-finite operators. In particular, I show that anomalous currents and the topological-charge density are finite in various models.

2 Removal of the η-ambiguity

The UV and IR limits of correlators containing insertions of the trace of the stress-tensor have been studied systematically in ref. [3]. General sum rules for the central charges a and a' have been written. Particularly meaningful is the notion of flow invariant, that is to say a flow integral, or combination of flow integrals, whose value depends only on the extrema of the flow.

The theory is embedded in external gravity. The definition of a and a' (and c) from the trace anomaly at criticality reads

$$\Theta_* = \frac{1}{(4\pi)^2}\left[c\,W^2 - \frac{a}{4}\,\mathrm{G} + \frac{2}{3}\,a'\,\Box R\right],$$

where W is the Weyl tensor and $\mathrm{G} = 4R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 16R_{\mu\nu}R^{\mu\nu} + 4R^2$ is the Euler density. The background metric is specialized to be conformally flat, i.e. $g_{\mu\nu} = \delta_{\mu\nu}\mathrm{e}^{2\phi}$. The Θ-correlators are related to the ϕ-derivatives of the induced action for the conformal factor, which I denote with $\Gamma[\phi]$. At criticality, $\Gamma[\phi]$ depends only on the quantities a and a'. The sum rules for $\Delta a = a_{\mathrm{UV}} - a_{\mathrm{IR}}$ and $\Delta a' = a'_{\mathrm{UV}} - a'_{\mathrm{IR}}$ can be written studying the critical limits of the Θ-correlators.

In this paper, I study specifically two sum rules for Δa, taken from section 7 of [3]. The first formula involves integrals of the two- and three-point functions:

$$\begin{aligned}\Delta a = {} & \frac{\pi^2}{48}\int \mathrm{d}^4x\,|x|^4\,\langle\widetilde\Theta(x)\,\widetilde\Theta(0)\rangle \\ & + \frac{\pi^2}{48}\int \mathrm{d}^4x\,\mathrm{d}^4y\;x^2\,y^2\left\{\langle\widetilde\Theta(x)\,\widetilde\Theta(y)\,\widetilde\Theta(0)\rangle + \left\langle\frac{\widetilde\delta\widetilde\Theta(x)}{\widetilde\delta\phi(y)}\,\widetilde\Theta(0)\right\rangle + 2\left\langle\frac{\widetilde\delta\widetilde\Theta(x)}{\widetilde\delta\phi(0)}\,\widetilde\Theta(y)\right\rangle\right\}. \end{aligned} \qquad (2.2)$$

The second formula involves integrals of the two-, three- and four-point functions:

$$\begin{aligned}\Delta a = {} & \frac{\pi^2}{48}\int \mathrm{d}^4x\,|x|^4\,\langle\widetilde\Theta(x)\,\widetilde\Theta(0)\rangle + \frac{\pi^2}{48}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\;(x\cdot y)\,(x\cdot z)\,\Big\langle\widetilde\Theta(x)\,\widetilde\Theta(y)\,\widetilde\Theta(z)\,\widetilde\Theta(0) \\ & + 2\frac{\widetilde\delta\widetilde\Theta(x)}{\widetilde\delta\phi(y)}\,\widetilde\Theta(z)\,\widetilde\Theta(0) + \frac{\widetilde\delta\widetilde\Theta(x)}{\widetilde\delta\phi(0)}\,\widetilde\Theta(y)\,\widetilde\Theta(z) + \frac{\widetilde\delta\widetilde\Theta(y)}{\widetilde\delta\phi(z)}\,\widetilde\Theta(x)\,\widetilde\Theta(0) + 2\frac{\widetilde\delta\widetilde\Theta(y)}{\widetilde\delta\phi(0)}\,\widetilde\Theta(x)\,\widetilde\Theta(z) \\ & + \frac{\widetilde\delta^2\widetilde\Theta(x)}{\widetilde\delta\phi(y)\,\widetilde\delta\phi(z)}\,\widetilde\Theta(0) + 2\frac{\widetilde\delta^2\widetilde\Theta(x)}{\widetilde\delta\phi(y)\,\widetilde\delta\phi(0)}\,\widetilde\Theta(z) + \frac{\widetilde\delta^2\widetilde\Theta(y)}{\widetilde\delta\phi(z)\,\widetilde\delta\phi(0)}\,\widetilde\Theta(x) \\ & + 2\frac{\widetilde\delta\widetilde\Theta(x)}{\widetilde\delta\phi(y)}\,\frac{\widetilde\delta\widetilde\Theta(z)}{\widetilde\delta\phi(0)} + \frac{\widetilde\delta\widetilde\Theta(x)}{\widetilde\delta\phi(0)}\,\frac{\widetilde\delta\widetilde\Theta(y)}{\widetilde\delta\phi(z)}\Big\rangle. \end{aligned} \qquad (2.3)$$

The notation is as follows. If φ denotes generically the dynamical fields of the theory, with conformal weight h, then the $\widetilde\delta/\widetilde\delta\phi$-derivatives are the derivatives with respect to ϕ at constant

$\widetilde{\varphi} \equiv \varphi\, \mathrm{e}^{h\phi}$. We have $\widetilde{\Theta} = -\widetilde{\delta} S/\widetilde{\delta}\phi$, where S denotes the action. It is understood that, after taking the ϕ-derivatives of $\widetilde{\Theta}$, ϕ is set to zero.

I also study the $\Delta a'$-sum rule

$$\Delta a' = \frac{\pi^2}{48} \int \mathrm{d}^4 x\, |x|^4\, \langle \widetilde{\Theta}(x)\, \widetilde{\Theta}(0) \rangle. \tag{2.4}$$

The central charge a in unambiguous at criticality, but a' is ambiguous. This ambiguity disappears in the difference $\Delta a'$, which is a physical quantity. These facts have important implications in the context of flow invariance and the dependence on the improvement ambiguity.

We can evaluate the above flow integrals using the one-parameter family of stress tensors $T_{\mu\nu}(\eta)$. Two situations can occur.

If the improvement term of the stress tensor does not vanish at both critical points, some non-trivial functions of η are generated, which depend also on the sum rule. I denote this dependence with a subscript i and write $\Delta_i a(\eta)$. Formulas (2.2) and (2.3) define the functions $\Delta_1 a(\eta)$ and $\Delta_2 a(\eta)$, respectively. Formula (2.4) defines $\Delta a'(\eta)$. Since, however, a is unambiguous at criticality, there must be a priviledged value of η which resolves the ambiguity and reproduces the correct Δa. This value can be found studying the RG equations for the parameter η, imposing conformal invariance at the critical points.

If the improvement term vanishes at both critical points, all values of η are in principle acceptable. The functions $\Delta_i a(\eta)$ do not depend on i and η and are identically equal to Δa. Instead, since a' has no unambiguous definition at criticality, the function $\Delta a'(\eta)$ can depend on η. We know from ref. [4] that the value $\bar{\eta}$ at which $\Delta a'(\eta)$ is minimum has particularly interesting properties. Using this, we can remove the improvement ambiguity also in this case.

Both situations are resolved by a universal criterion for the removal of the improvement ambiguity, encoded in a variational principle studied in [4], which expresses the independence of the flat-space theory from the embedding in external gravity. When both this principle and the analysis of the RG equations fix η, the results agree.

Criterion for the removal of the improvement ambiguity. Determine the (unique) $\bar{\eta}$ which satisfies

$$\left.\frac{\mathrm{d}\Delta a'(\eta)}{\mathrm{d}\eta}\right|_{\eta=\bar{\eta}} = 0, \qquad \left.\frac{\mathrm{d}\Delta_i a(\eta)}{\mathrm{d}\eta}\right|_{\eta=\bar{\eta}} = 0. \tag{2.5}$$

The functions $\Delta_i a(\eta)$ and $\Delta a'(\eta)$ are at most quadratic in η (this will be shown explicitly in the next section[2]), so the condition (2.5) has one solution for every sum rule. The solution $\bar{\eta}$ does not depend on the sum rule. The correct stress tensor is $T_{\mu\nu}(\bar{\eta})$ and the correct value of Δa is $\Delta_i a(\bar{\eta})$, independently of i. This criterion fixes also $\Delta a'$ unambiguously.

The integrals of (2.2) and (2.3) are assured to converge, when there is no improvement ambiguity. When the stress tensor admits improvement terms, instead, there can be a divergence in $\Delta a'(\eta)$. This divergence provides alternative criteria for the removal of the η-ambiguity (see

[2]In even dimension greater than four, the η-polynomials can have a higher degree. I am grateful to G. Festuccia for this remark.

below). If, on the other hand, the Θ-correlators are expanded perturbatively, the convergence of the term-by-term integration is not assured. Observe that the resolution of the η-ambiguity is intrinsically non-perturbative. A useful perturbative expansion can be defined, although computations are not simple.

Shortcuts and other criteria to remove the η-ambiguity. The value $\bar{\eta}$ does not depend on the sum rule and so it can be determined from the simplest of those, i.e. $\Delta a'(\eta)$. The Δa-sum rules involve more complicated flow integrals. In various cases, $\bar{\eta}$ can be fixed by conformal invariance at the critical points. In the nest sections I study the criteria for the removal of the improvement ambiguity in a variety of models. These cover essentially all cases. We can have the following behaviors:

i) the RG equations imply that the improvement term of $T_{\mu\nu}(\eta)$ survives at one of the critical points, where however the stress tensor is uniquely fixed by conformal invariance;

ii) the RG equations imply that the improvement term of $T_{\mu\nu}(\eta)$ diverges at one of the critical points and the divergence disappears if η is chosen appropriately;

iii) the improvement term vanishes at the critical points, but not sufficiently quickly. The quantity $\Delta a'$, should be finite, because it is physically meaningful (although it is not a flow invariant [4]). The finiteness of $\Delta a'(\eta)$ can fix η. This can also be seen as a consequence of (2.5).

In all cases, the $\bar{\eta}$ fixed with the criteria (*i*), (*ii*) and (*iii*) coincides with the $\bar{\eta}$ of (2.5). In the next section I present checks of this.

The forth situation is when the improvement term disappears quickly enough at both critical points. When this happens, we have $\Delta_i a(\eta) = \Delta a$ for every i and η. The variational principle (2.5) applies also to this case, in the sense that it outlines a noticeable value $\bar{\eta}$, such that $\Delta a'(\bar{\eta})$ has the properties studied in [4]. This behavior is studied in a higher-derivative model.

Modified, η-independent sum rules. Following [4], the criterion (2.5) is equivalent to the η-independence of more complicated sum rules. This illustrates that the removal of the $\bar{\eta}$-ambiguity fixed by (2.5) is compatible with the fact that the quantum field theory in flat space is independent of the non-minimal couplings to external gravity.

We proceed as follows. Using the fact that $\Delta_i a(\eta)$ is at most quadratic in η, we write

$$\Delta_i a(\eta) = \Delta_i a(\tilde{\eta}) + (\eta - \tilde{\eta}) \left.\frac{\mathrm{d}\Delta_i a(\eta)}{\mathrm{d}\eta}\right|_{\tilde{\eta}} + \frac{1}{2}(\eta - \tilde{\eta})^2 \left.\frac{\mathrm{d}^2\Delta_i a(\eta)}{\mathrm{d}\eta^2}\right|_{\tilde{\eta}}. \tag{2.6}$$

The right-hand side is clearly independent of $\tilde{\eta}$.

Finding $\bar{\eta}$ according to (2.5), inserting it in (2.6) and renaming $\tilde{\eta} \to \eta$, we get

$$\Delta a = \Delta_i a(\bar{\eta}) = \Delta_i a(\eta) - \frac{1}{2}\frac{\left(\frac{\mathrm{d}\Delta_i a(\eta)}{\mathrm{d}\eta}\right)^2}{\frac{\mathrm{d}^2\Delta_i a(\eta)}{\mathrm{d}\eta^2}}. \tag{2.7}$$

The final expression is an involved combination of flow integrals. It can be seen as a generalized sum rule for Δa, in the spirit of the formulas of [4]. The result is clearly independent of η

and gives the correct value of Δa. In the generalized sum rule, we can chose for $\Delta_i a(\eta)$ any equivalent Δa-formula from ref. [3]; for example, (2.2) and (2.3) of the present paper. The i-independence of the result can be rephrased in terms of equivalence relations among the flow integrals. These involve correlators of Θ and the improvement operator.

3 Checks and illustrative examples

In this section I study various examples, starting from simplest case, namely the massive free scalar. A richer structure is exhibited by gaussian non-unitary theories, where the issue of flow invariance is more apparent. This model describes some qualitative features of physical theories with several independent masses or dimensioned parameters. Then, I consider the φ^4-theory and asymptotically-free theories, supersymmetric and non-supersymmetric. Finally, I comment on the most general case.

Massive scalar field. The action in external gravity is

$$S=\frac{1}{2}\int \mathrm{d}^4x\ \sqrt{g}\left\{g^{\mu\nu}\partial_\mu\varphi\ \partial_\nu\varphi+\eta R\varphi^2+m^2\varphi^2\right\}.$$

Focusing on the conformal factor ϕ and eliminating a total derivative, we can simplify the action and write

$$S=\frac{1}{2}\int \mathrm{d}^4x\left\{(\partial_\mu\widetilde{\varphi})^2+m^2\widetilde{\varphi}^2\mathrm{e}^{2\phi}+(1-6\eta)\,\widetilde{\varphi}^2\left(\Box\phi+(\partial_\mu\phi)^2\right)\right\}, \qquad (3.1)$$

where $\widetilde{\varphi}=\varphi\,\mathrm{e}^{\phi}$. We need

$$\widetilde{\Theta}=-\frac{\widetilde{\delta}S}{\widetilde{\delta}\phi}=-m^2\widetilde{\varphi}^2\mathrm{e}^{2\phi}+\frac{1}{2}(1-6\eta)\left[\Box\left(\widetilde{\varphi}^2\right)-2\partial_\mu\left(\widetilde{\varphi}^2\partial_\mu\phi\right)\right], \qquad (3.2)$$

where $\widetilde{\delta}$ is the ϕ-derivative at fixed $\widetilde{\varphi}$ (check [3] for definitions), and the first two derivatives of $\widetilde{\Theta}$ with respect to ϕ.

The calculations give

$$\Delta_1 a(\eta)=-\frac{89}{360}+3\eta-9\eta^2, \qquad \Delta_2 a(\eta)=-\frac{37}{180}+\frac{5}{2}\eta-\frac{15}{2}\eta^2.$$

The condition (2.5) gives the (expected) value $\bar{\eta}=1/6$ in both cases and $\Delta_1 a(\bar{\eta})=\Delta_2 a(\bar{\eta})=1/360=\Delta a$. It is well-known that the value $\bar{\eta}=1/6$ is such that the action (3.1) is conformal at $m=0$. The correct stress tensor can be fixed, more simply, by requiring that $\widetilde{\Theta}$ be zero in the UV limit. This is a check that the criterion (2.5) gives the same result as conformal invariance at the critical points, expressed by shortcut (i). On the other hand, $\Delta_1 a(\eta)$ and $\Delta_2 a(\eta)$ do not coincide for $\eta\neq\bar{\eta}$.

The coincidences of the values of $\bar{\eta}$ determined by $\Delta_{1,2}a(\eta)$ and the equality of $\Delta_{1,2}a(\bar{\eta})$ are non-trivial. They are due to identities among the flow integrals. An illustrative example is

$$m^2\int \mathrm{d}^4x\,\mathrm{d}^4y\ x^2\langle\varphi^2(x)\,\varphi^2(y)\,\varphi^2(0)\rangle=2\int \mathrm{d}^4x\ x^2\langle\varphi^2(x)\,\varphi^2(0)\rangle,$$

which is relevant for the calculation of $\Delta_1 a(\eta)$. This identity can be verified directly or seen as a consequence of dimensional counting (each integral has the form const.$/m^2$), combined with the property that an insertion of $\int \mathrm{d}^4x\, m^2\varphi^2(x)$ is equivalent to the derivative $-m\partial/\partial m$. A similar cancellation takes place in $\Delta_2 a(\eta)$. The variational principle (2.5) "knows" about such relations.

Let us now study $\Delta a'$. The explicit calculation shows that a coefficient is infinite. Precisely:

$$\Delta a'(\eta) = -\frac{3}{40} + \frac{1}{2}\eta + (1-6\eta)^2\,\infty.$$

In the Δa-sum rule (2.2), the infinite term is compensated by a contribution coming from the flow integral of $\langle \widetilde{\delta\Theta}/\widetilde{\delta\phi}\ \Theta\rangle$ and the sum is finite for each value of η. An analogous compensation occurs in (2.3). Observe that (2.5), applied to $\Delta a'$, still fixes $\bar{\eta}$ to 1/6, so that, correctly, $\Delta a'(\bar{\eta}) = 1/120$ [5, 4].

The quantity $\Delta a'$ is much less restricted than Δa. For example, it can depend on the flow connecting the two fixed points [4]. Still, it is a physically meaningful quantity and characterizes the flow. The infinity of $\Delta a'(\eta)$ is not a "divergence" to be removed. The correct value of $\Delta a'$ must be finite. In the theory at hand (but also in the φ^4-theory and other models discussed below), finiteness of $\Delta a'$ fixes $\bar{\eta}$. We have a check that the $\bar{\eta}$s fixed with shortcut (*iii*) and any of the (2.5) coincide.

Flow invariance. Examples of flows interpolating between the same fixed points are easy to construct. An as illustration, take non-Abelian Yang-Mills theory with group $G = SU(N_c)$, N_f massless quarks and M_f massive quarks in the fundamental representation. In the large N_c limit, with $N_f/N_c \lesssim 11/2$ fixed, the theory is UV-free and has an interacting IR fixed point. Indeed, at low energies, the massive fermions decouple and the beta function

$$\beta = -\frac{1}{6\pi}(11N_c - 2N_f)\,\alpha^2 + \frac{25}{(4\pi)^2}N_c^2\,\alpha^3 + \mathcal{O}(\alpha^4)$$

has a second zero. The higher-loop terms can be neglected in the given large-N_c limit.

The UV and IR fixed points do not depend on the values of the masses of the M_f massive quarks. For each set of values of the masses we have a different flow interpolating between the same conformal field theories.

At the computational level, it is not easy to study the sum rules (2.2) and (2.3) in this model. A treatable perturbative expansion of the flow integrals of (2.2) and (2.3) has still to be developed. Gaussian higher-derivative theories, on the other hand, provide an interesting laboratory of flows interpolating between the same fixed points. Calculations are still lengthy, but doable.

Higher-derivative scalar field. The lagrangian of the theory is

$$\mathcal{L} = \frac{1}{2}\left[(\Box\varphi)^2 + \beta m^2(\partial_\mu\varphi)^2 + m^4\varphi^2\right]. \tag{3.3}$$

The embedding in external gravity gives

$$\mathcal{L} = \frac{1}{2}\sqrt{g}\left(\varphi\Delta_4\varphi + \beta m^2(\partial_\mu\varphi)(\partial_\nu\varphi)g^{\mu\nu} + \eta R m^2\varphi^2 + m^4\varphi^2\right), \tag{3.4}$$

where the differential operator

$$\Delta_4 = \nabla^2\nabla^2 + 2\nabla_\mu \left[R^{\mu\nu} - \frac{1}{3}g^{\mu\nu}R\right]\nabla_\nu \tag{3.5}$$

is such that $\sqrt{g}\Delta_4$ is conformally invariant (see for example [6]) .

I do not consider non-minimal couplings of the form $R^2\varphi^2$. Their coefficients can be set to zero imposing $\Theta = 0$ at criticality, as in the previous example. The non-minimal coupling $m^2R\varphi^2$, instead, disappears both in the UV limit ($m \to 0$) and IR limit ($m \to \infty$ and $\varphi \to 0$, keeping the mass term $m^4\varphi^2$ bounded).

I perform two calculations, with (2.2) and (2.3). The relevant operator is

$$\overline{\Theta} = -\frac{\delta S}{\delta\phi} = -\beta m^2(\partial_\mu\varphi)^2\mathrm{e}^{2\phi} - 2m^4\varphi^2\mathrm{e}^{4\phi} + 3\eta m^2\left[\mathrm{e}^{\phi}\Box\left(\varphi^2\mathrm{e}^{\phi}\right) + \mathrm{e}^{\phi}\varphi^2\Box\mathrm{e}^{\phi}\right].$$

Tilded quantities are equal to untilded quantities in this model, since the canonical weight of the higher-derivative scalar field is zero.

Cubic and quartic terms in η do not contribute to (2.2) and (2.3). The improvement term (in $\widetilde{\Theta}$ and it its ϕ-derivatives) carries a $\Box$. Using integrations by parts, the boxes can be moved and act on the degree-4 polynomials x^2y^2 or $(x\cdot y)(x\cdot z)$. Three boxes kill the polynomials and therefore the integral. This observation implies that the condition (2.5) always has a unique solution.

The sum rules (2.2) and (2.3) give

$$\Delta_1 a(\eta) = \Delta_2 a(\eta) = -\frac{7}{90} = \Delta a,$$

independently of η. I recall that in this model, $a_{\mathrm{UV}} = -7/90$ [7] and $a_{\mathrm{IR}} = 0$.

The calculations, lengthy and cumbersome, have been done with Mathematica. I do not report here intermediate results, because they do not seem to be particularly instructive.

The flow invariance of $\Delta_i a(\eta)$ and the cancellation of the $\mathcal{O}(\eta)$-terms and $\mathcal{O}(\eta^2)$-terms in (2.2) and (2.3) are consequences of non-trivial identities among flow integrals. Each term of (2.2) and (2.3) separately violates these properties. As for the quantity $\Delta a'$, we have

$$\Delta a'(\eta) = \frac{1 + 17r^2 - 17r^4 - r^6 + 10(1 + r^2 + r^4 + r^6)\ln r}{40(r^2 - 1)^3} + \eta U(r) + \eta^2 V(r),$$
$$U(r) = -3r\frac{1 - r^4 + 2(1 + r^4)\ln r}{2(r^2 - 1)^3}, \qquad V(r) = 9r^2\frac{1 - r^2 + (1 + r^2)\ln r}{(r^2 - 1)^3},$$

where r is defined by $\beta = r + 1/r$. r is the unique dimensionless parameter of the theory, besides the improvement coefficient η. Since $\Delta a'(\eta)$ is finite for every η, none of the shortcuts of the previous section applies. All values of η are in principle acceptable, but the value of η which minimizes $\Delta a'(\eta)$ is priviledged, in the sense that it has various interesting properties, outlined in [4]. We have

$$\bar{\eta}(r) = -\frac{U(r)}{2V(r)} = \frac{1 - r^4 + 2(1 + r^4)\ln r}{12r\left(1 - r^2 + (1 + r^2)\ln r\right)}$$

and [4]

$$\Delta a'(\bar{\eta}) = -\frac{(r^2-1)^2(3r^4-26r^2+3)+(r^8+18r^6-18r^2-1)\ln r^2 - 10r^2(r^4+1)\ln^2 r^2}{40(r^2-1)^3(r^2\ln r^2+\ln r^2-2r^2+2)}.$$

The value $\Delta a'(\bar{\eta})$ depends on r, which means that it not a flow invariant. Its minimum coincides with $\Delta c = -1/15$ [4].

In conclusion, the "good" stress tensor of the theory (3.3) is

$$\begin{aligned} T_{\mu\nu} = & -\partial_\nu \Box\varphi\,\partial_\mu\varphi - \partial_\mu\Box\varphi\,\partial_\nu\varphi + 2\Box\varphi\,\partial_\mu\partial_\nu\varphi + \frac{2}{3}\partial_\mu\partial_\nu\partial_\alpha\varphi\,\partial_\alpha\varphi - \frac{4}{3}\partial_\mu\partial_\alpha\varphi\,\partial_\nu\partial_\alpha\varphi \\ & +\delta_{\mu\nu}\left[\frac{1}{3}\partial_\alpha\Box\varphi\,\partial_\alpha\varphi + \frac{1}{3}\left(\partial_\alpha\partial_\beta\varphi\right)^2 - \frac{1}{2}\left(\Box\varphi\right)^2 - \frac{m^4}{2}\varphi^2\right] + \beta m^2\left(\partial_\mu\varphi\,\partial_\nu\varphi - \frac{\delta_{\mu\nu}}{2}(\partial_\alpha\varphi)^2\right) \\ & -\bar{\eta}(r)m^2\left(\partial_\mu\partial_\nu - \Box\delta_{\mu\nu}\right)\varphi^2. \end{aligned}$$

and does not contain any more parameters than the flat-space action (3.3).

The φ^4-theory. The renormalization mixing between the stress tensor and its improvement term in the φ^4-theory has been studied in detail by Brown and Collins [1] and Hathrell [2]. In the formulas below, the dimensional regularization technique and the minimal subtraction scheme are understood.

The parameter η satisfies the inhomogeneous RG equation [2]

$$\mu\frac{\mathrm{d}\eta}{\mathrm{d}\mu} - \delta(\lambda)\eta = \beta_\eta(\lambda) \equiv -\delta(\lambda)d(\lambda). \tag{3.6}$$

Here $\delta(\lambda)$ is the anomalous dimension of the composite operator φ^2, while $d(\lambda)$ is determined by the simple pole in the φ^4-$\Box\varphi^2$ renormalization mixing. Precisely,

$$\frac{\mu^{4-n}[\varphi^4]}{4!} = \frac{(n-4)}{\hat{\beta}}\left\{\frac{\lambda_0\varphi_0^4}{4!} - \frac{\gamma}{n-4}[\mathrm{E}] - \frac{d+L_d}{n-4}\Box[\varphi^2]\right\},$$

where n is the space-time dimension, [E] is the φ-field equation, γ is the φ-anomalous dimension, $\hat{\beta} = (n-4)\lambda + \beta(\lambda)$, $\beta(\lambda)$ is the beta function, and L_d denotes the poles higher that the simple one. The subscript 0 denotes bare quantities, and the square brakets denote renormalized operators. The trace of the stress tensor reads in four dimensions

$$\widetilde{\Theta} = -\beta\frac{[\varphi^4]}{4!} - \gamma[\mathrm{E}] + (\eta - d)\Box[\varphi^2].$$

The equation (3.6) can be decomposed in the following way:

$$\eta = \widetilde{\eta}(\lambda) + \eta' v(\lambda),$$

where η' is finite ($\mu\mathrm{d}\eta'/\mathrm{d}\mu = 0$), $\widetilde{\eta}$ is a particular solution of (3.6), fixed conventionally so that $\widetilde{\eta}(0) = 0$, and v satisfies the homogeneous equation:

$$v(\lambda) = \exp\left(\int^\lambda \frac{\delta(\lambda')}{\beta(\lambda')}\mathrm{d}\lambda'\right). \tag{3.7}$$

It is not necessary to specify the second extremum of integration, which can be absorbed in the factor η'. The function $v(\lambda)$ is related to the renormalization constant of the operator φ^2.

The surviving finite constant η' parametrizes the stress-tensor ambiguity, which reads

$$T_{\mu\nu}(\eta') = T_{\mu\nu}(0) - \frac{1}{3}\eta' v(\lambda)(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)[\varphi^2]. \tag{3.8}$$

It follows immediately from (3.7) that if $T_{\mu\nu}(0)$ is finite ($\mu\, \mathrm{d}T_{\mu\nu}(0)/\mathrm{d}\mu = 0$), then $T_{\mu\nu}(\eta')$ is also finite.

In [1] it was observed that η' can be consistently set to zero. In [2] it was remarked that η' should be fixed "by experiment", since it is the coefficient of the non-minimal coupling to external gravity. Here we want to see if there is a reason why η' should be set a priori to a particular value.

There is strong evidence that the φ^4-theory is non-perturbatively trivial. Even if we cannot view this theory as an RG interpolation between a UV and a IR fixed point, we can make a couple of general observations, which apply also to more general cases studied below.

To the lowest order, we have [2]

$$\delta(\lambda) = \frac{\lambda}{(4\pi)^2} + \mathcal{O}(\lambda^2), \qquad \beta(\lambda) = 3\frac{\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3), \qquad \beta_\eta(\lambda) = -\frac{1}{36}\frac{\lambda^4}{(4\pi)^8},$$

so that

$$\eta = -\frac{1}{288}\frac{\lambda^3}{(4\pi)^6} + \mathcal{O}(\lambda^4) + \eta'\lambda^{1/3}\left(1 + \mathcal{O}(\lambda)\right).$$

Let us consider the flow integral (2.4), which defines $\Delta a'(\eta)$. In the absence of information about the UV, we can study the convergence of this integral around the IR limit. Using the perturbative values given above and the Callan-Symanzik equations for the pair of operators $(\varphi^4, \Box\varphi^2)$, the behavior of the integral around the IR is

$$\Delta a'(\eta) \sim \int^\infty \mathrm{d}t \left(\frac{a_1}{t^4} + \eta'\frac{a_2}{t^{10/3}} + \eta'^2\frac{a_3}{t^{2/3}}\right),$$

where the a_i are numerical factors. We see that the $\mathcal{O}(\eta'^2)$-contribution diverges in the IR extremum of integration. Since the integrand is non-negative, this divergence cannot be cured by contributions from intermediate energies or by a hypothetical second fixed point (which exists in the models studied below, to which similar considerations apply). Therefore, the only value compatible with a finite $\Delta a'$ is $\eta' = 0$.

This case is different from the case of a free-massive scalar field. Here the improvement term of (3.8) does disappear at criticality (at a velocity $\lambda(t)^{1/3}$, where $\lambda(t) \sim 1/t$, $t = \ln|x|\mu)$, but it does not disappear sufficiently quickly for the sum rule to converge. This forces η' to be zero, by shortcut (*iii*).

The φ^4-interaction can be non-trivial in several models, which may admit conformal windows. In particular, interesting cases are the supersymmetric models, with or without superpotential. Supersymmetry is not necessary to the logic of the arguments below, but it simplifies the examples.

N=1 supersymmetric QCD. I consider now N=1 supersymmetric QCD with gauge group $G = SU(N_c)$ and N_f quark and antiquark superfields in the fundamental representation. The theory has no superpotential and a unique coupling constant g. For $N_f < 3N_c$ the theory is asymptotically free. The mass operator $\overline{\varphi}\varphi$, which is essential for the improvement term, is the lowest component of the Konishi superfield [8]. Since the axial currents have no anomalous dimension at the one-loop order (see the appendix), this is true also of $\overline{\varphi}\varphi$. The two-loop contribution to the anomalous dimension δ can be found in [9]:

$$\delta(g) = 4\left(N_c^2 - 1\right) N_f \left(\frac{g^2}{16\pi^2}\right)^2 + \mathcal{O}(g^6).$$

The structure of the RG equation for η and the η'-ambiguity of the stress tensor are the same as in (3.6) and (3.8). In particular, the function v is given by the analogue of (3.7). The one-loop beta function is $\beta = -g^3\left(3N_c - N_f\right)/(16\pi^2) + \mathcal{O}(g^5)$, so that the function

$$v(g) = \exp\left(-\frac{g^2}{8\pi^2}\frac{N_f(N_c^2-1)}{(3N_c - N_f)} + \mathcal{O}(g^4)\right) \tag{3.9}$$

tends to unity at $g \to 0$. Instead, $\beta_\eta(g)$ goes to zero at least as fast as g^8. We conclude that $\eta \to \eta'$ in the UV limit, so that the improvement term of (3.8) survives at criticality. As in the case of the free massive scalar field, this forces η' to be set to zero, by shortcut (i).

Supersymmetric theories with a superpotential. The superpotential gives a one-loop contribution to the anomalous dimension of the Konishi operator. An example of UV-free supersymmetric theory with superpotential and a well-defined IR fixed point is the theory obtained adding mesonic fields M_i^j to the N=1 supersymmetric QCD. The meson superfields interact with the quarks q_i and $\bar{q}^j$ by means of a superpotential $fM_j^i q_i \bar{q}^j$ [10, 11]. In complete generality, denoting the superpotential couplings by Y_{ijk}, the one-loop anomalous dimension of the mass operator $\bar{\varphi}\varphi$ is

$$\delta(Y) = \frac{3}{16\pi^2}|Y|^2, \tag{3.10}$$

$|Y|$ being defined by $Y_{ijk}Y^{ijl} = |Y|^2\delta_k^l$. Solving the RG equations around the UV fixed point [10] and applying (3.7), we have

$$v \sim |t|^c, \tag{3.11}$$

for $t \to -\infty$, with c positive numerical constant. The situation is even worse than in the previous model, where the superpotential was absent: the improvement term of the stress tensor diverges in the free-field limit. This forces again to set $\eta' = 0$, by shortcut (ii).

Asymptotically-free theories and flows with interacting UV fixed points. The arguments of the previous two cases apply to the most general asymptotically-free theory with scalar fields, supersymmetric or not. The anomalous dimension of the improvement term can have a vanishing one-loop contribution or a non-vanishing one-loop contribution. In either case, its first radiative correction is positive. On the other hand, the first term of the beta function is negative. Then, $v(t)$ behaves as in (3.9) or (3.11). In the free-field limit, the improvement term of the stress tensor is finite and non-vanishing or divergent. This fixes η'.

The same can be said of flows with interacting UV fixed points, where $\delta_{\rm UV}$ can be non-vanishing. We can conclude, in full generality, that in unitary models the UV behavior of $T_{\mu\nu}(\eta')$ is unacceptable, unless the η'-term of (3.8) is suitably fixed, according to the rules of sect. 2.

4 Conclusions

A proper RG interpolation between the UV and IR fixed points removes the improvement ambiguity of the stress tensor. The general criterion for this removal is encoded in the variational principle (2.5). In various cases suitable shortcuts can be more efficient. I have analysed concrete examples, one for each relevant situation. In particular, in asymptotically-free theories, the RG equations imply that the improvement term survives at one critical point or diverges there, unless the improvement parameter is suitably fixed. In IR free theories, the improvement term does disappear at criticality, but not sufficiently quickly. The behavior of gaussian higher-derivative theories shows that there are cases in which all improved stress tensors are in principle acceptable. Nevertheless, the criterion (2.5) outlines a priviledged stress tensor also in this case. In conclusion, we can always consistently remove the improvement ambiguity with the rules of section 2.

5 Appendix. Other remarks about the critical limits of correlators

In this appendix I consider other ambiguities in the critical limits of correlators. Let us assume that two observers study the same model using different renormalization schemes. We want to know what amount of information the observers can objectively compare and how many quantities they need to normalize before the comparison. I consider correlators of composite operators and distinguish the cases of finite and non-finite operators.

Let $\mathcal{O}$ be a multiplicatively renormalized operator. The Callan-Symanzik equations imply that the two-point function can be written in the form

$$\langle \mathcal{O}(x)\, \mathcal{O}(0)\rangle = \frac{1}{|x|^{2d}} Z^2\left(\alpha(1/|x|), \alpha(\mu)\right) G(\alpha(1/|x|)), \tag{5.12}$$

where d is the canonical dimension of $\mathcal{O}$, Z is the renormalization constant and $\alpha(\lambda)$ is the running coupling constant at the energy scale λ. Now, in the UV (respectively, IR) limit, namely $|x| \to 0$ ($|x| \to \infty$), $\alpha(1/|x|)$ tends to the critical value $\alpha_{\rm UV\,(IR)}$. If $\mathcal{O}$ is not finite (i.e. $Z \neq 1$), then the limit depends on $\alpha(\mu)$ and the subtraction scheme. The values $\alpha_{\rm UV\,(IR)}$ are themselves scheme dependent.

Among the finite operators, we distinguish conserved currents, anomalous (classically conserved) currents and others. Suppressing the space-time indices, if $\mathcal{O}$ is a conserved current, the UV and IR limits of (5.12) have the structure

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle_{\mathrm{UV\,(IR)}} \sim \frac{G(\alpha_{\mathrm{UV\,(IR)}})}{(x^2)^d}. \tag{5.13}$$

The quantities $G(\alpha_{\mathrm{UV\,(IR)}})$ (called primary central charges [12]) carry information about the conformal fixed points. The scheme dependence of $\alpha_{\mathrm{UV\,(IR)}}$ is compensated by an equal and opposite scheme dependence of G, so that $G(\alpha_{\mathrm{UV\,(IR)}})$ is scheme independent. Similar considerations extend to correlators with more insertions. When the two observers compare their results, they have to find the same answer.

Classically-conserved anomalous currents can be finite. In an asymptotically-free theory, for example, the renormalization constant Z_5 of the axial current J_5^μ resums non-perturbatively to a finite function $C(\alpha)$. Finiteness can be formally recovered multiplying J_5^μ by $C^{-1}(\alpha)$. Then, in (5.12) the renormalization constant can be non-perturbatively replaced by unity. Scalar operators can be finite as well. An example is the topological-charge density. Details are given below. The function $G(\alpha(1/|x|))$ tends to a constant in the free-field limit and behaves like a power of $x^2\mu^2$ around the interacting critical limit. Formula (5.13) is upgraded to the more general expression

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle_{\mathrm{UV\,(IR)}} \sim \frac{G_{\mathrm{UV\,(IR)}}}{(x^2)^d(x^2\mu^2)^{h_{\mathrm{UV\,(IR)}}}}. \tag{5.14}$$

Here the critical limits are unambiguous once the scale μ is normalized (μ plays the role of the RG invariant scale, e.g. Λ_{QCD}). Two observers can compare their results, once they agree on the definition of the reference scale. It is possible to define "secondary" central charges [12], where the μ-normalization is simplified away.

Finally, the critical limits of correlators containing insertions of non-finite operators provide one piece of information less [12], since a non-finite operator needs to be normalized at some reference energy. In (5.12) this is emphasized by the $\alpha(\mu)$-dependence surviving in the limits $|x| \to 0$ and $|x| \to \infty$.

These observations apply to operators whose correlators have power-behaved critical limits. Logarithmic behaviors are not unfrequent, however. The improvement term of the stress tensor often exhibits a logarithmic behavior: check the $\bar{\varphi}\varphi$-two-point function in *a*) the φ^4-theory around the IR and *b*) supersymmetric theories with superpotential around the UV (see (3.11)).

Anomalous currents. Anomalous currents can be finite operators, and therefore have unambiguous critical limits, of the form (5.14). This paragraph extends a discussion of Collins [13] to singlet currents and the topological-charge density.

I consider the axial current in an asymptotically-free gauge theory. I assume that the current is conserved at the classical level. The inclusion of mass terms is straightforward. The anomaly equation

$$\partial_\mu J_5^\mu - \frac{g^2 N_f}{16\pi^2} F\widetilde{F} = \bar{\psi}\gamma_5 \frac{\delta_l S}{\delta\bar{\psi}} + \frac{\delta_r S}{\delta\psi}\gamma_5\psi = \text{finite} \tag{5.15}$$

and the definition $[J_5^\mu] = Z_5 J_5^\mu$ imply the relations

$$[\partial_\mu J_5^\mu] = Z_5 \partial_\mu J_5^\mu, \qquad\qquad \frac{g^2 N_f}{16\pi^2}[F\widetilde{F}] = (Z_5 - 1)\,\partial_\mu J_5^\mu + \frac{g_0^2 N_f}{16\pi^2} F\widetilde{F}$$

Calling $\mathcal{O}_1 = \partial_\mu J_5^\mu$ and $\mathcal{O}_2 = g^2 F\widetilde{F}/(16\pi^2)$, we have

$$[\mathcal{O}_i] = Z_{ij}\mathcal{O}_j, \qquad Z_{ij} = \begin{pmatrix} Z_5 & 0 \\ Z_5 - 1 & 1 \end{pmatrix}.$$

Consider the two-point function $\langle [J_5]^\mu(x)\, [J_5]^\nu(0)\rangle$. At the one-loop order it has a conformal-invariant form, namely

$$\langle [J_5^\mu](x)\, [J_5^\nu](0)\rangle = A(g^2)\frac{\delta^{\mu\nu} - 2x^\mu x^\nu/x^2}{(x^2)^{3+\delta_5(g^2)}} + \mathcal{O}(g^4). \tag{5.16}$$

The one-loop conformal invariance is assured by the Callan-Symanzik equations. Indeed, the conformal-violating term in

$$\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\delta_5(g^2)$$

is $\beta\,\partial/\partial g$. Since $\beta = \mathcal{O}(g^3)$, this term is irrelevant at the one-loop order.

Taking two divergences of (5.16), using the anomaly equation (5.15) and excluding the coincident point, we get

$$\langle [\partial_\mu J_5^\mu](x)\ [\partial_\nu J_5^\nu](0)\rangle = \frac{g^4 N_f^2}{(4\pi)^4}\left\langle [F\widetilde{F}](x)\ [F\widetilde{F}](0)\right\rangle = -4A(g^2)\frac{\delta_5(g^2)\left(2+\delta_5(g^2)\right)}{(x^2)^{4+\delta_5(g^2)}}.$$

Since $A(g^2) = \mathcal{O}(1)$, we conclude $\delta_5(g^2) = \mathcal{O}(g^4)$. This result is unaffected by the presence of masses or other super-renormalizable parameters, but does not hold when the conservation of J_5^μ is violated at the classical level by marginal operators, such as in supersymmetric theories with a superpotential: see (3.10).

Now, we observe that the renormalization constant Z_5 has a finite limit when the cut-off is sent to infinity. We can see this using the dimensional-regularization technique, but it is more explicit to write the limit in the familiar cut-off notation. Precisely,

$$\lim_{\Lambda\to\infty} Z_5(g(\Lambda), g(\mu)) = \lim_{\Lambda\to\infty} \exp\left(-\int_{g(\mu)}^{g(\Lambda)} \frac{\delta_5(g')}{\beta(g')}\mathrm{d}g'\right) = C(g^2) = \text{finite}.$$

This property holds because in an asymptotically-free theory, $g(\Lambda)$ tends to zero when $\Lambda \to \infty$. The integral is convergent around zero, because $\delta_5(g^2) = \mathcal{O}(g^4)$ and $\beta(g) = \mathcal{O}(g^3)$.

The full matrix Z_{ij} has a finite limit $C_{ij}(g^2)$. Using the Callan-Symanzik equations, we conclude that the operators $J_5^{\mu R} \equiv C^{-1}(g^2)[J_5^\mu]$ and $\mathcal{O}_i^R \equiv C_{ij}^{-1}(g^2)[\mathcal{O}_j]$ have two-point functions of the form

$$\langle J_5^{\mu R}(x)\, J_5^{\nu R}(0)\rangle = \frac{A(g^2(1/|x|))\delta_{\mu\nu} + B(g^2(1/|x|))\, x^\mu x^\nu/x^2}{(x^2)^3}, \qquad \langle \mathcal{O}_i^R(x)\, \mathcal{O}_j^R(0)\rangle = \frac{A_{ij}(g^2(t))}{(x^2)^4},$$

and admit unambiguous critical limits, as in (5.14).

Acknowledgements

I am grateful to G.C. Rossi for discussions on the contents of the appendix, G. Festuccia for useful remarks, CERN for hospitality during the early stage of this work and MIT for hospitality during the final stage of this work.

References

[1] L.S. Brown and J.C. Collins, Dimensional renormalization of scalar field theory in curved space-time, Ann. Phys. (NY) 130 (1980) 215.

[2] S.J. Hathrell, Trace anomalies and $\lambda\phi^4$ theory in curved space, Ann. Phys. (N.Y.) 139 (1982) 136.

[3] D. Anselmi, Kinematic sum rules for trace anomalies, hepth/0107194.

[4] D. Anselmi, A universal flow invariant in quantum field theory, hep-th/0101088. To appear in Class. and Quantum Grav.

[5] D. Anselmi, Towards the classification of conrofmal field theories in arbitrary even dimensions, Phys. Lett. B 476 (2000) 182 and hep-th/9908014.

[6] R.J. Riegert, A non-local action for the trace anomaly, Phys. Lett. B 134 (1984) 56.

[7] I. Antoniadis, P. Mazur and E. Mottola, Conformal symmetry and central charges in four dimensions, Nucl. Phys. B 388 (1992) 627 and hep-th/9205015.

[8] D. Anselmi, M.T. Grisaru and A.A. Johansen, A critical behavior of anomalous currents, electric-magnetic universality and CFT_4, Nucl. Phys. B 491 (1997) 221 and hep-th/9601023.

[9] S.P. Martin and M.T. Vaughn, Two-loop renormalization group equations for soft supersymmetry-breaking couplings, Phys. Rev. D 50 (1984) 2282 and hep-ph/9311340.

[10] I.I. Kogan, M.A. Shifman and A.I. Vainshtein, Matching conditions and duality in N=1 susy gauge theories in the conformal window, Phys. Rev. D 53 (1996) 4526, Erratum-ibid. D 59 (1999) 109903, and hep-th/9507170.

[11] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, Universality of the operator product expansions of SCFT in four dimensions, Phys Lett B 394 (1997) 329 and hep-th/9608125.

[12] D. Anselmi, Central functions and their physical implications, JHEP 9805:005 (1998) and hep-th/9702056.

[13] J.C. Collins, *Renormalization*, Cambridge University Press, Cambridge 1984.